DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NIJMEGEN The Netherlands

IMPLEMENTATION OF LEAST-SQUARES FINITE ELEMENT METHOD FOR SOLVING THE GENERALIZED STOKES PROBLEM

I.O. Arushanian, G.M. Kobelkov

Report No. 9901 (January 1999)

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NIJMEGEN
Toernooiveld
6525 ED Nijmegen
The Netherlands

Implementation of Least-Squares Finite Element Method for Solving the Generalized Stokes Problem

I.O. Arushanian* G.M. Kobelkov†

Abstract

The least-squares finite element method is used for solving the generalized Stokes problem with model boundary conditions. A finite element approximation is proposed. This approximation allows us to solve efficiently the resulting system of linear algebraic equations by a direct method with the usage of the Fast Fourier transform (FFT). An approximate solution being obtained has the second order of accuracy in the L_2-norm both for the velocity and the pressure. In addition, finite element spaces are chosen independently of the "inf-sup" condition.

1 Problem statement

Let $\Omega \subset R^2$ be the square domain

$$\Omega = \big\{ (x,\, y) \,|\, 0 \le x \le \pi,\ 0 \le y \le \pi \,\big\}$$

with the boundary Γ. Let us consider the boundary value problem

$$\begin{aligned}
-\Delta w + w + \nabla p = \mathbf{f} \qquad & \text{in} \qquad \Omega\,, \\
\operatorname{div} w = 0 \qquad & \text{in} \qquad \Omega\,, \\
w \cdot n = 0 \qquad & \text{on} \qquad \Gamma\,, \\
\frac{\partial\,(w \cdot \tau)}{\partial n} = 0 \qquad & \text{on} \qquad \Gamma\,, \\
\int\limits_{\Omega} p\, dxdy = 0, \qquad &
\end{aligned} \tag{1}$$

where $w = (u,\, v)$.

*Permanent address: Faculty of Mechanics and Mathematics, Moscow State University, Moscow, 119899, Russia, E-mail: arush@apvk.srcc.msu.su

†Permanent address: Faculty of Mechanics and Mathematics, Moscow State University, Moscow, 119899, Russia, E-mail: kobelkov@inria.msu.ru

This work has been supported by INTAS-RFBR Grant (95–0098) and NWO grant (047.003.017).

This problem was chosen as model, since its solution can be found [8, 9] with the usage of Fourier techniques. Our aim is to construct a finite element approximation in such a way that this approximation would admit direct solving the problem with the help of the discrete Fourier transform.

The simplest approximation of such a kind is to use $\tilde{Q}_{22}$ elements on elementary cells to approximate the velocity w and Q_0 elements to approximate the pressure p [10]. For an approximate solution, this scheme yields the error estimate

$$\|w - w_h\|_1 + \|p - p_h\|_0 \leq ch \tag{2}$$

if the solution of the original problem is sufficiently smooth.

Nevertheless, attempts undertaken on the basis of the finite element method to construct higher order approximations admitting direct solving the resulting system of linear algebraic equations by the discrete Fourier transform cause serious difficulties up to now.

In this paper, an approximation for the Stokes problem (1) such that it can be solved by means of the discrete Fourier transform is constructed using the least-squares finite element method. It turns out that estimate (2) may be essentially improved.

Using the standard techniques [4-7], at first we obtain the following first order system equivalent to (1):

$$\begin{array}{llll} \dfrac{\partial\omega}{\partial y} + u + \dfrac{\partial p}{\partial x} = f_1 \quad \text{in} \quad \Omega\,, & & -\dfrac{\partial\omega}{\partial x} + v + \dfrac{\partial p}{\partial y} = f_2 \quad \text{in} \quad \Omega\,, & \\ \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} - \omega = 0 \quad \text{in} \quad \Omega\,, & & \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \quad \text{in} \quad \Omega\,, & \\ u = 0 \quad \text{on} \quad \Gamma_2 \bigcup \Gamma_4\,, & & v = 0 \quad \text{on} \quad \Gamma_1 \bigcup \Gamma_3\,, & \\ \omega = 0 \quad \text{on} \quad \Gamma\,, & & \displaystyle\int_\Omega p\, dxdy = 0\,, & \end{array} \tag{3}$$

where

$$\Gamma_1 = \{(x,\,y)\,|\ \ 0 \leq x \leq \pi, \ \ y = 0\}\,, \quad \Gamma_2 = \{(x,\,y)\,|\ \ x = \pi, \ \ 0 \leq y \leq \pi\}\,,$$
$$\Gamma_3 = \{(x,\,y)\,|\ \ 0 \leq x \leq \pi, \ \ y = \pi\}\,, \quad \Gamma_4 = \{(x,\,y)\,|\ \ x = 0, \ \ 0 \leq y \leq \pi\}\,.$$

Further, we introduce the functional spaces

$$\begin{aligned} H_0^1 &= \{u \in H^1(\Omega)\,|\, u = 0 \ \text{ on } \ \Gamma\}\,, \\ V^1 &= \{v \in H^1(\Omega)\,|\, v = 0 \ \text{ on } \ \Gamma_2 \cup \Gamma_4\}\,, \\ V^2 &= \{v \in H^1(\Omega)\,|\ \ v = 0 \text{ on } \ \Gamma_1 \cup \Gamma_3\}\,, \\ \widehat{V} &= \{v \in H^1(\Omega)\,|\ \ \int_\Omega v\, dxdy = 0\}\,, \end{aligned}$$

construct the residual functional

$$\Phi\big(\omega,\, u,\, v,\, p,\, \mathbf{f}\big) \;=\; \left\| \frac{\partial \omega}{\partial y} + u + \frac{\partial p}{\partial x} - f_1 \right\|^2 + \left\| -\frac{\partial \omega}{\partial x} + v + \frac{\partial p}{\partial y} - f_2 \right\|^2 + \left\| \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} - \omega \right\|^2 + \left\| \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right\|^2 ,$$

and consider the following minimization problem

$$\Phi\big(\omega,\, u,\, v,\, p,\, \mathbf{f}\big) \;\to\; \min \quad \text{on} \quad W \,, \tag{4}$$

where

$$W = H_0^1 \times V_1 \times V_2 \times \widehat{V} = (\omega,\, u,\, v,\, p)\,.$$

Note that

$$\Phi(\omega,\, u,\, v,\, p,\, \mathbf{0}) \;=\; \|\omega\|^2 + \|\nabla\omega\|^2 + \|\nabla p\|^2 + \|u\|^2 + \|v\|^2 + \|\nabla u\|^2 + \|\nabla v\|^2 + 2\left(u,\, \frac{\partial p}{\partial x}\right) + 2\left(v,\, \frac{\partial p}{\partial y}\right).$$

Let us prove that a solution to problem (4) exists and is unique.

Theorem 1. *There exist positive constants C_2 and C_3 such that the inequalities*

$$C_1\left(\|\omega\|^2 + \|\nabla\omega\|^2 + \|u\|^2 + \|v\|^2 + \|\nabla u\|^2 + \|\nabla v\|^2 + \|\nabla p\|^2\right) \le \Phi(\omega,\, u,\, v,\, p,\, \mathbf{0}) \le C_2\left(\|\omega\|^2 + \|\nabla\omega\|^2 + \|u\|^2 + \|v\|^2 + \|\nabla u\|^2 + \|\nabla v\|^2 + \|\nabla p\|^2\right)$$

hold for arbitrary functions $(\omega,\, u,\, v,\, p) \in W$.

