Symmetric functions, I:

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“How symmetric can a function be?”

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ABSTRACT

The symmetric complexity of a polynomial $f$ in $n$ variables is defined as the number of times the symmetric function theorem is applicable. In this paper a sharp upper bound on this measure is derived by a matrix method.

1 Introduction

Consider a field $K$ of characteristic 0, and let $R$ be the ring $K[x_1, \ldots, x_n]$ where $n$ is $> 0$.

A symmetric function is any element of $R$ invariant under the symmetric group acting as coordinate permutations. Examples are the elementary symmetric functions: $a_0 = 1$, $a_i = \sum_{1 \leq j_1 < j_2 < \cdots < j_i \leq n} x_{j_1} x_{j_2} \cdots x_j$, $(1 \leq i \leq n)$; $a_i = 0$ ($i < 0$ or $i > n$).
The Symmetric Function Theorem \cite{4,5} states that any symmetric function \( f \) can be uniquely written as \( g(a_1, \ldots, a_n) \) for some \( g = g(x_1, \ldots, x_n) \) from \( \mathbb{R} \), called the symmetric representation of \( f \).

Here we shall address the question of what happens when this \( g \) is symmetric again. This is of course perfectly possible and if it occurs \( k - 1 \) times, \( f \) is called \( k \)-fold symmetric. That is:

**Definition 1** A polynomial \( f \) in \( n \) variables is 0-fold symmetric if \( f \) is not symmetric; and \( k \)-fold symmetric with \( k > 0 \) if \( f \) is symmetric and the symmetric representation of \( f \) is \( k - 1 \)-fold symmetric. The number \( k \) is called the symmetric complexity of \( f \).

A \( k \)-fold symmetric function \( f \) possesses a high degree of symmetry indeed, and it is an interesting complexity problem to find a bound on \( k \) expressed in the coefficients and exponents of \( f \). Such a result is given in Theorem 1. Our method is based on term orderings and the like, familiar from Groebner basis theory \cite{3}. Thus it is possible to translate the problem into linear algebra, involving the explicit calculation of the spectrum and eigenvectors of a matrix.

Another interesting question that arises in a natural way in this context is: how can we describe the behavior (e.g., fixpoints) of the iteration \( (x_1, \ldots, x_n) \rightarrow (a_1, \ldots, a_n) \)? We shall restrict ourselves to a numerical example for \( n = 4 \) (see Section 4).
2 Notations and generalities

Put \( x = (x_1, \ldots, x_n) \); let \( a_i = a_i(x) \) be defined as above, and let \( a = (a_1, \ldots, a_n) \).

Stretching notation a bit, we can view \( \{c_1, \ldots, c_n\} \to a(c) \) as a mapping from the unordered lists of length \( n \) over \( K \) to \( K^n \), which is a bijection if \( K \) is algebraically closed. Indeed, one has

\[
\sum_{i=0}^{n} a_i(c_1, \ldots, c_n)T^i = \prod_{i=0}^{n}(c_i T + 1).
\]

Instead of this however we shall consider the simpler mappings \( c \to a(c) \) from \( K^n \) to \( K^n \) and \( a : x \to a(x) \) from \( R \) to \( R \).

Definition 2 Let \( a^0 = (x_1, \ldots, x_n) \); and for \( k > 0 \) define \( a^k = (a_1^k, a_2^k, \ldots, a_n^k) \)
where \( a_i^k = a_i(x) = a_i(a_1^{k-1}, a_2^{k-1}, \ldots, a_{n}^{k-1}), \ 1 \leq i \leq n. \)

The \( a_i^k \) are called the *iterated elementary symmetric functions* (iesf’s.) An interesting fact is given by

Lemma 1 For all \( k \geq 1 \), the iesf’s \( a_1^k, a_2^k, \ldots, a_n^k \) are algebraically independent.

Proof. Induction w.r.t. \( k \). For \( k = 1 \) this is well-known [5]. Now let \( f(y_1, \ldots, y_n) \) be such that \( f(a_1^{k+1}, a_2^{k+1}, \ldots, a_n^{k+1}) = 0 \) in \( R \).

