Euler's factorial series and global relations

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To the memory of Marc Huttner (1947–2015)

Abstract

Using Padé approximations to the series $E(z)=\sum_{k=0}^{\infty} k!\,(-z)^k$, we address arithmetic and analytical questions related to its values in both p-adic and Archimedean valuations.

1 Introduction

In his 1760 paper on divergent series [5], L. Euler introduced and studied the formal (hypergeometric) series

$$E(z) := \sum_{k=0}^{\infty} k!\,(-z)^k \tag{1}$$

(see also [1, 2] and, in particular, [8, Section 2.5]), which specializes at $z=1$ to Wallis' series

$$W := \sum_{k=0}^{\infty} (-1)^k k!\,, \tag{2}$$

which has teased people's imagination since the times of Euler. Notice that the series (1) is a perfectly convergent p-adic series in the disc $|z|_p \le 1$ for all primes p, so that one can discuss the arithmetic properties of its values, for example, at $z=1$ (the Wallis case) and at $z=-1$. The irrationality of the latter specialization,

$$K = K_p := \sum_{k=0}^{\infty} k!\,, \tag{3}$$

for any prime p is a folklore conjecture [12, p. 17] (see [11], also for a link of the problem to a combinatorial conjecture of H. Wilf). Already the expectation $|K_p|_p = 1$ (so that K_p is a p-adic unit) for all primes p, which is an equivalent form of Kurepa's conjecture [7] from 1971, remains open; see [4] for the latest achievements in this direction.

In what follows, we refer to the series in (1) as Euler's factorial series and label it $E_p(z)$ when treat it as the function in the p-adic domain for a given prime p.

Theorem 1. *Given $\xi \in \mathbb{Z} \setminus \{0\}$, let $\mathcal{P}$ be a subset of prime numbers such that*

$$\limsup_{n\to\infty} c^n n! \prod_{p\in\mathcal{P}} |n!|_p^2 = 0, \quad \text{where } c = c(\xi;\mathcal{P}) := 4|\xi| \prod_{p\in\mathcal{P}} |\xi|_p^2. \tag{4}$$

Then either there exists a prime $p \in \mathcal{P}$ for which $E_p(\xi)$ is irrational, or there are two distinct primes $p, q \in \mathcal{P}$ such that $E_p(\xi) \neq E_q(\xi)$ (while $E_p(\xi), E_q(\xi) \in \mathbb{Q}$).

Because $\prod_p |n!|_p = 1/n!$ when the product is taken over all primes, condition (4) is clearly satisfied for any subset $\mathcal{P}$ whose complement in the set of all primes is finite. This also suggests more exotic choices of $\mathcal{P}$. Furthermore, the conclusion of the theorem is contrasted with Euler's sum

$$\sum_{k=0}^{\infty} k \cdot k! = -1,$$

which is valid in any p-adic valuation (and follows from $\sum_{k=0}^{n-1} k \cdot k! = n! - 1$) — an example of what is called a *global* relation.

The idea behind the proof of Theorem 1 is to construct approximations p_n/q_n to the number $\omega_p = E_p(\xi)$ in question, which do not depend on p and approximate the number considerably well for each p-adic valuation: $q_n\omega_p - p_n = r_{n,p}$ for $n = 0, 1, 2, \dots$; $|r_{n,p}|_p \to 0$ as $n \to \infty$ and there are infinitely many indices n for which $r_{n,p} \neq 0$ for at least one prime $p \in \mathcal{P}$. Assume that $\omega_p = a/b$, the same rational number, for all $p \in \mathcal{P}$. Then $q_n a - p_n b \in \mathbb{Z} \setminus \{0\}$, so that $0 < |q_n a - p_n b|_p \le 1$, for infinitely many indices n and *all* primes p, hence

$$\begin{aligned} 1 = |q_n a - p_n b| \prod_p |q_n a - p_n b|_p &\le |q_n a - p_n b| \prod_{p\in\mathcal{P}} |q_n a - p_n b|_p \\ &\le (|a| + |b|) \max\{|q_n|, |p_n|\} \prod_{p\in\mathcal{P}} |b r_{n,p}|_p \\ &\le (|a| + |b|) \max\{|q_n|, |p_n|\} \prod_{p\in\mathcal{P}} |r_{n,p}|_p \end{aligned}$$

for those n. This means that the condition

$$\limsup_{n\to\infty} \max\{|q_n|, |p_n|\} \prod_{p\in\mathcal{P}} |r_{n,p}|_p = 0 \tag{5}$$

contradicts the latter estimate, thus making the $\mathcal{P}$-global linear relation $\omega_p = a/b$ impossible.

