Symmetric functions, I:

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“How symmetric can a function be?”

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ABSTRACT

The symmetric complexity of a polynomial $f$ in $n$ variables is defined as the number of times the symmetric function theorem is applicable. In this paper a sharp upper bound on this measure is derived by a matrix method.

1 Introduction

Consider a field $K$ of characteristic 0, and let $R$ be the ring $K[x_1, \ldots, x_n]$ where $n$ is $> 0$.

A symmetric function is any element of $R$ invariant under the symmetric group acting as coordinate permutations. Examples are the elementary symmetric functions: $a_0 = 1$, $a_i = \sum_{1 \leq j_1 < j_2 < \ldots < j_i \leq n} x_{j_1} x_{j_2} \cdots x_{j_i}$, $(1 \leq i \leq n)$; $a_i = 0$ ($i < 0$ or $i > n$).
The Symmetric Function Theorem [4, 5] states that any symmetric function \( f \) can be uniquely written as \( g(a_1, \ldots, a_n) \) for some \( g = g(x_1, \ldots, x_n) \) from \( \mathbb{R} \), called the symmetric representation of \( f \).

Here we shall address the question of what happens when this \( g \) is symmetric again. This is of course perfectly possible and if it occurs \( k - 1 \) times, \( f \) is called \( k \)-fold symmetric. That is:

**Definition 1** A polynomial \( f \) in \( n \) variables is 0-fold symmetric if \( f \) is not symmetric; and \( k \)-fold symmetric with \( k > 0 \) if \( f \) is symmetric and the symmetric representation of \( f \) is \( k - 1 \)-fold symmetric. The number \( k \) is called the symmetric complexity of \( f \).

A \( k \)-fold symmetric function \( f \) possesses a high degree of symmetry indeed, and it is an interesting complexity problem to find a bound on \( k \) expressed in the coefficients and exponents of \( f \). Such a result is given in Theorem 1. Our method is based on term orderings and the like, familiar from Groebner basis theory [3]. Thus it is possible to translate the problem into linear algebra, involving the explicit calculation of the spectrum and eigenvectors of a matrix.

Another interesting question that arises in a natural way in this context is: how can we describe the behavior (e.g., fixpoints) of the iteration \((x_1, \ldots, x_n) \mapsto (a_1, \ldots, a_n)\)? We shall restrict ourselves to a numerical example for \( n = 4 \) (see Section 4).
2 Notations and generalities

Put $x = (x_1, \ldots, x_n)$; let $a_i = a_i(x)$ be defined as above, and let $a = (a_1, \ldots, a_n)$.

Stretching notation a bit, we can view \( \{c_1, \ldots, c_n\} \to a(c) \) as a mapping from the unordered lists of length \( n \) over \( K \) to \( K^n \), which is a bijection if \( K \) is algebraically closed. Indeed, one has \( \sum_{i=0}^{n} a_i(c_1, \ldots, c_n)T^i = \prod_{i=0}^{n}(c_iT + 1) \).

Instead of this however we shall consider the simpler mappings \( c \to a(c) \) from \( K^n \) to \( K^n \) and \( a : \bar{x} \to a(x) \) from \( R \) to \( R \).

**Definition 2** Let \( a^0 = (x_1, \ldots, x_n) \); and for \( k > 0 \) define \( a^k = (a^k_1, a^k_2, \ldots, a^k_n) \) where \( a^k_i = a^k_i(x) = a_i(a^{k-1}_1, a^{k-1}_2, \ldots, a^{k-1}_n) \), \( 1 \leq i \leq n \).

The \( a^k_i \) are called the *iterated elementary symmetric functions* (iesf’s.) An interesting fact is given by

**Lemma 1** For all \( k \geq 1 \), the iesf’s \( a^k_1, a^k_2, \ldots, a^k_n \) are algebraically independent.

**Proof.** Induction w.r.t. \( k \). For \( k = 1 \) this is well-known [5]. Now let \( f(y_1, \ldots, y_n) \) be such that \( f(a^{k+1}_1, a^{k+1}_2, \ldots, a^{k+1}_n) = 0 \) in \( R \).