By definition of the \( a_i^k \)’s, there exists a symmetric polynomial \( g(z_1, \ldots, z_n) = f(a_1(z), a_2(z), \ldots, a_n(z)) \) with \( g(a_1^k, a_2^k, \ldots, a_n^k) = f(a_1^{k+1}, a_2^{k+1}, \ldots, a_n^{k+1}) = 0; \)
hence \( g(z_1, \ldots, z_n) = 0 \) by the induction hypothesis. But now we are in the case \( k = 1 \) again, since \( g(z_1, \ldots, z_n) = f(a_1(z), a_2(z), \ldots, a_n(z)) \) and it follows that \( f(y_1, \ldots, y_n) = 0. \)

\[\blacksquare\]
A term is any monomial \( t = x_1^{i_1}x_2^{i_2} \ldots x_n^{i_n} \). Its total degree is \( tdeg(t) = \sum_{j=1}^{n} i_j \) and the total degree \( tdeg(f) \) of \( f \in R \) is \( \max_{t \in f} tdeg(t) \) (which of course is equal to \( tdeg(t) \), any \( t \) in \( f \) if \( f \) is symmetric.)

An admissible ordering \([3]\) on the set \( T \) of terms in \( R \) is a total order on \( T \) that satisfies:

\[ 1 < t; \text{ and } t < t' \Rightarrow st < st' \] for all terms \( s, t, t' \).

The latter property is called monotonicity of term multiplication.

An admissible ordering is a well-ordering. Admissible orders abound and have been classified; well-known examples are the lexicographic orders and various total degree orderings like the "grevlex" \([3]\).

For a given ordering, the leading term \( lt(f) \) of \( f \) is the highest term occurring in \( f \).

### 3 The main theorem

Our main result is given by

**Theorem 1** Let \( f \) be any non-constant \( k \)-fold symmetric polynomial in \( n \geq 2 \) variables. Then the symmetric complexity \( k \) is bounded by:

\[
tdeg(f) \geq \frac{(2n + 1)^{k-1}}{\pi^{k-1}} \left\{ 1.149 - 1.048(0.53)^{k-1} \right\}
\]
Remark: This bound is fairly precise: it is an approximation of a more complex bound, which is sharp in the sense that it is reached by $f = a_k^k$. This will follow from the proof.

First let us give an outline of the proof. The idea is very simple and consists of three steps.

i. If $k$ increases, one observes that the iesf's $a_k^k$ grow very quickly in "size".

To measure this size, we consider the highest term $t_k^k$ of $a_k^k$ in an admissible ordering.

Remark: Explicit calculation of the complete $a_k^k$'s in Maple, say, leads to considerable memory problems. A piece of code to experiment with is given on the WWW at http://www.cs.kun.nl/bolke/ksymmaple.

ii. Next, we shall be able to estimate the exponents occurring in $t_k^k$; this is the technical part.

iii. Finally, for a given $f$ of complexity $k$ we shall show that for some $i$, a term $t_k^k$ actually occurs in $f$ as $lt(f)$. Hence, $k$ is bounded as a function of $lt(f)$, and this ends the proof.

As an admissible ordering on $T$, take the lexicographic order with $x_1 > x_2 > \ldots > x_n$. Let $t_1^k$ be $lt(a_k^k)$. We shall derive a recursion for $t_1^k$.

Lemma 2 a. $t_k^k = t_{n-1}^{k-1} \ldots t_1^{k-1} (k > 1)$ b. If $p > q$, $t_p^k > t_q^k$ ($k \geq 1$).

Proof:

For $k = 1$, statement b. holds. Indeed, $t_1^1 = lt(a_1) = x_1 x_2 \ldots x_i$. Also,
a. holds trivially. Now if for any \( k \) \( a \) and \( b \) are true, then by definition one has \( a_i^{k+1} = \sum_{1 \leq j_1 < j_2 < \ldots < j_i \leq n} a_{j_1}^k a_{j_2}^k \ldots a_{j_i}^k \). All coefficients are positive, so no terms cancel. By the monotonicity property, \( lt(a_{j_1}^k a_{j_2}^k \ldots a_{j_i}^k) = lt(a_{j_1}^k) lt(a_{j_2}^k) \ldots lt(a_{j_i}^k) = t_{j_1}^k t_{j_2}^k \ldots t_{j_i}^k \). Since \( b \) holds and, again, by monotonicity, this is maximal if \( j_i = n \), \( j_{i-1} = n - 1 \), \ldots, \( j_1 = n - i + 1 \). This proves \( a \) for index \( k + 1 \). But then, if \( p > q \) one has \( t_p^{k+1} > t_q^{k+1} \) since the r.h.s. divides the l.h.s. Hence \( b \) holds as well. ■

In part ii. of the proof, we shall estimate the size of the exponents in \( t_i^k \).