Proof. The upper-bound estimate follows directly from the triangle and Cauchy inequalities. To prove the lower-bound estimate, we use the ε-inequality. Then

$$\Phi(\omega,\, u,\, v,\, p,\, \mathbf{0}) \;\ge\; \|\omega\|^2 + \|\nabla\omega\|^2 + \|\nabla p\|^2 + \left(1 + \frac{1}{2c}\right)\left(\|u\|^2 + \|v\|^2\right) + 0.5\,\|\nabla u\|^2 + 0.5\,\|\nabla v\|^2 - \varepsilon^{-1}\left(\|u\|^2 + \|v\|^2\right) - \varepsilon\,\|\nabla p\|^2 \,, \tag{5}$$

where $\varepsilon > 0$ and c is a constant from the Friedrichs inequality

$$\|u\|^2 + \|v\|^2 \le c\left(\|\nabla u\|^2 + \|\nabla v\|^2\right).$$

If we choose ε satisfying the condition

$$\frac{1}{1 + (2c)^{-1}} < \varepsilon < 1\,,$$

then the conclusion of the theorem follows immediately from (5).

By H_h^1, $H_{0,h}^1$, $V_{1,h}$, $V_{2,h}$ we denote continuous piecewise linear subspaces of the spaces H^1, H_0^1, V_1, V_2, respectively. Here we assume that the grid is regular.

Theorem 2. *Suppose a solution of* (3) *belongs to the space* $\mathbf{H}^2(\Omega)\bigcap W$, f_1, $f_2 \in L_2(\Omega)$. *Then for an approximate solution* $(\omega_h, p_h, u_h, v_h)$ *the following estimates*

$$\|\omega-\omega_h\| + \|p-p_h\| + \|u-u_h\| + \|v-v_h\| \le C_4 h^2\big(\|\omega\|_2 + \|p\|_2 + \|u\|_2 + \|v\|_2\big),$$

$$\|\omega-\omega_h\|_1 + \|p-p_h\|_1 + \|u-u_h\|_1 + \|v-v_h\|_1 \le C_5 h\big(\|\omega\|_2 + \|p\|_2 + \|u\|_2 + \|v\|_2\big)$$

are valid with the constants C_4 *and* C_5 *independent of a choice of* h ([4, 5]).

2 Numerical solution

Now turn to the construction of a special triangulation of the domain Ω. Let $N = 2^m$, $m \in Z$. Denote $h = \pi(2N)^{-1}$. The set of lines of the form $x = 2ih$, $y = 2jh$ divides the domain $\overline{\Omega} = \Omega\bigcup\Gamma$ into elementary squares. In each of the elementary triangles being obtained, we draw diagonals and connect the midpoints of opposite sides with straight lines. As a result, a triangulation of the domain $\overline{\Omega}$ into right triangles with the legs h is constructed. Let

$$\Omega_h = \big\{(ih, jh)\,,\quad 0 \le i, j \le 2N\big\}$$

and

$$\begin{aligned} u &= \{u_{ij},\quad 1 \le i \le 2N-1,\ 0 \le j \le 2N\},\\ v &= \{v_{ij},\quad 0 \le i \le 2N,\ 1 \le j \le 2N-1\},\\ p &= \{p_{ij},\quad 0 \le i \le 2N,\ 0 \le j \le 2N\}. \end{aligned}$$

Further we consider the space of continuous functions V_h that are linear on each elementary triangle. We may construct the finite element basis functions φ_{ij} in the following way.

If $i + j = 2l$, then the support of φ_{ij} contains eight nearest elementary triangles. If $i + j = 2l + 1$, then the support of φ_{ij} contains four nearest elementary triangles. In both cases, it holds $\varphi_{ij}(kh, mh) = \delta_{ij}^{km}$.

By $H_{0,h}^1$, V_h^1, V_h^2, $\widehat{V}_h$ we denote the intersection of V_h with the spaces H_0^1, V^1, V^2, $\widehat{V}_h$, respectively.

Consider the minimization problem

$$\Phi\big(\omega_h,\, u_h,\, v_h,\, p_h,\, \mathbf{f}\big) \ \to\ \min \quad \text{on} \quad W_h\,,$$

where

$$W_h = H_{0,h}^1 \times V_h^1 \times V_h^2 \times \widehat{V}_h = (\omega_h,\, u_h,\, v_h,\, p_h)\,.$$

Let

$$\omega_h(x,\,y) = \sum_{\varphi_{ij}\in H^1_{0h}} \omega_{ij}\varphi_{ij}(x,\,y)\,, \quad p_h(x,\,y) = \sum_{\varphi_{ij}\in \widehat{V}_h} p_{ij}\varphi_{ij}(x,\,y)\,,$$
$$u_h(x,\,y) = \sum_{\varphi_{ij}\in V^1_h} u_{ij}\varphi_{ij}(x,\,y)\,, \quad v_h(x,\,y) = \sum_{\varphi_{ij}\in V^2_h} v_{ij}\varphi_{ij}(x,\,y).$$

In order to solve our minimization problem, we have to find a solution to the system

$$\frac{\partial\Phi}{\partial\omega_{ij}} = 0\,, \quad \frac{\partial\Phi}{\partial u_{ij}} = 0\,, \quad \frac{\partial\Phi}{\partial v_{ij}} = 0\,, \quad \frac{\partial\Phi}{\partial p_{ij}} = 0\,.$$

¿From here we get the following system of linear algebraic equations:

$$\begin{aligned}
&\left(G + D^0_{xx} + D^0_{yy}\right)\omega = F^1\,,\\
&\left(G^1 + D^1_{xx} + D^1_{yy}\right)u + K_x p = F^2\,,\\
&\left(G^2 + D^2_{xx} + D^2_{yy}\right)v + K_y p = F^3\,,\\
&K_x^{\mathrm{T}} u + K_y^{\mathrm{T}} v + \left(D_{xx} + D_{yy}\right)p = F^4\,,
\end{aligned} \tag{6}$$

where [1]

