Equilibrium prices in economies with indivisible goods and money.

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Abstract The aim of the paper is twofold. The first more theoretical part provides necessary and sufficient conditions for the existence of stable price equilibria in economies with indivisible goods, money and quasi-linear utility functions. It is an extension of a result of Bikhchandani and Mamer (1997). It also gives a necessary and sufficient condition for a core allocation to be a price equilibrium. The second part of the paper is more algorithmic. It gives a method to compute equilibrium prices and to check the conditions of the first part. It uses the 'nucleolus' concept, a well known solution rule for TU-games.

Key words: (stable) price equilibrium, stochastic redistribution, regular price vector, nucleolus.

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Introduction

Among economists it is widely believed that "finite TU-games do not matter for economic theory". In this paper, however, we prove that the nucleolus, a major solution concept in the theory of TU-games, can be used very elegantly to determine equilibrium prices and core allocations in economies with indivisible goods and money and—what is called—quasi-linear utility functions

The economies we consider in this paper have the following features:

- There is a finite set of *agents* N, $n = |N|$ and $i, j, k, \ldots \in N$.
- There is a finite set of *indivisible goods* Q, $q = |Q|$ and $\alpha, \beta, \gamma, \ldots \in Q$.
- Each agent $i \in N$ has an *initial endowment* (A_i, m_i); $A_i \subseteq Q$ denotes the set of indivisible goods initially hold by agent i and $m_i \geq 0$ is the amount of money agent i has in the beginning. We assume that $\{A_i\}_{i \in N}$ is a *distribution* of Q i.e., $A_i \cap A_j = \emptyset$ whenever $i \neq j$, and $\bigcup_{i \in N} A_i = Q$. We allow, however, that $A_i = \emptyset$ for some agents i.
- Each agent $i \in N$ has a *preference relation* $\preceq_i$ on the set of *commodity bundles* (B, x) with $\emptyset \subseteq B \subseteq Q$ and $x \in I\!R_+$. We assume that $\preceq_i$ can be represented by a utility function U_i of the form

$$U_i(B, x) := V_i(B) + x \quad (i \in N, \emptyset \subseteq B \subseteq Q, x \in I\!R_+), \quad \text{(separability for money)}$$
$$V_i(B) \leq V_i(C) \text{ whenever } B \subseteq C \text{ and } i \in N, \quad \text{(monotononicity)}$$
$$V_i(\emptyset) = 0.$$

Comment: Separability for money is the most restrictive condition. In fact, it consists of four properties, namely

– the *separability per se* saying that $U_i(B, x) = V_i(B) + W_i(x)$ for some function W_i, defined on $I\!R_+$,

– the *strict monotonicity in money* saying that W_i is strictly monotonic,

– the possibility of *interpersonal comparison of utility* expressed by $W_i = W_j$, and finally

– the property that money can be used as a physical means to *transfer utility* because the marginal utility for money is constant.

By trading a *coalition* $T \subseteq N$ can realize any redistribution of the goods in $\bigcup_{j \in T} A_j$ and any redistribution of the money supply $\sum_{j \in T} m_j$. We call such a twofold redistribution $\{C_j, y_j\}_{j\in T}$ a *T-reallocation*. So, a T-reallocation $\{C_j, y_j\}_{j\in T}$ must satisfy

$\bigcup_{j\in T} C_j = \bigcup_{j\in T} A_j$ and $\sum_{j\in T} y_j = \sum_{j\in T} m_j$

As usual in general equilibrium theory, we investigate, in this paper, the existence of core allocations and stable price equilibria and the relations between these concepts. This subject has been studied by so many authors that one cannot even hope to give a complete overview. For that reason, we mention only papers in the same spirit as the present one. This means, papers in which conditions for existence or non-existence of core allocations or price equilibria are formulated in terms of the primitives of the model, initial endowments and utility functions (preferences).

Most of the early papers put severe restrictions on the utility functions and the initial money supply. The most common restrictions are the following:

(a) There is a bipartition in the set of agents, sellers and buyers.

(b) The agents have initially at most one good and have also need for at most one good. We will call this the *one-one-condition*

(c) Money supply is so *abundant* that every agent can buy whatever he likes for his reservation value, the highest price he is ever willing to pay i.e.,

$V_i(Q) \leq m_i$ (or $m_i + V_i(A_i)$) $\;(i \in N)$.

Shapley and Shubik (1972) study the case satisfying bipartition, the one-one condition and abundance. In fact, money is not explicitly modeled but from the fact that an optimal assignment is never excluded because of lack of money, one may conclude that no agent has ever problems to 'pay the bill'.

In Henry (1970) all indivisible goods are substitutes of each other ($V_i(B) = f_i(b)$ where $b = |B|$) and f_i is a concave function.

In Gale (1984) and Quinzii (1984) we find the existence of price equilibria in economies satisfying the one-one-condition and abundance.

Wako (1986) proves the equivalence of strong core allocations and price equilibria in economies with the one-one-condition. He allows for scarcity of money. In the last three papers the utility functions are not necessarily quasi-linear (but continuously increasing in money).

The existence of price equilibria is also known in the case of additive reservation values i.e. when $V_i(B) = \sum_{\alpha\in B} V_i(\alpha)$ and an abundance of money. In such an economy each indivisible good can be sold separately by a second price auction.

Kelsó and Crawford (1982) designed a mechanism that generates equilibrium prices if the reservation values $\{V_i(C)\}_{C\subseteq Q}$ of each agent i satisfy the so called *gross substi-*

tutability property. This means, loosely speaking, that a good α in demand by player i under price p remains in demand by player i, if the price p is increased but p_α remains the same.

In a recent manuscript Beviá, Quinzii and Silva (1997) consider economies with an abundance of money ($V_i(Q) \leq V_i(A_i) + m_i$ for every agent i) and reservation prices of the form $V_i(B) = \sum_{\alpha \in B} V_i(\alpha) + f_i(b)$ where the functions f_i are concave functions of $b = |B|$. They prove that every redistribution $\{B_i\}_{i \in N}$ that *maximizes social welfare* is a price equilibrium and that the set of equilibrium prices form a complete lattice. The economies they consider satisfy the gross substitutability condition of Kelsó and Crawford.

Kaneko (1982) considers an economy with bipartition and the one-one condition (and later on the many-one condition). His utility functions are more general than quasi-linear. Furthermore, he requires that $U_i(\emptyset, m_i) > U_i(C, 0)$ for all $C \neq \emptyset$. His *interpretation* of this requirement is different, more psychological: 'agents do not like to spend all their income'. But formally, it is the abundance condition. The author proves the equivalence of core allocations and price equilibria and the existence of such allocations.

Finally, Bikhchandani and Mamer (1997) investigate general economies with indivisible goods, money and quasi-linear utility functions. They give a *necessary and sufficient* condition for the existence of price equilibria under the assumption of abundance. Their results cover most of the contents of our Theorems 1 and 2. We will, however, assume a much weaker form of abundance.

From this overview one can learn that the existence of price equilibria requires two conditions:

(1) a condition on the *reservation prices* $\{V_i(C)\}_{i \in N, C \subseteq Q}$ and

(2) a condition on the *money supply*.

Because of the separability of the utility functions for money, these conditions can be handled a good deal separately, as we shall see. The first condition will not depend on the initial endowments (we will call this the *social wellfare condition* or *SW-condition*) but the second condition (this will be called the *abundance condition* or *AB-condition*, for short) will do.

Before we can formulate these conditions we need some terminology.

An N-redistribution $\{B_i\}_{i \in N}$ *maximizes social welfare* or *is efficient* if $\sum_{i \in N} V_i(B_i)$ is maximal among all N-redistributions. The maximal social welfare is denoted by

$\mathcal{SW}_{\max}(\mathcal{E})$.

The next two concepts will be key concepts in our analysis.

A *stochastic redistribution* consists of a set of numbers $y_{i,C} \geq 0$, one for each agent i in N and each subset C of Q, with the property that $\sum_{C \subseteq Q} y_{i,C} = 1$ for all $i \in N$ and $\sum_{C:\alpha \in C} \sum_{i \in N} y_{i,C} = 1$ for every commodity $\alpha \in Q$. So, a stochastic redistribution is nonnegative solution of the vector equation $\displaystyle\sum_{i,C} y_{i,C}\,(e_i \oplus e_C) = e_N \oplus e_Q$.

Here e_i and e_C are the characteristic vectors of $\{i\} \subset N$ and $C \subseteq Q$ and $\oplus$ denotes the direct sum: $e_i \oplus e_C \in \mathbf{R}^N \oplus \mathbf{R}^Q$.

The numbers $\{y_{i,C}\}_{C \subseteq Q}$ can be understood as a lottery for agent i. The number $y_{i,C}$ is the chance that agent i obtains bundle C. The second condition says that the probability that object α is assigned to one of the agents is also one. Note that the integer-valued stochastic redistributions are exactly the N-redistributions: each agent obtains with probability 1 a bundle C_i and $\{C_i\}_{i \in N}$ is a redistribution.

Expected social welfare realized by the stochastic redistribution $\{y_{i,C}\}_{C \subseteq Q, i \in N}$ is, by definition, $\sum_{i \in N} \sum_{C \subseteq Q} y_{i,C}\, V_i(C)$.