**Definition 3** The exponents vector \( ev(t) \) of a term \( t = x_1^{i_1} \ldots x_n^{i_n} \) is \( i = (i_1, i_2, \ldots, i_n) \). We denote \( ev(t_i^k) \) by \( e_i^k = (e_{i_1}^k, \ldots, e_{i_n}^k) \).

One has \( e_i^k = (1, 1, \ldots, 1, 0, \ldots, 0) \) (\( i \) ones). Define \( E_k \) to be the matrix having the \( e_i^k \)'s as its columns; note that \( E_1 = U \), the upper triangular all-one matrix.

**Lemma 3** a. Let \( t = x_1^{i_1} \ldots x_n^{i_n} \) be any term; then for all \( k \geq 1 \) the exponents vector of \( lt(t_i^k) \) equals \( E_k(i) \).

b. Let \( D \) be the symmetric matrix with ones below and on the antidiagonal and zeroes above; put \( D^k = (d_{ij}) \), \( 1 \leq i, j \leq n \). Let \( U \) be the upper triangular all-one matrix. Then \( E_k = U D^k \). Hence \( E_k \) is nonsingular and for \( k \geq 1 \) one has: \( e_{i,a}^k = \sum_{j=1}^n d_{k-1,j}^a \).

**Proof:**

By monotonicity, \( lt((a_i^k)^{i_1} \ldots (a_n^k)^{i_n}) = (t_i^k)^{i_1} \ldots (t_n^k)^{i_n} \), the exponents vector of
which is $E_k(i)$ by linearity. This proves part $a$.

For part $b.$, note that statement $a.$ of Lemma 2 can be written as: $\xi^k = \xi^{k-1}_n + \xi^{k-1}_n + \ldots + \xi^{k-1}_{n-i+1}$. which is equivalent to $E_k = E_{k-1}D$. So $E_k = E_1D^{k-1} = UD^{k-1}$.

In the Corollary to Proposition 2 we shall find an explicit solution to this recursion.

Before analyzing this, let us first proceed to part $iii$. Suppose that $f$ is not constant and $k$-fold symmetric, $k \geq 1$. We wish to prove that some $t^k_i$ really occurs in $f$.

By definition, there exists $f_k \in R$ such that $f_k(a^1, \ldots, a^n) = f$ (though we shall not need it, note that $f_k$ is unique by Lemma 1. ) Let $t = x_1^{i_1} \ldots x_n^{i_n}$ be a term of the polynomial $f_k(x)$ such that $\tau = D_{i_1} \ldots D_{i_n}$ is maximal in the term ordering. By Lemma 3, $ev(\tau) = E_k(i_1, \ldots, i_n) = E_k(i)$.

First note that $\tau$ is unique. Indeed, suppose that besides $t$ there is another term $s = x_1^{j_1} \ldots x_n^{j_n}$ yielding the same $\tau$, then by lemma 3 one would have $E_k(\hat{i}) = E_k(\hat{j})$ (with $\hat{i} = (j_1, \ldots, j_n)$); hence $E_k(\hat{i} - \hat{j}) = 0$. But $E_k$ was nonsingular so $\hat{i} = \hat{j}$ and $s = t$. 7
Also, $\tau$ does not cancel when $f_k(a_1^k, \ldots, a_n^k)$ is expanded to $f$. Otherwise, there would be some term $s$ in $f_k$ and a term $\sigma$ from $s(a_1^k, \ldots, a_n^k)$ such that $\tau = \sigma$. (N.b. all these terms are in $R$, i.e. of the form $x_1^{p_1} \ldots x_n^{p_n}$.) Then however, $\sigma \leq \ell t(s(a^k)) < \ell t(t(a^k))$. This contradicts the unicity of $t$ and the maximality of $\tau$.

We conclude that $\tau = \ell t(f)$. This shows what we wanted, namely that some $t_k^i$ occurs in $t$, hence in $f$.  

In fact we have proven more, namely:

**Proposition 1** Let $U$ be the upper triangular all-one matrix and $D$ the (symmetric) lower antitriangular all-one matrix. Then for any $k$-fold symmetric function $f$ and $k \geq 1$,

$$\text{ev}(\ell t(f)) \in UD^{k-1}((\mathbb{N} \cup \{0\})^n).$$

How good is this? In order to answer this question let us give an estimate of the entries of powers of $D$.