$$\left(G\right)_{ij} = \left(\varphi_i,\,\varphi_j\right), \quad \varphi_i,\,\varphi_j \in H^1_{0,h}\,;$$
$$\left(G\right)^k_{ij} = \left(\varphi_i,\,\varphi_j\right), \quad \varphi_i,\,\varphi_j \in V^k_h\,, \quad k = 1, 2;$$
$$\left(D^k_{xx}\right)_{ij} = \left(\frac{\partial\varphi_i}{\partial x},\,\frac{\partial\varphi_j}{\partial x}\right), \quad \left(D^k_{yy}\right)_{ij} = \left(\frac{\partial\varphi_i}{\partial y},\,\frac{\partial\varphi_j}{\partial y}\right), \quad \varphi_i,\,\varphi_j \in V^k_h\,, \quad k = 1,\,2\,;$$
$$\left(D_{xx}\right)_{ij} = \left(\frac{\partial\varphi_i}{\partial x},\,\frac{\partial\varphi_j}{\partial x}\right), \quad \left(D_{yy}\right)_{ij} = \left(\frac{\partial\varphi_i}{\partial y},\,\frac{\partial\varphi_j}{\partial y}\right), \quad \varphi_i,\,\varphi_j \in V_h\,;$$
$$\left(K_x\right)_{ij} = \left(\varphi_i,\,\frac{\partial\varphi_j}{\partial x}\right), \quad \varphi_i \in V^1_h\,, \ \ \varphi_j \in H^1_{0,h}\,;$$
$$\left(K_y\right)_{ij} = \left(\varphi_i,\,\frac{\partial\varphi_j}{\partial y}\right), \quad \varphi_i \in V^2_h\,, \ \ \varphi_j \in H^1_{0,h}\,;$$
$$\left(D^0_{xx}\right)_{ij} = \left(\frac{\partial\varphi_i}{\partial x},\,\frac{\partial\varphi_j}{\partial x}\right), \quad \left(D^0_{yy}\right)_{ij} = \left(\frac{\partial\varphi_i}{\partial y},\,\frac{\partial\varphi_j}{\partial y}\right), \quad \varphi_i,\,\varphi_j \in H^1_{0,h}\,;$$
$$\left(F^1\right)_i = \left(\frac{\partial\varphi_i}{\partial y},\,f_1\right) - \left(\frac{\partial\varphi_i}{\partial x},\,f_2\right), \quad \varphi_i \in H^1_{0,h}\,;$$
$$\left(F^2\right)_i = \left(\varphi_i,\,f_1\right), \quad \varphi_i \in V^1_h\,; \ \left(F^3\right)_i = \left(\varphi_i,\,f_2\right), \quad \varphi_i \in V^2_h\,;$$
$$\left(F^4\right)_i = \left(\frac{\partial\varphi_i}{\partial x},\,f_1\right) + \left(\frac{\partial\varphi_i}{\partial y},\,f_2\right), \quad \varphi_i \in V_h$$

and $\sum_{\varphi_i\in V_h} p_i\varphi_i(x,\,y) \in \widehat{V}_h$. Here we denote $i = (i_1, i_2),\ j = (j_1, j_2)$.

Note that in (6) for ω we have a separate equation because of the chosen boundary conditions; hence, this vector can be found as a solution to the system

$$\left(G + D^0_{xx} + D^0_{yy}\right)\omega = F^1 .$$

Note also that if the standard Stokes problem (system (1) without the term w in the first equation) with the same boundary conditions is solved by the method being proposed, then the equation for p is separate of other unknowns [2].

To determine the other unknowns u, v, p, we obtain the system of linear algebraic equations

$$A\,z = F\,, \tag{7}$$

where

$$A = \begin{pmatrix} G^1 + D^1_{xx} + D^1_{yy} & 0 & K_x \\ 0 & G^2 + D^2_{xx} + D^2_{yy} & K_y \\ K_x^{\mathrm{T}} & K_y^{\mathrm{T}} & D_{xx} + D_{yy} \end{pmatrix},$$

$$z = \begin{pmatrix} u^1 \\ u^2 \\ p \end{pmatrix}, \quad F = \begin{pmatrix} F^2 \\ F^3 \\ F^4 \end{pmatrix}.$$

Denote

$$D = \begin{pmatrix} I + D^1_{xx} + D^1_{yy} & 0 & 0 \\ 0 & I + D^2_{xx} + D^2_{yy} & 0 \\ 0 & 0 & D_{xx} + D_{yy} \end{pmatrix}.$$

It follows from Theorem 1 that the matrices D and A are spectrally equivalent. Moreover, the linear system with the matrix D admits direct solving with the usage of FFT [8, 9]. Hence, for solving the system $A\,z = F$ an iterative process can be constructed in such a way that the matrix D is used as a preconditioner for the matrix A ([3]).

Let us write down system (7) at nodes of the grid Ω_h as follows.

1. First we consider "even" nodes of the form $\{(ih, jh)\}$, $i + j = 2l$.

1a. At the inner nodes $(0 < i, j < 2N)$ we have

$$\frac{8u_{ij} + u_{i+1,j} + u_{i+1,j+1} + u_{i,j+1} + u_{i-1,j-1} + u_{i-1,j} + u_{i-1,j-1} + u_{i,j-1} + u_{i+1,j-1}}{12}$$
$$+\frac{4u_{ij} - u_{i+1,j} - u_{i,j+1} - u_{i-1,j} - u_{i,j-1}}{h^2}$$
$$+\frac{2p_{i+1,j} - 2p_{i-1,j} + p_{i+1,j+1} - p_{i-1,j+1} + p_{i+1,j-1} - p_{i-1,j-1}}{6h} = F^1_{ij};$$

$$\frac{8v_{ij}+v_{i+1,j}+v_{i+1,j+1}+v_{i,j+1}+v_{i-1,j-1}+v_{i-1,j}+v_{i-1,j-1}+v_{i,j-1}+v_{i+1,j-1}}{12}$$
$$+\frac{4v_{ij}-v_{i+1,j}-v_{i,j+1}-v_{i-1,j}-v_{i,j-1}}{h^2}$$
$$+\frac{2p_{i,j+1}-2p_{i,j-1}+p_{i+1,j+1}-p_{i+1,j-1}+p_{i-1,j+1}-p_{i-1,j-1}}{6h}=F^2_{ij};$$
$$\frac{-2u_{i+1,j}+2u_{i-1,j}-u_{i+1,j+1}+u_{i-1,j+1}-u_{i+1,j-1}+u_{i-1,j-1}}{6h}$$
$$+\frac{-2v_{i,j+1}+2v_{i,j-1}-v_{i+1,j+1}+v_{i+1,j-1}-v_{i-1,j+1}+v_{i-1,j-1}}{6h}$$
$$+\frac{4p_{ij}-p_{i+1,j}-p_{i,j+1}-p_{i-1,j}-p_{i,j-1}}{h^2}=F^3_{ij}.$$

1b. For $i=0$ and $0<j<2N$, $j=2l$ we have

$$\frac{8v_{0j}+2v_{1j}+2v_{1,j+1}+v_{0,j+1}+v_{0,j-1}+2v_{1,j-1}}{12}+\frac{4v_{0j}-2v_{1j}-v_{0,j+1}-v_{0,j-1}}{h^2}$$
$$+\frac{2p_{0,j+1}-2p_{0,j-1}+2p_{1,j+1}-2p_{1,j-1}}{6h}=2F^2_{0j};$$
$$\frac{-4u_{0j}-2u_{0,j+1}-2u_{0,j-1}}{6h}+\frac{-2v_{0,j+1}+2v_{0,j-1}-2v_{0,j+1}+2v_{0,j-1}}{6h}$$
$$+\frac{4p_{0j}-2p_{1j}-p_{0,j+1}-p_{0,j-1}}{h^2}=2F^3_{0j}.$$

1c. For $i=2N$ and $0<j<2N$, $j=2l$ we have

$$\frac{8v_{2N,j}+v_{2N,j+1}+2v_{2N-1,j-1}+2v_{2N-1,j}+2v_{2N-1,j-1}+v_{2N,j-1}}{12}$$
$$+\frac{4v_{2N,j}-v_{2N,j+1}-2v_{2N-1,j}-v_{2N,j-1}}{h^2}$$
$$+\frac{2p_{2N,j+1}-2p_{2N,j-1}+2p_{2N-1,j+1}-2p_{2N-1,j-1}}{6h}=2F^2_{2N,j};$$
$$\frac{4u_{2N-1,j}+2u_{2N-1,j+1}+2u_{2N-1,j-1}}{6h}+\frac{-2v_{2N,j+1}+2v_{2N,j-1}-2v_{2N-1,j+1}+2v_{2N-1,j-1}}{6h}$$
$$+\frac{4p_{2N,j}-p_{2N,j+1}-2p_{2N-1,j}-p_{2N,j-1}}{h^2}=2F^3_{2N,j}.$$