Now we can formulate the **SW-condition**:

An economy $\mathcal{E}$ satisfies the *SW-condition* if no stochastic redistribution has a higher expected social welfare than $\mathcal{SW}_{\max}(\mathcal{E})$.

As maximal expected social welfare is determined by the following linear program (LP):

max $\quad \sum_{i \in N} \sum_{C \subseteq Q} y_{i,C}\, V_i(C)$ under the conditions
$y_{i,C} \geq 0$ for $\emptyset \subseteq C \subseteq Q$ and $i \in N$,
$\sum_{C \subseteq Q} y_{i,C} = 1$ for all $i \in N$,
$\sum_{C:\alpha \in C} \sum_{i \in N} y_{i,C} = 1$ for all $\alpha \in Q$,

the SW-condition says that (LP) has an integer-valued optimal solution. Note that the SW-condition is not dependent on the initial endowments.

The **AB-condition** is a weaker form of the abundance condition we found in the literature.

An economy $\mathcal{E}$ satisfies the *AB-condition* if there is an N-redistribution $\{B_i\}_{i \in N}$ that maximizes social welfare and satisfies the inequalities $V_i(B_i) \leq V_i(A_i) + m_i$ $(i \in N)$.

The AB-condition depends on initial endowments. If e.g., the initial distribution of the indivisible goods $\{A_i\}_{i \in N}$ maximizes social welfare, it is even an empty condition.

The AB-condition stipulates that each agent has enough money to sell A_i for the price $V_i(A_i)$ (the lowest price for which he is willing to sell A_i) and to buy B_i for the price $V_i(B_i)$ (the highest price he is willing to pay for B_i). This makes clear that the AB-condition might be too restrictive. If the price for A_i is higher than $V_i(A_i)$ or the price of B_i is lower than $V_i(B_i)$, a smaller amount of money is sufficient. We will frequently use the phrase '$\{B_i\}_{i\in N}$ satisfies the AB-condition'.

Note that the AB-condition is much weaker than the usual abundance condition $V_i(C) \leq V_i(A_i) + m_i$ $(i \in N, C \subseteq Q)$ (or even $V_i(C) \leq m_i$) (see Bevià et al (1997), Bikhchandani and Mamer (1997)). These conditions are, in our opinion, unreasonably restrictive: *every* agent must be able to buy *all* indivisible goods for the highest price he is willing to pay.

The second key concept is that of a *regular price vector.*

A vector $p \in I\!R^Q$ is called a *regular price vector* if there is a vector $q \in I\!R^N$ such that (p,q) is an optimal solution of the dual linear program (LP)* that reads as:

> min $p(Q) + q(N)$ under the conditions
>
> $p(C) + q_i \geq V_i(C)$ for all $i \in N$ and $C \subseteq Q$.

The first result we will prove (Section 2) is that the AB-condition and the SW-condition are *sufficient* for the existence of price equilibria. We can even make a more explicit statement: for any N-distribution $\{B_i\}_{i\in N}$ satisfying the AB-condition and for any regular price vector p the N-reallocation $\{B_i, x_i\}_{i\in N}$ is a price equilibrium with equilibrium price p if $x_i{:}= p(A_i) + m_i - p(B_i)$ $\ (i \in N)$ (to satisfy the budget constraint). If the intial money supply is increased ($\bar{m}_i{:}= m_i + \Delta_i$ with $\Delta_i \geq 0$ for every agent $i \in N$), the AB-condition remains true and $\{B_i, \bar{x}_i = x_i + \Delta_i\}_{i\in N}$ gives also an equilibrium. We call such an equilibria—stable under an increase of the initial money supply—a *stable price equilibria.*

The second result in Section 2 says that the SW-condition is *necessary* for the existence of *stable* price equilibria, that every stable price equilibrium maximizes social welfare and equilibrium prices are regular price vectors.

The remaining part of Section 2 investigates the relation between core allocations of an economy $\mathcal{E}$ and the core elements of the TU-game associated with the economy $\mathcal{E}$ as well as the question which core elements are associated with price equilibria.

The second part of the paper (Section 3 and 4) looks for methods to find regular price vectors (potential equilibrium prices) and to check the SW-condition. In Section 3 we use the nucleolus concept from the theory of TU-games to compute regular price vectors. We will connect the problem to solve (LP)* with the computation of the nucleolus of the TU-game $(N \cup Q, W)$ with player set $N \cup Q$ and coalition values

$$W(T) := \max_{i \in T \cap N} V_i(T \cap Q) \text{ whenever } T \cap N \neq \emptyset \text{ and } W(T) = 0 \text{ if } T \cap N = \emptyset.$$

We will prove that, under very mild conditions, a regular price vector can be found by *two* nucleolus calculations, one in $(N \cup Q, W)$ and one in $(N \cup Q, \bar{W})$, a TU-game that differs from the first one in the value of the grand coalition only.
Because the nucleolus can only be calculated for moderate values of $|N| + |Q|$ and this number tends to be large, Section 4 investigates an alternative method to find a regular price vector. It is inspired by the way Solymosi and Raghavan (1994) compute the nucleolus of the assignment game of Shapley and Shubik (1972).

Finally, if we have a regular price vector p (or an optimal solution (p, q) of (LP)*), one can compute the set $\mathcal{B}$ of pairs (i, C) with $p(C) + q_i = V_i(C)$. The SW-condition is satisfied if and only if this collection $\mathcal{B}$ contains a N-redistribution.

1. Basic definitions and statement of the theorems.

Let us first repeat some definitions. Let $(B_i, x_i)_{i \in N}$ be an N-feasible reallocation.
A T-feasible reallocation $\{C_j, y_j\}_{j \in T}$ is a *weak improvement* (or *strict improvement*) upon $(B_i, x_i)_{i \in N}$ if $U_j(C_j, y_j) \geq U_j(B_j, x_j)$ for all $j \in T$ and at least one strict inequality (or $U_j(C_j, y_j) > U_j(B_j, x_j)$ for all $j \in T$).

As usual, we call an N-reallocation $(B_i, x_i)_{i \in N}$ *Pareto-optimal* (or *weakly Pareto-optimal*), if coalition N does not have a weak improvement (or strict improvement). It is called a *strong core allocation* (or *core allocation*) if no coalition T has a weak improvement (or a strict improvement). A weak improvement $\{C_j, y_j\}_{j \in T}$ can be transformed into a strict improvement, if there is an agent i^* with $V_{i^*}(C_{i^*}) + y_{i^*} > V_{i^*}(B_{i^*}) + x_{i^*}$ **and** $y_{i^*} > 0$. This is always the case if the abundance condition $V_i(Q) \leq V_i(A_i) + m_i$ $(i \in N)$ holds. In this case the difference between weak Pareto optimality/Pareto optimality and between core/strong core disappears.

An N-feasible reallocation $(B_i, x_i)_{i \in N}$ is a *price equilibrium*, if there exists a *price vector* $p \in \mathbb{R}^Q$ with the following properties:

$$p(B_i) + x_i \leq p(A_i) + m_i, \quad (i \in N) \qquad \text{(budget constraints)}$$

$$\text{if } U_i(C, y) > U_i(B_i, x_i) \text{ for some } C \subset Q \text{ and } i \in N, \text{ then } p(C) + y > p(A_i) + m_i. \qquad \text{(maximality conditions)}$$

By the strict monotonicity of the utility functions U_i, the maximality conditions imply that the budget constraints are, in fact, equalities. Furthermore, by the monotonicity of the reservation prices, an equilibrium price is nonnegative.

Definition: A N-reallocation $\{B_i, x_i\}_{i\in N}$ is a *stable price equilibrium* if there is a price vector $p \in I\!R^Q$ such that the reallocation $(\{B_i, x_i+\Delta_i\}_{i\in N}$ is a price equilibrium for equilibrium price p, if the initial money supply becomes $m + \Delta$ and $\Delta \in \mathbf{R}^N_+$.

For an economy $\mathcal{E}$ we define a TU-game $(N, v_{\mathcal{E}})$ with coalition values

$$v_{\mathcal{E}}(S) := \mathcal{SW}_{\max}(\mathcal{E}, S) = \max\{\textstyle\sum_{j\in S} V_j(C_j) : \{C_j\}_{j\in S} \text{ is an } S\text{-redistribution }\}.$$

So, $v_{\mathcal{E}}(S)$ is the maximal social welfare that can be realized in the *sub-economy* in which only the actions of agents in S are considered.

Before we formulate the first two theorems of this section we give two examples.

Example 1. Let $N = \{1, 2\}$ and $Q = \{\alpha, \beta\}$. Let $A_1 = \{\alpha\}$ and $A_2 = \{\beta\}$. The reservation values are *additive*

	α	β
1	10	70
2	20	50

It is easy to see that social welfare is optimized if the agents switch their endowments. The set of regular price vectors is $10 \le p_\alpha \le 20$ and $50 \le p_\beta \le 70$.

To support the redistribution $\beta \to 1$ and $\alpha \to 2$ by a regular price vector, player 1 has, after payment, $m_1 + p_\alpha - p_\beta \in m_1 + [-60, -30]$ and we see that *lack of money* may block the existence of regular equilibrium prices (if $m_1 < 30$) or may block some regular equilibrium prices (if $30 \le m_1 < 60$).