For $p = 1, 2, \ldots, n$ let us define the following quantities:

$$w_p = -e^{-\frac{2\pi p}{2n+1}};$$
 $$\alpha_p = w_p + w_p^{-1} = -2\cos\left(\frac{2\pi p}{2n+1}\right);$$
 $$V_p = w_p - w_p^{-1} = 2\sin\left(\frac{2\pi p}{2n+1}\right);$$
 $$\lambda_p = 4\cos^2\left(\frac{\pi p}{2n+1}\right);$$
 $$\mu_p = (-1)^p / 2\cos\left(\frac{2 \pi p}{2n+1}\right);$$
 $$x_m^p = 2(-1)^{m+1} \sin\left(\frac{2 \pi m p}{2n+1}\right) \ (m = 1, 2, \ldots, n);$$
 $$x^p = (x_1^p, x_2^p, \ldots, x_n^p).$$
These numbers satisfy the relations:

\[ \lambda_p = 2 + \alpha_p; \quad V_p^2 = \alpha_p^2 - 4; \quad w_p^{2n+1} = -1; \]

\[ \mu_p = \frac{1}{(w_p^2 + \alpha_p^2)}; \quad \mu_p^{-2} = \lambda_p; \quad x_p^m = \left(\frac{w_p^m - \alpha_p^{-m}}{\sqrt{2m+1}}\right) (m = 1, 2, \ldots, n); \quad \text{also,} \]

\[ w_p = \frac{(\alpha_p + V_p)}{2} \quad \text{and} \quad w_p^{-1} = \frac{(\alpha_p - V_p)}{2} \]

are the roots of \( X^2 - \alpha_p X + 1 = 0 \).

Let \( \langle x, y \rangle = \sum_{i=1}^n x_i \overline{y}_i \) be the standard Hermitian inner product. It is elementary to verify that the \( x_p \) are perpendicular of length 1. Now one has:

**Proposition 2** The vectors \( x_p \) form an orthonormal basis upon which the matrix \( D \) assumes a diagonal form \( \Delta = \text{Diag}(\mu_1, \mu_2, \ldots, \mu_n) \).

**Proof.**

Since the proof is fairly standard, let us just outline it. One easily verifies that the inverse of \( D \) is the matrix with ones on the antidiagonal, -1’s just above it, and zeroes elsewhere. Next, its square \( D^{-2} \) is seen to be tridiagonal:

\[ (D^{-2})_{i,i} = 2 \quad (i < n); \quad (D^{-2})_{n,n} = 1; \quad (D^{-2})_{i,j} = -1 \quad (|i-j| = 1). \]

Tridiagonal matrices have been studied extensively in the theory of orthogonal polynomials [2] and the numerical theory of parabolic differential equations.

\( D^{-2} \), being symmetric, can be diagonalized on a real orthonormal basis. Let \( \tilde{z} = (z_1, z_2, \ldots, z_n) \) be an eigenvector of \( D^{-2} \) with eigenvalue \( \lambda \). Put \( \tilde{z} = \tilde{z}(\alpha) \), again with \( \alpha = 2 - \lambda \). W.l.o.g, let \( z_1 = 1 \) and let \( z_0 =_{D_{\alpha}f} 0 \). Then \( (D^{-2} - \lambda I)\tilde{z} = 0 \) amounts to the recursion

\[ \tilde{z} = (z_1, z_2, \ldots, z_n) \]
\( z_0 = 0; z_1 = 1; \)
\[ z_m = \alpha z_{m-1} - z_{m-2} \quad (1 < m \leq n); \]
\[ -z_{n-1} + (\alpha - 1)z_n = 0 \]

(the latter being the characteristic equation.)

**Remark:** this is the familiar recursion of the Tchebycev polynomials \( T_{m-1}(x) \) in \( x = \frac{a}{2} \), though these have initial values \( T_0 = 1, T_1 = x \). In fact it is not difficult to prove that \( z_m = \frac{(T_m(\frac{a}{2}) - T_{m-1}(\frac{a}{2}))}{((\frac{a}{2})^2 - 1)}. \)

Let \( V = \sqrt{(a^2 - 4)} \) and \( w = (\alpha + V)/2, w' = (\alpha - V)/2 \), the roots of \( X^2 - \alpha X + 1 = 0 \). If \( w = w' \), \( \alpha = \pm 2 \); but then \( z_m = (\pm 1)^{m-1}m, -z_{n-1} + (\alpha - 1)z_n \neq 0 \), and there are no eigenvalues. So suppose \( w \neq w' \).