1d. For $0<i=2l<2N$ and $j=0$ we have

$$\frac{8u_{i0}+u_{i+1,0}+2u_{i+1,1}+2u_{i1}+2u_{i-1,1}+u_{i-1,0}}{12}+\frac{4u_{i0}-u_{i+1,0}-2u_{i1}-u_{i-1,0}}{h^2}$$
$$+\frac{2p_{i+1,0}-2p_{i-1,0}+2p_{i+1,1}-2p_{i-1,1}}{6h}=2F^1_{i0};$$
$$\frac{-2u_{i+1,j}+2u_{i-1,j}-2u_{i+1,j+1}+2u_{i-1,j+1}}{6h}+\frac{-4v_{i,j+1}-2v_{i+1,j+1}-2v_{i-1,j+1}}{6h}$$
$$+\frac{4p_{ij}-p_{i+1,j}-2p_{i,j+1}-p_{i-1,j}}{h^2}=2F^3_{ij}.$$

1e. For $0 < i = 2l < 2N$ and $j = 2N$ we have

$$\frac{8u_{i,2N} + u_{i+1,2N} + 2u_{i+1,2N-1} + 2u_{i,2N-1} + 2u_{i-1,2N-1} + u_{i-1,2N}}{12} + \frac{4u_{i,2N} - u_{i+1,2N} - 2u_{i,2N-1} - u_{i-1,2N}}{h^2} + \frac{2p_{i+1,2N} - 2p_{i-1,2N} + 2p_{i+1,2N-1} - 2p_{i-1,2N-1}}{6h} = 2F^1_{i,2N};$$

$$\frac{-2u_{i+1,2N} + 2u_{i-1,j} - 2u_{i+1,2N-1} + 2u_{i-1,2N-1}}{6h} + \frac{4v_{i,2N-1} + 2v_{i+1,2N-1} + 2v_{i-1,2N-1}}{6h} + \frac{4p_{i,2N} - p_{i+1,2N} - 2p_{i,2N-1} - p_{i-1,2N}}{h^2} = 2F^3_{i,2N}.$$

1g. In the angle nodes we have

$$\frac{-4u_{10} - 4u_{11} - 4v_{01} - 4v_{11}}{6h} + \frac{4p_{00} - 2p_{10} - 2p_{01}}{h^2} = 4F^3_{00}$$
$$\frac{4u_{2N-1,0} + 4u_{2N-1,1} - 4v_{2N,1} - 4v_{2N-1,1}}{6h} + \frac{4p_{2N,0} - 2p_{2N-1,0} - 2p_{2N,1}}{h^2} = 4F^3_{2N,0}$$
$$\frac{-4u_{1,2N} - 4u_{1,2N-1} + 4v_{0,2N-1} + 4v_{1,2N-1}}{6h} + \frac{4p_{0,2N} - 2p_{1,2N} - 2p_{0,2N-1}}{h^2} = 4F^3_{0,2N}$$
$$\frac{4u_{2N-1,2N} + 4u_{2N-1,2N-1} + 4v_{2N,2N-1} + 4v_{2N-1,2N-1}}{6h} + \frac{4p_{2N,2N} - 2p_{2N-1,2N} - 2p_{2N,2N-1}}{h^2} = 4F^3_{2N,2N}.$$

2. Now we consider "odd" nodes of the form $\{(ih, jh)\}$, $i + j = 2l + 1$.
2a. At the inner nodes ($0 < i, j < 2N$) we have

$$\frac{4u_{ij} + u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1}}{12} + \frac{4u_{ij} - u_{i+1,j} - u_{i,j+1} - u_{i-1,j} - u_{i,j-1}}{h^2} + \frac{p_{i+1,j} - p_{i-1,j}}{3h} = F^1_{ij};$$

$$\frac{4v_{ij} + v_{i+1,j} + v_{i,j+1} + v_{i-1,j} + v_{i,j-1}}{12} + \frac{4v_{ij} - v_{i+1,j} - v_{i,j+1} - v_{i-1,j} - v_{i,j-1}}{h^2} + \frac{p_{i,j+1} - p_{i,j-1}}{3h} = F^2_{ij};$$

$$\frac{-u_{i+1,j} + u_{i-1,j}}{3h} + \frac{-v_{i,j+1} + v_{i,j-1}}{3h} + \frac{4p_{ij} - p_{i+1,j} - p_{i,j+1} - p_{i-1,j} - p_{i,j-1}}{h^2} = F^3_{ij}.$$

2b. For $i = 0$, $0 < j = 2l + 1 < 2N$ we have

$$\frac{4v_{0j} + 2v_{1j} + v_{0,j+1} + v_{0,j-1}}{12} + \frac{4v_{0j} - 2v_{1j} - v_{0,j+1} - v_{0,j-1}}{h^2} + \frac{p_{0,j+1} - p_{0,j-1}}{3h} = F^2_{0j};$$

$$\frac{-2u_{1j} - v_{0,j+1} + v_{0,j-1}}{3h} + \frac{4p_{0j} - 2p_{1j} - p_{0,j+1} - p_{0,j-1}}{h^2} = F^3_{0j}.$$

2c. For $i = 2N$, $0 < j = 2l + 1 < 2N$ we have

$$\frac{4v_{2N,j} + 2v_{2N-1,j} + v_{2N,j+1} + v_{2N,j-1}}{12} + \frac{4v_{2N,j} - 2v_{2N-1,j} - v_{2N,j+1} - v_{2N,j-1}}{h^2}$$
$$+ \frac{p_{2N,j+1} - p_{2N,j-1}}{3h} = 2F^2_{2N,j};$$

$$\frac{2u_{2N-1,j} - v_{2N,j+1} + v_{2N,j-1}}{3h} + \frac{4p_{2N,j} - 2p_{2N-1,j} - p_{2N,j+1} - p_{2N,j-1}}{h^2} = 2F^3_{2N,j}.$$

2d. For $0 < i = 2l + 1 < 2N$ and $j = 0$ we have

$$\frac{4u_{i0} + u_{i+1,0} + 2u_{i1} + u_{i-1,0}}{12} + \frac{4u_{i0} - u_{i+1,0} - 2u_{i1} - u_{i-1,0}}{h^2}$$
$$+ \frac{p_{i+1,0} - p_{i-1,0}}{3h} = 2F^1_{i0};$$

$$\frac{-u_{i+1,0} + u_{i-1,0} - 2v_{i1}}{3h} + \frac{4p_{i0} - p_{i+1,0} - 2p_{i1} - p_{i-1,0}}{h^2} = 2F^3_{i0}.$$

