

# Critical (Chiral) Heisenberg Model with the Functional Renormalisation Group

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We discuss the Heisenberg model and its chiral extension in an extended truncation with the help of functional methods. Employing computer algebra to derive the beta functions, and pseudo-spectral methods to solve them, we are able to go significantly beyond earlier approximations, and provide new estimates on the critical quantities of both models. The fixed point of the Heisenberg model is mostly understood, and our results are in agreement with estimates from various other approaches, including Monte Carlo and conformal bootstrap studies. By contrast, in the chiral case, the formerly known disagreement with lattice studies persists, raising the question whether actually the same universality class is described.

## I. INTRODUCTION

Many magnetic materials can be efficiently described by the Heisenberg model, which consists of a vector invariant under  $O(3)$  rotations. Examples are the Curie transition in isotropic ferromagnets and antiferromagnets at the Néel transition point<sup>1</sup>. The price to pay for the simplicity of this model is the negligence of some interactions that are present in real materials, for example dipolar interactions. Even though such interactions can be relevant perturbations<sup>2-4</sup>, studies show that their impact is small<sup>3,5-7</sup>.

Another interesting case where the Heisenberg model, and its chiral extension, play a role is the description of graphene<sup>8-11</sup>, and related materials<sup>12-34</sup>. Graphene is a very interesting material. Due to its honeycomb structure, it behaves very differently when compared with standard materials with a Bravais lattice structure. A direct consequence of the lattice structure of graphene is that the Fermi surface consists of two points only, and in principle invalidates the Landau Fermi liquid construction. Expanding the dispersion relation around the Fermi points, continuum models for the fermionic non-interacting low-energy excitations with relativistic symmetry can be constructed<sup>35-69</sup>.

A particular model for Dirac materials is given in Refs. 70-72. In Ref. 73, we dealt with the Ising-like subset of this model, which corresponds to a 3d Gross-Neveu model for four-component Dirac fermions in a reducible representation. Supersymmetric aspects of this model are considered *e.g.* in Refs. 74-76. The aim of the present work is to consider the Heisenberg-like subset, where an  $O(3)$ -invariant vector is coupled to these fermions via Pauli matrices.

Monolayered graphene is an effectively (2+1)-dimensional material. Since the upper (lower) critical dimension of the model that we consider here is  $d = 4(2)$ , the accuracy of perturbative results obtained with  $\epsilon$ -expansions<sup>47,77</sup> around one of the critical dimensions is an open question. On the other hand, lattice studies involving fermions might suffer from sign problems. Here, we will treat our model with the continuous realisation of the exact renormalisation group by Wetterich<sup>78</sup>.

There are several difficulties one encounters in the

study of critical phenomena with functional methods. The first step is to decide on an approximation (often called truncation), and to determine the renormalisation group (RG) flow of the operators present in this truncation. We will do this in complete analogy to the earlier study of the Ising counterpart<sup>73</sup> of this work with the help of xAct<sup>79-83</sup>. Consequently, the resulting differential equations have to be solved. We will use pseudo-spectral methods to do so, which were systematically put forward in the present context in Refs. 84 and 85, and applications of these methods to functional renormalisation group (FRG) studies can be found in Refs. 73, 74, 86, and 87.

The aim of the present work is to provide new estimates on critical quantities for both the Heisenberg model and its chiral equivalent. Several methods agree quite well on the Heisenberg model, whereas for the model involving fermions, the situation is not settled. In particular, there is a clash between FRG results and quantum Monte Carlo simulations on the values of the relevant critical exponent and the bosonic anomalous dimension<sup>47</sup>. Our study shows that this clash persists, even when a truncation retaining 12 operators is considered.

This paper is organised as follows. We start with a short recap on the FRG in section II, followed by the introduction of our model in section III. Subsequently, we discuss the results of the model, first without fermions in section IV, then with fermions in section V. We finally summarise the results in section VI.

## II. FUNCTIONAL RENORMALISATION GROUP

A convenient way to investigate quantum fluctuations in a non-perturbative manner is the effective average action,  $\Gamma_k$ . It interpolates between the classical action and the full quantum effective action, and fulfils the functional equation<sup>78</sup>

$$\partial_t \Gamma_k = \frac{\mathbf{i}}{2} \text{STr} \left[ \left( \Gamma_k^{(2)} + R_k \right)^{-1} (\partial_t R_k) \right], \quad t = \ln \left( \frac{k}{\Lambda} \right), \quad (1)$$

where  $\Gamma_k^{(2)}$  is the second functional derivative of the effective average action with respect to the fields that are con-

sidered, and  $t$  is the RG “time”, measuring momenta in units of a fixed momentum scale  $\Lambda$ . The super-trace  $\text{STr}$  indicates a sum over discrete and an integration over continuous indices, and includes a minus sign for fermions. The equation is well-defined due to the regulator  $R_k$ , which provides both ultraviolet (UV) and infrared (IR) regularisation. For reviews on the FRG, see Refs. 88–92.