The result in Theorem 1 can be put in a general context of global relations for Euler-type series; the corresponding settings can be found in the paper [3]. We do not pursue this route here as our principal motivation is a sufficiently elementary arithmetic treatment of an analytical function that have some historical value. The rational approximations to $E(z)$ we construct in Section 2 are Padé approximations; in spite of being known for centuries, implicitly from the continued fraction for $E(z)$ given by Euler himself [5] and explicitly from the work of T. J. Stieltjes [13], these Padé approximations remain a useful source for arithmetic and analytical investigations. In Section 3 we revisit Euler's summation of Wallis' series (2) using the approximations; we also complement the derivation by providing 'Archimedean analogue(s)' for the divergent series $K = E(-1)$ from (3) — the case when the classical strategy does not work.

Our way of constructing the Padé approximations is inspired by a related Padé construction of M. Hata and M. Huttner in [6]. The construction was a particular favourite of Marc Huttner who, for the span of his mathematical life, remained a passionate advocate of interplay between Picard–Fuchs linear differential equations and Padé approximations. We dedicate this work to his memory.

2 Hypergeometric series and Padé approximations

Euler's factorial series (1) is the particular $a = 1$ instance of the hypergeometric series

$$ {}_2F_0(a, 1 \mid z) = \sum_{k=0}^{\infty} (a)_k z^k. \tag{6} $$

Here and in what follows, we use the Pochhammer notation $(a)_k$ which is defined inductively by $(a)_0 = 1$ and $(a)_{k+1} = (a+k)(a)_k$ for $k \in \mathbb{Z}_{\ge 0}$. Our Padé approximations below are given more generally for the function (6).

Theorem 2. *For $n, \lambda \in \mathbb{Z}_{\ge 0}$, take*

$$ B_{n,\lambda}(z) = \sum_{i=0}^{n} \binom{n}{i} \frac{(-1)^i z^{n-i}}{(a)_{i+\lambda}}. $$

Then $\deg_z B_{n,\lambda} = n$ and for a polynomial $A_{n,\lambda}(z)$ of degree $\deg_z A_{n,\lambda} \le n + \lambda - 1$ we have

$$ B_{n,\lambda}(z)\, {}_2F_0(a, 1 \mid z) - A_{n,\lambda}(z) = L_{n,\lambda}(z), \tag{7} $$

where $\operatorname{ord}_{z=0} L_{n,\lambda}(z) = 2n + \lambda$. Explicitly,

$$ L_{n,\lambda}(z) = (-1)^n n!\, z^{2n+\lambda} \sum_{k=0}^{\infty} k! \binom{n+k}{k} \binom{n+k+a+\lambda-1}{k} z^k. \tag{8} $$

Proof. Relation (7) means that there is a 'gap' of length n in the power series expansion

$$ B_{n,\lambda}(z)\, {}_2F_0(a, 1 \mid z) = A_{n,\lambda}(z) + L_{n,\lambda}(z). $$