By definition of the \( a^k_i \)’s, there exists a symmetric polynomial \( g(z_1, \ldots, z_n) = f(a_1(z), a_2(z), \ldots, a_n(z)) \) with \( g(a^k_1, a^k_2, \ldots, a^k_n) = f(a^{k+1}_1, a^{k+1}_2, \ldots, a^{k+1}_n) = 0 \); hence \( g(z_1, \ldots, z_n) = 0 \) by the induction hypothesis. But now we are in the case \( k = 1 \) again, since \( g(z_1, \ldots, z_n) = f(a_1(z), a_2(z), \ldots, a_n(z)) \) and it follows that \( f(y_1, \ldots, y_n) = 0 \). ■
A term is any monomial \( t = x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n} \). Its total degree is \( t\text{deg}(t) = \sum_{j=1}^n i_j \) and the total degree \( t\text{deg}(f) \) of \( f \in R \) is \( \max_{t \in f} \ t\text{deg}(t) \) (which of course is equal to \( t\text{deg}(t) \), any \( t \) in \( f \) if \( f \) is symmetric.)

An admissible ordering [3] on the set \( T \) of terms in \( R \) is a total order on \( T \) that satisfies:

1. \( t < t' \) and \( t' < t' \) for all terms \( s, t, t' \).

The latter property is called monotonicity of term multiplication.

An admissible ordering is a well-ordering. Admissible orders abound and have been classified; well-known examples are the lexicographic orders and various total degree orderings like the "grevlex" [3].

For a given ordering, the leading term \( lt(f) \) of \( f \) is the highest term occurring in \( f \).

3 The main theorem

Our main result is given by

**Theorem 1** Let \( f \) be any non-constant \( k \)-fold symmetric polynomial in \( n \geq 2 \) variables. Then the symmetric complexity \( k \) is bounded by:

\[
\text{tdeg}(f) \geq \frac{(2n + 1)^k - 1}{\pi^{k-1}} \{1.149 - 1.048(0.53)^{k-1}\}
\]
Remark: This bound is fairly precise: it is an approximation of a more complex bound, which is sharp in the sense that it is reached by \( f = a_k^1 \). This will follow from the proof.

First let us give an outline of the proof. The idea is very simple and consists of three steps.

i. If \( k \) increases, one observes that the iesf's \( a_k^i \) grow very quickly in "size".

To measure this size, we consider the highest term \( t_k^1 \) of \( a_k^i \) in an admissible ordering.

Remark: Explicit calculation of the complete \( a_k^i \)'s in Maple, say, leads to considerable memory problems. A piece of code to experiment with is given on the WWW at http://www.cs.kun.nl/bolke/ksymmaple.

ii. Next, we shall be able to estimate the exponents occurring in \( t_k^1 \); this is the technical part.

iii. Finally, for a given \( f \) of complexity \( k \) we shall show that for some \( i \), a term \( t_k^1 \) actually occurs in \( f \) as \( lt(f) \). Hence, \( k \) is bounded as a function of \( lt(f) \), and this ends the proof.

As an admissible ordering on \( T \), take the lexicographic order with \( x_1 > x_2 > \ldots > x_n \). Let \( t_k^1 \) be \( lt(a_k^i) \). We shall derive a recursion for \( t_k^1 \).

**Lemma 2** a. \( t_k^1 = t_{n-1}^{k-1} t_{n-1}^{k-1} \ldots t_{n-i+1}^{k-1} (k > 1) \) b. If \( p > q \), \( t_p^k > t_q^k (k \geq 1) \).

**Proof:**

For \( k = 1 \), statement b. holds. Indeed, \( t_1^1 = lt(a_1) = x_1 x_2 \ldots x_i \). Also,
a. holds trivially. Now if for any $k$ $a.$ and $b.$ are true, then by definition one has $a_i^{k+1} = \sum_{1 \leq j \leq n} a_i^j a_{j_2}^k \ldots a_{j_n}^k$. All coefficients are positive, so no terms cancel. By the monotonicity property, $lt(a_{j_1}^k a_{j_2}^k \ldots a_{j_n}^k) = lt(a_{j_1}^k) lt(a_{j_2}^k) \ldots lt(a_{j_n}^k) = t_{j_1}^k t_{j_2}^k \ldots t_{j_n}^k$. Since $b.$ holds and, again, by monotonicity, this is maximal if $j_i = n$, $j_{i-1} = n-1$, ..., $j_1 = n-i+1$. This proves $a.$ for index $k+1$. But then, if $p > q$ one has $t_p^{k+1} > t_q^{k+1}$ since the r.h.s. divides the l.h.s. Hence $b.$ holds as well. 

In part ii. of the proof, we shall estimate the size of the exponents in $t_i^k$.

