LOGARITHMIC TOPOLOGICAL HOCHSCHILD HOMOLOGY OF TOPOLOGICAL K-THEORY SPECTRA

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ABSTRACT. In this paper we continue our study of logarithmic topological Hochschild homology. We show that the inclusion of the connective Adams summand ℓ into the p-local complex connective K-theory spectrum $ku_{(p)}$, equipped with suitable log structures, is a formally log THH-étale map, and compute the $V(1)$-homotopy of their logarithmic topological Hochschild homology spectra. As an application, we recover Ausoni's computation of the $V(1)$-homotopy of the ordinary THH of ku.

1. INTRODUCTION

Logarithmic topological Hochschild homology (log THH) is an extension of the usual topological Hochschild homology, which is defined on more general objects than ordinary rings or ordinary structured ring spectra. Its input is a *pre-log ring spectrum* (A, M), consisting of a commutative symmetric ring spectrum A together with certain extra data. We recall the precise definition below. One reason for considering this theory is that the log THH of appropriate pre-log ring spectra participates in interesting localization homotopy cofiber sequences that do not exist for ordinary THH. In the first part [RSS15] of this series of papers, we have shown that there is a localization homotopy cofiber sequence

$$\mathrm{THH}(e) \to \mathrm{THH}(e, j_*\mathrm{GL}_1^{\mathcal{J}}(E)) \to \Sigma\,\mathrm{THH}(e[0, d\rangle) \tag{1.1}$$

associated with the connective cover map $j\colon e \to E$ of a d-periodic commutative symmetric ring spectrum E. Here $(e, j_*\mathrm{GL}_1^{\mathcal{J}}(E))$ is a pre-log ring spectrum with underlying ring spectrum e, and $e[0, d\rangle$ is the $(d-1)$-st Postnikov section of e. Real and complex topological K-theory spectra give rise to examples of this homotopy cofiber sequence.

In the present paper, we compute log THH in some important examples. One reason this is interesting is that the homotopy groups of $\mathrm{THH}(A, M)$ sometimes have a simpler structure than those of $\mathrm{THH}(A)$. By means of the homotopy cofiber sequence (1.1), one can then use knowledge about $\mathrm{THH}(A, M)$ to determine $\mathrm{THH}(A)$. Specifically, we implement this strategy in the case of the p-local complex K-theory spectrum $ku_{(p)}$ for an odd prime p. Let $V(1)$ denote the Smith–Toda complex of type 2 (see Notation 7.1 below). We shall then first determine the $V(1)$-homotopy of $\mathrm{THH}(ku_{(p)}, j_*\mathrm{GL}_1^{\mathcal{J}}(KU_{(p)}))$ and based on this compute $V(1)_*\,\mathrm{THH}(ku_{(p)})$. This realizes the approach to $V(1)_*\,\mathrm{THH}(ku) \cong V(1)_*\,\mathrm{THH}(ku_{(p)})$ outlined by Ausoni in [Aus05, §10] and gives an independent proof of the main result of [Aus05]. One key ingredient for this is that the tame ramification of the inclusion of the Adams summand $\ell \to ku_{(p)}$ is detected by log THH. This strategy is motivated by related results for discrete rings obtained by Hesselholt–Madsen [HM03, §2]. The idea of extending it to the topological K-theory example was first promoted by Hesselholt.

Date: April 1, 2016.

2010 *Mathematics Subject Classification.* 55P43; 14F10, 19D55.

Key words and phrases. Logarithmic ring spectrum, Adams summand, complex topological K-theory spectrum.

1.1. **Definition of log** THH. We briefly recall the definition of log THH and refer the reader to [RSS15] or Sections 2 to 4 of the present paper for more details, and to [RSS15] and [Rog09] for background and motivation. Let A be a commutative symmetric ring spectrum. It has an underlying *graded* multiplicative E_∞ space $\Omega^{\mathcal{J}}(A)$. This object $\Omega^{\mathcal{J}}(A)$ is a *commutative* $\mathcal{J}$*-space monoid* in the sense of [SS12, Section 4], i.e., a symmetric monoidal functor from a certain indexing category $\mathcal{J}$ to spaces. A pre-log ring spectrum (A, M) is a commutative symmetric ring spectrum A together with a commutative $\mathcal{J}$-space monoid M and a map of commutative $\mathcal{J}$-space monoids $\alpha\colon M \to \Omega^{\mathcal{J}}(A)$. The following direct image construction is a source of non-trivial pre-log ring spectra: If $j\colon e \to E$ is the connective cover map of a periodic commutative symmetric ring spectrum E, then one can form a diagram of commutative $\mathcal{J}$-space monoids

$$\mathrm{GL}_1^{\mathcal{J}}(E) \to \Omega^{\mathcal{J}}(E) \leftarrow \Omega^{\mathcal{J}}(e), \tag{1.2}$$

where $\mathrm{GL}_1^{\mathcal{J}}(E)$ is the commutative $\mathcal{J}$-space monoid of *graded* units associated with the ring spectrum E. The pullback $j_*\mathrm{GL}_1^{\mathcal{J}}(E)$ of (1.2) comes with a canonical map to $\Omega^{\mathcal{J}}(e)$ and defines a pre-log ring spectrum $(e, j_*\mathrm{GL}_1^{\mathcal{J}}(E))$.

Let (A, M) be a pre-log ring spectrum. By definition, its logarithmic topological Hochschild homology $\mathrm{THH}(A, M)$ is the homotopy pushout of the following diagram of cyclic commutative symmetric ring spectra:

$$\mathbb{S}^{\mathcal{J}}[B^{\mathrm{rep}}(M)] \leftarrow \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)] \to \mathrm{THH}(A). \tag{1.3}$$

The right hand term is the ordinary THH of A, given by the cyclic bar construction on A. The middle term is the graded suspension spectrum associated with the cyclic bar construction $B^{\mathrm{cy}}(M)$ on the commutative $\mathcal{J}$-space monoid M. The right hand map is induced by the adjoint $\mathbb{S}^{\mathcal{J}}[M] \to A$ of the structure map $\alpha\colon M \to \Omega^{\mathcal{J}}(A)$ of (A, M). The left hand map is induced by the map $B^{\mathrm{cy}}(M) \to B^{\mathrm{rep}}(M)$ to the *replete* bar construction $B^{\mathrm{rep}}(M)$ of M. The latter can be viewed as a variant of $B^{\mathrm{cy}}(M)$ that is formed relative to the group completion of M.

1.2. **Log** THH **of the Adams summand.** Let p be an odd prime and let ℓ be the Adams summand of the p-local complex connective K-theory spectrum $ku_{(p)}$. It is known that the map $j\colon \ell \to L$ to the periodic version L of ℓ can be represented by a map of commutative symmetric ring spectra. Hence we can form the pre-log ring spectrum $(\ell, j_*\mathrm{GL}_1^{\mathcal{J}}(L))$. In this case, the homotopy cofiber sequence (1.1) relates $\mathrm{THH}(\ell, j_*\mathrm{GL}_1^{\mathcal{J}}(L))$ to the ordinary THH of ℓ and $H\mathbb{Z}_{(p)}$.

Writing E and P for exterior and polynomial algebras over $\mathbb{F}_p$, respectively, we can formulate our first main result.

Theorem 1.3. *There is an algebra isomorphism*

$$V(1)_* \,\mathrm{THH}(\ell, j_*\mathrm{GL}_1^{\mathcal{J}}(L)) \cong E(\lambda_1, d\log v) \otimes P(\kappa_1)\,,$$

with $|\lambda_1| = 2p-1$, $|d\log v| = 1$ *and* $|\kappa_1| = 2p$. *The suspension operator* σ *arising from the circle action on* $\mathrm{THH}(\ell, j_*\mathrm{GL}_1^{\mathcal{J}}(L))$ *satisfies* $\sigma(\kappa_1) = \kappa_1 \cdot d\log v$, *and is zero on* λ_1 *and* $d\log v$.

The strategy for the proof of Theorem 1.3 is as follows. In a first step, we use the invariance of log THH under logification established in [RSS15, Theorem 4.24] to replace $(\ell, j_*\mathrm{GL}_1^{\mathcal{J}}(L))$ by a pre-log ring spectrum $(\ell, D(v))$ with equivalent log THH. This $(\ell, D(v))$ was also considered in [Sag14]. Its advantage is that the E_∞ space $\mathrm{hocolim}_{\mathcal{J}} D(v)$ associated with $D(v)$ is equivalent to $Q_{\geq 0}S^0$, the non-negative components of $QS^0 = \Omega^\infty\Sigma^\infty S^0$. So the homology of $\mathrm{hocolim}_{\mathcal{J}} D(v)$ is well understood and independent of ℓ. Using the graded Thom isomorphism established by the last two authors in [SS14], this allows us to determine the homology of $\mathbb{S}^{\mathcal{J}}[D(v)]$, $\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(D(v))]$, and $\mathbb{S}^{\mathcal{J}}[B^{\mathrm{rep}}(D(v))]$. Combining this with the computation of $V(1)_* \operatorname{THH}(\ell)$ by McClure and Staffeldt [MS93], an application of the Künneth spectral sequence associated with the homotopy pushout (1.3) leads to our computation of $V(1)_* \operatorname{THH}(\ell, j_*\mathrm{GL}_1^{\mathcal{J}}(L))$.

1.4. **The inclusion of the Adams summand.** In analogy with the notion of *formally* THH-*étale* maps in [Rog08, §9.2], we say that $(A, M) \to (B, N)$ is a *formally log* THH-*étale* map of pre-log ring spectra if $B \wedge_A \operatorname{THH}(A, M) \to \operatorname{THH}(B, N)$ is a stable equivalence. Our second main theorem verifies this property in an example:

Theorem 1.5. *The inclusion of the Adams summand $\ell \to ku_{(p)}$ induces a formally log* THH-*étale map $(\ell, j_*\mathrm{GL}_1^{\mathcal{J}}(L)) \to (ku_{(p)}, j_*\mathrm{GL}_1^{\mathcal{J}}(KU_{(p)}))$.*

Here $KU_{(p)}$ is the periodic version of $ku_{(p)}$, and $j_*\mathrm{GL}_1^{\mathcal{J}}(KU_{(p)})$ denotes the direct image pre-log structure on $ku_{(p)}$, constructed as above. We stress that the proof of this theorem does not depend on computations of $\operatorname{THH}(\ell, j_*\mathrm{GL}_1^{\mathcal{J}}(L))$ and $\operatorname{THH}(ku_{(p)}, j_*\mathrm{GL}_1^{\mathcal{J}}(KU_{(p)}))$. Instead, it is based on a certain decomposition of the replete bar construction, and on the graded Thom isomorphism and the invariance of log THH under logification mentioned above.