Write $B_{n,\lambda}(z)=\sum_{h=0}^{n} b_h t^h$ and consider the series expansion of the product

$$B_{n,\lambda}(z)\,{}_2F_0(a,1\mid z)=\sum_{l=0}^{\infty} r_l z^l,$$

where $r_l=\sum_{h+k=l} b_h (a)_k$; in particular,

$$r_l=\sum_{i=0}^{n}(-1)^i\binom{n}{i}\frac{(a)_{i+\lambda+m}}{(a)_{i+\lambda}}=\sum_{i=0}^{n}(-1)^i\binom{n}{i}(a+i+\lambda)_m$$

with $m=l-n-\lambda$ for $l>n+\lambda-1$. To verify the desired 'gap' condition,

$$r_{n+\lambda}=r_{n+\lambda+1}=\cdots=r_{n+\lambda+n-1}=0, \tag{9}$$

we introduce the shift operators $N=N_a$ and $\Delta=\Delta_a=N-\mathrm{id}$ defined on functions $f(a)$ by

$$Nf(a)=f(a+1) \quad \text{and} \quad \Delta f(a)=f(a+1)-f(a).$$

It follows from

$$\Delta^n(a+\lambda)_m=(-1)^n(-m)_n(a+\lambda+n)_{m-n} \quad \text{for } n\in\mathbb{Z}_{\ge 0}$$

that

$$\begin{aligned} r_l&=\sum_{i=0}^{n}(-1)^i\binom{n}{i}N^i(a+\lambda)_m=(\mathrm{id}-N)^n(a+\lambda)_m\\ &=(-\Delta)^n(a+\lambda)_m=(-m)_n(a+\lambda+n)_{m-n}, \end{aligned}$$

which, in turn, implies (9) because of the vanishing of $(-m)_n$ for $m=0,1,\dots,n-1$. The explicit expression for r_l just found also gives

$$r_{2n+\lambda+k}=(-n-k)_n(a+\lambda+n)_k=(-1)^n n!\binom{n+k}{k}k!\binom{n+k+a+\lambda-1}{k},$$

hence the closed form (8). We also have

$$A_{n,\lambda}(z)=\sum_{l=0}^{n+\lambda-1} r_l z^l=\sum_{l=0}^{n+\lambda-1} z^l\sum_{\substack{i=0\\ i\ge n-l}}^{n}(-1)^i\binom{n}{i}\frac{(a)_{i+l-n}}{(a)_{i+\lambda}}.$$

This concludes our proof of the theorem. $\square$

From now on we choose $a=1$, $\lambda=0$, change z to $-z$ and renormalize the corresponding Padé approximations produced by Theorem 2 by multiplying them by

$(-1)^n n!$:

$$Q_n(z) := (-1)^n n!\, B_{n,0}(-z) = \sum_{i=0}^{n} \binom{n}{i} \frac{n!}{i!}\, z^{n-i} = \sum_{i=0}^{n} i!\, \binom{n}{i}^2 z^i,$$

$$P_n(z) := (-1)^n n!\, A_{n,0}(-z) = (-1)^n \sum_{l=0}^{n-1} (-z)^l \sum_{i=n-l}^{n} (-1)^{n-i} \binom{n}{i} \frac{n!\,(i+l-n)!}{i!}$$

$$= (-1)^n \sum_{l=0}^{n-1} (-z)^l \sum_{i=0}^{l} (-1)^i i!\,(l-i)!\, \binom{n}{i}^2$$

and

$$R_n(z) := (-1)^n n!\, L_{n,0}(-z) = n!^2 z^{2n} \sum_{k=0}^{\infty} (-1)^k k!\, \binom{n+k}{k}^2 z^k.$$

In this notation, the Padé approximation formula (7) may be rewritten as

$$Q_n(z)E(z) - P_n(z) = R_n(z) \tag{10}$$

for $n \in \mathbb{Z}_{>0}$. Observe that

$$Q_n(z)P_{n+1}(z) - Q_{n+1}(z)P_n(z) = n!^2 z^{2n}. \tag{11}$$

Indeed, the standard Padé walkabout proves the identity

$$Q_n(z)P_{n+1}(z) - Q_{n+1}(z)P_n(z) = Q_{n+1}(z)R_n(z) - Q_n(z)R_{n+1}(z),$$

in which the degree of the left-hand side is at most $2n$ while the order of the right-hand side is at least $2n$.