2e. For $0 < i = 2l + 1 < 2N$ and $j = 2N$ we have

$$\frac{4u_{i,2N} + u_{i+1,2N} + 2u_{i,2N-1} + u_{i-1,2N}}{12} + \frac{4u_{i,2N} - u_{i+1,2N} - 2u_{i,2N-1} - u_{i-1,2N}}{h^2}$$
$$+ \frac{p_{i+1,2N} - p_{i-1,2N}}{3h} = 2F^1_{i,2N};$$

$$\frac{-u_{i+1,2N} + u_{i-1,2N} + 2v_{i,2N-1}}{3h} + \frac{4p_{i,2N} - p_{i+1,2N} - 2p_{i,2N-1} - p_{i-1,2N}}{h^2} = 2F^3_{i,2N}.$$

Note that in all the equations given above we should set

$$u_{0j} = u_{2N,j} = v_{i0} = v_{i,2N} = 0 \quad \text{for all} \;\; i, j.$$

Now we continue p evenly from Ω_h relative to the coordinate axes. Then the continued difference function $\tilde{p}$ will be defined within the grid square, which is four times larger than Ω_h. If the function u is evenly continued relative to the x-axis and oddly relative to the y-axis and the function v is evenly continued relative to the y-axis and oddly relative to the x-axis, then the difference equations given above are fulfilled for the continued difference functions $\tilde{u}$, $\tilde{v}$, $\tilde{p}$ at the nodes where these equations are meaningful. After the construction of the 2π-periodic continuation for the functions $\tilde{u}$, $\tilde{v}$, $\tilde{p}$ over the entire plane, we obtain the functions for which equations **1a** and **2a** are valid. Thus, in this case the discrete problem is periodically continuable over the entire plane.

Let us consider the grid domains

$$\Omega_{1,h} = \{(ih, jh), \;\; i = 2m, \; j = 2l\},$$
$$\Omega_{2,h} = \{(ih, jh), \;\; i = 2m+1, \; j = 2l\},$$
$$\Omega_{3,h} = \{(ih, jh), \;\; i = 2m, \; j = 2l+1\},$$
$$\Omega_{4,h} = \{(ih, jh), \;\; i = 2m+1, \; j = 2l+1\}.$$

Now we are able to represent $\{\tilde{u},\, \tilde{v},\, \tilde{p}\}$ in the form $\{u^i,\, v^i,\, p^i,\, i = 1,2,3,4\}$, where the difference functions u^i, v^i, p^i are defined on the domains $\Omega_{i,h}$, respectively. Since each of the grid domains $\Omega_{i,h}$ is a grid rectangle, a solution on $\Omega_{i,h}$ can be expanded in a Fourier discrete series.

We shall use the following expansions ensuring the satisfaction of the main boundary conditions:

$$
\begin{aligned}
u^1(x,\, y) &= \sum_{k_1=1}^{N-1} \sum_{k_2=0}^{N} d_{k_2} u^1_{k_1 k_2} \sin k_1 x \cos k_2 y, && (x,\, y) \in \Omega_{1,h}, \\
u^2(x,\, y) &= \sum_{k_1=1}^{N} \sum_{k_2=0}^{N} d_{k_1} d_{k_2} u^2_{k_1 k_2} \sin k_1 x \cos k_2 y, && (x,\, y) \in \Omega_{2,h}, \\
u^3(x,\, y) &= \sum_{k_1=1}^{N-1} \sum_{k_2=0}^{N-1} d_{k_2} u^3_{k_1 k_2} \sin k_1 x \cos k_2 y, && (x,\, y) \in \Omega_{3,h}, \\
u^4(x,\, y) &= \sum_{k_1=1}^{N} \sum_{k_2=0}^{N-1} d_{k_1} d_{k_2} u^4_{k_1 k_2} \sin k_1 x \cos k_2 y, && (x,\, y) \in \Omega_{4,h}, \\
v^1(x,\, y) &= \sum_{k_1=0}^{N} \sum_{k_2=1}^{N-1} d_{k_1} v^1_{k_1 k_2} \cos k_1 x \sin k_2 y, && (x,\, y) \in \Omega_{1,h}, \\
v^2(x,\, y) &= \sum_{k_1=0}^{N-1} \sum_{k_2=1}^{N-1} d_{k_1} v^2_{k_1 k_2} \cos k_1 x \sin k_2 y, && (x,\, y) \in \Omega_{2,h}, \\
v^3(x,\, y) &= \sum_{k_1=0}^{N} \sum_{k_2=1}^{N} d_{k_1} d_{k_2} v^3_{k_1 k_2} \cos k_1 x \sin k_2 y, && (x,\, y) \in \Omega_{3,h}, \\
v^4(x,\, y) &= \sum_{k_1=0}^{N-1} \sum_{k_2=1}^{N} d_{k_1} d_{k_2} v^4_{k_1 k_2} \cos k_1 x \sin k_2 y, && (x,\, y) \in \Omega_{4,h}, \\
p^1(x,\, y) &= \sum_{k_1=0}^{N} \sum_{k_2=0}^{N} d_{k_1} d_{k_2} p^1_{k_1 k_2} \cos k_1 x \cos k_2 y, && (x,\, y) \in \Omega_{1,h} \\
p^2(x,\, y) &= \sum_{k_1=0}^{N-1} \sum_{k_2=0}^{N} d_{k_1} d_{k_2} p^2_{k_1 k_2} \cos k_1 x \cos k_2 y, && (x,\, y) \in \Omega_{2,h}, \\
p^3(x,\, y) &= \sum_{k_1=0}^{N} \sum_{k_2=0}^{N-1} d_{k_1} d_{k_2} p^3_{k_1 k_2} \cos k_1 x \cos k_2 y, && (x,\, y) \in \Omega_{3,h}, \\
p^4(x,\, y) &= \sum_{k_1=0}^{N-1} \sum_{k_2=0}^{N-1} d_{k_1} d_{k_2} p^4_{k_1 k_2} \cos k_1 x \cos k_2 y, && (x,\, y) \in \Omega_{4,h}.
\end{aligned}
\tag{8}
$$

Here $d_j = 1$ for all $0 < j < N$ and $d_0 = d_N = 0.5$. Note that for the grid functions F^2,

F^3, F^4 from (7) we can construct their expansions in the same way with the Fourier coefficients F_k^i, Φ_k^i, Θ_k^i, $i = 1,2,3,4$, where k stands for the multiindex $k = (k_1, k_2)$.