It is shown in [Sag14, Theorem 1.6] that $(\ell, j_*\mathrm{GL}_1^{\mathcal{J}}(L)) \to (ku_{(p)}, j_*\mathrm{GL}_1^{\mathcal{J}}(KU_{(p)}))$ is also formally étale with respect to logarithmic topological André-Quillen homology. Since the logarithmic Kähler differentials of algebraic geometry can be used to measure ramification beyond *tame* ramification of discrete valuation rings, these results show that $\ell \to ku_{(p)}$ behaves as a tamely ramified extension of ring spectra. By analogy with Emmy Noether's correspondence between tame ramification and the existence of normal bases, these results are compatible with the fact that $ku_{(p)}$ is a retract of a finite cell $\ell[\Delta]$-module spectrum, where $\Delta = (\mathbb{Z}/p)^\times$ is the Galois group of $L \to KU_{(p)}$.

1.6. THH **of the connective complex K-theory spectrum.** Combining the last two theorems leads to the following result. Here P_{p-1} denotes a height $p-1$ truncated polynomial algebra.

Theorem 1.7. *There is an algebra isomorphism*

$$V(1)_* \operatorname{THH}(ku_{(p)}, j_*\mathrm{GL}_1^{\mathcal{J}}(KU_{(p)})) \cong P_{p-1}(u) \otimes E(\lambda_1, d\log u) \otimes P(\kappa_1)$$

with $|u| = 2$, $|\lambda_1| = 2p-1$, $|d\log u| = 1$ and $|\kappa_1| = 2p$. The suspension operator σ arising from the circle action on $\operatorname{THH}(ku_{(p)}, j_*\mathrm{GL}_1^{\mathcal{J}}(KU_{(p)}))$ satisfies $\sigma(u) = u \cdot d\log u$ and $\sigma(\kappa_1) = -\kappa_1 \cdot d\log u$, and is zero on λ_1 and $d\log u$.

In logarithmic algebraic geometry, the passage from Kähler differentials to logarithmic Kähler differentials allows one to have differentials with logarithmic poles, i.e., it introduces elements $d\log x$ satisfying $x \cdot d\log x = dx$. By analogy with the Hochschild–Kostant–Rosenberg correspondence between (HH, σ) and (Ω, d), one may expect similar phenomena for logarithmic THH. In view of this, the above relation $\sigma(u) = u \cdot d\log u$ is a justification for denoting the relevant homotopy class by $d\log u$.

Using the homotopy cofiber sequence (1.1), the previous theorem allows us to recover Ausoni's computation of the rather complicated finitely presented $\mathbb{F}_p$-algebra $V(1)_* \operatorname{THH}(ku_{(p)})$ [Aus05]. For this application of Theorem 1.7 it is important that already in the case of the Adams summand, the explicit definition of logarithmic THH allows us to determine the homomorphisms in the long exact sequence of $V(1)$-homotopy groups induced by (1.1). It is not clear if the construction of a localization homotopy cofiber sequence for THH by Blumberg–Mandell [BM11] provides such an explicit understanding of the resulting long exact sequence. Nonetheless, we expect our sequence to be equivalent to theirs, in the special cases they consider. If true, this would be one way to relate our homotopy cofiber sequence (1.1) to the corresponding K-theoretical localization sequence.

1.8. **Organization.** In Section 2 we briefly review commutative $\mathcal{J}$-space monoids and their cyclic bar construction and prove a decomposition result for the cyclic bar construction of grouplike commutative $\mathcal{J}$-space monoids. In Section 3 we review the replete bar construction and prove a similar decomposition formula. In Section 4 we briefly recall the definition of log THH. In Section 5 we explain how the graded Thom isomorphism established in [SS14] can be used to compute the homology of $\mathbb{S}^{\mathcal{J}}[M]$ for certain commutative $\mathcal{J}$-space monoids M. Section 6 contains the proof of Theorem 1.5. In Section 7 we compute the $V(1)$-homotopy of the log THH of the Adams summand and prove Theorem 1.3. In the final Section 8 we prove Theorem 1.7 about the log THH of $ku_{(p)}$ and explain how to use this for computing the $V(1)$-homotopy of $\mathrm{THH}(ku_{(p)})$.

1.9. **Acknowledgments.** The authors would like to thank the referee for useful comments.

2. The cyclic bar construction for commutative $\mathcal{J}$-space monoids

We briefly recall some terminology that is needed to state the definition of logarithmic THH in Section 4. More details on these foundations can be found in [SS12, Section 4].

2.1. **$\mathcal{J}$-spaces.** Let $\mathcal{J}$ be the category given by Quillen's localization construction on the permutative category Σ of finite sets and bijections. It is a symmetric monoidal category whose classifying space $B\mathcal{J}$ is weakly equivalent to QS^0. The objects of $\mathcal{J}$ are pairs $(\mathbf{m_1}, \mathbf{m_2})$ of finite sets of the form $\mathbf{m_i} = \{1, \dots, m_i\}$, where each $m_i \geq 0$. A *$\mathcal{J}$-space* is a functor from $\mathcal{J}$ to simplicial sets. For each object $(\mathbf{m_1}, \mathbf{m_2})$ of $\mathcal{J}$ there is a functor from the category $\mathcal{S}$ of simplicial sets to $\mathcal{J}$-spaces

$$F^{\mathcal{J}}_{(\mathbf{m}_1,\mathbf{m}_2)} \colon \mathcal{S} \to \mathcal{S}^{\mathcal{J}}, \qquad K \mapsto \mathcal{J}((\mathbf{m}_1, \mathbf{m}_2), -) \times K,$$

which is left adjoint to the evaluation of a $\mathcal{J}$-space at $(\mathbf{m_1}, \mathbf{m_2})$.

The category of $\mathcal{J}$-spaces admits a Day type convolution product $\boxtimes$ induced by the ordered concatenation of finite sets and the cartesian product of simplicial sets. The unit $U^{\mathcal{J}}$ of this symmetric monoidal product is the functor $\mathcal{J}((\mathbf{0}, \mathbf{0}), -)$, corepresented by the monoidal unit $(\mathbf{0}, \mathbf{0})$ of $\mathcal{J}$. A *commutative $\mathcal{J}$-space monoid* is a commutative monoid object in $(\mathcal{S}^{\mathcal{J}}, \boxtimes, U^{\mathcal{J}})$, and we write $\mathcal{CS}^{\mathcal{J}}$ for the category of commutative $\mathcal{J}$-space monoids.

The category $\mathcal{CS}^{\mathcal{J}}$ admits a proper simplicial positive projective model structure where $M \to N$ is a weak equivalence if it induces a weak equivalence of spaces $M_{h\mathcal{J}} \to N_{h\mathcal{J}}$ [SS12, Proposition 4.10]. Here $M_{h\mathcal{J}} = \mathrm{hocolim}_{\mathcal{J}} M$ denotes the Bousfield–Kan homotopy colimit of M over $\mathcal{J}$, which is an associative (but not commutative) simplicial monoid. In the following we will refer to this model structure as the *positive $\mathcal{J}$-model* structure and call its weak equivalences the *$\mathcal{J}$-equivalences.* Unless otherwise stated, the notions of cofibrations or fibrations in $\mathcal{CS}^{\mathcal{J}}$ will refer to this model structure. Equipped with the positive $\mathcal{J}$-model structure, $\mathcal{CS}^{\mathcal{J}}$ is Quillen equivalent to the category of E_∞ spaces over $B\mathcal{J}$. We therefore think of commutative $\mathcal{J}$-space monoids as (QS^0-)graded E_∞ spaces. The category $\mathcal{J}$ is closely related to symmetric spectra. In particular, there is a Quillen adjunction

$$\mathbb{S}^{\mathcal{J}} \colon \mathcal{CS}^{\mathcal{J}} \rightleftarrows \mathcal{C}\mathrm{Sp}^{\Sigma} \colon \Omega^{\mathcal{J}}$$

relating $\mathcal{CS}^{\mathcal{J}}$ to the category of commutative symmetric ring spectra $\mathcal{C}\mathrm{Sp}^{\Sigma}$ with the positive projective stable model structure. If A is a commutative symmetric ring spectrum, we view $\Omega^{\mathcal{J}}(A)$ as a model for the underlying graded multiplicative E_∞ space of A. We say that a commutative $\mathcal{J}$-space monoid M is *grouplike* if the monoid $\pi_0(M_{h\mathcal{J}})$ is a group. If A is positive fibrant, then $\Omega^{\mathcal{J}}(A)$ has a subobject $\mathrm{GL}_1^{\mathcal{J}}(A)$ of *graded units* such that inclusion $\mathrm{GL}_1^{\mathcal{J}}(A) \to \Omega^{\mathcal{J}}(A)$ corresponds to the inclusion $\pi_*(A)^\times \to \pi_*(A)$ of the units of the graded ring of stable homotopy groups of A.

2.2. **The cyclic bar construction.** Let M be a commutative $\mathcal{J}$-space monoid. The *cyclic bar construction* $B^{\mathrm{cy}}(M)$ of M is the realization of a simplicial object $[q] \mapsto M^{\boxtimes(q+1)}$. Its structure maps are defined using the unit and the multiplication of M and the twist isomorphism for the symmetric monoidal product $\boxtimes$; see [RSS15, Section 3] for details. The object $B^{\mathrm{cy}}(M)$ will be one of the three building blocks of log THH. We note that since M is commutative, the iterated multiplication maps induce a natural augmentation $\epsilon\colon B^{\mathrm{cy}}(M) \to M$.