**Definition 3** The exponents vector $ev(t)$ of a term $t = x_1^{i_1} \ldots x_n^{i_n}$ is $i = (i_1, i_2, \ldots, i_n)$. We denote $ev(t_i^k)$ by $e_i^k = (e_{i,1}^k, \ldots, e_{i,n}^k)$.

One has $e_i^k = (1, 1, \ldots, 1, 0, \ldots, 0)$ ($i$ ones). Define $E_k$ to be the matrix having the $e_i^k$’s as its columns; note that $E_1 = U$, the upper triangular all-one matrix.

**Lemma 3** a. Let $t = x_1^{i_1} \ldots x_n^{i_n}$ be any term; then for all $k \geq 1$ the exponents vector of $lt(t(a_1^k, \ldots, a_n^k))$ equals $E_k(i)$.

b. Let $D$ be the symmetric matrix with ones below and on the antidiagonal and zeroes above; put $D^k = (d_{i,j}), 1 \leq i, j \leq n$. Let $U$ be the upper triangular all-one matrix. Then $E_k = U D^k - 1$. Hence $E_k$ is nonsingular and for $k \geq 1$ one has: $e_{i,a}^k = \sum_{j=a}^a d_{k-1,j} e_{j,a}$.

**Proof:**

By monotonicity, $lt((a_1^k)^{i_1} \ldots (a_n^k)^{i_n}) = (t_1^k)^{i_1} \ldots (t_n^k)^{i_n}$, the exponents vector of
which is $E_k(i)$ by linearity. This proves part $a$.

For part $b$, note that statement $a$ of Lemma 2 can be written as: 

$$\xi^k = \xi^{k-1}_n + \xi^{k-1}_{n-1} + \ldots + \xi^{k-1}_{n-i+1},$$

which is equivalent to $E_k = E_{k-1}D$. So $E_k = E_1D^{k-1} = UD^{k-1}$. 

In the Corollary to Proposition 2 we shall find an explicit solution to this recursion.

Before analyzing this, let us first proceed to part $iii$. Suppose that $f$ is not constant and $k$-fold symmetric, $k \geq 1$. We wish to prove that some $t^k_i$ really occurs in $f$.

By definition, there exists $f_k \in R$ such that $f_k(a_1^k, \ldots, a_n^k) = f$ (though we shall not need it, note that $f_k$ is unique by Lemma 1.) Let $t = x_1^{i_1} \ldots x_n^{i_n}$ be a term of the polynomial $f_k(x)$ such that $\tau = D_{ij} \text{lt}((a_1^k)^{i_1}, \ldots, (a_n^k)^{i_n})$ is maximal in the term ordering. By Lemma 3, $ev(\tau) = E_k(i_1, \ldots, i_n) = E_k(i)$.

First note that $\tau$ is unique. Indeed, suppose that besides $t$ there is another term $s = x_1^{j_1} \ldots x_n^{j_n}$ yielding the same $\tau$, then by lemma 3 one would have $E_k(i) = E_k(j)$ (with $j = (j_1, \ldots, j_n)$); hence $E_k(i - j) = 0$. But $E_k$ was nonsingular so $i = j$ and $s = t$.
Also, $\tau$ does not cancel when $f_k(a_1^k, \ldots, a_n^k)$ is expanded to $f$. Otherwise, there would be some term $s$ in $f_k$ and a term $\sigma$ from $s(a_1^k, \ldots, a_n^k)$ such that $\tau = \sigma$. (N.b. all these terms are in $R$, i.e. of the form $x_1^{p_1} \ldots x_n^{p_n}$.) Then however, $\sigma < lt(s(a^k)) < lt(t(a^k))$. This contradicts the unicity of $t$ and the maximality of $\tau$.

We conclude that $\tau = lt(f)$. This shows what we wanted, namely that some $t_i^k$ occurs in $t$, hence in $f$. $\blacksquare$

In fact we have proven more, namely:

**Proposition 1** Let $U$ be the upper triangular all-one matrix and $D$ the (symmetric) lower antitriangular all-one matrix. Then for any $k$-fold symmetric function $f$ and $k \geq 1$,

$$ev(lt(f)) \in UD^{k-1}(((\mathbb{N} \cup \{0\})^n).$$

How good is this? In order to answer this question let us give an estimate of the entries of powers of $D$.