Solving the flow equation (1) exactly is typically very difficult, and approximations have to be introduced. As an example, in gauge theories it is often important to resolve the full momentum dependence of vertices, thus a vertex expansion is employed. By contrast, in scalar and fermionic models, it seems to be the case that retaining momenta up to a fixed power, but arbitrary field dependence, captures the most important physics. In the present work, we will investigate an approximation which retains operators with at most two derivatives and two fermion fields. In the bosonic case, all operators of this class are included, and this approximation is commonly called next-to-leading order (NLO) in a derivative expansion. By contrast, in the chiral case, the full NLO approximation is intricate due to the complicated tensor structure, thus we will focus on a subset of all possible operators, concentrating on momentum-dependent corrections to the Yukawa coupling. We further discuss the effect of two rank 2 tensor couplings, showing that their impact is suppressed.

### III. THE MODEL

The model that we will describe shares essential features with classical four-fermion models such as the Gross-Neveu model<sup>93</sup>. We will deal with two flavours of massless relativistic Dirac fermions in a four-dimensional

reducible representation. In the conventions of Ref. 73, the Minkowskian microscopic action of this model reads

$$S = \int \left( \bar{\psi} (\mathbb{1}_2 \otimes \not{\partial}) \psi + \frac{\bar{g}}{4} (\bar{\psi} (\sigma^a \otimes \mathbb{1}_4) \psi)^2 \right), \quad (2)$$

where  $\sigma^a$  are the Pauli matrices. The crucial symmetry of this theory is the invariance under  $SU(2)$  spin rotations. By a partial bosonisation, we can reformulate the action in terms of a Yukawa theory with action

$$S^{\text{pb}} = \int \left( \bar{\psi} ((\mathbb{1}_2 \otimes \not{\partial}) + \bar{h} \phi^a (\sigma^a \otimes \mathbb{1}_4)) \psi - \bar{m}^2 (\phi^a)^2 \right). \quad (3)$$

Here,  $\bar{g} = \bar{h}^2 / \bar{m}^2$ , and  $\phi^a$  is a vector field invariant under  $SU(2) \simeq O(3)$  rotations. This will be the starting point for our investigation. In general, once quantum fluctuations are included, all further operators that are allowed by the symmetries will be generated, and have to be taken into account. In the following, we will include operators with at most two fermions and two derivatives.

Let us start with the purely bosonic part of our ansatz for the effective average action,

$$\Gamma_k^{\text{bos}} = \int \left( \frac{1}{2} Z_\phi(\rho) (\partial_\mu \phi^a)^2 + \frac{1}{2} Y_\phi(\rho) (\partial_\mu \rho)^2 - V(\rho) \right), \quad (4)$$

which includes the wave function renormalisation  $Z_\phi$ , a correction term to the radial propagator,  $Y_\phi$ , and the potential  $V$ . We also introduced  $\rho = \phi^a \phi^a / 2$  for convenience. In the subsequent section, where we discuss the Heisenberg model, this ansatz will be discussed.

For the fermions, we first introduce the kinetic term with fermion wave function renormalisation  $Z_\psi$ , and the standard Yukawa coupling,  $g_1$ ,

$$\Gamma_k^{\text{ferm}} = \int \left( \frac{1}{2} Z_\psi(\rho) (\bar{\psi} (\mathbb{1}_2 \otimes \not{\partial}) \psi - (\partial^\mu \bar{\psi}) (\mathbb{1}_2 \otimes \gamma_\mu) \psi) + g_1(\rho) \phi^a \bar{\psi} (\sigma^a \otimes \mathbb{1}_4) \psi \right). \quad (5)$$

There are 7 further operators that we will consider here. Most come with the tensor structure  $\sigma^a \otimes \mathbb{1}_4$  and carry two derivatives, thus they are momentum-dependent extensions of the Yukawa coupling and will be labelled by a  $g$ . To study the effect of tensorial interactions, we further study two operators which couple via  $\Sigma_{\mu\nu} = 2[\gamma_\mu, \gamma_\nu]$ :

$$\begin{aligned} \Gamma_k^{\text{int}} = \int \left[ & - \left( g_2(\rho) - \frac{1}{2} g_6(\rho) \right) (\partial^2 \phi^a) \bar{\psi} (\sigma^a \otimes \mathbb{1}_4) \psi - \left( g_3(\rho) - \frac{1}{2} g'_6(\rho) \right) (\partial^2 \rho) \phi^a \bar{\psi} (\sigma^a \otimes \mathbb{1}_4) \psi \right. \\ & + \frac{1}{2} g_4(\rho) (\partial^\mu \phi^b)^2 \phi^a \bar{\psi} (\sigma^a \otimes \mathbb{1}_4) \psi + \{ g_5(\rho) - g'_2(\rho) - g_3(\rho) + g'_6(\rho) \} (\partial_\mu \phi^a) (\partial^\mu \rho) \bar{\psi} (\sigma^a \otimes \mathbb{1}_4) \psi \\ & - \frac{1}{2} g_6(\rho) \phi^a (\bar{\psi} (\sigma^a \otimes \mathbb{1}_4) \partial^2 \psi + (\partial^2 \bar{\psi}) (\sigma^a \otimes \mathbb{1}_4) \psi) \\ & \left. + \frac{1}{2} \{ T_1(\rho) (\partial_\mu \phi^a) + T_2(\rho) \phi^a (\partial_\mu \rho) \} (\bar{\psi} (\sigma^a \otimes \Sigma^{\mu\nu}) \partial_\nu \psi - (\partial_\nu \bar{\psi}) (\sigma^a \otimes \Sigma^{\mu\nu}) \psi) \right]. \quad (6) \end{aligned}$$

The specific linear combinations in front of the invariants are chosen for convenience, and simplify the calculation.