Let us consider the case that $m = (10, 20)$ and the price vector is $p = (15, 40)$. Then the reallocation $(\emptyset, 25) \to 1$ and $(\{\alpha, \beta\}, 5) \to 2$ is a price equilibrium that *does not maximize social welfare* and the equilibrium price is *not regular*. The better assignment $\beta \to 1$ and $\alpha \to 2$ cannot be realized because agent 1 does not have enough money to buy β.

Example 2. In the second example the reservation values $\{V_i(B)\}_{i\in N, B\subseteq Q}$ exclude the existence of equilibrium prices. The example comes from Bevià et al. (1997). $N = \{1, 2, 3\}$ and $Q = \{\alpha, \beta, \gamma\}$. The reservation values are given in the table and $A_1 = \{\alpha\}$, $A_2 = \{\beta\}$ and $A_3 = \{\gamma\}$.:

	α	β	γ	$\alpha\beta$	$\alpha\gamma$	$\beta\gamma$	$\alpha\beta\gamma$
1	10	8	2	13	11	9	13
2	8	5	10	13	14	13	15
3	1	1	8	2	9	9	10

The authors show that the unique social optimum $[\alpha \to 2, \beta \to 1$ and $\gamma \to 3]$ is not supported by regular equilibrium prices. The reason is that a *stochastic redistribution* has a higher value. If agent 1 obtains α and β each with chance $\frac{1}{2}$, agent 2 obtains γ or $\alpha\beta$ with equal chances and agent 3 obtains γ or $\emptyset$ each with chance $\frac{1}{2}$, the total *expected* utility is 24.5, higher than the social optimum 24. And indeed if we increase e.g. $V_2(\alpha)$ to 8.5, the price vector $p = (6.5, 4.5, 8)$ supports the socially optimal redistribution, if there is enough money ($m_2 \geq 2$ is sufficient). Note that, in the original economy (with $V_2(\alpha) = 8$) the social optimal redistribution $\beta \to 1$, $\alpha \to 2$ and $\gamma \to 3$ is a price equilibrium, if the prices are $(8, 6, 8)$ (not regular) and $m_2 = 2$ (not abundant).

Let us formulate the first two theorems of this section. In the next section we will give all the proofs.

Theorem 1. [cf. Bikhchandani and Mamer (1997)] *An exchange economy $\mathcal{E}$ with quasi-linear utility functions, indivisible goods and money has a price equilibrium if the SW-condition and the AB-condition are satisfied.*

In fact, we shall see that every redistribution $\{B_i\}_{i \in N}$ for which the AB-condition holds can be extended to a price equilibrium $\{B_i, x_i\}_{i \in N}$ and that the set of equilibrium prices consists of all regular price vectors.
In example 1 we have seen that an economy may have equilibria that do not maximize social welfare and that equilibrium prices need not be regular. These equilibria arise *from a lack of money*. They are *unstable* in the sense that a (sufficiently high) increase of the initial money supply upsets the equilibrium character of the reallocation.
In the following theorem we prove that the SW-condition is a necessary condition for the existence of *stable* price equilibria.

Theorem 2. [cf. Bikhchandani and Mamer (1997)] *If an economy $\mathcal{E}$ with quasi-linear utility functions, indivisible goods and money has a* stable *price equilibrium, then the SW-condition holds, every stable equilibrium allocation maximizes social welfare and every equilibrium price is regular.*

The third theorem of this section describes a relation between core allocations of $\mathcal{E}$ and the core elements of the TU-game $(N, v_{\mathcal{E}})$. In this theorem a stronger Abundance condition occurs, namely:

An economy $\mathcal{E}$ satisfies the Total Abundance condition if every coalition S has an redistribution $\{C_j\}_{j\in S}$ such that

$$\sum_{j\in S} V_j(C_j) = \mathcal{SW}_{\max}(\mathcal{E}, S) \text{ and } V_j(C_j) \leq V_j(A_j) + m_j \ (j \in S).$$

We say that a core element of $(N, v_{\mathcal{E}})$ is *realized* by a core allocation $\{B_i, x_i\}_{i\in N}$ if $z_i + m_i = V_i(B_i) + x_i$. The core allocation $\{B_i, x_i\}_{i\in N}$ gives the utility level $z + m$, the same as the core allocation.

Theorem 3. *If the AB-condition holds, then every core allocation $z \in \mathrm{Core}\,(N, v_{\mathcal{E}}))$ can be realized by a core reallocation $\{B_i, x_i\}_{i\in N} \in \mathrm{Core}\,(\mathcal{E})$. If the Total Abundance condition holds, then every core reallocation of $\mathcal{E}$ defines a core element of the TU-game $(N, v_{\mathcal{E}})$.*

In the last theorem of this series we give a necessary and sufficient condition that a core element of $(N, v_{\mathcal{E}})$ can be realized by a *stable price equilibrium.*

Theorem 4. *In an exchange economy $\mathcal{E}$ satisfying the AB-condition a core allocation $z \in \mathrm{Core}\,(N, v_{\mathcal{E}}))$ can be realized by a stable price equilibrium if and only if for every stochastic redistribution $\{y_{i,C}\}_{i\in N, C\subseteq Q}$ the following inequality holds:*

$$\sum_{i,C} y_{i,C}\, V_i(C) + \sum_{i\in N} y_{i,A_i}\,(z_i - V_i(A_i)) \leq \mathcal{SW}_{\max}(\mathcal{E}). \qquad (1)$$

If we compare inequality (1) with the inequality in the SW-condition, the simularity is striking. Only the numbers $V_i(A_i)$ $(i \in N)$ are replaced by the larger numbers z_i for $i \in N$.

2. The existence of price equilibria: necessary and sufficient conditions

We start by giving a proof of the fact that the SW-condition and the AB-condition guarantee the existence of price equilibria (Theorem 1).

Proof of Theorem 1: Let $\{B_i\}_{i\in N}$ be *any* redistribution of the indivisible goods that maximizes social welfare and with the property $V_i(B_i) \leq V_i(A_i) + m_i$ $(i \in N)$. Let (p, q) be *any* optimal solution of the linear program (LP)*:

$$\min\ p(Q) + q(N) \text{ under the conditions}$$
$$p \in I\!R^Q,\ q \in I\!R^N,$$
$$p(B) + q_i \geq V_i(B) \text{ for } B \subseteq Q \text{ and } i \in N.$$

Define $x_i := p(A_i) + m_i - p(B_i)$ for every agent $i \in N$.

By the SW-condition the integer-valued stochastic reallocation $y_{i,C} = 1$ if and only if $C = B_i$ and $y_{i,C} = 0$ if $C \neq B_i$, is an optimal solution of (LP). By complementary slackness, we find $p(B_i) + q_i = V_i(B_i)$ $(i \in N)$.

Since $p(A_i) \geq V_i(A_i) - q_i = V_i(A_i) + p(B_i) - V_i(B_i)$, we find $x_i := p(A_i) + m_i - p(B_i) \geq V_i(A_i) + m_i - V_i(B_i) \geq 0$ by the AB-condition. So, $x \geq 0$.

If $V_i(C) + y > V_i(B_i) + x_i$ for some commodity bundle $C \subseteq Q$, $y \geq 0$ and $i \in N$, then

$$p(C) + y \geq V_i(C) - q_i + y > V_i(B_i) + x_i - q_i = p(B_i) + x_i = p(A_i) + m_i.$$

The first inequality follows from the feasibility condition $p(C) + q_i \geq V_i(C)$. As this inequality is an equality for $C = B_i$, we find the third relation. The last equality follows from the definition of x_i. The N-reallocation $\{B_i, x_i\}_{i \in N}$ is a price equilibrium with equilibrium price p. ◁

Comments: If we reconsider the proof of Theorem 1, we see that *every* N-reallocation $\{B_i\}_{i \in N}$ satisfying the AB-condition and *every* regular price vector can be matched to a price equilibrium.

The SW-condition is also necessary for the existence of *stable* price equilibria (Theorem 2).

Proof of Theorem 2: Let $\{B_i, x_i\}_{i \in N}$ be a stable price equilibrium with equilibrium price p. Define $q_i := \max_{C \subseteq Q} [V_i(C) - p(C)]$ for each agent $i \in N$. The pair (p, q) is a feasible point of (LP)*.

We prove that $p(B_i) + q_i = V_i(B_i)$ for each agent $i \in N$.

Let $C \subseteq Q$ be any commodity bundle and let $i \in N$ be any agent. Let y be the real number $y := V_i(B_i) + x_i - V_i(C)$ and let δ be any positive number. Then $V_i(C) + y + \delta > V_i(B_i) + x_i$.

If $y \geq 0$, the maximality condition and the budget constraint generate the inequality $p(C) + y + \delta > p(A_i) + m_i = p(B_i) + x_i$. So, $p(C) + y \geq p(B_i) + x_i$. Substitution of y gives $p(C) + V_i(B_i) + x_i - V_i(C) \geq p(B_i) + x_i$ and therefore, $V_i(C) - p(C) \leq V_i(B_i) - p(B_i)$.