Our first goal is to decompose $B^{\mathrm{cy}}(M)$ as a coproduct of commutative $\mathcal{J}$-space monoids, in the case when M is grouplike. For this we fix a factorization of the unit of M into an acyclic cofibration followed by a positive $\mathcal{J}$-fibration as indicated in the bottom row of the diagram

$$\begin{tikzcd} & V(M) \arrow[r, two heads] \arrow[d] & B^{\mathrm{cy}}(M) \arrow[d, "\epsilon"] \\ U^{\mathcal{J}} \arrow[r, tail, "\sim"] & U(M) \arrow[r, two heads] & M. \end{tikzcd}$$

We define $V(M)$ to be the pullback of the augmentation $\epsilon\colon B^{\mathrm{cy}}(M) \to M$ and $U(M) \to M$. It is a model for the homotopy fiber of the augmentation over the unit. Using the multiplicative structure of $B^{\mathrm{cy}}(M)$ we get a map of commutative $\mathcal{J}$-space monoids

$$M \boxtimes V(M) \to B^{\mathrm{cy}}(M) \boxtimes B^{\mathrm{cy}}(M) \to B^{\mathrm{cy}}(M).$$

Proposition 2.3. *The map $M \boxtimes V(M) \to B^{\mathrm{cy}}(M)$ is a $\mathcal{J}$-equivalence provided that M is grouplike and cofibrant.*

Proof. Consider the commutative diagram of homotopy colimits

$$\begin{tikzcd} V(M)_{h\mathcal{J}} \arrow[r] \arrow[d, equal] & M_{h\mathcal{J}} \times V(M)_{h\mathcal{J}} \arrow[r] \arrow[d, "\sim"'] & M_{h\mathcal{J}} \times U(M)_{h\mathcal{J}} \arrow[d, "\sim"'] \\ V(M)_{h\mathcal{J}} \arrow[r] \arrow[d, equal] & (M \boxtimes V(M))_{h\mathcal{J}} \arrow[r] \arrow[d] & (M \boxtimes U(M))_{h\mathcal{J}} \arrow[d, "\sim"'] \\ V(M)_{h\mathcal{J}} \arrow[r] & B^{\mathrm{cy}}(M)_{h\mathcal{J}} \arrow[r] & M_{h\mathcal{J}} \end{tikzcd}$$

in which the map in the lemma induces the middle lower vertical arrow. The bottom part is obtained by passing to homotopy colimits from the corresponding diagram of commutative $\mathcal{J}$-space monoids, and the vertical equivalences in the upper part of the diagram arise from the monoidal structure map of the homotopy colimit.

We must show that the vertical composition $M_{h\mathcal{J}} \times V(M)_{h\mathcal{J}} \to B^{\mathrm{cy}}(M)_{h\mathcal{J}}$ is a weak homotopy equivalence. Notice that the latter is equivariant as a map of spaces with left $M_{h\mathcal{J}}$-action. Furthermore, the assumption that M is commutative and grouplike implies that $V(M)_{h\mathcal{J}}$ is path connected and that $M_{h\mathcal{J}} \to B^{\mathrm{cy}}(M)_{h\mathcal{J}}$ induces an isomorphism of path components. Since $M_{h\mathcal{J}}$ is assumed to be grouplike it therefore suffices to show that the map in question restricts to a weak homotopy equivalence on the path components containing the unit elements. For this we observe that the corresponding restriction of the above diagram is a diagram of horizontal homotopy fiber sequences, which gives the result. $\square$

The next proposition identifies the homotopy type of $V(M)$ for grouplike M.

Proposition 2.4. *Suppose that M is a grouplike and cofibrant commutative $\mathcal{J}$-space monoid. Then there is a chain of natural $\mathcal{J}$-equivalences of $\mathcal{J}$-spaces augmented over M relating $V(M)$ and $U^{\mathcal{J}} \times B(M_{h\mathcal{J}})$.*

The proof of this proposition needs some preparation and will be given at the end of this section.

2.5. **The bar resolution of $\mathcal{J}$-spaces.** Let X be a $\mathcal{J}$-space. We define the *bar resolution* $\overline{X}$ of X to be the $\mathcal{J}$-space given by the bar construction

$$\overline{X}(\mathbf{n}_1, \mathbf{n}_2) = B(\mathcal{J}(-, (\mathbf{n}_1, \mathbf{n}_2)), \mathcal{J}, X),$$

where we view $\mathcal{J}(-, (\mathbf{n}_1, \mathbf{n}_2))$ as a $\mathcal{J}^{\mathrm{op}}$-space in the obvious way. (See e.g. [HV92] for a discussion of the bar construction in the context of diagram spaces.) By definition, this is the same as the homotopy left Kan extension of X along the identity functor on $\mathcal{J}$. Equivalently, we may describe $\overline{X}(\mathbf{n}_1, \mathbf{n}_2)$ as the homotopy colimit

$$\overline{X}(\mathbf{n}_1, \mathbf{n}_2) = \mathrm{hocolim}_{\mathcal{J}\downarrow(\mathbf{n}_1,\mathbf{n}_2)} X \circ \pi_{(\mathbf{n}_1,\mathbf{n}_2)}$$

over the comma category $\mathcal{J} \downarrow (\mathbf{n}_1, \mathbf{n}_2)$ of objects in $\mathcal{J}$ over $(\mathbf{n}_1, \mathbf{n}_2)$. Here $\pi_{(\mathbf{n}_1,\mathbf{n}_2)}$ denotes the forgetful functor from $\mathcal{J} \downarrow (\mathbf{n}_1, \mathbf{n}_2)$ to $\mathcal{J}$. Each of the categories $\mathcal{J} \downarrow (\mathbf{n}_1, \mathbf{n}_2)$ has a terminal object and hence the projection of the homotopy colimit onto the colimit defines an evaluation map of $\mathcal{J}$-spaces $\overline{X} \to X$ that is a level equivalence.

Lemma 2.6. *There is a natural isomorphism* $\mathrm{colim}_{\mathcal{J}} \overline{X} \cong X_{h\mathcal{J}}$. □

As a consequence of the lemma there is a natural map of $\mathcal{J}$-spaces $\overline{X} \to X_{h\mathcal{J}}$ when we view $X_{h\mathcal{J}}$ as a constant $\mathcal{J}$-space. (Notice that this is not a $\mathcal{J}$-equivalence since $B\mathcal{J}$ is not contractible). The lemma suggests that $\overline{X}$ is a kind of cofibrant replacement of X. More precisely we have the following result, which can be deduced from the skeletal filtration of the bar construction.

Lemma 2.7. *Let X be a $\mathcal{J}$-space. As a $\mathcal{J}$-space, the bar resolution $\overline{X}$ is cofibrant in the absolute projective model structure of* [SS12, Proposition 4.8]. □

Clearly the bar resolution $X \mapsto \overline{X}$ is functorial in X and we claim that it canonically has the structure of a lax monoidal functor. Indeed, the monoidal product $\overline{X} \boxtimes \overline{Y} \to \overline{X \boxtimes Y}$ is induced by the natural map of $\mathcal{J} \times \mathcal{J}$-spaces

$$\begin{aligned}
&B(\mathcal{J}(-, (\mathbf{m}_1, \mathbf{m}_2)), \mathcal{J}, X) \times B(\mathcal{J}(-, (\mathbf{n}_1, \mathbf{n}_2)), \mathcal{J}, Y) \\
&\cong B(\mathcal{J}(-, (\mathbf{m}_1, \mathbf{m}_2)) \times \mathcal{J}(-, (\mathbf{n}_1, \mathbf{n}_2)), \mathcal{J} \times \mathcal{J}, X \times Y) \\
&\to B(\mathcal{J}(-, (\mathbf{m}_1, \mathbf{m}_2) \sqcup (\mathbf{n}_1, \mathbf{n}_2)), \mathcal{J}, X \boxtimes Y).
\end{aligned}$$

Here we use that the bar construction preserves products. The second map is induced by the monoidal structure map $\sqcup\colon \mathcal{J} \times \mathcal{J} \to \mathcal{J}$, the canonical map of $\mathcal{J} \times \mathcal{J}$-spaces $X \times Y \to \sqcup^*(X \boxtimes Y)$, and the map of $(\mathcal{J} \times \mathcal{J})^{\mathrm{op}}$-spaces in the first variable determined by $\sqcup$. The monoidal unit is the unique map of $\mathcal{J}$-spaces from the unit $U^{\mathcal{J}}$ for the $\boxtimes$-product to its bar resolution. Furthermore, it is easy to check that the evaluation $\overline{X} \to X$ is a monoidal natural transformation. This implies that the bar resolution of a $\mathcal{J}$-space monoid M is again a $\mathcal{J}$-space monoid and that $\overline{M} \to M$ is a map of $\mathcal{J}$-space monoids. By Lemma 2.6 there also is a natural map of $\mathcal{J}$-space monoids $\overline{M} \to M_{h\mathcal{J}}$.

Remark 2.8. The bar resolution functor is not lax symmetric monoidal, and consequently it does not take commutative $\mathcal{J}$-space monoids to commutative $\mathcal{J}$-space monoids.

Lemma 2.9. *Let M be a cofibrant commutative $\mathcal{J}$-space monoid. Then evaluation $\overline{M} \to M$ induces a level equivalence $B^{\mathrm{cy}}(\overline{M}) \to B^{\mathrm{cy}}(M)$.*

Proof. By Lemma 2.7 above and [SS12, Proposition 4.28], the underlying $\mathcal{J}$-spaces of $\overline{M}$ and M are flat in the sense of [SS12, Section 4.27]. Since $\overline{M} \to M$ is a level equivalence, [SS12, Proposition 8.2] implies that $B^{\mathrm{cy}}_q(\overline{M}) \to B^{\mathrm{cy}}_q(M)$ is a level equivalence in every simplicial degree q. The claim follows by the realization lemma for bisimplicial sets. □

We now use the bar resolution to analyze the homotopy colimit of $B^{\mathrm{cy}}(M)$ under suitable assumptions on M. For this we note that one can also define the cyclic bar construction B^{cy} in $(\mathcal{S}, \times, *)$ and apply it to associative simplicial monoids.

Lemma 2.10. *There is a natural weak equivalence $B^{\mathrm{cy}}(\overline{M})_{h\mathcal{J}} \to B^{\mathrm{cy}}(M_{h\mathcal{J}})$.*

Proof. Notice first that $B^{\mathrm{cy}}(\overline{M})_{h\mathcal{J}}$ is isomorphic to the realization of the cyclic space $B^{\mathrm{cy}}_\bullet(\overline{M})_{h\mathcal{J}}$ obtained by evaluating the homotopy colimit in each simplicial degree. Now we use that the colimit functor from $\mathcal{S}^{\mathcal{J}}$ to $\mathcal{S}$ is strong symmetric monoidal, with respect to the $\boxtimes$-product on $\mathcal{S}^{\mathcal{J}}$ and the usual categorical product on $\mathcal{S}$, to get a natural map of cyclic spaces

$$\operatorname{hocolim}_{\mathcal{J}} B^{\mathrm{cy}}_\bullet(\overline{M}) \to \operatorname{colim}_{\mathcal{J}} B^{\mathrm{cy}}_\bullet(\overline{M}) \cong B^{\mathrm{cy}}_\bullet(\operatorname{colim}_{\mathcal{J}} \overline{M}) \cong B^{\mathrm{cy}}_\bullet(M_{h\mathcal{J}}). \tag{2.1}$$

By the realization lemma for bisimplicial sets it is enough to show that the first map is a weak homotopy equivalence in each simplicial degree. Since $\overline{M}$ and hence $B^{\mathrm{cy}}_q(\overline{M})$ are cofibrant $\mathcal{J}$-spaces, this follows from [SS12, Lemma 6.22]. □

Corollary 2.11. *For a cofibrant commutative $\mathcal{J}$-space monoid M there is a chain of natural weak homotopy equivalences*

$$B^{\mathrm{cy}}(M)_{h\mathcal{J}} \xleftarrow{\sim} B^{\mathrm{cy}}(\overline{M})_{h\mathcal{J}} \xrightarrow{\sim} B^{\mathrm{cy}}(M_{h\mathcal{J}}). \qquad \square$$

In the next lemma we consider the classifying space $B(M_{h\mathcal{J}})$ and the levelwise cartesian product $M \times B(M_{h\mathcal{J}})$. The latter may be interpreted as either the tensor of M with the space $B(M_{h\mathcal{J}})$, the $\boxtimes$-product of M with $F^{\mathcal{J}}_{(\mathbf{0},\mathbf{0})}(B(M_{h\mathcal{J}}))$, or the cartesian product of M with the constant $\mathcal{J}$-space defined by $B(M_{h\mathcal{J}})$.