Our conventions on the Clifford algebra are the same as in Ref. 73. The full ansatz for the chiral Heisenberg

model combines all of this,

$$\Gamma_k^{\text{HGN}} = \Gamma_k^{\text{bos}} + \Gamma_k^{\text{ferm}} + \Gamma_k^{\text{int}}. \quad (7)$$

All functions depend on the renormalisation group scale  $k$ , and have to be real in order that the Minkowskian ansatz for the action is real. All algebraic manipulations are done in Minkowski space, and only the final integration over the loop momentum is done after a Wick rotation. The symmetries of the above model are discussed in Ref. 47, and the most constraining symmetry for the construction of the ansatz is the discrete  $\mathbb{Z}_2$  reflection symmetry,

$$\psi \rightarrow (\mathbb{1}_{N_f} \otimes \gamma_2)\psi, \bar{\psi} \rightarrow -\bar{\psi}(\mathbb{1}_{N_f} \otimes \gamma_2), \phi^a \rightarrow -\phi^a, \quad (8)$$

together with a parity transformation of spacetime, *e.g.*  $x_1 \rightarrow -x_1$ . We don't expect further accidental symmetries as in the case of the chiral Ising model<sup>73</sup>, where a symmetry related to a reality constraint is present, constraining the occurrence of a certain operator class. This expectation comes from the explicit occurrence of a factor of  $\mathbf{i}$  in the commutator of the Pauli matrices.

The truncation (7) goes significantly beyond any FRG calculation of this model. All calculations so far only included a field-dependent potential together with field independent but scale-dependent wave functions  $Z_\phi, Z_\psi$  and Yukawa coupling  $g_\phi$ <sup>47,72</sup>.

To discuss the critical behaviour of a given model, dimensionless or renormalised quantities are introduced. Fixed points, which describe *e.g.* phase transitions, are then characterised by the vanishing of the flow of these renormalised couplings. The relation between bare and renormalised quantities is straightforward, and we will not write it down explicitly. It is in complete analogy to Ref. 73, and we encounter the same ambiguity: at which  $\rho = \bar{\rho}$  do we normalise the wave function renormalisations? Possible choices include the vacuum expectation value (vev), or zero. This ambiguity can be used to check the stability of our results. The running of this normalisation is encoded in the anomalous dimensions,

$$\begin{aligned} \eta_\phi &= -\partial_t \ln Z_\phi(\bar{\rho}), \\ \eta_\psi &= -\partial_t \ln Z_\psi(\bar{\rho}). \end{aligned} \quad (9)$$

Now, let us specify the regulator that we employ. In complete analogy to Ref. 73, we regularise the action by adding

$$\begin{aligned} \Delta S_\chi &= \int \left( \frac{1}{2} \phi^a R_\phi \left( \frac{p^2}{k^2} \right) \phi^a \right. \\ &\quad \left. + \bar{\psi} R_\psi \left( \frac{p^2}{k^2} \right) \frac{(\mathbb{1}_2 \otimes \gamma_\mu) \partial^\mu}{p} \psi \right). \end{aligned} \quad (10)$$

Here, momentum arguments are to be understood as those after Wick rotation. To be able to optimise results with the principle of minimum sensitivity (PMS), we will employ several regulator kernels. Optimisation aspects in the context of the FRG are discussed *e.g.* in

Refs. 89, 94–103. On the one hand, we discuss the linear regulator<sup>95</sup>,

$$R_\phi(x) = k^2(1-x)\theta(1-x), \quad (11)$$

with  $\theta$  being the Heaviside step function. On the other hand, we also study a one-parameter family of exponential regulators given by

$$\begin{aligned} R_\phi(x) &= \frac{k^2}{2e^{x^a} - 1}, \\ R_\psi(x) &= \frac{k}{2e^{x^a} - 1}. \end{aligned} \quad (12)$$

The numerical integration of the threshold functions is performed via an adaptive Gauss-Kronrod 7-15 rule, with the same parameters as chosen in Ref. 73.

Finally, some words on the derivation of the actual flow equations are in order. Clearly, it is very tedious to calculate the flow equations for all functions in (7) by hand. To minimise the danger of errors, we used the Mathematica package xAct<sup>79–83</sup> to derive them. The calculation proceeds similarly to the standard Gross-Neveu model<sup>73</sup>, except that the additional  $SU(2)$  index makes the tensor structure richer, and thus the flavour structure cannot be treated abstractly. Still, in principle the calculation is straightforward, but very lengthy<sup>104</sup>.