For $p = 1, 2, \ldots, n$ let us define the following quantities:

$$w_p = -e^{-\frac{2\pi p}{2n+1}};$$

$$\alpha_p = w_p + w_p^{-1} = -2\cos\left(\frac{2p\pi}{2n+1}\right);$$

$$V_p = w_p - w_p^{-1} = 2\sin\left(\frac{2p\pi}{2n+1}\right);$$

$$\lambda_p = 4\cos^2\left(\frac{p\pi}{2n+1}\right);$$

$$\mu_p = (-1)^n / 2\cos\left(\frac{p\pi}{2n+1}\right);$$

$$x_p^m = 2(-1)^{m+1} \sin\left(\frac{2p\pi}{2n+1}\right) (m = 1, 2, \ldots, n);$$

$$x^p = (x_1^p, x_2^p, \ldots, x_n^p).$$
These numbers satisfy the relations:

\[
\lambda_p = 2 + \alpha_p; \quad V_p^2 = \alpha_p^2 - 4; \quad w_p^{2n+1} = -1;
\]

\[
\mu_p = \frac{1}{(w_p^2 + w_p^{-2})}; \quad \mu_p^{-2} = \lambda_p; \quad x_p^m = \frac{(w_p^m - w_p^{-m})}{i\sqrt{2m+1}} (m = 1, 2, \ldots, n); \text{ also,}
\]

\[
w_p = \frac{(\alpha_p + V_p)}{2} \text{ and } w_p^{-1} = \frac{(\alpha_p - V_p)}{2}
\]
are the roots of \(X^2 - \alpha_pX + 1 = 0\).

Let \(\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y}_i\) be the standard Hermitian inner product. It is elementary to verify that the \(x_p\) are perpendicular of length 1. Now one has:

**Proposition 2** The vectors \(x_p\) form an orthonormal basis upon which the matrix \(D\) assumes a diagonal form \(\Delta = \text{Diag}(\mu_1, \mu_2, \ldots, \mu_n)\).

**Proof.**

Since the proof is fairly standard, let us just outline it. One easily verifies that the inverse of \(D\) is the matrix with ones on the antidiagonal, -1's just above it, and zeroes elsewhere. Next, its square \(D^{-2}\) is seen to be tridiagonal:

\[
(D^{-2})_{i,i} = 2 \ (i < n); \quad (D^{-2})_{n,n} = 1; \quad (D^{-2})_{i,j} = -1 \ (|i - j| = 1).
\]

Tridiagonal matrices have been studied extensively in the theory of orthogonal polynomials [2] and the numerical theory of parabolic differential equations.

\(D^{-2}\), being symmetric, can be diagonalized on a real orthonormal basis. Let \(\tilde{z} = (z_1, z_2, \ldots, z_n)\) be an eigenvector of \(D^{-2}\) with eigenvalue \(\lambda\). Put \(\tilde{z} = \tilde{z}(\alpha)\), again with \(\alpha = 2 - \lambda\). W.l.o.g, let \(z_1 = 1\) and let \(z_0 = D_{ef} 0\). Then \((D^{-2} - \lambda I)\tilde{z} = 0\) amounts to the recursion
Let $z_0 = 0; z_1 = 1$;

$$z_m = az_{m-1} - z_{m-2} \quad (1 < m \leq n);$$

$$-z_{n-1} + (\alpha - 1)z_n = 0$$

(the latter being the characteristic equation.)

**Remark:** this is the familiar recursion of the Tchebycev polynomials $T_{m-1}(x)$ in $x = \frac{\alpha}{2}$, though these have initial values $T_0 = 1, T_1 = x$. In fact it is not difficult to prove that $z_m = \frac{(\frac{\alpha}{2})^m - T_{m-1}(\frac{\alpha}{2})}{((\frac{\alpha}{2})^2 - 1)}$.

Let $V = \sqrt{(\alpha^2 - 4)}$ and $w = (\alpha + V)/2, w' = (\alpha - V)/2$, the roots of $X^2 - \alpha X + 1 = 0$. If $w = w', \alpha = \pm 2$, but then $z_m = (\pm 1)^{m-1} m, -z_{n-1} + (\alpha - 1)z_n \neq 0$, and there are no eigenvalues. So suppose $w \neq w'$.

Solving the recursion by standard techniques yields $z_m = \frac{(w^{m} - w^{l-m})}{V}$; $1 \leq m \leq n$. By some easy calculations, the eigenvalue equation $-z_{n-1} + (\alpha - 1)z_n = 0$ reduces to $w^{2n+1} = -1$ (where $w \neq -1$ since $w \neq w'$). From this, $w = -e^{\frac{2\pi p}{n+1}}, p = 1, 2, \ldots, n$. We shall now take this $p$ as an index (i.e., use $\alpha_p, \lambda_p, \mu_p, w_p, V_p, z_p^m, \tilde{z}_p^m, x_p^m, \bar{x}_p^m$).