Lemma 2.12. *Let M be a commutative $\mathcal{J}$-space monoid. There is a natural map of $\mathcal{J}$-spaces $(\overline{\epsilon}, \overline{\pi})\colon B^{\mathrm{cy}}(\overline{M}) \to M \times B(M_{h\mathcal{J}})$, which is a $\mathcal{J}$-equivalence if M is grouplike.*

Proof. Consider the map of $\mathcal{J}$-spaces $B^{\mathrm{cy}}(\overline{M}) \to B^{\mathrm{cy}}(M) \to M$ induced by the evaluation $\overline{M} \to M$ and the augmentation of $B^{\mathrm{cy}}(M)$, using that M is commutative. This gives the first factor $\overline{\epsilon}$ of the map in the lemma. The second factor $\overline{\pi}$ is defined by the composition $B^{\mathrm{cy}}(\overline{M}) \to \mathrm{const}_{\mathcal{J}} B^{\mathrm{cy}}(M_{h\mathcal{J}}) \to \mathrm{const}_{\mathcal{J}} B(M_{h\mathcal{J}})$ where the first map is the adjoint of the isomorphism $\operatorname{colim}_{\mathcal{J}} B^{\mathrm{cy}}(\overline{M}) \to B^{\mathrm{cy}}(M_{h\mathcal{J}})$ used in the definition of the map (2.1) and the second map is given by the projection away from the zeroth coordinate in each simplicial degree. The induced map of homotopy colimits $B^{\mathrm{cy}}(\overline{M})_{h\mathcal{J}} \to (M \times B(M_{h\mathcal{J}}))_{h\mathcal{J}} \cong M_{h\mathcal{J}} \times B(M_{h\mathcal{J}})$ fits into a commutative diagram

$$\begin{tikzcd}
M_{h\mathcal{J}} \arrow[r] & B^{\mathrm{cy}}(M_{h\mathcal{J}}) \arrow[r] & B(M_{h\mathcal{J}}) \\
\overline{M}_{h\mathcal{J}} \arrow[r] \arrow[u, "\sim"] \arrow[d, "\sim"'] & B^{\mathrm{cy}}(\overline{M})_{h\mathcal{J}} \arrow[r] \arrow[u, "\sim"] \arrow[d] & B(M_{h\mathcal{J}}) \arrow[u, equal] \arrow[d, equal] \\
M_{h\mathcal{J}} \arrow[r] & M_{h\mathcal{J}} \times B(M_{h\mathcal{J}}) \arrow[r] & B(M_{h\mathcal{J}})
\end{tikzcd}$$

as the middle lower vertical arrow. The equivalences $\overline{M}_{h\mathcal{J}} \to M_{h\mathcal{J}}$ are defined as follows: In the lower part of the diagram it is induced by the evaluation $\overline{M} \to M$, whereas in the upper part of the diagram it is given by the projection from the homotopy colimit to the colimit using the identification in Lemma 2.6, see also

[HV92, Theorem 5.5]. (These two equivalences are canonically homotopic but that is not relevant for the argument.) The assumption that M is grouplike implies that the upper row is a homotopy fiber sequence in the sense that the map from $M_{h\mathcal{J}}$ to the homotopy fiber of the second map is a weak homotopy equivalence. This follows from standard results on geometric realization of simplicial quasifibrations as in the proof of [Goo85, Lemma V.I.3]. Hence the middle row is also a homotopy fiber sequence by Lemma 2.10, which gives the result. □

Proof of Proposition 2.4. Let M be a grouplike and cofibrant commutative $\mathcal{J}$-space monoid and let $\overline{V}(M)$ be defined as the pullback of the diagram

$$U(M) \twoheadrightarrow M \xleftarrow{\overline{\epsilon}} B^{\mathrm{cy}}(\overline{M}),$$

where the map on the right is defined as in Lemma 2.12. We claim that there is a chain of natural $\mathcal{J}$-equivalences of $\mathcal{J}$-spaces over M

$$V(M) \xleftarrow{\sim} \overline{V}(M) \xrightarrow{\sim} U(M) \times B(M_{h\mathcal{J}}) \xleftarrow{\sim} U^{\mathcal{J}} \times B(M_{h\mathcal{J}}). \tag{2.2}$$

To obtain the maps in (2.2), we note that the evaluation map $\overline{M} \to M$ induces a $\mathcal{J}$-equivalence $B^{\mathrm{cy}}(\overline{M}) \to B^{\mathrm{cy}}(M)$ by Lemma 2.9. There is an induced $\mathcal{J}$-equivalence $\overline{V}(M) \to V(M)$ since the positive $\mathcal{J}$-model structure is right proper. For the second equivalence we use the given map to $U(M)$ in the first factor and the second factor is the composition

$$\overline{V}(M) \to B^{\mathrm{cy}}(\overline{M}) \xrightarrow{\overline{\pi}} \mathrm{const}_{\mathcal{J}} B(M_{h\mathcal{J}})$$

of the given map to $B^{\mathrm{cy}}(\overline{M})$ with the map from Lemma 2.12. Now consider the homotopy cartesian square of $\mathcal{J}$-spaces

$$\begin{array}{ccc} U(M) \times B(M_{h\mathcal{J}}) & \twoheadrightarrow & M \times B(M_{h\mathcal{J}}) \\ \downarrow & & \downarrow \\ U(M) & \twoheadrightarrow & M. \end{array}$$

Together with the $\mathcal{J}$-equivalence from Lemma 2.12 the map just described defines a map from the square defining $\overline{V}(M)$ to the latter square. Hence the result again follows from right properness of the positive $\mathcal{J}$-model structure. □

3. The replete bar construction for commutative $\mathcal{J}$-space monoids

Let M be a commutative $\mathcal{J}$-space monoid. As the second building block of the logarithmic THH to be defined in Section 4, we now recall the definition of the *replete bar construction* $B^{\mathrm{rep}}(M)$ from [RSS15, Section 3.3]. Let $M \to M^{\mathrm{gp}}$ be a chosen functorial group completion of M and let

$$M \overset{\sim}{\rightarrowtail} M' \twoheadrightarrow M^{\mathrm{gp}} \tag{3.1}$$

be a (functorial) factorization of $M \to M^{\mathrm{gp}}$ into an acyclic cofibration followed by a fibration. The replete bar construction $B^{\mathrm{rep}}(M)$ is defined as the pullback of the diagram of commutative $\mathcal{J}$-space monoids

$$M' \twoheadrightarrow M^{\mathrm{gp}} \xleftarrow{\epsilon} B^{\mathrm{cy}}(M^{\mathrm{gp}})$$

provided by the above map $M' \to M^{\mathrm{gp}}$ and the augmentation $\epsilon\colon B^{\mathrm{cy}}(M^{\mathrm{gp}}) \to M^{\mathrm{gp}}$. By construction, there is a natural *repletion* map $\rho\colon B^{\mathrm{cy}}(M) \to B^{\mathrm{rep}}(M)$.

Proposition 3.1. *Let M be a cofibrant commutative $\mathcal{J}$-space monoid and view M^{gp} as a left M-module via $M \to M^{\mathrm{gp}}$. There is a chain of natural $\mathcal{J}$-equivalences of left M-modules over M^{gp} relating $B^{\mathrm{rep}}(M)$ and $M \times B(M_{h\mathcal{J}})$.*

Proof. Given a factorization $M \to M' \to M^{\mathrm{gp}}$ as in (3.1), we factor the unit map $U^{\mathcal{J}} \to M'$ as an acyclic cofibration $U^{\mathcal{J}} \to U(M^{\mathrm{gp}})$ followed by a fibration $U(M^{\mathrm{gp}}) \to M'$. This provides the lower part of the following commutative diagram:

$$\begin{tikzcd}
V(M^{\mathrm{gp}}) \arrow[r] \arrow[d] & B^{\mathrm{rep}}(M) \arrow[r] \arrow[d] & B^{\mathrm{cy}}(M^{\mathrm{gp}}) \arrow[d] \\
U(M^{\mathrm{gp}}) \arrow[r, two heads] & M' \arrow[r, two heads] & M^{\mathrm{gp}}. \\
U^{\mathcal{J}} \arrow[u, tail, "\sim"] \arrow[r] & M \arrow[u, tail, "\sim"'] &
\end{tikzcd}$$

Using the resulting factorization of the unit of M^{gp} into an acyclic cofibration $U^{\mathcal{J}} \to U(M^{\mathrm{gp}})$ followed by a fibration $U(M^{\mathrm{gp}}) \to M^{\mathrm{gp}}$ for the definition of the commutative $\mathcal{J}$-space monoid $V(M^{\mathrm{gp}})$ studied in the last section, we obtain $V(M^{\mathrm{gp}})$ as an iterated pullback as indicated in the previous diagram. The above maps induce the following commutative cube:

$$\begin{tikzcd}
& B^{\mathrm{rep}}(M) \arrow[rr] \arrow[dd] & & B^{\mathrm{cy}}(M^{\mathrm{gp}}) \arrow[dd] \\
M \boxtimes V(M^{\mathrm{gp}}) \arrow[ur] \arrow[rr] \arrow[dd] & & M^{\mathrm{gp}} \boxtimes V(M^{\mathrm{gp}}) \arrow[ur] \arrow[dd] & \\
& M' \arrow[rr] & & M^{\mathrm{gp}}. \\
M \boxtimes U(M^{\mathrm{gp}}) \arrow[ur] \arrow[rr] & & M^{\mathrm{gp}} \boxtimes U(M^{\mathrm{gp}}) \arrow[ur] &
\end{tikzcd}$$

The back face is homotopy cartesian by definition, and the front face is homotopy cartesian by [Sag14, Lemma 2.11] and [SS12, Corollary 11.4]. The map in the upper right hand corner is a $\mathcal{J}$-equivalence by Proposition 2.3 (applied to M^{gp}), and the maps in the lower corners are $\mathcal{J}$-equivalences by construction. It follows that $M \boxtimes V(M^{\mathrm{gp}}) \to B^{\mathrm{rep}}(M)$ is also a $\mathcal{J}$-equivalence.