Solving the recursion by standard techniques yields \( z_m = \frac{(w^m - w'^{-m})}{V}; 1 \leq m \leq n. \) By some easy calculations, the eigenvalue equation \(-z_{n-1} + (\alpha - 1)z_n = 0\) reduces to \( w^{2n+1} = -1 \) (where \( w \neq -1 \) since \( w \neq w' \)). From this, \( w = -\frac{e^{\frac{2\pi i}{2n+1}}}{p = 1, 2, \ldots, n} \). We shall now take this \( p \) as an index (i.e., use \( \alpha_p, \lambda_p, \mu_p, w_p, V_p, z_m^p, x_m^p, x^p \)).

The numbers and vectors \( \alpha_p, \lambda_p, \mu_p, w_p, V_p, x_m^p, x^p \) are in fact those defined earlier. Normalization of \( V_p z^p \) yields the \( p^{th} \) eigenvector \( z^p \) as \( x_m^p = 2(-1)^{m+1}\frac{\sin((\frac{2\pi n + 1}{2n+1})p)}{\sqrt{2n+1}} \). Similarly, one finds the formulas for \( \alpha_p, \lambda_p \) etc.

The \( x^p(\alpha) \) form an orthogonal eigenbasis over which the symmetric matrix \( D^{-2} \) diagonalizes. But in fact by an easy calculation, \( D^{-1}x^p = \mu_p^{-1}z^p \); hence
Note that the eigenvalues \( \mu_p \) of \( D \) are all different and \( \max_p |\mu_p| = |\mu_n| = \frac{1}{2 \cos \left( \frac{\pi}{2n+1} \right)} \). Also, \( \text{sign} \mu_p = (-1)^{n+p} \) (consider \( pn \mod 2n + 1 \) for \( p \) odd and \( p \) even).

**Corollary**

The (nonnegative integral) entries of \( D^k \) are given in closed form by the formula

\[
(D^k)_{i,j} = \sum_{p=1}^{n} (-1)^{i+j+(n+p)k} \frac{\sin \left( \frac{2pi\pi}{2n+1} \right) \sin \left( \frac{2pj\pi}{2n+1} \right)}{(2n+1)2^{k-2}\cos \left( \frac{p\pi}{2n+1} \right)}
\]

**Proof:**

As before, let \( \Delta = \text{Diag}(\mu_1, \mu_2, \ldots, \mu_n) \). Let \( S \) be the orthogonal basis transformation matrix with columns \( x^1, x^2, \ldots, x^n \) and let \( S^T \) (\( = S^{-1} \)) be its transpose. Then \( D^k = S\Delta^k S^T \) and, thus, \( (D^k)_{i,j} = \sum_{p=1}^{n} (\mu_p^k x_i^p x_j^p) \). Substitution of our earlier expressions now yields the desired formula.

This also is the explicit solution of the recursion for the exponents vectors \( e^k \).

**Remark:** The following very nice graph-theoretic argument to find the eigenvalues of the matrix \( D \) was communicated by A. Blokhuis, A.E. Brouwer and R. Riebeek [?].
Let \( N = (-1)^n D^{-1} \). We can write \( N = A - B \), where both \( A \) and \( B \) are 0–1 matrices (and \( A \) and \( B \) are zero wherever \( N \) is zero). With \( P = \begin{pmatrix} A & B \\ B & A \end{pmatrix} \)

we see that \( P \) is the adjacency matrix of a path of length \( 2n \). Each eigenvector \( u \) of \( N \) with eigenvalue \( \theta \) yields an antisymmetric eigenvector \( \begin{pmatrix} u \\ -u \end{pmatrix} \) of \( P \) with eigenvalue \( \theta \), and conversely. But the antisymmetric eigenvectors of \( P \) are precisely those that can be extended to eigenvectors of a \((2n + 1)\)-cycle by defining it to be zero on the additional point. It follows that the eigenvalues are \( \theta = 2 \cos \frac{2\pi j}{2n+1} \), where \( 1 \leq j \leq n \) from which those of \( D \) follow.

All calculations involving the matrix \( D \) have been checked for specific cases using Maple. A collection of appropriate Maple statements can be found on the WWW at http://www.cs.kun.nl/ bolke/ksymmaple.

In order to prove Theorem 1, we have to estimate the total degree of \( h(f) \), which in view of Proposition 1 can be written as \( < U D^{k-1}(i), j > \) for some nonzero vector \( i \) over \( \mathbb{N} \cup \{0\} \) and with \( j \) the all-one vector.