Substituting (8) into system (7) of grid equations, we obtain the following relation between u_k^i, v_k^i, p_k^i and F_k^i, Φ_k^i, Θ_k^i. For all $0 < k_1,\, k_2 < N$ we get:

$$\left(\frac{2}{3}+\frac{4}{h^2}\right)u_k^1+\left(\frac{1}{6}-\frac{2}{h^2}\right)u_k^2\cos k_1h+\left(\frac{1}{6}-\frac{2}{h^2}\right)u_k^3\cos k_2h+\frac{1}{3}u_k^4\cos k_1h\cos k_2h$$
$$-\frac{2}{3h}p_k^2\sin k_1h-\frac{2}{3h}p_k^4\sin k_1h\cos k_2h=F_k^1,$$
$$\left(\frac{1}{6}-\frac{2}{h^2}\right)u_k^1\cos k_1h+\left(\frac{1}{3}+\frac{4}{h^2}\right)u_k^2+\left(\frac{1}{6}-\frac{2}{h^2}\right)u_k^4\cos k_2h-\frac{2}{3h}p_k^1\sin k_1h=F_k^2,$$
$$\left(\frac{1}{6}-\frac{2}{h^2}\right)u_k^1\cos k_2h+\left(\frac{1}{3}+\frac{4}{h^2}\right)u_k^3+\left(\frac{1}{6}-\frac{2}{h^2}\right)u_k^4\cos k_1h-\frac{2}{3h}p_k^4\sin k_1h=F_k^3,$$
$$\frac{1}{3}u_k^1\cos k_1h\cos k_2h+\left(\frac{1}{6}-\frac{2}{h^2}\right)u_k^2\cos k_2h+\left(\frac{1}{6}-\frac{2}{h^2}\right)u_k^3\cos k_1h+\left(\frac{2}{3}+\frac{4}{h^2}\right)u_k^4$$
$$-\frac{2}{3h}p_k^1\sin k_1h\cos k_2h-\frac{2}{3h}p_k^3\sin k_1h=F_k^4;$$

$$\left(\frac{2}{3}+\frac{4}{h^2}\right)v_k^1+\left(\frac{1}{6}-\frac{2}{h^2}\right)v_k^2\cos k_1h+\left(\frac{1}{6}-\frac{2}{h^2}\right)v_k^3\cos k_2h+\frac{1}{3}v_k^4\cos k_1h\cos k_2h$$
$$-\frac{2}{3h}p_k^3\sin k_2h-\frac{2}{3h}p_k^4\cos k_1h\sin k_2h=\Phi_k^1,$$
$$\left(\frac{1}{6}-\frac{2}{h^2}\right)v_k^1\cos k_1h+\left(\frac{1}{3}+\frac{4}{h^2}\right)v_k^2+\left(\frac{1}{6}-\frac{2}{h^2}\right)v_k^4\cos k_2h-\frac{2}{3h}p_k^4\sin k_2h=\Phi_k^2,$$
$$\left(\frac{1}{6}-\frac{2}{h^2}\right)v_k^1\cos k_2h+\left(\frac{1}{3}+\frac{4}{h^2}\right)v_k^3+\left(\frac{1}{6}-\frac{2}{h^2}\right)v_k^4\cos k_1h-\frac{2}{3h}p_k^1\sin k_2h=\Phi_k^3,$$
$$\frac{1}{3}v_k^1\cos k_1h\cos k_2h+\left(\frac{1}{6}-\frac{2}{h^2}\right)v_k^2\cos k_2h+\left(\frac{1}{6}-\frac{2}{h^2}\right)v_k^3\cos k_1h+\left(\frac{2}{3}+\frac{4}{h^2}\right)v_k^4$$
$$-\frac{2}{3h}p_k^1\sin k_2h\cos k_1h-\frac{2}{3h}p_k^2\sin k_2h=\Phi_k^4;$$

$$\frac{2}{3h}u_k^2 \sin k_1 h + \frac{2}{3h}u_k^4 \sin k_1 h \cos k_2 h + \frac{2}{3h}v_k^3 \sin k_2 h + \frac{2}{3h}v_k^4 \cos k_1 h \sin k_2 h$$
$$+\frac{4}{h^2}p_k^1 - \frac{2}{h^2}p_k^2 \cos k_1 h - \frac{2}{h^2}p_k^3 \cos k_2 h = \Theta_k^1,$$
$$\frac{2}{3h}u_k^1 \sin k_1 h + \frac{2}{3h}v_k^4 \sin k_2 h - \frac{2}{h^2}p_k^1 \cos k_1 h + \frac{4}{h^2}p_k^2 - \frac{2}{h^2}p_k^4 \cos k_2 h = \Theta_k^2,$$
$$\frac{2}{3h}u_k^4 \sin k_1 h + \frac{2}{3h}v_k^1 \sin k_2 h - \frac{2}{h^2}p_k^1 \cos k_2 h + \frac{4}{h^2}p_k^3 - \frac{2}{h^2}p_k^4 \cos k_1 h = \Theta_k^3,$$
$$\frac{2}{3h}u_k^1 \sin k_1 h \cos k_2 h + \frac{2}{3h}u_k^3 \sin k_1 h + \frac{2}{3h}v_k^1 \sin k_2 h \cos k_1 h + \frac{2}{3h}v_k^2 \sin k_2 h$$
$$-\frac{2}{h^2}p_k^2 \cos k_2 h - \frac{2}{h^2}p_k^3 \cos k_1 h + \frac{4}{h^2}p_k^4 = \Theta_k^4.$$

For $0 < k_1 < N$ and $k_2 = 0$ we get:

$$\left(\frac{2}{3} + \frac{4}{h^2}\right) u_{k_1,0}^1 + \left(\frac{1}{6} - \frac{2}{h^2}\right) u_{k_1,0}^2 \cos k_1 h + \left(\frac{1}{6} - \frac{2}{h^2}\right) u_{k_1,0}^3 + \frac{1}{3}u_{k_1,0}^4 \cos k_1 h$$
$$-\frac{2}{3h}p_{k_1,0}^2 \sin k_1 h - \frac{2}{3h}p_{k_1,0}^4 \sin k_1 h = F_{k_1,0}^1,$$
$$\left(\frac{1}{6} - \frac{2}{h^2}\right) u_{k_1,0}^1 \cos k_1 h + \left(\frac{1}{3} + \frac{4}{h^2}\right) u_{k_1,0}^2 + \left(\frac{1}{6} - \frac{2}{h^2}\right) u_{k_1,0}^4 - \frac{2}{3h}p_{k_1,0}^1 \sin k_1 h = F_{k_1,0}^2,$$
$$\left(\frac{1}{6} - \frac{2}{h^2}\right) u_{k_1,0}^1 + \left(\frac{1}{3} + \frac{4}{h^2}\right) u_{k_1,0}^3 + \left(\frac{1}{6} - \frac{2}{h^2}\right) u_{k_1,0}^4 \cos k_1 h - \frac{2}{3h}p_{k_1,0}^4 \sin k_1 h = F_{k_1,0}^3,$$
$$\frac{1}{3}u_{k_1,0}^1 \cos k_1 h + \left(\frac{1}{6} - \frac{2}{h^2}\right) u_{k_1,0}^2 + \left(\frac{1}{6} - \frac{2}{h^2}\right) u_{k_1,0}^3 \cos k_1 h + \left(\frac{2}{3} + \frac{4}{h^2}\right) u_{k_1,0}^4$$
$$-\frac{2}{3h}p_{k_1,0}^1 \sin k_1 h - \frac{2}{3h}p_{k_1,0}^3 \sin k_1 h = F_{k_1,0}^4;$$

$$\frac{2}{3h}u_{k_1,0}^2 \sin k_1 h + \frac{2}{3h}u_{k_1,0}^4 \sin k_1 h + \frac{4}{h^2}p_{k_1,0}^1 - \frac{2}{h^2}p_{k_1,0}^2 \cos k_1 h - \frac{2}{h^2}p_{k_1,0}^3 = \Theta_{k_1,0}^1,$$
$$\frac{2}{3h}u_{k_1,0}^1 \sin k_1 h + \frac{2}{h^2}p_{k_1,0}^1 \cos k_1 h + \frac{4}{h^2}p_{k_1,0}^2 - \frac{2}{h^2}p_{k_1,0}^4 = \Theta_{k_1,0}^2,$$
$$\frac{2}{3h}u_{k_1,0}^4 \sin k_1 h + \frac{2}{h^2}p_{k_1,0}^1 + \frac{4}{h^2}p_{k_1,0}^3 - \frac{2}{h^2}p_{k_1,0}^4 \cos k_1 h = \Theta_{k_1,0}^3,$$
$$\frac{2}{3h}u_{k_1,0}^1 \sin k_1 h + \frac{2}{3h}u_{k_1,0}^3 \sin k_1 h - \frac{2}{h^2}p_{k_1,0}^2 - \frac{2}{h^2}p_{k_1,0}^3 \cos k_1 h + \frac{4}{h^2}p_{k_1,0}^4 = \Theta_{k_1,0}^4.$$