Extending the $\mathcal{J}$-space maps in the chain of $\mathcal{J}$-equivalences of Proposition 2.4 (applied to M^{gp}) to M-module maps shows that there is a chain of $\mathcal{J}$-equivalences of M-modules over M^{gp} relating $M \boxtimes V(M^{\mathrm{gp}})$ and $M \times B((M^{\mathrm{gp}})_{h\mathcal{J}})$. Since $B(M_{h\mathcal{J}}) \to B((M^{\mathrm{gp}})_{h\mathcal{J}})$ is a weak equivalence, the claim follows. □

Let $f \colon M \to N$ be a map of commutative $\mathcal{J}$-space monoids. In Section 6 we shall be interested in the diagram of commutative symmetric ring spectra

$$\begin{tikzcd}
\mathbb{S}^{\mathcal{J}}[M] \arrow[r] \arrow[d] & \mathbb{S}^{\mathcal{J}}[B^{\mathrm{rep}}(M)] \arrow[d] \\
\mathbb{S}^{\mathcal{J}}[N] \arrow[r] & \mathbb{S}^{\mathcal{J}}[B^{\mathrm{rep}}(N)]
\end{tikzcd} \tag{3.2}$$

induced by f. In order to measure to what extent this square is homotopy cocartesian in $\mathcal{C}\mathrm{Sp}^{\Sigma}$, we use the following terminology: Given a symmetric spectrum E, a commutative diagram of commutative symmetric ring spectra

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}$$

is *E_*-homotopy cocartesian* if whenever we factor $A \to C$ as a cofibration $A \to C'$ followed by a stable equivalence $C' \to C$ of commutative symmetric ring spectra, the induced map $C' \wedge_A B \to D$ is an E_*-equivalence.

Proposition 3.2. *For a symmetric spectrum E and a map of cofibrant commutative $\mathcal{J}$-space monoids $f \colon M \to N$, the diagram (3.2) is E_*-homotopy cocartesian if and only if f gives rise to an E_*-equivalence*

$$\mathbb{S}^{\mathcal{J}}[N] \wedge B(M_{h\mathcal{J}})_+ \to \mathbb{S}^{\mathcal{J}}[N] \wedge B(N_{h\mathcal{J}})_+.$$

Proof. Without loss of generality, we may assume that f is a cofibration. Then the diagram (3.2) is E_*-homotopy cocartesian if and only if the map

$$\mathbb{S}^{\mathcal{J}}[N] \wedge_{\mathbb{S}^{\mathcal{J}}[M]} \mathbb{S}^{\mathcal{J}}[B^{\mathrm{rep}}(M)] \to \mathbb{S}^{\mathcal{J}}[B^{\mathrm{rep}}(N)] \tag{3.3}$$

is an E_*-equivalence. By the argument given in the proof of [RSS15, Lemma 4.8], the extension of scalars functor $\mathbb{S}^{\mathcal{J}}[N] \wedge_{\mathbb{S}^{\mathcal{J}}[M]} (-)$ preserves stable equivalences. Since the $\mathcal{J}$-equivalences of M-modules in Proposition 3.1 are augmented over the cofibrant commutative $\mathcal{J}$-space monoid M^{gp}, it follows from [RSS15, Corollary 8.8] that $\mathbb{S}^{\mathcal{J}}$ maps these $\mathcal{J}$-equivalences to stable equivalences. Hence the map (3.3) is stably equivalent to the map

$$\mathbb{S}^{\mathcal{J}}[N] \wedge_{\mathbb{S}^{\mathcal{J}}[M]} (\mathbb{S}^{\mathcal{J}}[M] \wedge B(M_{h\mathcal{J}})_+) \to \mathbb{S}^{\mathcal{J}}[N] \wedge B(N_{h\mathcal{J}})_+,$$

and the domain of this map is isomorphic to $\mathbb{S}^{\mathcal{J}}[N] \wedge B(M_{h\mathcal{J}})_+$. □

Notation 3.3. For each integer n, let $\mathcal{J}_n \subset \mathcal{J}$ be the full subcategory generated by the objects $(\mathbf{m}_1, \mathbf{m}_2)$ with $m_2 - m_1 = n$. Then $B\mathcal{J} = \coprod_n B\mathcal{J}_n$ and we refer to the part of a $\mathcal{J}$-space X that maps to $B\mathcal{J}_n$ as the *$\mathcal{J}$-degree n part* of X. If M is a commutative $\mathcal{J}$-space monoid, we let $M_{\{0\}}$ and $B^{\mathrm{rep}}_{\{0\}}(M)$ denote the $\mathcal{J}$-degree 0 parts of M and $B^{\mathrm{rep}}(M)$, respectively. We also use the notations $M_{\geq 0}$ and $M_{>0}$ for the non-negative and positive $\mathcal{J}$-degree parts of M, respectively, cf. [RSS15, Definition 6.1].

Remark 3.4. Let M be a commutative $\mathcal{J}$-space monoid that is *repetitive* in the sense of [RSS15, Definition 6.4]. By definition, this means that $M \neq M_{\{0\}}$ and that the group completion map $M \to M^{\mathrm{gp}}$ induces a $\mathcal{J}$-equivalence $M \to (M^{\mathrm{gp}})_{\geq 0}$. Propositions 2.3 and 3.1 can be used to identify the homotopy cofiber of the map of $\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M_{\{0\}})]$-module spectra

$$\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M_{\{0\}})] \xrightarrow{\mathbb{S}^{\mathcal{J}}[\sigma]} \mathbb{S}^{\mathcal{J}}[B^{\mathrm{rep}}_{\{0\}}(M)] \tag{3.4}$$

with $\Sigma\, \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M_{\{0\}})]$. The idea for this is to use that the homotopy cofiber in question is equivalent to the homotopy cofiber of the map

$$\mathbb{S}^{\mathcal{J}}[M_{\{0\}}] \wedge B((M_{\{0\}})_{h\mathcal{J}})_+ \to \mathbb{S}^{\mathcal{J}}[M_{\{0\}}] \wedge B(M_{h\mathcal{J}})_+.$$

An application of the Bousfield–Friedlander theorem shows that there is a homotopy fiber sequence

$$B((M_{\{0\}})_{h\mathcal{J}}) \to B(M_{h\mathcal{J}}) \to B(d\mathbb{N}_0),$$

where d is the period of M, as in [RSS15, Definition 6.5]. Hence we can recognize the homotopy cofiber of (3.4), as claimed.

Using this argument in place of [RSS15, Proposition 6.11] leads to a slightly different proof for the localization homotopy cofiber sequences established in [RSS15]. However, the disadvantage of the alternative approach outlined here is that it does not identify the homotopy cofiber as a cyclic object.

4. Logarithmic topological Hochschild homology

Let A be a commutative symmetric ring spectrum. A *pre-log structure* (M, α) on A is a commutative $\mathcal{J}$-space monoid M together with a map $\alpha\colon M \to \Omega^{\mathcal{J}}(A)$; see [RSS15, Definition 4.1]. The ring spectrum A together with a chosen pre-log structure is called a *pre-log ring spectrum* and will be denoted by (A, M, α) or just (A, M). (As explained in [RSS15, Remark 4.2], this terminology differs from the one used in [Rog09] in that we use $\mathcal{J}$-spaces, and from [SS12, §4.30] and [Sag14] in that we skip the additional word *graded* used there.)

A basic example of a pre-log structure is the free pre-log structure generated by a 0-simplex $x \in \Omega^{\mathcal{J}}(A)(\mathbf{d_1}, \mathbf{d_2})$. It is given by the map

$$\mathbb{C}\langle \mathbf{d_1}, \mathbf{d_2} \rangle = \textstyle\coprod_{k \geq 0} \left(F^{\mathcal{J}}_{(\mathbf{d_1}, \mathbf{d_2})}(*)^{\boxtimes k} / \Sigma_k \right) \to \Omega^{\mathcal{J}}(A)$$

from the free commutative $\mathcal{J}$-space monoid $\mathbb{C}\langle \mathbf{d_1}, \mathbf{d_2} \rangle$ on a generator in bidegree $(\mathbf{d_1}, \mathbf{d_2})$ determined by x. We often write $\mathbb{C}\langle x \rangle$ for $\mathbb{C}\langle \mathbf{d_1}, \mathbf{d_2} \rangle$ when discussing this map.

A more interesting kind of pre-log structure arises as follows: If $j\colon e \to E$ is the connective cover map of a positive fibrant commutative symmetric ring spectrum, then the pullback $j_*\mathrm{GL}^{\mathcal{J}}_1(E)$ of $\mathrm{GL}^{\mathcal{J}}_1(E) \to \Omega^{\mathcal{J}}(E) \leftarrow \Omega^{\mathcal{J}}(e)$ defines a pre-log structure $j_*\mathrm{GL}^{\mathcal{J}}_1(E) \to \Omega^{\mathcal{J}}(e)$ on e. We call this the *direct image* pre-log structure on e induced by the trivial pre-log structure on E.

Definition 4.1. [RSS15, Definition 4.6] Let (A, M) be a pre-log ring spectrum. Its *logarithmic topological Hochschild homology* $\mathrm{THH}(A, M)$ is the commutative symmetric ring spectrum given by the pushout of the diagram

$$\mathrm{THH}(A) \leftarrow \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M^{\mathrm{cof}})] \to \mathbb{S}^{\mathcal{J}}[B^{\mathrm{rep}}(M^{\mathrm{cof}})]$$

of commutative symmetric ring spectra. Here $(A^{\mathrm{cof}}, M^{\mathrm{cof}}) \to (A, M)$ is a cofibrant replacement and $\mathrm{THH}(A) = B^{\mathrm{cy}}(A^{\mathrm{cof}})$ is the topological Hochschild homology of A, defined as the cyclic bar construction of A^{cof} with respect to the smash product $\wedge$. The left hand map is induced by the identification $\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M^{\mathrm{cof}})] \xrightarrow{\cong} \mathrm{THH}(\mathbb{S}^{\mathcal{J}}[M^{\mathrm{cof}}])$ and the adjoint structure map $\mathbb{S}^{\mathcal{J}}[M^{\mathrm{cof}}] \to A^{\mathrm{cof}}$ of $(A^{\mathrm{cof}}, M^{\mathrm{cof}})$. The right hand map is induced by the repletion map $\rho\colon B^{\mathrm{cy}}(M^{\mathrm{cof}}) \to B^{\mathrm{rep}}(M^{\mathrm{cof}})$.