If y happens to be negative, we use the fact that $\{B_i, x_i + \Delta_i\}$ is also an equilibrium if the initial amount of money is $m + \Delta$. Then we define $y := V_i(B_i) + x_i + \Delta_i - V_i(C)$ and now $y \geq 0$ if Δ_i is large enough. We can proceed as before and find again $V_i(C) - p(C) \leq V_i(B_i) - p(B_i)$. So, $p(B_i) + q_i = V_i(B_i)$ for all $i \in N$.

Then we have a feasible point (p, q) of (LP)* and the *integer-valued* stochastic redistribution $y_{i,C} = 1$ if and only if $C = B_i$ $(i \in N)$ and if $y_{i,C} > 0$ we have an equality in the dual program. From complementary slackness follows that (p, q) is optimal in (LP)* (a regular price vector) and the *integer-valued redistribution* $\{y_{i,C}\}$ maximizes the linear program (LP) (the SW-condition). Finally the redistribution $\{B_i\}_{i\in N}$ maximizes social welfare. $\triangleleft$

Summarizing the results of Theorem 1 and 2 we find that the SW-condition is a necessary and sufficient condition for the existence of stable price equilibria, as soon as the money supply satisfies the AB-condition, a stable price equilibrium maximizes social welfare and equilibrium prices are regular price vectors. For *unstable* price equilibria the last two statements need not be true. In example 1 the equilibrium price $p = (15, 40)$ is not regular and the reallocation $\emptyset \to 1$ and $\{\alpha\beta\} \to 2$ does not maximize social welfare. Comparing this result with the results of Bikhchandani and Mamer (1997) we find the following difference. B & M assume the stronger AB-condition by which every efficient distribution satisfies our AB-condition. Under this assumption they prove the equivalence of the SW-condition and the existence of price equilibria.

Proof of Theorem 3: Let z be a core allocation of the TU-game $(N, v_{\mathcal{E}})$. Let $\{B_i\}_{i\in N}$ be an N-redistribution satisfying the AB-condition. We define $x_i := z_i + m_i - V_i(B_i)$ for each agent $i \in N$. Since z is a core element, we have $z_i \geq v_{\mathcal{E}}(i) = V_i(A_i)$ and $x_i \geq V_i(A_i) + m_i - V_i(B_i) \geq 0$ by the AB-condition. Then $\{B_i, x_i\}_{i\in N}$ is an N-reallocation because of $z(N) = v_{\mathcal{E}}(N) = \mathcal{SW}_{\max}(\mathcal{E}) = \sum_{i\in N} V_i(B_i)$.

Suppose T is a coalition and $\{C_j, y_j\}_{j\in T}$ is a T-reallocation and $V_j(C_j)+y_j \geq V_j(B_j)+x_j$ for all agents $j \in T$ and there is at least one strict inequality. Then

$$\sum_{j\in T}[V_j(C_j)+y_j] > \sum_{j\in T}[V_j(B_j)+x_j] = \sum_{j\in T}[z_j+m_j] \text{ by the definition of } x_j. \text{ Then}$$

$$v_{\mathcal{E}}(T) \geq \sum_{j\in T} V_j(C_j) > \sum_{j\in T} z_j = z(T) \text{ in contradiction with } z \in \mathrm{Core}(N, v_{\mathcal{E}}).$$

Conversely, let $\{B_i, x_i\}_{i\in N}$ be a core reallocation of $\mathcal{E}$ and define z_i by the equality $z_i + m_i = V_i(B_i) + x_i$. Then $z(N) = \sum_{i\in N} V_i(B_i)$. Suppose that $z(T) < v_{\mathcal{E}}(T)$ for some coalition T. Note that also $T = N$ is one of the possibilities. Because of the *Total AB-condition* there is a T-reallocation $\{C_j\}_{j\in T}$ with $\sum_{j\in T} V_j(C_j) = v_{\mathcal{E}}(T)$ and $V_j(C_j) \leq V_j(A_j) + m_j$.

From the definition of z_j and $z(T) < \sum_{j\in T} V_j(C_j)$ we infer

$$\sum_{j\in T}[V_j(B_j)+x_j] = \sum_{j\in T}[z_j+m_j] < \sum_{j\in T}[V_j(C_j)+m_j].$$

For every agent $j \in T$ there is a number y_j such that $V_j(C_j) + y_j = V_j(B_j) + x_j$ and by the core-relations for coalition $\{j\}$, we have

$$y_j = [V_j(B_j) + x_j] - V_j(C_j) = (z_j + m_j) - V_j(C_j) \geq [V_j(A_j) + m_j] - V_j(C_j) \geq 0.$$

Furthermore, $\sum_{j \in T} y_j < m(T)$. So, there is a number $\delta > 0$ such that $\{C_j, y_j + \delta\}_{j \in T}$ is a T-reallocation and an improvement upon $\{B_i, x_i\}_{i \in N}$. This is not possible and therefore, $z \in \text{Core}\,(N, v_{\mathcal{E}})$. $\triangleleft$

Proof of Theorem 4: Let z be a core element of $(N, v_{\mathcal{E}})$ and $\{B_i\}_{i \in N}$ an N-reallocation satisfying the AB-condition.

We prove that the core element z can be realized by a stable price equilibrium if and only if the following linear program has minimum value 0.

min $p(Q) - r(N)$ under the conditions
$p \in I\!R^Q$ and $r \in I\!R^N$,
$p(C) - r_i \geq V_i(C) - V_i(B_i)$ for $i \in N$ and $C \neq A_i$,
$p(A_i) - r_i \geq z_i - V_i(B_i)$ for $i \in N$.

If the linear program has minimum value 0, we have, for the case $C = B_i$, $p(B_i) - r_i = 0$ i.e., $r_i = p(B_i)$. So, we find $V_i(B_i) - p(B_i) \geq V_i(C) - p(C)$ for every agent $i \in N$ and every bundle $C \neq A_i$. For A_i we find $V_i(B_i) - p(B_i) \geq z_i - p(A_i) \geq V_i(A_i) - p(A_i)$. Then (p, q) with $q_i = V_i(B_i) - p(B_i)$ is an optimal point of (LP)*. If we take $x_i := z_i + m_i - V_i(B_i)$ we get $p(B_i) + x_i = z_i + m_i - V_i(B_i) + p(B_i) \leq p(A_i) + m_i$.
In fact the last inequality is an equality because of

$$p(A_i) - r_i \geq z_i - V_i(B_i), \quad z(N) = \textstyle\sum_{i \in N} V_i(B_i) \text{ and } p(Q) - r(N) = 0.$$

The proof of the maximality condition is the same as in the proof of Theorem 1. So we find a price equilibrium $\{B_i, x_i\}_{i \in N}$ and in fact a *stable* price equilibrium.

Conversely, if z is realized by a stable price equilibrium $\{B_i, x_i\}_{i \in N}$ and p is a (regular) equilibrium price, then $z_i + m_i = V_i(B_i) + x_i$ (z is realized by $\{B_i, x_i\}_{i \in N}$) and $p(B_i) + x_i = p(A_i) + m_i$ (the budget constraint) gives $p(A_i) - p(B_i) = z_i - V_i(B_i)$. Furthermore, p is a regular price vector and therefore, there exists a vector $q \in I\!R^N$ such that $p(C) + q_i \geq V_i(C)$ ($i \in N, C \subseteq Q$) and $p(B_i) + q_i = V_i(B_i)$. Then, $p(C) - p(B_i) \geq V_i(C) - V_i(B_i)$ for $i \in N$ and $C \neq A_i$. If we define $r_i := p(B_i)$ we have a feasible point of the linear program with $p(Q) - r(N) = 0$.

The duality theory of linear programming gives that z can be realized by a stable price equilibrium if and only if the dual program has maximal value 0. The dual program has the following form

$$\max \sum_{i\in N}\sum_{C\neq A_i} y_{i,C}\,[V_i(C)-V_i(B_i)] + \sum_{i\in N} y_{i,A_i}\,[z_i - V_i(B_i)] \text{ under the conditions}$$

$$y_{i,C} \geq 0,\ (i \in N, C \subseteq Q)$$

$$\sum_{i,C} y_{i,C}\,(e_i \oplus e_C) = e_N \oplus e_Q.$$

The Theorem follows from $\sum_{i,C} y_{i,C}\, V_i(B_i) = \sum_{i\in N} V_i(B_i) = \mathcal{SW}_{\max}(\mathcal{E})$. ◁

Example 2 (continued): If we take in example 2 the value of $V_2(\alpha) = 8.5$ (instead of 8), we have two stochastic redistributions maximizing social welfare, namely

$y_{1,\beta} = y_{2,\alpha} = y_{3,\gamma} = 1$ and all other $y_{i,C} = 0$ and

$\bar{y}_{1,\alpha} = \bar{y}_{1,\beta} = \bar{y}_{2,\gamma} = \bar{y}_{2,\alpha\beta} = \bar{y}_{3,\emptyset} = \bar{y}_{3,\gamma} = 0.5$ and all other $\bar{y}_{i,C} = 0$.

An optimal solution of (LP)* must have equalities where $y_{i,C} > 0$ or $\bar{y}_{i,C} > 0$. Then there is only one solution: $(p; q) = (6.5,\ 4.5,\ 8;\ 3.5,\ 2,\ 0)$.