Write this as \( < D(i), D^{k-2} U^T(j) > = < D(i), D^{k-2}(1, 2, \ldots, n) > \). Note that \( D(i) \) has at least one positive entry, namely the \( n^{th} \). Hence,

\[
tdeg f \geq \sum_{q=1}^{n} q (D^{k-2})_{n,q}.
\]

(equality occurs if \( i = (1, 0, \ldots, 0) \); e.g., if \( f = a_i^1 \).)
Put $t = k - 2$. By the Corollary,

$$
\sum_{q=1}^{n} q(D^{k-2})_{n,q} = \sum_{q=1}^{n} q \sum_{p=1}^{n} (-1)^{n+q+(n+p)t} \frac{\sin\left(\frac{2n\pi}{2n+1}\right) \sin\left(\frac{2\pi (n+p)}{2n+1}\right)}{(2n+1)2^{t-2}\cos\left(\frac{\pi}{2n+1}\right)}.
$$

The summation over the index $q$ can easily, though tediously, be calculated explicitly (e.g., using the complex form of the sine or with the help of a computer algebra package like Maple).

The double sum then reduces to:

$$
\frac{(-1)^{nt}}{(2n+1)2^t} \sum_{p=1}^{n} (-1)^{or} \frac{\sin\left(\frac{2n\pi}{2n+1}\right)^2}{\cos^t + 2\left(\frac{\pi}{2n+1}\right)}
$$

The largest term occurs for $p = n$ and we shall see that in fact this term dominates. Indeed, since $\cos x \geq 1 - \frac{2\pi}{x^2}$ on the interval $[0, \pi]$, one has $\cos\left(\frac{\pi}{2n+1}\right) \geq \frac{2(n-\pi) + 1}{2n+1}$. Also, $\sin\left(\frac{2n\pi}{2n+1}\right)^2 \leq 1$. Hence, the sum of the first $n - 1$ terms can be estimated as

$$
\sum_{p=1}^{n-1} \frac{(-1)^{n+p}(p\pi}{2n+1)} \sin^2\left(\frac{\pi}{2n+1}\right) \leq \sum_{r=1}^{n-1} \frac{(2n+1)^{t+1}}{2^t (2r + 1)^{t+2}} \quad \text{(where } r = n - p \text{)}
$$

and

$$
\frac{(2n+1)^{t+1}}{2^t} \sum_{r=1}^{\infty} \frac{1}{(2r + 1)^{t+2}} \leq \frac{(2n+1)^{t+1}}{2^t} \left\{ \frac{1}{3^{t+2}} + \int_{1}^{\infty} \frac{dx}{(2x + 1)^{t+2}} \right\} \leq \frac{(2n+1)^{t+1}}{2^t 3^{t+1}}
$$

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Let $H = \frac{\sin^2(\frac{n\pi}{2(2n+1)})}{2(2n+1)\cos^2(\frac{n\pi}{2n+1})}$ be the largest ($n^{th}$) term. By Taylor expansion around $\frac{\pi}{2}$ one has, for some $|e| \leq 1$, $\sin(\frac{n\pi}{2n+1}) = 1 - \left(\frac{\pi}{2(2n+1)}\right)^2 \frac{e}{2!} \geq \frac{19}{20}$ if $n \geq 2$. Similarly, $\cos(\frac{n\pi}{2n+1}) \leq \left(\frac{\pi}{2(2n+1)}\right)$. Thus, $H \geq \frac{4(2n+1)^{t+1}}{\pi^{t+2}}\left(\frac{19}{20}\right)^2$.

Taking into account our estimate for the small terms we finally find $tdeg(f) \geq \frac{4(2n+1)^{t+1}}{\pi^{t+2}}\left(\frac{19}{20}\right)^2$ from which Theorem 1 immediately follows. ■

4 An example of a "fixpoint polynomial"

In the introduction we mentioned the fixpoints of the iteration $(x_1, \ldots, x_n) \rightarrow (a_1, \ldots, a_n)$. An amusing and perhaps intriguing numerical example for $n=4$ is the following:

$$(-T + 1)(-1.324717957T + 1)(.7548776668T + 1)(.5698402906T + 1) \approx 1 - .9999999994T - 1.324717957T^2 + .7548776668T^3 + .5698429127T^4$$

The relevant equations were solved in the obvious way using Maple, by first constructing a Groebner basis of the ideal $I(x_1 + x_2 + x_3 + x_4 - x_1, x_1x_2 + x_1x_3 + x_2x_3 + x_1x_4 + x_2x_4 + x_3x_4 - x_2, x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 - x_3, x_1x_2x_3x_4 - x_4)$.

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