When computing $\mathrm{THH}(A, M)$ in examples, it will be convenient to work with pre-log structures (M, α) such that the homology of the space $M_{h\mathcal{J}}$ associated with M is well understood. To obtain interesting examples of such pre-log structures, we review [Sag14, Construction 4.2]:

Construction 4.2. Let E be a positive fibrant commutative symmetric ring spectrum that is d-periodic, i.e., $\pi_*(E)$ has a unit of positive degree and the natural number d is the minimal positive degree of a unit in $\pi_*(E)$. We also assume not to be in the degenerate case where $\pi_*(E)$ is the zero ring. Let $j\colon e \to E$ be the connective cover map and assume that e is also positive fibrant. Then there exists an object $(\mathbf{d_1}, \mathbf{d_2})$ of $\mathcal{J}$ with $d_1 > 0$ and a map $x\colon S^{d_2} \to e_{d_1}$ such that $d = d_2 - d_1$ and the homotopy class $[x] \in \pi_d(e)$ represented by x is mapped to a periodicity element in $\pi_*(E)$.

In this general situation we will build a pre-log structure $D(x) \to \Omega^{\mathcal{J}}(e)$. The next diagram outlines its construction:

$$\begin{tikzcd}
\mathbb{C}\langle x \rangle \arrow[dr, tail] \arrow[ddrrrr, bend left] \arrow[dddrr, bend right] & & & & \\
& D(x) \arrow[dr, two heads, "\sim"] & & & \\
& & D'(x) \arrow[rr] \arrow[d] & & \Omega^{\mathcal{J}}(e) \arrow[d] \\
& & \mathbb{C}\langle x \rangle^{\mathrm{gp}} \arrow[r] & \mathrm{GL}^{\mathcal{J}}_1(E) \arrow[r] & \Omega^{\mathcal{J}}(E).
\end{tikzcd} \tag{4.1}$$

We start with the free pre-log structure $\mathbb{C}\langle x \rangle$ on e generated by x. The composite of its structure map $\mathbb{C}\langle x \rangle \to \Omega^{\mathcal{J}}(e)$ with $\Omega^{\mathcal{J}}(e) \to \Omega^{\mathcal{J}}(E)$ factors through $\mathrm{GL}^{\mathcal{J}}_1(E) \to \Omega^{\mathcal{J}}(E)$ because x becomes a unit in $\pi_*(E)$. We then factor the resulting map $\mathbb{C}\langle x \rangle \to \mathrm{GL}^{\mathcal{J}}_1(E)$ in the group completion model structure of [Sag15] as an acyclic cofibration $\mathbb{C}\langle x \rangle \to \mathbb{C}\langle x \rangle^{\mathrm{gp}}$ followed by a fibration $\mathbb{C}\langle x \rangle^{\mathrm{gp}} \to \mathrm{GL}^{\mathcal{J}}_1(E)$.

The intermediate object $\mathbb{C}\langle x\rangle^{\mathrm{gp}}$ is fibrant in the group completion model structure because, by construction, it comes with a fibration to the fibrant object $\mathrm{GL}_1^{\mathcal{J}}(E)$. Hence the acyclic cofibration $\mathbb{C}\langle x\rangle \to \mathbb{C}\langle x\rangle^{\mathrm{gp}}$ is indeed a model for the group completion of $\mathbb{C}\langle x\rangle$. The commutative $\mathcal{J}$-space monoid $D'(x)$ is defined to be the pullback of

$$\mathbb{C}\langle x\rangle^{\mathrm{gp}} \to \Omega^{\mathcal{J}}(E) \leftarrow \Omega^{\mathcal{J}}(e).$$

In a final step, we define $D(x)$ by the indicated factorization of $\mathbb{C}\langle x\rangle \to D'(x)$, now in the positive $\mathcal{J}$-model structure. We call $D(x) \to \Omega^{\mathcal{J}}(e)$ the *direct image pre-log structure* generated by x. We note that $D(x)$ is cofibrant since $\mathbb{C}\langle x\rangle$ is cofibrant. Moreover, $D(x)$ is repetitive in the sense of Remark 3.4 since $\Omega^{\mathcal{J}}(e)_{\geq 0} \to \Omega^{\mathcal{J}}(E)_{\geq 0}$ is a $\mathcal{J}$-equivalence.

It follows that there is a sequence of maps of pre-log ring spectra

$$(e, \mathbb{C}\langle x\rangle) \to (e, D(x)) \to (e, j_*\mathrm{GL}_1^{\mathcal{J}}(E)) \to (E, \mathrm{GL}_1^{\mathcal{J}}(E)). \tag{4.2}$$

The significance of Construction 4.2 for log THH stems from the following results.

Proposition 4.3. *The map* $(e, D(x)) \to (e, j_*\mathrm{GL}_1^{\mathcal{J}}(E))$ *in* (4.2) *induces a stable equivalence* $\mathrm{THH}(e, D(x)) \xrightarrow{\sim} \mathrm{THH}(e, j_*\mathrm{GL}_1^{\mathcal{J}}(E))$.

Proof. By [Sag14, Lemma 4.7], the map $(e, D(x)) \to (e, j_*\mathrm{GL}_1^{\mathcal{J}}(E))$ is stably equivalent to the logification map. So [RSS15, Theorem 4.24] implies that it induces a stable equivalence when applying log THH. □

Theorem 4.4. *In the situation of Construction 4.2, there is a natural homotopy cofiber sequence*

$$\mathrm{THH}(e) \xrightarrow{\rho} \mathrm{THH}(e, D(x)) \xrightarrow{\partial} \Sigma\,\mathrm{THH}(e[0, d\rangle) \tag{4.3}$$

of $\mathrm{THH}(e)$*-modules with circle action, where* ρ *is a map of commutative symmetric ring spectra and* $e[0, d\rangle$ *is the* $(d-1)$*-st Postnikov section of* e.

Proof. This follows by combining [RSS15, Theorem 6.10 and Lemma 6.16] or by combining [RSS15, Theorem 6.18] with Proposition 4.3. □

For later use we record a more explicit description of the homotopy type of $D(x)$. Since e and E are assumed to be positive fibrant and $j\colon e \to E$ is the connective cover map, the induced map $\Omega^{\mathcal{J}}(e \to E)(\mathbf{m_1}, \mathbf{m_2})$ is a weak equivalence if $m_2 - m_1 \geq 0$ and $m_1 > 0$. Moreover, $D(x)(\mathbf{m_1}, \mathbf{m_2})$ is empty if $m_2 - m_1 < 0$ because the negative-dimensional units of $\pi_*(E)$ are not in the image of the map from $\pi_*(e)$. This argument implies [Sag14, Lemma 4.6] which we reproduce here for the reader's convenience:

Lemma 4.5. *The chain of maps* $\mathbb{C}\langle x\rangle_{h\mathcal{J}} \to D(x)_{h\mathcal{J}} \to (\mathbb{C}\langle x\rangle^{\mathrm{gp}})_{h\mathcal{J}}$ *is weakly equivalent to* $\coprod_{k\geq 0} B\Sigma_k \to Q_{\geq 0}S^0 \to QS^0$. *The latter chain is the canonical factorization of the group completion map through the inclusion of the non-negative components of* QS^0. *In particular,* $D(x) \to \mathbb{C}\langle x\rangle^{\mathrm{gp}}$ *induces a* $\mathcal{J}$*-equivalence* $D(x) \to (\mathbb{C}\langle x\rangle^{\mathrm{gp}})_{\geq 0}$. □

Hence the homotopy type of $D(x)_{h\mathcal{J}} \simeq Q_{\geq 0}S^0$ does not depend on the map of spectra $e \to E$ used to construct $D(x)$. The structure map

$$Q_{\geq 0}S^0 \xrightarrow{\sim} D(x)_{h\mathcal{J}} \to (\mathrm{const}_{\mathcal{J}}*)_{h\mathcal{J}} \xrightarrow{\cong} B\mathcal{J} \xrightarrow{\sim} QS^0$$

is multiplication by the degree $d = d_2 - d_1$ of $x\colon S^{d_2} \to e_{d_1}$.

5. The graded Thom isomorphism

As another preparatory step for computing $\mathrm{THH}(A, M)$, we explain how to compute the homology of the spectrum $\mathbb{S}^{\mathcal{J}}[M]$ for $M = D(x)$ and related examples. The key idea for this, worked out by the last two authors in [SS14], is to express $\mathbb{S}^{\mathcal{J}}[M]$ as the Thom spectrum of the virtual vector bundle classified by the composite

$$M_{h\mathcal{J}} \to B\mathcal{J} \xrightarrow{\sim} QS^0 \to \mathbb{Z} \times BO$$

of the structure map $M_{h\mathcal{J}} \to B\mathcal{J}$ induced by applying $(-)_{h\mathcal{J}}$ to the map from M to the terminal $\mathcal{J}$-space, the weak equivalence $B\mathcal{J} \to QS^0$, and the map of infinite loop spaces $QS^0 \to \mathbb{Z} \times BO$ induced by the unit $\mathbb{S} \to ko$. In the case where $M = \mathbb{C}\langle \mathbf{d_1}, \mathbf{d_2} \rangle^{\mathrm{gp}}$ with $d_2 - d_1$ even it is proved in [SS14] that M is orientable in the strong sense that there exists a map of commutative symmetric ring spectra $\mathbb{S}^{\mathcal{J}}[M] \to H\mathbb{Z}P$. Here $H\mathbb{Z}P$ denotes a cofibrant and even periodic version of the integral Eilenberg–Mac Lane spectrum, i.e., the underlying symmetric spectrum of $H\mathbb{Z}P$ decomposes as $H\mathbb{Z}P = \bigvee_{n \in 2\mathbb{Z}} H\mathbb{Z}P_{\{n\}}$ where $H\mathbb{Z}P_{\{0\}} = H\mathbb{Z}$ and $H\mathbb{Z}P_{\{n\}} = \Sigma^n H\mathbb{Z}$. If M is a commutative $\mathcal{J}$-space monoid that is concentrated in even $\mathcal{J}$-degrees, then the monoid structure of $M_{h\mathcal{J}}$ and the multiplication of $H\mathbb{Z}P$ equip $\bigvee_{n \in 2\mathbb{Z}} (M_{h\mathcal{J}_n})_+ \wedge H\mathbb{Z}P_{\{n\}}$ with the structure of a symmetric ring spectrum.