The system of 12 flow equations for the ansatz (7) has been solved with pseudo-spectral methods, which were systematically adapted to the present case in Ref. 84, applications in the context of the FRG can be found in Refs. 73, 74, 85–87. For the handling of linear algebra, we employed the library Eigen<sup>105</sup>.

#### IV. RESULTS FOR THE HEISENBERG MODEL

We will now discuss the results for the Heisenberg model, *i.e.* we switch off the fermions. First the result obtained with the Litim regulator will be presented, and afterwards optimised values with the help of the exponential regulators are given.

From here on, we discuss dimensionless quantities only, which are obtained by suitable rescalings with the RG scale  $k$ . In the following,  $\rho_0$  will denote the vev, so that  $V'(\rho_0) = 0$ .

Two possible solutions are compared, where we use the aforementioned ambiguity in normalising the wave function renormalisation to one at an arbitrary point. For scheme A, we fix  $Z_\phi$  to unity at the vev, *i.e.*  $\bar{\rho} = \rho_0$ , whereas for scheme B, we fix  $Z_\phi(0) = 1$ , such that  $\bar{\rho} = 0$ . The nomenclature follows Ref. 73.

All three fixed point functions are shown in Figure 1. In contrast to the case of the Ising model<sup>73</sup>, for the Heisenberg model, both projection schemes A and B deliver a consistent picture. A small difference can only be seen (naturally) in the wave function renormalisation. This also settles in the values for the vev and the anomalous dimension,

$$\rho_0^A = 0.056838, \quad \eta^A = 0.052347,$$

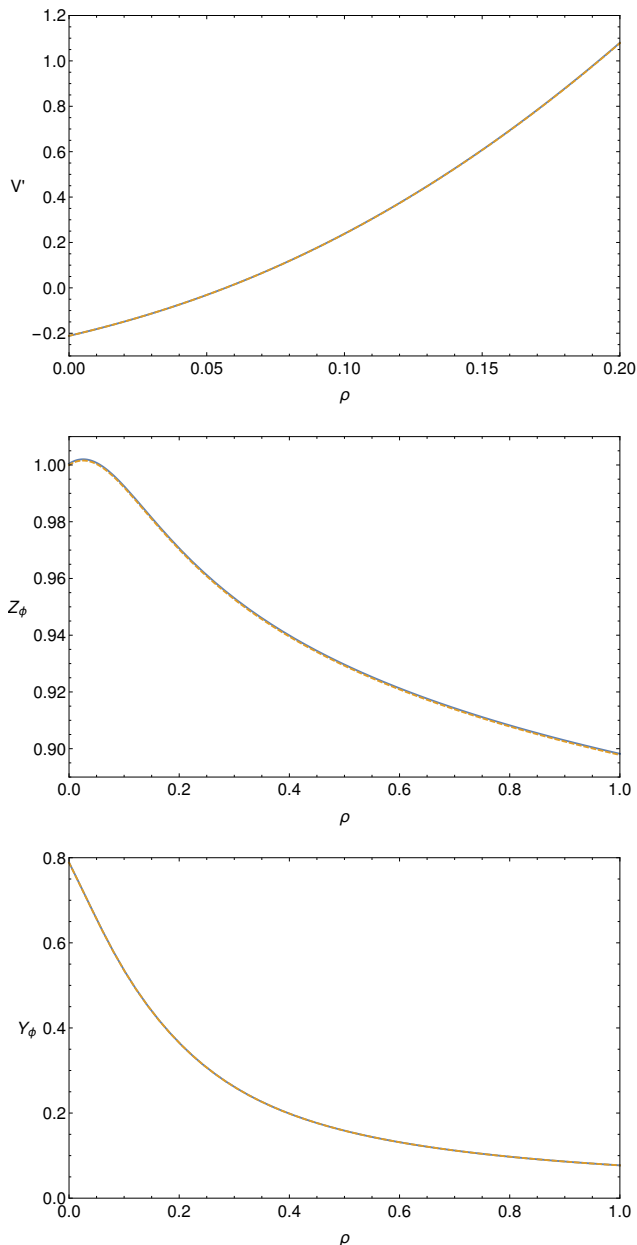


FIG. 1. Fixed point solution to the Heisenberg model in NLO. Shown are the derivative of the effective potential (top), the wave function renormalisation (middle), and the correction to the radial propagator (bottom). Blue solid lines correspond to the solution where  $Z_\phi(\rho_0) = 1$ , whereas orange, dashed lines indicate that  $Z_\phi(0) = 1$  was chosen. In contrast to the Ising model, here both projection schemes deliver virtually the same fixed point solution. Even the small shift between the two wave function renormalisations can hardly be seen.

$$\rho_0^B = 0.056868, \quad \eta^B = 0.052356. \quad (13)$$

The difference between the two projections is at the per mille level.