When $0 < k_1 < N$ and $k_2 = N$, we get (note that in this case $k_2 h = \pi/2$):

$$\left(\frac{2}{3}+\frac{4}{h^2}\right)u^1_{k_1,N} + \left(\frac{1}{6}-\frac{2}{h^2}\right)u^2_{k_1,N}\cos k_1 h - \frac{2}{3h}p^2_{k_1,N}\sin k_1 h = F^1_{k_1,N},$$

$$\left(\frac{1}{6}-\frac{2}{h^2}\right)u^1_{k_1,N}\cos k_1 h + \left(\frac{1}{3}+\frac{4}{h^2}\right)u^2_{k_1,N} - \frac{2}{3h}p^1_{k_1,N}\sin k_1 h = F^2_{k_1,N},$$

$$\left(\frac{1}{3}+\frac{4}{h^2}\right)v^3_{k_1,N} + \left(\frac{1}{6}-\frac{2}{h^2}\right)v^4_{k_1,N}\cos k_1 h - \frac{2}{3h}p^1_{k_1,N} = \Phi^3_{k_1,N},$$

$$\left(\frac{1}{6}-\frac{2}{h^2}\right)v^3_{k_1,N}\cos k_1 h + \left(\frac{2}{3}+\frac{4}{h^2}\right)v^4_{k_1,N} - \frac{2}{3h}p^1_{k_1,N}\cos k_1 h - \frac{2}{3h}p^2_{k_1,N} = \Phi^4_{k_1,N};$$

$$\frac{2}{3h}u^2_{k_1,N}\sin k_1 h + \frac{2}{3h}v^3_{k_1,N} + \frac{2}{3h}v^4_{k_1,N}\cos k_1 h + \frac{4}{h^2}p^1_{k_1,N} - \frac{2}{h^2}p^2_{k_1,N}\cos k_1 h = \Theta^1_{k_1,N},$$

$$\frac{2}{3h}u^1_{k_1,N}\sin k_1 h + \frac{2}{3h}v^4_{k_1,N} - \frac{2}{h^2}p^1_{k_1,N}\cos k_1 h + \frac{4}{h^2}p^2_{k_1,N} = \Theta^2_{k}.$$

When $k_1 = 0$ and $0 < k_2 < N$, we get

$$\begin{aligned}&\left(\frac{2}{3}+\frac{4}{h^2}\right)v^1_{0,k_2} + \left(\frac{1}{6}-\frac{2}{h^2}\right)v^2_{0,k_2} + \left(\frac{1}{6}-\frac{2}{h^2}\right)v^3_{0,k_2}\cos k_2 h + \frac{1}{3}v^4_{0,k_2}\cos k_2 h\\ &\quad -\frac{2}{3h}p^3_{0,k_2}\sin k_2 h - \frac{2}{3h}p^4_{0,k_2}\sin k_2 h = \Phi^1_{k},\end{aligned}$$

$$\left(\frac{1}{6}-\frac{2}{h^2}\right)v^1_{0,k_2} + \left(\frac{1}{3}+\frac{4}{h^2}\right)v^2_{0,k_2} + \left(\frac{1}{6}-\frac{2}{h^2}\right)v^4_{0,k_2}\cos k_2 h - \frac{2}{3h}p^4_{0,k_2}\sin k_2 h = \Phi^2_{0,k_2},$$

$$\left(\frac{1}{6}-\frac{2}{h^2}\right)v^1_{0,k_2}\cos k_2 h + \left(\frac{1}{3}+\frac{4}{h^2}\right)v^3_{0,k_2} + \left(\frac{1}{6}-\frac{2}{h^2}\right)v^4_{0,k_2} - \frac{2}{3h}p^1_{0,k_2}\sin k_2 h = \Phi^3_{0,k_2},$$

$$\begin{aligned}&\frac{1}{3}v^1_{0,k_2}\cos k_2 h + \left(\frac{1}{6}-\frac{2}{h^2}\right)v^2_{0,k_2}\cos k_2 h + \left(\frac{1}{6}-\frac{2}{h^2}\right)v^3_{0,k_2} + \left(\frac{2}{3}+\frac{4}{h^2}\right)v^4_{0,k_2}\\ &\quad -\frac{2}{3h}p^1_{0,k_2}\sin k_2 h - \frac{2}{3h}p^2_{0,k_2}\sin k_2 h = \Phi^4_{0,k_2};\end{aligned}$$

$$\frac{2}{3h}v^3_{0,k_2}\sin k_2 h + \frac{2}{3h}v^4_{0,k_2}\sin k_2 h + \frac{4}{h^2}p^1_{0,k_2} - \frac{2}{h^2}p^2_{0,k_2} - \frac{2}{h^2}p^3_{0,k_2}\cos k_2 h = \Theta^1_{0,k_2},$$

$$\frac{2}{3h}v^4_{0,k_2}\sin k_2 h - \frac{2}{h^2}p^1_{0,k_2} + \frac{4}{h^2}p^2_{0,k_2} - \frac{2}{h^2}p^4_{0,k_2}\cos k_2 h = \Theta^2_{0,k_2},$$

$$\frac{2}{3h}v^1_{0,k_2}\sin k_2 h - \frac{2}{h^2}p^1_{0,k_2}\cos k_2 h + \frac{4}{h^2}p^3_{0,k_2} - \frac{2}{h^2}p^4_{0,k_2} = \Theta^3_{0,k_2},$$

$$\frac{2}{3h}v^1_{0,k_2}\sin k_2 h + \frac{2}{3h}v^2_{0,k_2}\sin k_2 h - \frac{2}{h^2}p^2_{0,k_2}\cos k_2 h - \frac{2}{h^2}p^3_{0,k_2} + \frac{4}{h^2}p^4_{0,k_2} = \Theta^4_{k_2}.$$