In [SS14], the following statement is derived from a more general graded Thom isomorphism theorem (in [SS14], also E_∞ structures are addressed):

Proposition 5.1. [SS14, Proposition 8.3] *Let $(\mathbf{d_1}, \mathbf{d_2})$ be an object of $\mathcal{J}$ of even $\mathcal{J}$-degree $d_2 - d_1$, and let $M \to \mathbb{C}\langle \mathbf{d_1}, \mathbf{d_2} \rangle^{\mathrm{gp}}$ be a map of commutative $\mathcal{J}$-space monoids. Then there is a chain of stable equivalences of symmetric ring spectra that relates $\mathbb{S}^{\mathcal{J}}[M] \wedge H\mathbb{Z}$ and $\bigvee_{n \in 2\mathbb{Z}} (M_{h\mathcal{J}_n})_+ \wedge H\mathbb{Z}P_{\{n\}}$. The chain of maps is natural with respects to maps of commutative $\mathcal{J}$-space monoids over $\mathbb{C}\langle \mathbf{d_1}, \mathbf{d_2} \rangle^{\mathrm{gp}}$.* □

In the proposition we do not need to assume that M is cofibrant since the existence of an augmentation to the cofibrant object $\mathbb{C}\langle \mathbf{d_1}, \mathbf{d_2} \rangle^{\mathrm{gp}}$ ensures that $\mathbb{S}^{\mathcal{J}}[M]$ captures the correct homotopy type (see [RSS15, Section 8]).

When working with the isomorphism of homology algebras resulting from Proposition 5.1, it will be convenient to view the homology of $M_{h\mathcal{J}}$ as a $\mathbb{Z}$-graded algebra in a way that takes the $\mathcal{J}$-grading into account. We use a $\circledast$ as a subscript to denote this new grading and set

$$H_{\circledast}(M_{h\mathcal{J}}; \mathbb{Z}) = \bigoplus_{n \in \mathbb{Z}} \Sigma^n H_*(M_{h\mathcal{J}_n}; \mathbb{Z}), \tag{5.1}$$

and similarly for other coefficient rings. In this notation, Proposition 5.1 implies the following statement.

Proposition 5.2. *Let $(\mathbf{d_1}, \mathbf{d_2})$ be an object of $\mathcal{J}$ of even $\mathcal{J}$-degree $d_2 - d_1$, and let M be a commutative $\mathcal{J}$-space monoid over $\mathbb{C}\langle \mathbf{d_1}, \mathbf{d_2} \rangle^{\mathrm{gp}}$. Then $H_*(\mathbb{S}^{\mathcal{J}}[M]; \mathbb{Z})$ and $H_{\circledast}(M_{h\mathcal{J}}; \mathbb{Z})$ are naturally isomorphic as $\mathbb{Z}$-graded algebras.* □

If $x \colon S^{d_2} \to e_{d_1}$ has even degree $d = d_2 - d_1$, then the two previous propositions apply for example to the commutative $\mathcal{J}$-space monoids $D(x)$, $D(x)^{\mathrm{gp}}$, $B^{\mathrm{cy}}(D(x))$, $B^{\mathrm{rep}}(D(x))$, $B^{\mathrm{cy}}(D(x)_{\{0\}})$, and $B^{\mathrm{cy}}_{\{0\}}(D(x)^{\mathrm{gp}})$. Here $B^{\mathrm{cy}}(D(x)_{\{0\}})$ and $B^{\mathrm{cy}}_{\{0\}}(D(x)^{\mathrm{gp}})$ denote the $\mathcal{J}$-degree zero parts; see Notation 3.3. In view of later applications, we formulate the following results for homology with $\mathbb{F}_p$-coefficients.

Corollary 5.3. *Let x have even degree d. There are algebra isomorphisms*

$$H_*(\mathbb{S}^{\mathcal{J}}[D(x)]; \mathbb{F}_p) \cong H_{\circledast}(D(x)_{h\mathcal{J}}; \mathbb{F}_p) \cong P(x) \otimes H_*(D(x)_{h\mathcal{J}_0}; \mathbb{F}_p)$$

$$H_*(\mathbb{S}^{\mathcal{J}}[D(x)^{\mathrm{gp}}]; \mathbb{F}_p) \cong H_{\circledast}(D(x)^{\mathrm{gp}}_{h\mathcal{J}}; \mathbb{F}_p) \cong P(x^{\pm 1}) \otimes H_*(D(x)_{h\mathcal{J}_0}; \mathbb{F}_p)$$

with $H_*(D(x)_{h\mathcal{J}_0}; \mathbb{F}_p) \cong H_*(Q_0 S^0; \mathbb{F}_p)$ in $\mathcal{J}$-degree 0.

Proof. The first isomorphisms follow from the above observations. For a general simplicial monoid A that is grouplike and homotopy commutative, A and $\pi_0(A)\times A_0$ are weakly equivalent as H-spaces. Here A_0 denotes the connected component of the unit element in A. This provides the second isomorphisms. The last statement follows from Lemma 4.5. $\square$

5.4. **Homology of the cyclic and replete bar constructions.** To describe the homology of $\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(D(x))]$ and $\mathbb{S}^{\mathcal{J}}[B^{\mathrm{rep}}(D(x))]$, we write

$$C_* = H_*(B^{\mathrm{cy}}(D(x)_{\{0\}})_{h\mathcal{J}}; \mathbb{F}_p) \tag{5.2}$$

for the homology algebra of the E_∞ space $B^{\mathrm{cy}}(D(x)_{\{0\}})_{h\mathcal{J}}$. If k is a positive integer, we say that a $\mathbb{Z}$-graded $\mathbb{F}_p$-algebra A is k-connected if $\mathbb{F}_p \cong A_0$ and $A_i = 0$ if $i < 0$ or $0 < i \leq k$. In view of Corollary 2.11, the underlying $\mathbb{Z}$-graded $\mathbb{F}_p$-vector space of C_* can be identified with $H_*(B^{\mathrm{cy}}(D(x)_{h\mathcal{J}_0}); \mathbb{F}_p)$ when x has non-zero degree. Since we have not defined a multiplicative structure on $B^{\mathrm{cy}}(D(x)_{h\mathcal{J}_0})$, we cannot view this vector space isomorphism as an algebra isomorphism. Nonetheless, the isomorphism implies that C_* is $(2p-4)$-connected, since $B^{\mathrm{cy}}(D(x)_{h\mathcal{J}_0})$ is weakly equivalent to $D(x)_{h\mathcal{J}_0} \times B(D(x)_{h\mathcal{J}_0})$ by the argument given in the proof of Lemma 2.12, and $H_*(D(x)_{h\mathcal{J}_0}; \mathbb{F}_p)$ is $(2p-4)$-connected since $D(x)_{h\mathcal{J}_0} \simeq Q_0S^0$ by Lemma 4.5.

Proposition 5.5. *Let $p \geq 3$ be an odd prime and assume that x has positive even degree. There are algebra isomorphisms*

$$H_*(\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(D(x))]; \mathbb{F}_p) \cong H_{\circledast}(B^{\mathrm{cy}}(D(x))_{h\mathcal{J}}; \mathbb{F}_p) \cong P(x) \otimes E(dx) \otimes C_*$$
$$H_*(\mathbb{S}^{\mathcal{J}}[B^{\mathrm{rep}}(D(x))]; \mathbb{F}_p) \cong H_{\circledast}(B^{\mathrm{rep}}(D(x))_{h\mathcal{J}}; \mathbb{F}_p) \cong P(x) \otimes E(d\log x) \otimes C_*$$
$$H_*(\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}_{\{0\}}(D(x)^{\mathrm{gp}})]; \mathbb{F}_p) \cong H_{\circledast}(B^{\mathrm{cy}}(D(x)^{\mathrm{gp}})_{h\mathcal{J}_0}; \mathbb{F}_p) \cong E(d\log x) \otimes C_*$$

with $|dx| = |x|+1$, $|d\log x| = 1$, and dx mapping to $x \cdot d\log x$ under the repletion map. The suspension operator satisfies $\sigma(x) = dx$, $\sigma(dx) = 0$ in the first case, and $\sigma(x) = x\cdot d\log x$, $\sigma(d\log x) = 0$ in the second case.

We need some preparation to prove the proposition. First we recall from [RSS15, Section 7] that for a commutative $\mathcal{J}$-space monoid M concentrated in $\mathcal{J}$-degrees divisible by d, there is a natural augmentation map $B^{\mathrm{cy}}(M)_{h\mathcal{J}} \to B^{\mathrm{cy}}(d\mathbb{Z})$ which is defined as the realization of the map

$$B^{\mathrm{cy}}_s(M)_{h\mathcal{J}} = \left(\textstyle\coprod_{(d_0,\dots,d_s)\in B^{\mathrm{cy}}_s(d\mathbb{Z})} M_{\{d_0\}} \boxtimes \cdots \boxtimes M_{\{d_s\}}\right)_{h\mathcal{J}} \to B^{\mathrm{cy}}_s(d\mathbb{Z})$$

that collapses the summand indexed by $(d_0,\dots,d_s)$ to the point $(d_0,\dots,d_s)$.

The category of simplicial monoids has a model structure in which a map is a fibration or weak equivalence if and only if the underlying map of simplicial sets is. Specializing to the case $M = D(x)^{\mathrm{gp}}$, we choose a factorization

$$B^{\mathrm{cy}}(D(x)^{\mathrm{gp}})_{h\mathcal{J}} \xrightarrow{\sim} B^{\mathrm{cy}}(D(x)^{\mathrm{gp}})^{\mathrm{fib}}_{h\mathcal{J}} \xrightarrow{q} B^{\mathrm{cy}}(d\mathbb{Z})$$

of the augmentation in this model structure.

Lemma 5.6. *There is a basepoint preserving space level section to q.*

Proof. Let d be the degree of x. Since $D(x)^{\mathrm{gp}}_{h\mathcal{J}} \simeq QS^0$, the canonical augmentation $D(x)^{\mathrm{gp}}_{h\mathcal{J}} \times B(D(x)^{\mathrm{gp}}_{h\mathcal{J}}) \to d\mathbb{Z} \times B(d\mathbb{Z})$ admits a section in $\mathrm{Ho}(\mathcal{S}_*)$, the homotopy category of based simplicial sets. The chain of weak equivalences between $B^{\mathrm{cy}}(D(x)^{\mathrm{gp}})_{h\mathcal{J}}$ and $D(x)^{\mathrm{gp}}_{h\mathcal{J}} \times B(D(x)^{\mathrm{gp}}_{h\mathcal{J}})$ resulting from Corollary 2.11 and the proof of Lemma 2.12 is basepoint preserving and compatible with the augmentation to $B^{\mathrm{cy}}(d\mathbb{Z}) \simeq d\mathbb{Z} \times B(d\mathbb{Z})$. Hence the augmentation $B^{\mathrm{cy}}(D(x)^{\mathrm{gp}})_{h\mathcal{J}} \to B^{\mathrm{cy}}(d\mathbb{Z})$ also admits a section in $\mathrm{Ho}(\mathcal{S}_*)$.