Let us now discuss the critical exponents. In the context of the FRG, they are determined as (minus) the

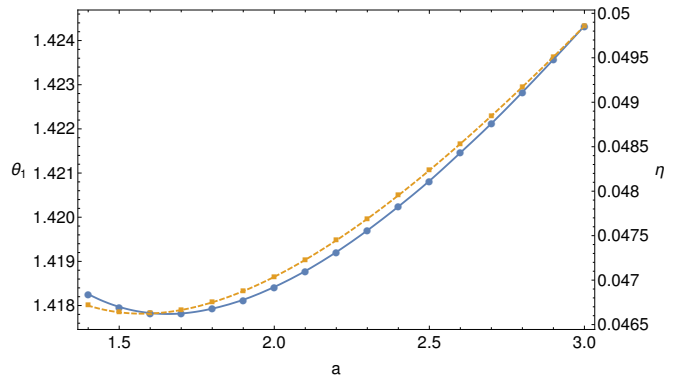


FIG. 2. Dependence of the first critical exponent  $\theta_1$  (blue dots) and the anomalous dimension  $\eta$  (orange boxes) of the Heisenberg model on the regulator parameter  $a$ . An interpolation helps to guide the eye.

eigenvalues of the differential operator which is obtained by linearising the flow equations around the fixed point. Since trivial rescalings of the field are of no physical interest, we further have to demand that the variation of the wave function renormalisation at  $\bar{\rho}$  vanishes. The above picture carries over to the critical exponents, both schemes deliver quantitatively well agreeing values,

$$\begin{aligned} \theta_1^A &= 1.42965, & \theta_1^B &= 1.42987, \\ \theta_2^A &= -0.73398, & \theta_2^B &= -0.73358, \end{aligned} \quad (14)$$

the difference being in the sub per mille level.

Let us switch to the one-parameter family of exponential regulators, and for definiteness we only discuss projection scheme A. As an example, the dependence of  $\theta_1$  and  $\eta$  on  $a$  is plotted in Figure 2. It seems that the regulator dependence is a bit stronger compared to the one in the Ising model<sup>73</sup>. Still, optimised values for the first two critical exponents and the anomalous dimension can be inferred with PMS,

$$\begin{aligned} \theta_1^{\text{opt}} &= 1.4178, & a^{\text{opt}} &= 1.66, \\ \theta_2^{\text{opt}} &= -0.7473, & a^{\text{opt}} &= 1.73, \\ \eta^{\text{opt}} &= 0.04662, & a^{\text{opt}} &= 1.57. \end{aligned} \quad (15)$$

The optimal values for the parameter  $a$  are quite close to the ones in the Ising model<sup>73</sup>, and consistently optimise the first two critical exponents as well as the anomalous dimension. More general optimisation criteria can be found in Refs. 89, 94–96, and 103. For comparison, we give recent Monte Carlo<sup>106</sup> (MC) and conformal bootstrap<sup>107</sup> (CBS) results:

$$\begin{aligned} \theta_1^{\text{MC}} &= 1.4053(20), \\ \eta^{\text{MC}} &= 0.0378(3), \\ \theta_1^{\text{CBS}} &= 1.4043(55), \\ \eta^{\text{CBS}} &= 0.03856(124). \end{aligned} \quad (16)$$