The numbers and vectors $\alpha_p, \lambda_p, \mu_p, w_p, V_p, x_p^m, \bar{x}_p^m$ are in fact those defined earlier. Normalization of $V_pz_p^m$ yields the $p^{th}$ eigenvector $\vec{z}_p^m$ as $x_p^m = 2(-1)^{m+1} \sin\left(\frac{\pi p \pm \pi}{2n+1}\right)$. Similarly, one finds the formulas for $\alpha_p, \lambda_p$ etc.

The $\vec{z}_p^m(\alpha)$ form an orthogonal eigenbasis over which the symmetric matrix $D^{-2}$ diagonalizes. But in fact by an easy calculation, $D^{-1}\vec{z}_p^m = \mu_p^{-1} \vec{z}_p^m$; hence
$D^{-1}$ and $D$ diagonalize as well. This ends the proof.

Note that the eigenvalues $\mu_p$ of $D$ are all different and $\max_p |\mu_p| = |\mu_n| = \frac{1}{2 \cos(\frac{\pi}{2n+1})}$. Also, sign $\mu_p = (-1)^{p+1}$ (consider $pn \mod 2n + 1$ for $p$ odd and $p$ even).

**Corollary**

The (nonnegative integral) entries of $D^k$ are given in closed form by the formula

$$(D^k)_{i,j} = \sum_{p=1}^{n} (-1)^{i+j+(n+p)k} \frac{\sin\left(\frac{2p\pi}{2n+1}\right) \sin\left(\frac{2j\pi}{2n+1}\right)}{(2n+1)2^{k-2}\cos\left(\frac{\pi}{2n+1}\right)}$$

**Proof:**

As before, let $\Delta = \text{Diag}(\mu_1, \mu_2, \ldots, \mu_n)$. Let $S$ be the orthogonal basis transformation matrix with columns $e^1, e^2, \ldots, e^n$ and let $S^T$ ($= S^{-1}$) be its transpose.

Then $D^k = S\Delta^k S^T$ and, thus, $(D^k)_{i,j} = \sum_{p=1}^{n} (\mu_p^k e_i^p e_j^p)$. Substitution of our earlier expressions now yields the desired formula.

This also is the explicit solution of the recursion for the exponents vectors $\mathbf{c}_k$. 

**Remark:** The following very nice graph-theoretic argument to find the eigenvalues of the matrix $D$ was communicated by A. Blokhuis, A.E. Brouwer and R. Riebeek [?].
Let $N = (-1)^n D^{-1}$. We can write $N = A - B$, where both $A$ and $B$ are $0-1$ matrices (and $A$ and $B$ are zero wherever $N$ is zero). With $P = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$ we see that $P$ is the adjacency matrix of a path of length $2n$. Each eigenvector $u$ of $N$ with eigenvalue $\theta$ yields an antisymmetric eigenvector $\begin{pmatrix} u \\ -u \end{pmatrix}$ of $P$ with eigenvalue $\theta$, and conversely. But the antisymmetric eigenvectors of $P$ are precisely those that can be extended to eigenvectors of a $(2n + 1)$-cycle by defining it to be zero on the additional point. It follows that the eigenvalues are $\theta = 2 \cos \frac{2\pi j}{2n + 1}$, where $1 \leq j \leq n$ from which those of $D$ follow.

All calculations involving the matrix $D$ have been checked for specific cases using Maple. A collection of appropriate Maple statements can be found on the WWW at http://www.cs.kun.nl/bolke/ksymmaple.

In order to prove Theorem 1, we have to estimate the total degree of $\mathcal{H}(f)$, which in view of Proposition 1 can be written as $\langle Ud^{k-1}(i), j \rangle$ for some nonzero vector $i$ over $\mathbb{N} \cup \{0\}$ and with $j$ the all-one vector.