Proof of Theorem 1. We may use the formal series identity (10) to get appropriate numerical approximations for any at $z = \xi \in \mathbb{Z} \setminus \{0\}$. For $n = 1, 2, \ldots$, take $p_n = P_n(\xi) \in \mathbb{Z}$, $q_n = Q_n(\xi) \in \mathbb{Z}$ and define

$$r_{n,p} = R_n(\xi) = q_n E_p(\xi) - p_n$$

for each prime $p \in \mathcal{P}$. Using elementary summation formulas and trivial estimates for binomials we have

$$|q_n| \le |\xi|^n n! \sum_{i=0}^{n} \binom{n}{i}^2 = |\xi|^n n! \binom{2n}{n} < 4^n |\xi|^n n!\,,$$

$$|p_n| \le |\xi|^{n-1} n \sum_{i=0}^{n-1} i!\,(n-1-i)!\, \binom{n}{i}^2 \le |\xi|^n n! \sum_{i=0}^{n} \binom{n}{i}^2 < 4^n |\xi|^n n!$$

and

$$|r_{n,p}|_p = |\xi|_p^{2n} |n!|_p^2 \left| \sum_{k=0}^{\infty} (-1)^k k! \binom{n+k}{k}^2 \xi^k \right|_p \le |\xi|_p^{2n} |n!|_p^2.$$

Therefore, condition (5) reads

$$\limsup_{n\to\infty} 4^n |\xi|^n n! \prod_{p\in\mathcal{P}} |\xi|_p^{2n} |n!|_p^2 = 0$$

and because for at least one $p \in \mathcal{P}$ we have either $r_{n,p} \neq 0$ or $r_{n+1,p} \neq 0$ from (11), the theorem follows. □

3 Summation of divergent series

Fix a prime p. If we restrict, for simplicity, the sequence of indices n to the arithmetic progression $n \equiv 0 \bmod p$, then it is not hard to see from the calculation in Section 2 that the sequence of rational approximations $p_n/q_n = P_n(\xi)/Q_n(\xi)$ converges p-adically to $E(\xi)$. Indeed,

$$q_n = 1 + \sum_{i=1}^{p-1} i! \binom{n}{i}^2 \xi^i + \sum_{i=p}^{n} i! \binom{n}{i}^2 \xi^i \equiv 1 \bmod p$$

is a p-adic unit (all the binomials are divisible by p in the first sum, while $i!$ is divisible by p in the second one), and

$$\left| E(\xi) - \frac{p_n}{q_n} \right|_p = |q_n|_p^{-1} |q_n E(\xi) - p_n|_p = |r_{n,p}|_p \to 0 \quad \text{as } n \to \infty.$$

When $\xi > 0$, the same sequence of rational numbers $p_n/q_n = P_n(\xi)/Q_n(\xi)$ converges to the value at $z = \xi$ of the integral

$$\tilde{E}(z) = \int_0^\infty \frac{e^{-s}}{1+zs} \, \mathrm{d}s \tag{12}$$

with respect to the Archimedean norm. The integral itself converges for any $z \in \mathbb{C} \setminus (-\infty, 0]$, though its formal Taylor expansion at $z = 0$ (obtained by expanding $1/(1+zs)$ into the power series under the integral sign) is precisely Euler's factorial series $E(z)$ as in (1). In particular, relation (10) remains valid with $E(z)$ replaced by $\tilde{E}(z)$ and $R_n(z)$ by

$$\tilde{R}_n(z) = n!\, z^{2n} \int_0^\infty \frac{s^n e^{-s}}{(1+zs)^{n+1}} \, \mathrm{d}s \quad \text{for } n = 0, 1, 2, \dots,$$

so that

$$\left| \tilde{E}(\xi) - \frac{p_n}{q_n} \right| = \frac{\widetilde{R}_n(\xi)}{Q_n(\xi)} \le \frac{n^n}{(n+1)^{n+1}} \to 0 \quad \text{as } n \to \infty,$$

where the estimates $Q_n(\xi) \ge n!\,\xi^n$ and

$$\tilde{R}_n(\xi) \le n!\,\xi^{2n}\biggl(\max_{s>0}\frac{s^n}{(1+\xi s)^{n+1}}\biggr)\int_0^\infty e^{-s}\,\mathrm{d}s = n!\,\xi^n\frac{n^n}{(n+1)^{n+1}}$$