The game $(N, v_{\mathcal{E}})$ has the values:

$v_{\mathcal{E}}(1) = 10,\ v_{\mathcal{E}}(2) = 5,\ v_{\mathcal{E}}(3) = 8,$

$v_{\mathcal{E}}(12) = 16.5,\ v_{\mathcal{E}}(13) = 18,\ v_{\mathcal{E}}(23) = 13,$

$v_{\mathcal{E}}(N) = 24.5.$

The core consists of the segment between (11.5, 5, 8) and (10, 6.5, 8) and only the second point can be realized by a price equilibrium.

3. Computation of equilibrium price vectors

This section and the subsequent section investigate the possibilities to check the SW-condition and to compute stable price equilibria and their equilibrium price vectors. In fact we start with the last problem, the computation of *regular price vectors* i.e., an optimal solution of (LP)*.

A reader, familiar with the theory of TU-games, will read the linear relations in (LP)

$$y_{i,C} \geq 0 \text{ and } \sum_{i,C} y_{i,C}\,(e_i \oplus e_C) = e_N \oplus e_Q$$

as a 'balancedness relation' and the relations in (LP)*

$$p(C) + q_i \geq V_i(C) \text{ for } i \in N \text{ and } C \subseteq Q$$

as 'core inequalities' for coalitions $\{i\} \cup C \subset N \cup Q$.

Therefore, it seems to be interesting to introduce a cooperative game with player set $N \cup Q$ and coalition values

$$W(T) := \max_{i\in T\cap N} V_i(T \cap Q) \text{ if } T \cap N \neq \emptyset \text{ and } W(T) = 0 \text{ if } T \cap N = \emptyset.$$

To solve (LP)* the computation of the prenucleolus of the game $(N \cup Q, W)$ will be helpful.

For convenience of the reader we repeat the definition. Let (N, v) be a TU-game. If $x \in I\!R^N$ with $x(N) = v(N)$ we call the vector x a *pre-imputation*. For a coalition $S \subseteq N$ and a pre-imputation x the *excess of* S is defined by $\text{exc}\,(S, x \,|\, v) := v(S) - x(S)$. Let Exc be the map that assigns to a pre-imputation x the vector

$$\text{Exc}(x) := \{\text{exc}\,(S, x \,|\, v)\}_{S \subseteq N} \in I\!R^{2^N}.$$

The map $\theta: I\!R^{2^N} \to I\!R^{2^n}$ orders the coordinates of vectors in $I\!R^{2^N}$ in weakly decreasing order. A pre-imputation x is a point in the *prenucleolus of the game* (N, v) if

$$\theta \circ \text{Exc}\,(x) \preceq_{\text{lex}} \theta \circ \text{Exc}\,(y) \text{ for all pre-imputations } y$$

where $\preceq_{\text{lex}}$ is the lexicographic ordering of $I\!R^{2^n}$ (Sobolev (1975)). If we restrict the candidates x to the *imputations* i.e. pre-imputations also satisfying the inequalities $x_i \geq v(i)$ for all $i \in N$, we obtain the *nucleolus* (Schmeidler (1969)).

The following facts are known about the (pre-)nucleolus:

(a) The nucleolus as well as the prenucleolus consists of one point. We denote this point by $\text{nu}\,(N, v)$ and $\text{nu}^*(N, v)$, respectively (Schmeidler (1969)).

(b) If the game (N, v) is *zero-monotonic* i.e. $v(S) + \sum_{i \in T \setminus S} v(i) \leq v(T)$ whenever $S \subset T$, then the nucleolus and the prenucleolus are the same (Maschler, Peleg and Shapley (1979)).

(c) If x is the prenucleolus of (N, v) and $\mathcal{B}$ is the set of coalitions with maximal excess (w.r.t. x), then $\mathcal{B}$ is *balanced*. A collection of coalitions is called *balanced* if there are positive numbers λ_T, one for each $T \in \mathcal{B}$, such that $\sum_{T \in \mathcal{B}} \lambda_T \, e_T = e_N$ (Sobolev (1975), Kohlberg (1971)).

(d) There are several methods to compute the prenucleolus of a TU-game (see Kopelowitz (1967), Dragan (1974),Sankaran (1991), Solymosi (1993), Potters et al. (1996), Derks and Kuipers (1996)), if the number of players is not too large, say $|N| \leq 20$.

About the present game $(N \cup Q, W)$ the following observations can be made:

- $(N \cup Q, W)$ is a zero-normalized monotonic game i.e., $W(i) = W(\alpha) = 0$ for all $i \in N$ and $\alpha \in Q$ and $W(T \cup i) \geq W(T)$ if $i \notin T$, and $W(T \cup \alpha) \geq W(T)$ if $\alpha \notin T$.
- The prenucleolus and the nucleolus of $(N \cup Q, W)$ are the same (point (b)). So, if (x, y) is the prenucleolus of W, we have, in particular, $x \geq 0$ and $y \geq 0$.
- The maximal excess $E_1 := \max_T \text{exc}\,(T, (x, y) | W) \geq 0$, because $W(N \cup Q) = W(\{i^*\} \cup Q)$ for some agent $i^* \in N$ and $x(N \setminus i^*) \geq 0$. Then $\{i^*\} \cup Q$ has an ex-

cess ≥ 0.

The first step in the procedure to find a regular price vector consists of the computation of the nucleolus of $(N \cup Q, W)$: $(x, y) \in I\!R_+^N \times I\!R_+^Q$.

If the maximal excess $E_1 = 0$, the pair $p := y$ and $q := x$ is a feasible point of $(\mathrm{LP})^*$. If $W(N \cup Q) = V_{i^*}(Q) = W(i^* \cup Q)$, we have $p(Q) + q_i^* \geq V_{i^*}(Q) = W(N \cup Q) = p(Q) + q(N)$. Hence, $q_j = 0$ for $j \neq i^*$ and $p(Q) + q_{i^*} = V_{i^*}(Q)$. For any other feasible point $(\bar{p}, \bar{q})$ of $(\mathrm{LP})^*$, we also have $\bar{q}_j \geq V_j(\emptyset) = 0$ and $\bar{p}(Q) + \bar{q}_{i^*} \geq V_{i^*}(Q)$. Conclusion: (p, q) is an optimal point of $(\mathrm{LP})^*$ and $p := y$ is a regular price vector, whenever $E_1 = 0$.

If $E_1 > 0$, we increase, *in the second step,* the value of x_i ($\bar{x}_i := x_i + \delta_i$) till all inequalities $\bar{x}_i + y(C) \geq V_i(C)$ are satisfied. Then $\bar{x}(N) + y(Q) = W(N \cup Q) + \sum_{i \in N} \delta_i$. The vector $(\bar{x}, y)$ is a *core allocation* in the TU-game $(N \cup Q, \bar{W})$ where $\bar{W}(T) = W(T)$ if $T \neq N \cup Q$ and $\bar{W}(N \cup Q) = W(N \cup Q) + \sum_{i \in N} \delta_i$.

The third step is the computation of the nucleolus of $(N \cup Q, \bar{W})$: $(u, z) \in I\!R_+^N \times I\!R_+^Q$.

Let $-E_2$ be the maximal excess of (z, u) in the game $(N \cup Q, \bar{W})$. Then $E_2 \geq 0$ because $(N \cup Q, \bar{W})$ has a core element, namely $(\bar{x}, y)$. In Theorem 5 we will prove that $p := z$ is a regular price vector, if $z_\alpha > E_2$ for all commodities $\alpha \in Q$.

Theorem 5. *If the vector (u, z) is the (pre-)nucleolus of the game $(N \cup Q, \bar{W})$ and no one-good-coalition $\{\alpha\}$ has maximal excess, then $p = z$ is a regular price vector.*
Proof: Let $-E_2$ be the maximal excess of the point (u, z) in the game $(N \cup Q, \bar{W})$ (for coalitions $T \neq \emptyset, N \cup Q$). As the core of this game is nonempty ($(\bar{x}, y)$ is a core element), we have $E_2 \geq 0$. As $u_i \geq W(\{i\} + E_2$ for all $i \in N$, we have $\bar{u}_i := u_i - E_2 \geq 0$. As no one-good-coalition $\{\alpha\}$ has highest excess, we have $z_\alpha > E_2$ for all $\alpha \in Q$.
We prove that $p := z$ and $q := \bar{u}$ is an optimal point of $(\mathrm{LP})^*$.
Since $u_i + z(C) \geq W(\{i\} \cup C) + E_2 = V_i(C) + E_2$, the point $(z, \bar{u})$ is a feasible point of $(\mathrm{LP})^*$. Let $\mathcal{B}$ be the collection of coalitions T with excess $-E_2$ with respect to (u, z). Then, there is a balancedness relation $\sum_{T \in \mathcal{B}} \lambda_T \, e_T = e_N \oplus e_Q$ with $\lambda_T > 0$ for all $T \in \mathcal{B}$.

Case (i): $E_2 > 0$. In this case $T \cap N$ consists of one agent for every coalition $T \in \mathcal{B}$. By assumption, $T \cap N$ cannot be empty. If $T \cap N$ contains more then one agent, there is an agent $i^* \in T \cap N$ such that $W(T) = W((T \cap Q) \cup \{i^*\})$. As $u_j \geq E_2 > 0$ for $j \in T \cap N$ and $j \neq i^*$, we get $W(T) - u(T \cap N) - z(T \cap Q) \leq -2\, E_2$.
So, every coalition $T \in \mathcal{B}$ is of the form $\{i\} \cup C$ with $i \in N$ and $C \subseteq Q$ and the

balancedness relation is $\sum_{\{i\}\cup C\in\mathcal{B}} \lambda_{\{i\}\cup C}\, e_i \oplus e_C = e_N \oplus e_Q$. This defines a stochastic redistribution.