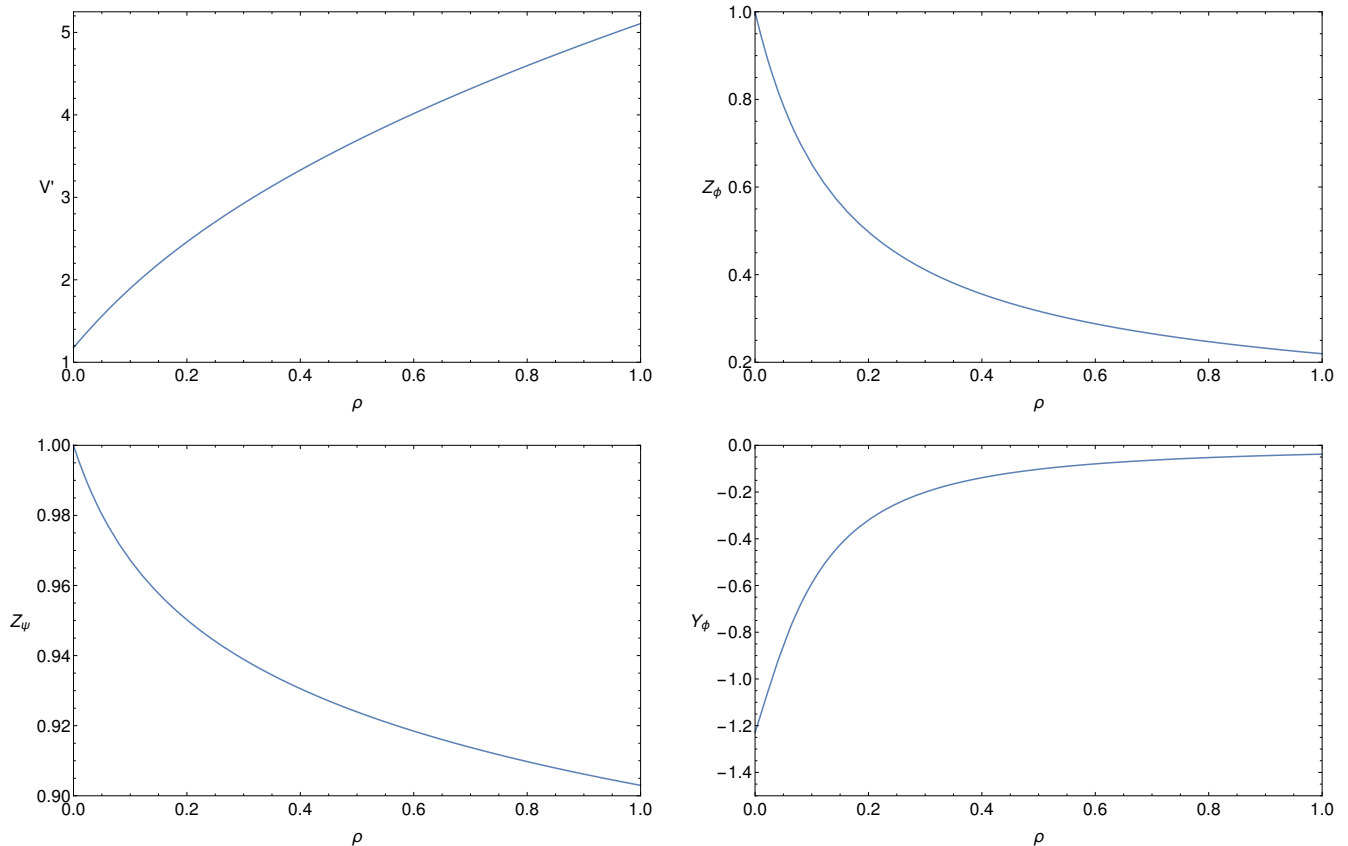


FIG. 3. The first derivative of the potential and the wave function renormalisations at the fixed point of the chiral Heisenberg model, for the regulator parameter  $a = 2$ . In contrast to the Heisenberg model, the correction to the radial wave function,  $Y_\phi$ , is negative. Still, the combination of  $Z_\phi$  and  $Y_\phi$  that appears in the propagator is strictly positive, and thus the propagator is well-defined for all field values.

The optimised value for the leading critical exponent is in good agreement with these estimates, differing by only 1%. As expected, the anomalous dimension needs further improvement by enhancing the truncation. In comparison to a truncation that only retains the potential and the anomalous dimension<sup>72</sup>,  $\theta_1$  changes by about 4%.

## V. RESULTS FOR THE CHIRAL HEISENBERG MODEL

Let us now switch on fermions, and study the full system (7) at criticality. The fixed point lies in the symmetric regime, and thus, schemes A and B are the same. For definiteness, we only discuss the family of exponential regulators.

As exemplary case, we show the fixed point solution for the specific regulator parameter choice  $a = 2$  in Figure 3 and Figure 4. The first of the two figures displays the first derivative of the potential, the two wave function renormalisations and the correction term to the radial propagator of the bosons. Since  $V' > 0$ , we are in the symmetric regime. In contrast to the purely bosonic case,

$Y_\phi$  is negative. This is not a problem, since the combination of  $Z_\phi$  and  $Y_\phi$  that appears in the propagator is strictly positive. Figure 4 shows the standard as well as the generalised Yukawa interactions. As the bosonic anomalous dimension is larger than 1, the Yukawa interaction decreases for increasing  $\rho$ , contrary to the case of the standard Gross-Neveu model<sup>73</sup>. All generalised Yukawa couplings and the tensorial couplings are suppressed, as their mass dimension suggests.

From the dependence on the regulator parameter  $a$ , we can optimise our estimates for physical quantities. The dependence of the first critical exponent and the two anomalous dimensions on  $a$  is shown in Figure 5. From that, we obtain the optimised values

$$\begin{aligned} \theta^{\text{opt}} &= 0.795, & a^{\text{opt}} &= 3.03, \\ \eta_\phi^{\text{opt}} &= 1.032, & a^{\text{opt}} &= 1.68. \end{aligned} \quad (17)$$

The fermionic anomalous dimension doesn't display an extremum in the considered parameter range. Taking the value at  $a = 3$  and accounting for the dependence on  $a$  by varying it by  $\pm 1$ , we estimate