When $k_1 = N$ and $0 < k_2 < N$, we get

$$\left(\frac{1}{3}+\frac{4}{h^2}\right) u^2_{N,k_2} + \left(\frac{1}{6}-\frac{2}{h^2}\right) u^4_{N,k_2} \cos k_2 h - \frac{2}{3h} p^1_{N,k_2} = F^2_{N,k_2},$$

$$\left(\frac{1}{6}-\frac{2}{h^2}\right) u^2_{N,k_2} \cos k_2 h + \left(\frac{2}{3}+\frac{4}{h^2}\right) u^4_{N,k_2} - \frac{2}{3h} p^1_{N,k_2} \cos k_2 h - \frac{2}{3h} p^3_{N,k_2} = F^4_{N,k_2};$$

$$\left(\frac{2}{3}+\frac{4}{h^2}\right) v^1_{N,k_2} + \left(\frac{1}{6}-\frac{2}{h^2}\right) v^3_{N,k_2} \cos k_2 h - \frac{2}{3h} p^3_{N,k_2} \sin k_2 h = \Phi^1_k,$$

$$\left(\frac{1}{6}-\frac{2}{h^2}\right) v^1_{N,k_2} \cos k_2 h + \left(\frac{1}{3}+\frac{4}{h^2}\right) v^3_{N,k_2} - \frac{2}{3h} p^1_{N,k_2} \sin k_2 h = \Phi^3_{N,k_2},$$

$$\frac{2}{3h} u^2_{N,k_2} + \frac{2}{3h} u^4_{N,k_2} \cos k_2 h + \frac{2}{3h} v^3_{N,k_2} \sin k_2 h + \frac{4}{h^2} p^1_{N,k_2} - \frac{2}{h^2} p^3_{N,k_2} \cos k_2 h = \Theta^1_{N,k_2},$$

$$\frac{2}{3h} u^4_{N,k_2} + \frac{2}{3h} v^1_{N,k_2} \sin k_2 h - \frac{2}{h^2} p^1_{N,k_2} \cos k_2 h + \frac{4}{h^2} p^3_{N,k_2} = \Theta^3_{N,k_2}.$$

When $k_1 = k_2 = N$, we have only the one equation

$$\frac{4}{h^2} p^1_{NN} = \Theta^1_{NN}.$$

For $k_1 = k_2 = 0$ we obtain the system

$$\begin{aligned}
\frac{4}{h^2} p^1_{00} - \frac{2}{h^2} p^2_{00} - \frac{2}{h^2} p^3_{00} &= \Theta^1_{00},\\
-\frac{2}{h^2} p^1_{00} + \frac{4}{h^2} p^2_{00} - \frac{2}{h^2} p^4_{00} &= \Theta^2_{00},\\
-\frac{2}{h^2} p^1_{00} + \frac{4}{h^2} p^3_{00} - \frac{2}{h^2} p^4_{00} &= \Theta^3_{00},\\
-\frac{2}{h^2} p^2_{00} - \frac{2}{h^2} p^3_{00} + \frac{4}{h^2} p^4_{00} &= \Theta^4_{00}.
\end{aligned} \tag{9}$$

All these systems are uniquely solvable, since their matrices are positive definite. We can solve each system of this group using finite (independent of N) number of arithmetic operations.

Note that the matrix of system (9) has one zero eigenvalue with the eigenvector $e = (1, 1, 1, 1)^T$. System (9) is solvable, since the vector $(\Theta^1_{00}, \Theta^2_{00}, \Theta^3_{00}, \Theta^4_{00})^T$ is orthogonal to e.

Now we are able to determine the coefficients u^i_k, v^i_k, p^i_k using $O\left(N^2\right)$ arithmetic operations and, as a result, to construct the solution u, v, p from (8) using FFT. Thus, the total number of arithmetic operations is $O\left(N^2 \log_2 N\right)$.

3 Numerical example

For the numerical experiment in the square domain

$$\Omega = \left\{ (x,\, y) \,|\, 0 \le x \le \pi,\ 0 \le y \le \pi \right\}$$

we consider the following boundary value problem

$$
\begin{aligned}
-\Delta u + u + \frac{\partial p}{\partial x} &= 3\sin x \cos y + \frac{24(2x+y^3)}{\pi^4(8+3\pi^2)} && \text{in} \quad \Omega\,, \\
-\Delta v + v + \frac{\partial p}{\partial y} &= -3\cos x \sin y + \frac{72xy^2}{\pi^4(8+3\pi^2)} && \text{in} \quad \Omega\,, \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 && \text{in} \quad \Omega\,, \\
w\cdot n &= 0 && \text{on} \quad \partial\Omega\,, \\
\frac{\partial\,(w\cdot\tau)}{\partial n} &= 0 && \text{on} \quad \partial\Omega\,, \\
\int\limits_{\Omega} p\,dxdy &= 0,
\end{aligned}
\tag{10}
$$

where $w = (u,\,v)$. The exact solution of (10) is

$$
u = \sin x \cos y, \quad v = -\cos x \sin y,
$$
$$
p = \frac{24(x^2+xy^3)}{\pi^4(8+3\pi^2)} - 1.
$$

Using the method described in the previous section, we obtained the approximate solution $(\omega_h, u_h, v_h, p_h)$. Denote

$$
\begin{aligned}
\|e\| &= \|\omega-\omega_h\| + \|p-p_h\| + \|u-u_h\| + \|v-v_h\|\,, \\
\|e\|_1 &= \|\omega-\omega_h\|_1 + \|p-p_h\|_1 + \|u-u_h\|_1 + \|v-v_h\|_1\,.
\end{aligned}
$$

The following table demonstrates the behavior of the error with respect to $L_2(\Omega)$- and $H^1(\Omega)$-norms.

π/h	$\|e\|$	$\|e\|_1$
8	0.10576	0.37965
16	0.02752	0.20099
32	0.00736	0.10152
64	0.00190	0.05119
128	0.00047	0.02579

These numerical results illustrate the second-order convergence in the L_2-norm and the first-order convergence in the H^1-norm as $h \to 0$.

References

[1] I.O. Arushanian, O. Axelsson, and G.M. Kobelkov, *Implementation of a least-squares finite element method for solving the Poisson equation.* Department of Mathematics, University of Nijmegen, the Netherlands, Report No. 9717 (June 1997).

[2] I.O. Arushanian and G.M. Kobelkov, *Implementation of a least-squares finite element method for solving the Stokes problem with periodic boundary conditions.* Subroutine Libraries and Application Program Packages, Science Research Computer Center, Moscow State University, Moscow, 1998, 57–67 (in Russian).

[3] O. Axelsson, *Iterative Solution Methods.* Cambridge University Press, 1994.

[4] P. Bochev and M. Gunzburger, *Analysis of least-squares finite element methods for the Stokes equations.* Math. Comp., 63 (1994), 479–506.

[5] Z. Cai, T.A. Manteuffel, and S.F. McCormick, *First-order system least squares for the Stokes equations, with application to linear elasticity.* SIAM J. Numer. Anal., V. 35, No 5(1997), 1727-1741.

[6] B.-N. Jiang and L.A. Povinelli, *Least-squares finite element method for fluid dynamics.* Computer Methods in Appl. Mech. and Engineering, 81 (1990), 13–37.

[7] B.-N. Jiang and L.A. Povinelli, *Optimal least-squares finite element method for elliptic problems.* Computer Methods in Appl. Mech. and Engineering, 102 (1993), 199–212.

[8] G.M. Kobelkov, *Numerical methods for solving Navier–Stokes equations in velocity-pressure variables.* Computational Processes and Systems, Nauka, Moscow, Issue 8, 1991, 204–236 (in Russian).

[9] G.M. Kobelkov, *Solution of the problem on stiff contact for an incompressible material.* Zhurnal Vychislitel'noi Matem. i Matem. Fiz., 4 (1976), 987–995 (in Russian).

[10] S. Turek, *On discrete projection methods for the incompressible Navier-Stokes equations: an algorithmical approach.* Computer Methods in Appl. Mech. and Engineering, 143 (1997), 271–288.