Taking the inner product with $(\bar{u}, z)$ gives

$$\bar{u}(N) + z(Q) = \sum_{\{i\}\cup C\in\mathcal{B}} \lambda_{\{i\}\cup C}\,(\bar{u}_i + z(C)) = \sum_{\{i\}\cup C\in\mathcal{B}} \lambda_{\{i\}\cup C}\, V_i(C).$$

By the duality theorem $(\bar{u}, z)$ is an optimal point of (LP)* and z is a regular price vector.

Case (ii): $E_2 = 0$. In this case $\bar{u} = u$. Suppose that $(\bar{p}, \bar{q})$ is an optimal point of (LP)* and $\bar{p}(Q) + \bar{q}(N) < u(N) + z(Q)$.

We prove first that $\bar{p} \geq 0$. If some price $\bar{p}_\alpha < 0$, we find for every agent $i \in N$ and every bundle $C \subset Q$ with $\alpha \notin C$

$$\bar{p}(C) + \bar{q}_i > \bar{p}(C \cup \{\alpha\}) + \bar{q}_i \geq V_i(C \cup \{\alpha\}) \geq V_i(C).$$

If $\eta := \min_{i\in N, C:\alpha\notin C} [\bar{p}(C) + \bar{q}_i - V_i(C)] > 0$, we can replace $\bar{p}_\alpha$ by $\bar{p}_\alpha + \eta$ and all $\bar{q}_i$ by $\bar{q}_i - \eta$. We keep a feasible point of (LP)* with a lower value for the objective function.

Then the point $(\bar{p}, \bar{q})$ satisfies the inequalities $\bar{p}(T \cap Q) + \bar{q}(T \cap N) \geq W(T)$ for all coalitions $T \subset N \cup Q$. If we divide the difference between $u(N) + z(Q) = \bar{W}(N \cup Q)$ and $\bar{p}(Q) + \bar{q}(N)$ equally among the players in $N \cup Q$, we get an imputation with all excesses < 0. Then (u, z) is not the nucleolus. $\triangleleft$

Remark: In the second case $E_2 = 0$ we did not use the inequality $z_\alpha > E_2 = 0$. Therefore, we can replace the condition in Theorem 5 by the weaker condition '$z_\alpha > E_2$ if $E_2 > 0$'. So, if Theorem 5 applies i.e., if $E_2 = 0$ or $z_\alpha > E_2 > 0$, we find a regular price vector by computing two nucleoli in TU-games with $n + q$ players.

If $(\bar{u}, z)$ is an optimal solution of (LP)*, every stochastic reallocation with maximal expected social welfare $\{y_{i,C}\}_{\subset\in N, C\subseteq Q}$ has $y_{i,C} = 0$ if $\bar{u}_i + z(C) > V_i(C)$ (complementary slackness). To find an equilibrium allocation one has to compute all pairs (i, C) with $\bar{u}_i + z(C) = V_i(C)$ and look for a subset $\{(i, C_i)\}_{i\in N}$ where $\{C_i\}_{i\in}$ is a redistribution of Q. Is this possible then the SW-condition is satisfied and we have a price equilibrium. If such a redistribution does not exist, then the SW-condition does not hold and there is no stable price equilibrium. The reader should not underestimate the difficulty of the last problem. In general it is an $\mathcal{NP}$-hard problem. If the number of pairs (i, C) with $\bar{u}_i + z(C) = V_i(C)$ is not too large, it can be done by considering all possible combinations or by any other ad-hoc method.

Example 2 (continued) In example 2 we find for the nucleolus of $(N \cup Q, W)$:

$x = (1.33, 1.67, 0)$ and $y = (4.67, 3, 4.33)$ and $W(N \cup Q) = 15$. We used an algorithm based on Potters et al. (1996) to compute the nucleolus.

The coalitions $(1,\alpha)$, $(2,\gamma)$, $(2,\beta\gamma)$ have highest excess $E_1=\delta_1=\delta_2=4$ and $(3,\gamma)$ has excess $\delta_3=3.67$. Therefore, $\bar{W}(N\cup Q)+15+4+4+3.67=26.67$.

The nucleolus of $(N\cup Q,\bar{W})$ is $u=(4.22,\ 2.72,\ 0.72)$ and $z=(6.5,\ 4.5,\ 8)$. The following coalitions have the highest excess $E_2=-0.72$: $(1,\alpha)$, $(1,\beta)$, $(2,\gamma)$, $(2,\alpha\beta)$ $(3,\emptyset)$ and $(3,\gamma)$. All z-coordinates are larger than 0.72. From Proposition 5 follows that $z=(6.5,\ 4.5,\ 8)$ is a regular price vector. Furthermore, $\mathcal{B}$ does not contain a partition and the regular equilibrium price $(6.5,\ 4.5,\ 8)$ does not allow a price equilibrium. There is no stable price equilibrium.

4. The computation of equilibrium prices: an alternative approach

In this section we give an alternative method to compute a solution of the linear program $(\mathrm{LP})^*$ and a regular price vector. We found our inspiration in the method used by Solymosi and Raghavan (1994) to compute the nucleolus of assignment games.

$(\mathrm{LP})^*$ min $p(Q)+q(N)$ under the conditions

$p\in I\!R^Q$, $q\in I\!R^N$,

$p(B)+q_i\geq V_i(B)$ for $B\subseteq Q$ and $i\in N$.

We describe an iterative process that leads to a solution in finitely many iterations. The idea is the following.

We start with a feasible point (p,q) of $(\mathrm{LP})^*$ such that every agent i has a bundle C with $p(C)+q_i=V_i(C)$. Let $\mathcal{T}=\mathcal{T}(p,q)$ be the set of pairs (i,C) satisfying the equality $p(C)+q_i=V_i(C)$. Then we look for a vector $(u,v)\in I\!R^Q\times I\!R^N$ satisfying the inequalities $v_i+u(C)\geq 0$ for $(i,C)\in\mathcal{T}(p,q)$ and $v(N)+u(Q)<0$. If such a vector exists, $(p+t\,u,q+t\,v)$ is a feasible point of $(\mathrm{LP})^*$ for small positive values of t. If t_0 is the largest value of t for which $(p+t\,u,q+t\,v)$ is a feasible point of $(\mathrm{LP})^*$, we have a new feasible point of $(\mathrm{LP})^*$ $(\bar{p},\bar{q}):=(p+t_0\,u,q+t_0\,v)$ with $\bar{p}(Q)+\bar{q}(N)<p(Q)+q(N)$. We repeat the procedure with $(\bar{p},\bar{q})$ instead of (p,q). If such a vector (u,v) does not exist, then (p,q) is an optimal solution of $(\mathrm{LP})^*$.

Let us describe the procedure in more detail.

Initialization: To start with we give p_α the *second highest value* among the numbers $\{V_i(\alpha)\}_{i\in N}$ (at least two agents are interested in $\{\alpha\}$, can make a nonnegative profit by buying α) and $q_i:=\max_{C\subseteq Q}[V_i(C)-p(C)]$. Then $p_\alpha\geq 0$ and $q_i\geq V_i(\emptyset)=0$. (p,q) is a feasible point of $(\mathrm{LP})^*$. Each agent i has a commodity bundle $C\subseteq Q$ with $p(C)+q_i=V_i(C)$. If the reservation values are additive, the initial price vector is already a regular price vector.

Iterations: In each iteration we look for a direction $(u,v) \in I\!R^Q \times I\!R^N$ that satisfies the inequalities

$v_i + u(C) \geq 0$ for $(i,C) \in \mathcal{T}$ and

$v(N) + u(Q) < 0$

To find a direction (u,v) we introduce 'slack' variables $\bar{z}_{i,C} \geq 0$ and write the relations of (LP)* as $v_i + u(C) - \bar{z}_{i,C} = 0$, $\bar{z}_{i,C} \geq 0$ for $(i,C) \in \mathcal{T}$.

From these linear equations we eliminate the variables v_i and as many variables u_α as possible. Let Q_1 be the set of α's that are eliminated and $Q_2 := Q \backslash Q_1$.

Then we have linear equalities of the form

$v_i + F_i(u_\beta, \bar{z}_{i,C}) = 0$ for $i \in N$

$u_\alpha + G_\alpha(u_\beta, \bar{z}_{i,C}) = 0$ for $\alpha \in Q_1$

some linear equations of the form $H_\rho(\bar{z}_{i,C}) = 0$ (in the slack variables only).

The functions F_i, G_α and H_ρ are linear functions of u_β ($\beta \in Q_2$) and $\bar{z}_{i,C}$ ($(i,C) \in \mathcal{T}$). Finally, we can eliminate from the equations $H_\rho = 0$ some slack variables $\bar{z}_{i,C}$ as functions of the remaining slack variables. Let $\mathcal{T}_1$ be the set of slack variables that are eliminated and $\mathcal{T}_2 := \mathcal{T} \backslash \mathcal{T}_1$. The variables $\bar{z}_{i,C}$ with $(i,C) \in \mathcal{T}_1$ are also eliminated from the F_i- and G_α-equations.