$$\eta_\psi = 0.071(2). \quad (18)$$

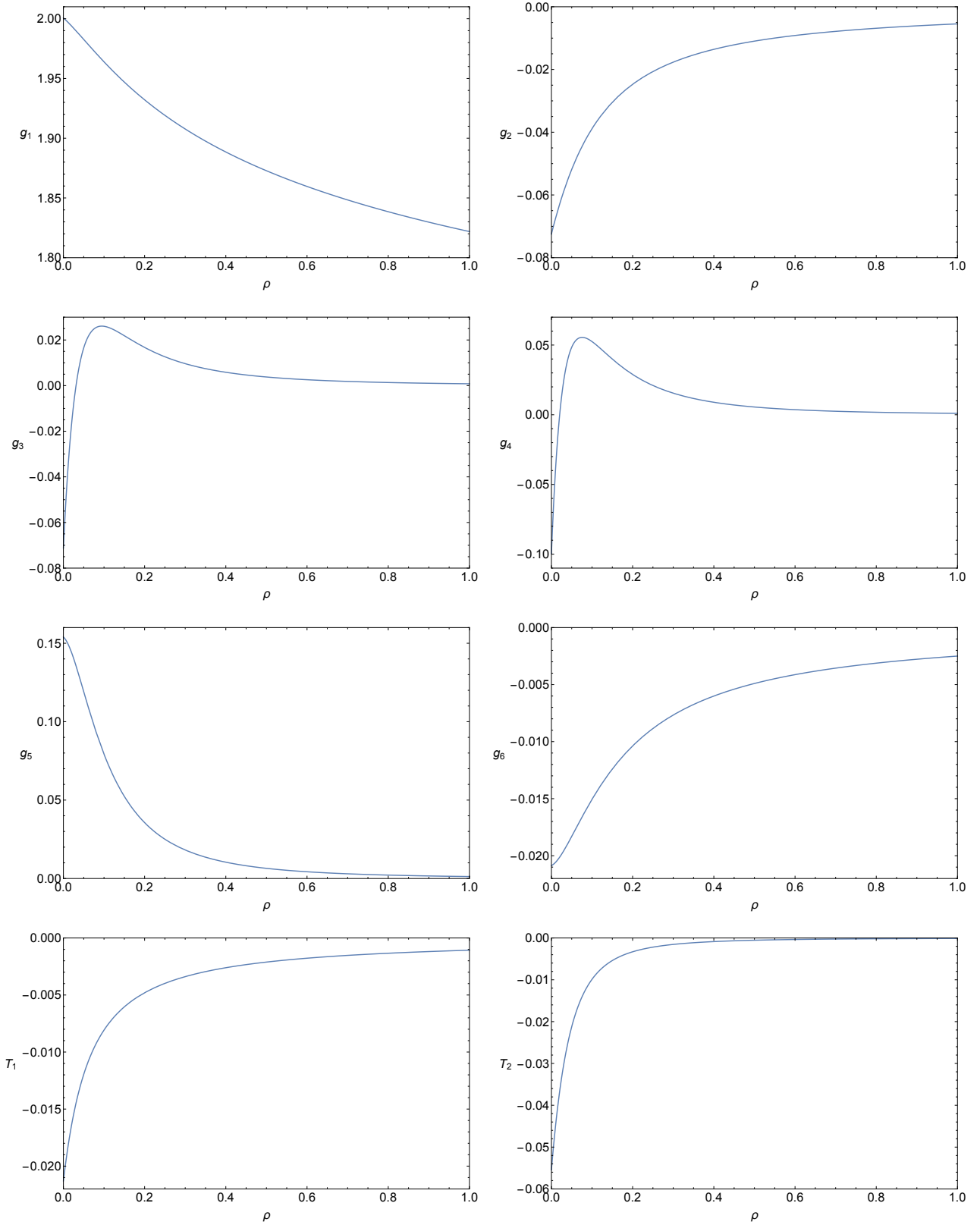


FIG. 4. Fixed point solution of the generalised Yukawa, and tensorial interactions of the chiral Heisenberg model, for the regulator parameter  $a = 2$ . Since  $\eta_\phi$  is larger than 1, the standard Yukawa coupling  $g_1$  falls off to zero for large  $\rho$ , in contrast to the Yukawa coupling of the chiral Ising model. The other couplings are suppressed, as expected from their high mass dimension.

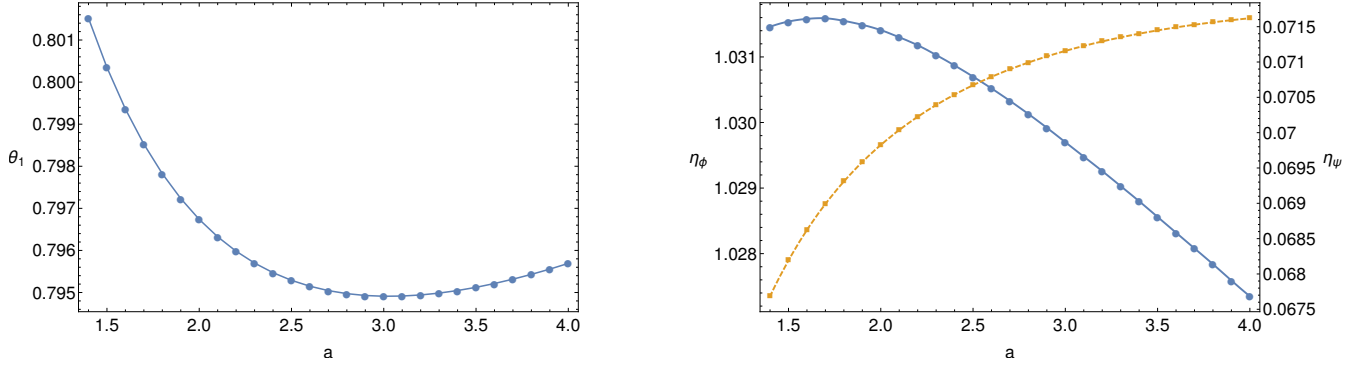


FIG. 5. Regulator dependence of physical quantities of the chiral Heisenberg model. On the left panel, the first critical exponent is shown. On the right panel, the bosonic (blue dots) and fermionic (orange boxes) anomalous dimensions are plotted. Interpolations help to guide the eye. Whereas both the critical exponent and the bosonic anomalous dimension show an extremum, the fermionic anomalous dimension doesn't, and cannot be optimised by PMS.