were used. This computation reveals us that

$$\lim_{n\to\infty}\frac{p_n}{q_n} = \tilde{E}(\xi) = -xe^x\biggl(\gamma+\log x+\sum_{k=1}^\infty\frac{(-x)^k}{k\cdot k!}\biggr)\bigg|_{x=1/\xi},$$

where $\gamma = 0.5772156649\dots$ is Euler's constant. In particular,

$$W = \tilde{E}(1) = e\biggl(-\gamma+\sum_{k=1}^\infty\frac{(-1)^{k-1}}{k\cdot k!}\biggr) = 0.5963473623\dots$$

for Wallis' series (2). The resulted quantity is known as the Euler–Gompertz constant [8].

The strategy in the previous paragraph does not apply to $z=\xi<0$, somewhat already observed by Stieltjes in [13]. The analytical continuation of the the function $\tilde{E}(z)$ depends on whether we perform the integration along the upper or lower banks of the ray $[0,\infty)$ in (12); denote the corresponding values by $\tilde{E}_+(z)$ and $\tilde{E}_-(z)$, respectively. By considering the integration of $e^{-s}/(1+zs)$ along the curvilinear triangle that consists of the segment $[0,R]$ (along a particular bank), the arc $[R, R\,e^{\sqrt{-1}\theta}]$ followed by the segment $[R\,e^{\sqrt{-1}\theta},0]$, where $0<\theta<\pi/2$ for the upper bank and $-\pi/2<\theta<0$ for the lower one, and then taking the limit as $R\to\infty$ (so that the integral along the arc tends to 0) we conclude that

$$\tilde{E}_\pm(z) = \int_0^{e^{\sqrt{-1}\theta}\infty}\frac{e^{-s}}{1+zs}\,\mathrm{d}s = -xe^x\biggl(\gamma+\log|x|\pm\sqrt{-1}\pi+\sum_{k=1}^\infty\frac{(-x)^k}{k\cdot k!}\biggr)\bigg|_{x=1/z}, \quad (13)$$

with the choice of θ arbitrary in the interval $0<\theta<\pi/2$ for $\tilde{E}_+(z)$ and in the interval $-\pi/2<\theta<0$ for $\tilde{E}_-(z)$. In particular,

$$\begin{aligned} K = \tilde{E}_\pm(-1) &= \frac{1}{e}\biggl(\gamma+\sum_{k=1}^\infty\frac{1}{k\cdot k!}\mp\sqrt{-1}\pi\biggr) \\ &= 0.6971748832\dots\mp\sqrt{-1}\cdot 1.1557273497\dots \end{aligned}$$

for the series in (3).

4 Final remarks

In [14] we outline a different strategy of proving a result analogous to Theorem 1 on using the Hankel determinants generated by the tails of Euler's factorial series (1). As the condition on a subset of primes $\mathcal{P}$ in that result is spiritually similar to (4), we do not detail the derivation here. However we stress that a potential combination of the two methods, namely, using the Hankel determinants generated by the Padé approximations of Euler's factorial series, may be a source of further novelties on the topic. A discussion on this type of construction in the Archimedean setting can be found in [15].

One consequence of the formula in (13), which uncovers a pair of *complex* conjugate values for $E(\xi)$ when $\xi < 0$, is that the *rational* approximations $p_n/q_n = P_n(\xi)/Q_n(\xi)$ do not converge at all in such cases. Interestingly enough, the Hankel determinants 'see' those complex values (13) as experimentally observed in [14].

Finally, we would like to note that nothing is known about the irrationality and transcendence of the Archimedean valuations of Euler's factorial series (1) at rational $z = \xi$ (see the discussion in [8, Sections 3.15, 3.16]). This is in contrast with its q-analogue

$$\sum_{k=0}^{\infty} z^k \prod_{i=1}^{k} (1 - q^i),$$

for which irrationality and linear independence results are known in Archimedean and non-Archimedean places alike — see [9]. Further details on a nice q-counterpart of the Padé approximation analysis can be found in [10].

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