Substituting the F_i- and G_α-equations in the *objective function* $v(N) + u(Q)$ gives the equation $v(N) + u(Q) = -\sum_{i \in N} F_i(u_\beta, \bar{z}_{i,C}) - \sum_{\alpha \in Q_1} G_\alpha(u_\beta, \bar{z}_{i,C}) + \sum_{\beta \in Q_2} u_\beta.$

This is a linear expression in the variables u_β ($\beta \in Q_2$) and the variables $\bar{z}_{i,C}$ with $(i,C) \in \mathcal{T}_2$. After these eliminations we have linear equalitions of the following form:

$$\begin{bmatrix} I_N & & & A_{Q_2} & A_{\mathcal{T}_2} \\ & I_{Q_1} & & B_{Q_2} & B_{\mathcal{T}_2} \\ & & I_{\mathcal{T}_1} & C_{Q_2}(=0) & C_{\mathcal{T}_2} \\ 0_N & 0_{Q_1} & 0_{\mathcal{T}_1} & d_{Q_2} & d_{\mathcal{T}_2} \end{bmatrix} \begin{bmatrix} v_N \\ u_{Q_1} \\ \bar{z}_{\mathcal{T}_1} \\ u_{Q_2} \\ \bar{z}_{\mathcal{T}_2} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ - \end{bmatrix}$$

As usual, I_X is the identity matrix of size X and 0_X is the zero vector of size X. The matrices A_X, B_X and C_X are matrices with columns indexed by X and d_X is an X-vector.

Case (0): If all coefficients of the u_β-variables vanish ($d_{Q_2} = 0$) and the coefficients of the $\bar{z}_{i,C}$-variables are nonnegative ($d_{T_2} \geq 0$), there is no direction (u, v) in which $p(Q) + q(N)$ decreases. Then (p, q) is optimal in (LP)*.

Case (1): If $d_{Q_2} \neq 0$, we take one good β with $|d_\beta|$ is maximal. We define $u_\beta := +1$, if $d_\beta < 0$ is negative and $u_\beta := -1$, if $d_\beta > 0$. We take all other variables $u_{\beta'} := 0$ if $\beta' \in Q_2 \backslash \{\beta\}$ as well as the slack variables $\bar{z}_{i,C} = 0$. The values for v_i $(i \in N)$ and u_α $(\alpha \in Q_1)$ are computed from the F_i-equation and the G_α-equation respectively. Then we have a direction (u, v) in which $p(Q) + q(N)$ decreases and all equalities $p(C) + q_i = V_i(C)$ are kept. The new pair $(\bar{p}, \bar{q})$ will be $\bar{p} = p + t_0\, u$ and $\bar{q} = q + t_0\, v$ where

$$t_0 := \min\{(V_k(D) - p(D) - q_k)/(v_k + u(D)) \text{ for the pairs } (k, D) \text{ with } v_k + u(D) < 0\}.$$

The pairs (k, D) for which the minimum is obtained, enter the collection $\mathcal{T}$. We add the corresponding equations $p(D) + q_k - \bar{z}_{k,D} = 0$ and eliminate the variables q_i $(i \in N)$, p_α $(\alpha \in Q)$ and $\bar{z}_{i,C}$ with $(i, C) \in \mathcal{T}_1$ i.e., we restore the basis we had before. As no new vectors $e_k \oplus e_D$ is linear dependent on $\{e_i \oplus e_C : (i, C) \in \mathcal{T}\}$, the rank of these vectors increases. So, at least one new p_β variable can be eliminated and after at most $|Q_2|$ iterations of this kind, we have $d_Q = 0$.

Case (1'): In exceptional cases we find $d_Q = 0$ and $Q_1 \neq Q$. In this case we perform the same iteration as in case (1) at any place $\beta \in Q_2$. In this case we increase the price of p_β, keep the other prices in Q_2 constant and the slack variables zero. The value of q_i $(i \in N)$ and the prices p_α $(\alpha \in Q_1)$ are adapted. Also in this case the rank of $\mathcal{T}$ increases.

After at most q iterations the collection of vectors $\{e_i \oplus e_C : (i, C) \in \mathcal{T}\}$ has rank $n + q$ and $Q_1 = Q$. $n = |N|$ and $q = |Q|$. The equations have the form:

$$\begin{bmatrix} I_N & & & A_{\mathcal{T}_2} \\ & I_Q & & B_{\mathcal{T}_2} \\ & & I_{\mathcal{T}_1} & C_{\mathcal{T}_2} \\ 0_N & 0_Q & 0_{\mathcal{T}_1} & d_{\mathcal{T}_2} \end{bmatrix} \begin{bmatrix} v_N \\ u_Q \\ \bar{z}_{\mathcal{T}_1} \\ \bar{z}_{\mathcal{T}_2} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ - \end{bmatrix}$$

Lemma 6. *If no iterations of type (1) are possible anymore, then*

$$\sum_{(i,C)\in\mathcal{T}_2} d_{i,C}\,(e_i \oplus e_C) = e_N \oplus e_Q.$$

Proof: In all intial equations $p(C) + q_i - \bar{z}_{i,C} = 0$ we have the following identity

$$\sum_{\alpha \in Q} [\text{coefficient of } p_\alpha]\,(0 \oplus e_\alpha) + \sum_{i \in N} [\text{coefficient of } q_i]\,(e_i \oplus 0) + \sum_{(i,C) \in \mathcal{T}} [\text{coefficient of } \bar{z}_{i,C}]\,(e_i \oplus e_C) = 0$$

This does not change under elimination of variables and in the goal function we start with (and keep) the equality:

$$\sum_{\alpha \in Q} [\text{coefficient of } p_\alpha]\,(0 \oplus e_\alpha) + \sum_{i \in N} [\text{coefficient of } q_i]\,(e_i \oplus 0) + \sum_{(i,C) \in \mathcal{T}} [\text{coefficient of } \bar{z}_{i,C}]\,(e_i \oplus e_C) = (e_N \oplus e_Q).$$

If $d_Q = 0$, we have $\sum_{(i,C) \in \mathcal{T}} [\text{coefficient of } \bar{z}_{i,C}]\,(e_i \oplus e_C) = (e_N \oplus e_Q)$. $\triangleleft$

Note that Lemma 6 gives a stochastic redistribution if $d_{\mathcal{T}_2} \geq 0$. This will be the case in many situations and we are done. If not, we have to proceed with case (2).

Case (2): If $Q_1 = Q$ but some of the $\bar{z}_{i,C}$-variables still have a negative coefficient, we have to proceed more carefully.

The idea is to give a variable $\bar{z}_{i,C}$ occurring in the objective function with a negative coefficient the value $+1$. But, if we do so and compute the values of $\bar{z}_{i,C}$ $((i,C) \in \mathcal{T}_1)$, we may find negative values for some of the variables in $\mathcal{T}_1$ and this is not allowed (in the direction (u,v) we would leave the set of feasible points of (LP)* immediately i.e., for all $t > 0$). Therefore, before we choose the direction we have to take care that a $\bar{z}_{i,C}$-variable occurring in the objective function with a negative coefficient has only *nonpositive coefficients in its column in* $C_{\mathcal{T}_2}$. Here we make the transition from simple linear algebra to simplex method techniques but only in the submatrix

$$\begin{bmatrix} I_{\mathcal{T}_1} & C_{\mathcal{T}_2} \\ 0_{\mathcal{T}_1} & d_{\mathcal{T}_2} \end{bmatrix}.$$

We start with an attempt to make $d_{\mathcal{T}_2} \geq 0$.

case (2'): As long as there is a negative coefficient in $d_{\mathcal{T}_2}$, say at place (i,C) and in the $\bar{z}_{i,C}$-column of $C_{\mathcal{T}_2}$ there is a positive entry we stay in case (2'). Notice that the rows of the matrix $[I_{\mathcal{T}_1}, C_{\mathcal{T}_2}]$ are lexicographically larger than the 0-vector. In the $\bar{z}_{i,C}$-column we perform a pivot operation such that all rows in the matrix remain lexicographically larger than the 0-vector. Then the objective function becomes lexicographically *larger* and we cannot come in an infinite loop. So, after finitly many steps the objective function has only nonnegative coefficients or the entries in the $C_{\mathcal{T}_2}$-matrix above the places (i, C) where $d_{\mathcal{T}_2}$ is negative are all ≤ 0. If the last statement is true we go to case (2").

case (2"): Let (i, C) be a position in the objective function with a negative coefficient. As the variable $\bar{z}_{i,C}$ is not in the present basis, we can find a vector $(v, u, \bar{z})$ that solves the system of linear equations and gives $\bar{z}_{i,C} = +1$. $\bar{z}_{k,D}$ gets only a positive value if $\bar{z}_{k,D}$ is in the basis and the unique equation containing $\bar{z}_{k,D}$ has a *negative coefficient* at place (i, C). By going in the direction (u, v) these pairs (k, D) (and the pair (i, C)) leave the set $\mathcal{T}$. Furthermore, the objective function decreases in the direction (u, v). Therefore, there is at least one position in the (total) $\bar{z}_{i,C}$-column that is positive. We perform a pivot operation at such a position and take the precaution that all rows remain lexicographically larger than the 0-vector (i.e. we choose the equation with the highest index). So, *exactly one* v_i- or u_α-variable leaves the basis and the objective function gets a positive coefficient for this variable. We delete this equation, the equations containing the variables $\bar{z}_{k,D} > 0$ and also the column $\bar{z}_{i,C}$. So, we lose one equation in the v_i– or u_α–part (the upper part of the matrix) and some equations of the $\bar{z}$-part (the lower part of the matrix). The rank of the upper part is $n+q-1$ after deletion. Along the half-line $(p, q)+t(u, v)$, $t \geq 0$ and $\bar{z} = 0$ the remaining equations are satisfied and the objective function decreases. After a while one (or more than one) new pair (l, E) becomes tight and by adding the equations $p(E) + q_l - \bar{z}_{l,E} = 0$ the rank of the u- and v-part of the matrix is $n + q$ again, as $e_l \oplus e_E$ is linearly independent from the vectors $e_i \oplus e_C$ that remained. We restore the basis and return to case (2).