	$\theta_1$	$\eta_\phi$	$\eta_\psi$
FRG (this work)	0.795	1.032	0.071(2)
FRG <sup>47</sup>	0.772	1.015	0.084
quantum Monte Carlo <sup>108</sup>	1.19(6)	0.70(15)	—
quantum Monte Carlo <sup>109</sup>	0.98(1)	0.49(5)	0.20(2)
(4 - $\epsilon$ ) 2nd order <sup>47,77</sup>	0.834	0.959	0.242

TABLE I. Comparison of the first critical exponent and the anomalous dimensions of the chiral Heisenberg model with the literature. To estimate the bosonic anomalous dimension from the results of Ref. 109, we employed the hyperscaling relation  $\eta_\phi = 2\beta\theta_1 + 2 - d$ .

Table I shows a comparison to results obtained by different methods. The present results are in good agreement with earlier FRG studies that only retain the field dependence of the potential together with field-independent wave functions and standard Yukawa coupling  $g_1$ <sup>47</sup>. The critical exponent and the bosonic anomalous dimension only change by a few percent, the fermionic anomalous dimension by about 17%. Except for  $\eta_\psi$ , our results are in reasonable agreement with results obtained with an  $\epsilon$ -expansion<sup>47,77</sup>, and in mutual disagreement with quantum Monte Carlo results<sup>108,109</sup>. Taking our results as point of reference, their estimate for the critical exponent disagrees by 23% and 50%, respectively. For the bosonic anomalous dimension, the discrepancy is about 32% and 53%, respectively, whereas the fermionic anomalous dimension from the quantum Monte Carlo simulation of Ref. 109 differs by about a factor of 3. This raises the question whether the same universality class is studied.

## VI. SUMMARY

The present work completes our study of the two subsystems of the particular model of graphene put forward in Refs. 70–72, including the presumably most important operators at next-to-leading order in the derivative

expansion. In the first part, we investigated the model without fermions, *i.e.* the Heisenberg model. The ambiguity in the normalisation of the wave function renormalisation was shown to be not a problem at all, rather both schemes that have been investigated deliver a consistent picture of the model at criticality. This is in contrast to the Ising model at the same level of truncation, and might indicate that  $O(N)$ -symmetric models with  $N = 3$  are already well described by a large- $N$  approximation, as there the influence of the wave function renormalisation is parametrically suppressed.

Regarding the chiral model, we studied a very extensive truncation, including 12 operators. In particular, we focussed on the momentum-dependent corrections to the standard Yukawa coupling, and discussed the effect of tensorial couplings. The stability of critical quantities, as the leading critical exponent or the anomalous dimensions, is remarkable, if one compares to a truncation that only retains minimal information, similarly to the case of the chiral Ising model<sup>73</sup>. This is taken as a strong hint to an exceptionally good convergence behaviour of the derivative expansion in such models, and strengthens our trust in the quantitative accuracy of the present results.

The previously found disagreement<sup>47</sup> with quantum Monte Carlo studies<sup>108</sup> persists, and now includes also the more recent quantum Monte Carlo results of Ref. 109. The first critical exponent and the bosonic anomalous dimension differ by 20 to 50%, the fermionic anomalous dimension disagrees by a factor of about 3. The present work suggests that including more operators with more derivatives won't change the results by a lot. By comparison with results from both the  $\epsilon$ -expansion<sup>47,77</sup> and quantum Monte Carlo studies<sup>109</sup>, we however expect the fermionic anomalous dimension to become larger upon inclusion of further operators.

There are at least two possibilities for improvement of the present calculation. On the one hand, four-fermion terms can be included, ideally by a dynamical bosonisa-

tion along the lines of Refs. 89, 90, 110–114. On the other hand, to go beyond the derivative expansion, also the momentum dependence can be resolved, see *e.g.* Refs. 115 and 116 for works resolving both momentum and field dependences with the FRG.

In principle, now we are in the situation to study the combined system of (7) and Ref. 73, and give precision estimates on critical quantities for Dirac materials. Unfortunately, even for the uncoupled fixed points that can be constructed directly from the solutions of the two subsystems, no estimate on the decisive third critical exponent can be made, as no scaling relation is known, in contrast to the situation in *e.g.* the  $O(N) \oplus O(M)$ -model. Also, further operators appear that mix the two bosonic order parameters, and these will be important both for the determination of the third critical exponent of the uncoupled fixed points, and for the determination of fully coupled fixed points. Still, both the technology put forward and the experience with the subsystems will

be helpful to study the coupled system. In particular, it seems that a truncation which resolves the potential, all kinetic terms of the bosons, the kinetic term of the fermions and the standard Yukawa coupling are already reliable quantitatively.

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