Write this as $\langle D(i), D^{k-2}U_T(j) \rangle = \langle D(i), D^{k-2}(1, 2, \ldots, n) \rangle$. Note that $D(i)$ has at least one positive entry, namely the $n^{th}$. Hence,

$$tdeg f \geq \sum_{q=1}^{n} q(D^{k-2})_{i,q}.$$ 

(equality occurs if $i = (1, 0, \ldots, 0)$; e.g., if $f = a_1^n$.)
Put \( t = k - 2 \). By the Corollary,

\[
\sum_{q=1}^{n} q(D^{k-2})_{n,q} = \sum_{q=1}^{n} q \sum_{p=1}^{n} (-1)^{q+p+(n+p)t} \frac{\sin\left(\frac{2\pi p}{2n+1}\right) \sin\left(\frac{2\pi q}{2n+1}\right)}{(2n + 1)^{2t-2} \cos^t\left(\frac{p\pi}{2n+1}\right)}.
\]

The summation over the index \( q \) can easily, though tediously, be calculated explicitly (e.g., using the complex form of the sine or with the help of a computer algebra package like Maple).

The double sum then reduces to:

\[
\frac{(-1)^{nt}}{(2n + 1)^{2t}} \sum_{p=1}^{n} (-1)^{pt} \frac{\sin\left(\frac{2\pi p}{2n+1}\right)^2}{\cos^{t+2}\left(\frac{p\pi}{2n+1}\right)}
\]

The largest term occurs for \( p = n \) and we shall see that in fact this term dominates. Indeed, since \( \cos x \geq 1 - \frac{2x^2}{2} \) on the interval \([0, \frac{\pi}{2}]\), one has \( \cos\left(\frac{\pi p}{2n+1}\right) \geq \frac{2(n-p+1)}{2n+1} \). Also, \( \sin\left(\frac{2\pi p}{2n+1}\right)^2 \leq 1 \). Hence, the sum of the first \( n - 1 \) terms can be estimated as

\[
\left| \sum_{p=1}^{n-1} (-1)^{(n+p)t} \frac{\sin\left(\frac{2\pi p}{2n+1}\right)^2}{\cos^{t+2}\left(\frac{p\pi}{2n+1}\right)} \right| \leq \sum_{r=1}^{n-1} \frac{(2n + 1)^{t+1}}{2^t (2n + 1)\cos^{t+2}\left(\frac{p\pi}{2n+1}\right)} \text{ (where } r = n - p \text{)} \leq \frac{(2n + 1)^{t+1}}{2^t (2n + 1)^{t+2}} \int_{1}^{\infty} \frac{dx}{(2x + 1)^{t+2}} \leq \frac{(2n + 1)^{t+1}}{2^t (2n + 1)^{t+2}} \int_{1}^{\infty} \frac{dx}{(2x + 1)^{t+2}} \leq \frac{(2n + 1)^{t+1}}{2^t (2n + 1)^{t+2}}
\]
Let $H = \frac{\sin^2\left(\frac{n\pi}{2(n+1)}\right)}{2^{(2n+1)}\cos^2\left(\frac{n\pi}{2(n+1)}\right)}$ be the largest $(n^{th})$ term. By Taylor expansion around $\frac{\pi}{2}$ one has, for some $|e| \leq 1$, $\sin\left(\frac{n\pi}{2(n+1)}\right) = 1 - \left(\frac{e}{2(n+1)}\right)^2 \geq \frac{1}{20}$ if $n \geq 2$. Similarly, $\cos\left(\frac{n\pi}{2n+1}\right) \leq \left(\frac{\pi}{2(2n+1)}\right)$. Thus, $H \geq \frac{4(2n+1)^{t+1}}{\pi^{t+2}}\left(\frac{19}{20}\right)^2$.

Taking into account our estimate for the small terms we finally find $tdeg(f) \geq \frac{4(2n+1)^{t+1}}{\pi^{t+2}}(\frac{19}{20})^2$ from which Theorem 1 immediately follows. ■

4 An example of a "fixpoint polynomial"

In the introduction we mentioned the fixpoints of the iteration $(x_1, \ldots, x_n) \rightarrow (a_1, \ldots, a_n)$ An amusing and perhaps intriguing numerical example for $n = 4$ is the following:

$$(-T + 1)(-1.324717957T + 1)(.7548776668T + 1)(.5698402912T + 1) \approx 1 - .9999999994T - 1.324717957T^2 + .7548776668T^3 + .5698402912T^4$$

The relevant equations were solved in the obvious way using Maple, by first constructing a Groebner basis of the ideal $I(x_1 + x_2 + x_3 + x_4 - x_1, x_1x_2 + x_1x_3 + x_2x_3 + x_1x_4 + x_2x_4 + x_3x_4 - x_2, x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 - x_3, x_1x_2x_3x_4 - x_4)$.

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