About the performance of the algorithm the following remarks can be made:

(i) Case (1) requires at most q pivot operations.

(ii) Every time we enter case (2) we have a collection $\mathcal{T}(p, q) = \mathcal{T}$ of tight pairs (i, C) of rank $n + q$ and never the same one as before, as $p(C) + q_i = V_i(C)$ for $(i, C) \in \mathcal{T}$ has a unique solution and the value of the objective function increases every time. This means that the algorithm is finite and the number of possible $\mathcal{T}$'s gives a (very coarse) bound for the number of iterations in case (2).

(iii) Case (2') requires only finitely many pivot operations, since the objective function is increasing lexicographically.

Let us show the algorithm in an example.

Example 3. Let $\mathcal{E}$ be an economy with four agents and four goods. The reservation values are given in the following table.

	α	β	γ	δ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$
1	4	6	4	7	11	6	7
2	5	5	6	9	12	10	11
3	2	4	3	5	7	6	9
4	7	10	7	9	10	10	10

	$\beta\gamma$	$\beta\delta$	$\gamma\delta$	$\alpha\beta\gamma$	$\alpha\beta\delta$	$\alpha\gamma\delta$	$\beta\gamma\delta$	N
1	13	14	10	13	16	14	16	18
2	10	14	12	16	15	14	18	19
3	10	12	8	11	13	10	12	17
4	13	12	14	17	16	16	18	19

Initialization: We start with $p = (5, 6, 6, 9)$, the second highest reservation prices. Computing $V_i(C) - p(C)$ we get the following table The asterisks $*$ denote the places where $V_i(C) - p(C)$ is maximal.

	α	β	γ	δ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$
1	–1	0	–2	–2	0	–5	–7
2	0	–1	0	0	1*	–1	–3
3	–3	–2	–3	–4	–4	–5	–5
4	2	4*	1	0	–1	–1	–4

	$\beta\gamma$	$\beta\delta$	$\gamma\delta$	$\alpha\beta\gamma$	$\alpha\beta\delta$	$\alpha\gamma\delta$	$\beta\gamma\delta$	N
1	1*	–1	–5	–4	–4	–6	–5	–8
2	–2	–1	–3	–1	–5	–6	–3	–7
3	–2	–3	–7	–6	–7	–10	–9	–9
4	1	–3	–1	0	–4	–4	–3	–7

We see that $q = (1, 1, 0, 4)$ and $\mathcal{T}$ consists of the pairs: $(1, \beta\gamma)$, $(2, \alpha\beta)$, $(3, \emptyset)$, $(4, \beta)$. The value of the objective function is 32. The tableau for the first iteration becomes

1	2	3	4	α	β	γ	δ				
1	0	0	0	0	1	1	0	–1	0	0	0
0	1	0	0	1	1	0	0	0	–1	0	0
0	0	1	0	0	0	0	0	0	0	–1	0
0	0	0	1	0	1	0	0	0	0	0	–1
1	1	1	1	1	1	1	1	0	0	0	0

After the elimination of the v- variables from the goal function we have

1	2	3	4	α	β	γ	δ				
1	0	0	0	0	1	1	0	–1	0	0	0
0	1	0	0	1	1	0	0	0	–1	0	0
0	0	1	0	0	0	0	0	0	0	–1	0
0	0	0	1	0	1	0	0	0	0	0	–1
0	0	0	0	0	–2	0	1	1	1	1	1

Good β has the absolutely highest coefficient: $u_\beta = 1$ and $v_1 = v_2 = v_4 = -1$. The vector $u = (0, 1, 0, 0)$ and $v = (-1, -1, 0, -1)$.

The new vectors become $p = (5,\ 6 + t,\ 6,\ 9)$ and $q = (1 - t,\ 1 - t,\ 0,\ 4 - t)$. If we take $t = 1$, we get the additional tight pairs $(2, \alpha)$, $(2, \gamma)$ and $(2, \delta)$.

So, $p = (5,\ 7,\ 6,\ 9)$ and $q = (0,\ 0,\ 0,\ 3)$. The value of the objective function is 30. Adding the equation for $(2, \alpha)$, $(2, \gamma)$, $(2, \delta)$ and its slack variables, we get

1	2	3	4	α	β	γ	δ							
1	0	0	0	0	1	1	0	–1	0	0	0	0	0	0
0	1	0	0	1	1	0	0	0	–1	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	–1	0	0	0	0
0	0	0	1	0	1	0	0	0	0	0	–1	0	0	0
0	1	0	0	1	0	0	0	0	0	0	0	–1	0	0
0	1	0	0	0	0	1	0	0	0	0	0	0	–1	0
0	1	0	0	0	0	0	1	0	0	0	0	0	0	–1
0	0	0	0	0	–2	0	1	1	1	1	1	0	0	0

We restore the basis and eliminate the variables p_α, p_β and p_γ and find:

1	2	3	4	α	β	γ	δ							
1	0	0	0	0	0	0	1	–1	1	0	0	–1	1	–1
0	0	0	0	0	1	0	0	0	–1	0	0	1	0	0
0	0	1	0	0	0	0	0	0	0	–1	0	0	0	0
0	0	0	1	0	0	0	0	0	1	0	–1	–1	0	0
0	0	0	0	1	0	0	–1	0	0	0	0	–1	0	1
0	0	0	0	0	0	1	–1	0	0	0	0	0	–1	1
0	1	0	0	0	0	0	1	0	0	0	0	0	0	–1
0	0	0	0	0	0	0	1	1	–1	1	1	2	0	0

Then $u = (-1,\, 0,\, -1,\, -1)$ and $v = (1,\, 1,\, 0,\, 0)$. We get $p = (5-t,\, 7,\, 6-t,\, 9-t)$ and $q = (t,\, t,\, 0,\, 3)$.

We find $t = 1$ and $(4, \alpha)$ and $(2, \alpha\gamma)$ enter $\mathcal{T}$. The new vectors p and q are:

$p = (4,\, 7,\, 5,\, 8)$ and $q = (1,\, 1,\, 0,\, 3)$.

We are now in an optimal point as the next stage would show. One also can come to this conclusion from the fact that

$\mathcal{T} = \{(1, \alpha\beta),\, (1, \beta\gamma),\, (2, \alpha),\, (2, \gamma),\, (2, \delta),\, (2, \alpha\gamma),\, (3, \emptyset),\, (4, \alpha),\, (4, \beta)\}$

contains the N-reallocation $\beta\gamma \to 1$, $\delta \to 2$, $\emptyset \to 3$ and $\alpha \to 4$ with total social welfare 29, equal to $p(Q) + q(N)$.

If we apply the 'nucleolus'-method of Section 2 we find for the first nucleolus:

$x = (0,\, \frac{5}{14},\, 0,\, \frac{8}{7})$ and $y = (\frac{18}{7},\, \frac{39}{7},\, \frac{59}{14},\, \frac{36}{7})$.

The vector δ is $(\frac{23}{7},\, \frac{7}{2},\, \frac{9}{7},\, \frac{7}{2})$. We increase $W(N \cup Q)$ to $\bar{W}(N \cup Q) = 19 + 11\frac{4}{7} = 30\frac{4}{7}$. The second nucleolus computation gives

$u = (1\frac{11}{28},\, 2\frac{61}{84},\, \frac{11}{28},\, 3\frac{61}{84})$ and $z = (3\frac{2}{3},\, 7\frac{1}{3},\, 4\frac{2}{3},\, 6\frac{2}{3})$.

Furthermore, $E_2 = \frac{11}{28}$ and this is lower than any of the z-coordinates. By Proposition 5, z is a regular price vector and $\bar{u} = (1,\, 2\frac{1}{3},\, 0,\, 3\frac{1}{3})$. The collection $\mathcal{T}$ consists of the pairs $(1, \beta\gamma)$, $(2, \delta)$, $(3, \emptyset)$ and $(4, \alpha)$ only. This is the advantage of the 'nucleolus'-method. The collection $\mathcal{T}$ is minimal: what is tight in the nucleolus, is tight in *every* regular price vector. This makes it easier to decide if there is an N-reallocation in $\mathcal{T}$.

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