It follows that the map q is a fibration of cofibrant and fibrant based simplicial sets, which admits a section in the homotopy category. By the homotopy lifting property it therefore admits a section in the category $\mathcal{S}_*$ of based simplicial sets. $\square$

Proof of Proposition 5.5. The graded Thom isomorphism provides the first isomorphism in each line of the statement. For the second, we recall from [Rog09, Propositions 3.20 and 3.21] or [RSS15, Section 5.2] that there are algebra isomorphisms

$$\begin{aligned} H_*(B^{\mathrm{cy}}(\mathbb{N}_0);\mathbb{F}_p) &\cong H_*(* \sqcup \textstyle\coprod_{k\geq 1} S^1(k);\mathbb{F}_p) \cong P(x)\otimes E(dx),\\ H_*(B^{\mathrm{rep}}(\mathbb{N}_0);\mathbb{F}_p) &\cong H_*(\textstyle\coprod_{k\geq 0} S^1(k);\mathbb{F}_p) \cong P(x)\otimes E(d\log x)\,, \text{ and}\\ H_*(B^{\mathrm{cy}}_{\{0\}}(\mathbb{Z});\mathbb{F}_p) &\cong H_*(S^1(0);\mathbb{F}_p) \cong E(d\log x). \end{aligned}$$

Here each $S^1(k)$ is a 1-sphere, and we have $x \in H_0(S^1(1);\mathbb{F}_p)$, $dx \in H_1(S^1(1);\mathbb{F}_p)$, and $d\log x \in H_1(S^1(0);\mathbb{F}_p)$.

We first treat the case of $H_*(B^{\mathrm{cy}}(D(x))_{h\mathcal{J}};\mathbb{F}_p)$. Writing d for the degree of x, we observe that the augmentations induce a commutative diagram

$$\begin{tikzcd} B^{\mathrm{cy}}(D(x)_{\{0\}})_{h\mathcal{J}} \arrow[r] \arrow[d] & B^{\mathrm{cy}}(D(x))_{h\mathcal{J}} \arrow[r] \arrow[d] & B^{\mathrm{cy}}(D(x)^{\mathrm{gp}})_{h\mathcal{J}} \arrow[d] \\ * \arrow[r] & B^{\mathrm{cy}}(d\mathbb{N}_0) \arrow[r] & B^{\mathrm{cy}}(d\mathbb{Z})\,. \end{tikzcd} \tag{5.3}$$

Using the weak equivalences from Corollary 2.11 and applying the Bousfield–Friedlander theorem as in the proof [RSS15, Proposition 7.1] shows that that the outer rectangle and the right hand square in this diagram are homotopy cartesian. Hence so is the left hand square.

Let $\pi\colon B^{\mathrm{cy}}(D(x))^{\mathrm{fib}}_{h\mathcal{J}} \twoheadrightarrow B^{\mathrm{cy}}(d\mathbb{N}_0)$ be the fibration of simplicial monoids obtained by forming the base change of the fibration q considered in Lemma 5.6 along $B^{\mathrm{cy}}(d\mathbb{N}_0) \to B^{\mathrm{cy}}(d\mathbb{Z})$. Then the canonical map $\iota\colon B^{\mathrm{cy}}(D(x))_{h\mathcal{J}} \to B^{\mathrm{cy}}(D(x))^{\mathrm{fib}}_{h\mathcal{J}}$ is a weak equivalence since the right hand square in (5.3) is homotopy cartesian, and it follows from Lemma 5.6 that π admits a basepoint preserving space level section τ.

For a 0-simplex $c \in B^{\mathrm{cy}}(d\mathbb{N}_0)$, we now consider the diagram

$$\begin{tikzcd} B^{\mathrm{cy}}(D(x)_{\{0\}})_{h\mathcal{J}} \arrow[r, "\mathrm{incl}_c"] \arrow[d, equal] & B^{\mathrm{cy}}(D(x)_{\{0\}})_{h\mathcal{J}} \times B^{\mathrm{cy}}(d\mathbb{N}_0) \arrow[r, "\mathrm{proj}"] \arrow[d, "\mu"] & B^{\mathrm{cy}}(d\mathbb{N}_0) \arrow[d, equal] \\ B^{\mathrm{cy}}(D(x)_{\{0\}})_{h\mathcal{J}} \arrow[r, "\nu_c"] & B^{\mathrm{cy}}(D(x))^{\mathrm{fib}}_{h\mathcal{J}} \arrow[r, "\pi"] & B^{\mathrm{cy}}(d\mathbb{N}_0)\,. \end{tikzcd} \tag{5.4}$$

where $\mathrm{incl}_c(a) = (a,c)$, $\mu(a,b) = \iota(a)\cdot\tau(b)$, and $\nu_c(a) = \iota(a)\cdot\tau(c)$. By construction, both squares commute. We want to show that μ is a weak equivalence. Since the left hand square in (5.3) is homotopy cartesian, the lower sequence in (5.4) is a homotopy fiber sequence if c is the basepoint. A five-lemma argument with the long exact sequences and a consideration of path components shows that μ induces an isomorphism on homotopy groups with basepoints in the zero component. Since in the diagram (5.3), both squares are homotopy cartesian, the rightmost map is a homomorphism of grouplike simplicial monoids, and $B^{\mathrm{cy}}_{>0}(d\mathbb{N}_0) \to B^{\mathrm{cy}}_{>0}(d\mathbb{Z})$ is a weak equivalence, it follows that the lower sequence in (5.4) is also a homotopy fiber sequence if c lies in a positive path component. Arguing again with the long exact sequence completes the proof of μ being a weak equivalence.

With the notation $C_* = H_*(B^{\mathrm{cy}}(D(x)_{\{0\}})_{h\mathcal{J}};\mathbb{F}_p)$ from (5.2), the Künneth isomorphism and the weak equivalence μ induce an isomorphism

$$C_* \otimes H_*(B^{\mathrm{cy}}(d\mathbb{N}_0);\mathbb{F}_p) \cong H_*(B^{\mathrm{cy}}(D(x))_{h\mathcal{J}};\mathbb{F}_p).$$

It is an isomorphism of C_*-modules under C_* since μ is a map of $B^{\mathrm{cy}}(D(x)_{\{0\}})_{h\mathcal{J}}$-modules under $B^{\mathrm{cy}}(D(x)_{\{0\}})_{h\mathcal{J}}$. We need to verify that this isomorphism is multiplicative, and it suffices to check this on each tensor factor on the left. For C_*, this holds by construction. The homomorphism

$$\tau_*\colon H_*(B^{\mathrm{cy}}(d\mathbb{N}_0);\mathbb{F}_p) \to H_*(B^{\mathrm{cy}}(D(x))_{h\mathcal{J}};\mathbb{F}_p)$$

is induced by the space level section τ. Hence it is an additive section to the algebra homomorphism $\pi_*\colon H_*(B^{\mathrm{cy}}(D(x))_{h\mathcal{J}};\mathbb{F}_p) \to H_*(B^{\mathrm{cy}}(d\mathbb{N}_0);\mathbb{F}_p)$. The algebra C_* is $(2p-4)$-connected and therefore at least 2-connected for $p \geq 3$. Hence π_* is an isomorphism in degrees $* \leq 2$. Since $H_*(B^{\mathrm{cy}}(d\mathbb{N}_0);\mathbb{F}_p)$ is concentrated in degrees $* \leq 1$, this implies, as an algebraic fact, that the additive section τ_* is multiplicative. It also implies that the suspension operator satisfies $\sigma(x) = dx$ and $\sigma(dx) = 0$, since these relations hold in $H_*(B^{\mathrm{cy}}(d\mathbb{N}_0);\mathbb{F}_p)$ and are preserved by π_*.

The claims about $H_*(B^{\mathrm{rep}}(D(x))_{h\mathcal{J}};\mathbb{F}_p)$ and $H_*(B^{\mathrm{cy}}_{\{0\}}(D(x)^{\mathrm{gp}})_{h\mathcal{J}};\mathbb{F}_p)$ follow by a similar argument with $B^{\mathrm{cy}}_{\geq 0}(d\mathbb{Z})$ (respectively $B^{\mathrm{cy}}_{\{0\}}(d\mathbb{Z})$) replacing $B^{\mathrm{cy}}(d\mathbb{N}_0)$. $\square$

We do not know if the statement of Proposition 5.5 holds for $p = 2$. Since we are interested in $V(1)$-homotopy calculations later on, we shall not pursue this question further.

5.7. **Homology of the group completion of free commutative $\mathcal{J}$-space monoids.** We give another application of the graded Thom isomorphism that will become relevant for the proof of Theorem 6.2 below. Let $(\mathbf{e}_1, \mathbf{e}_2)$ be an object with $e_2 - e_1 > 0$ and let $(\mathbf{d}_1, \mathbf{d}_2) = (\mathbf{e}_1, \mathbf{e}_2)^{\sqcup k}$ for a positive integer k. We have a map of commutative $\mathcal{J}$-space monoids $\mathbb{C}\langle \mathbf{d}_1, \mathbf{d}_2\rangle \to \mathbb{C}\langle \mathbf{e}_1, \mathbf{e}_2\rangle$ defined by mapping the generator $\mathrm{id}_{(\mathbf{d}_1,\mathbf{d}_2)}$ to $\mathrm{id}^{\boxtimes k}_{(\mathbf{e}_1,\mathbf{e}_2)}$. Consider the commutative diagram of commutative $\mathcal{J}$-space monoids

$$\begin{array}{ccccc} \mathbb{C}\langle \mathbf{d}_1, \mathbf{d}_2\rangle & \longrightarrow & \mathbb{C}\langle \mathbf{d}_1, \mathbf{d}_2\rangle^{\mathrm{gp}}_{\geq 0} & \longrightarrow & \mathbb{C}\langle \mathbf{d}_1, \mathbf{d}_2\rangle^{\mathrm{gp}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{C}\langle \mathbf{e}_1, \mathbf{e}_2\rangle & \longrightarrow & \mathbb{C}\langle \mathbf{e}_1, \mathbf{e}_2\rangle^{\mathrm{gp}}_{\geq 0} & \longrightarrow & \mathbb{C}\langle \mathbf{e}_1, \mathbf{e}_2\rangle^{\mathrm{gp}} \end{array}$$

in which $\mathbb{C}\langle \mathbf{d}_1, \mathbf{d}_2\rangle^{\mathrm{gp}}$ and $\mathbb{C}\langle \mathbf{e}_1, \mathbf{e}_2\rangle^{\mathrm{gp}}$ are cofibrant, the horizontal composites are group completions and the right hand horizontal maps are the canonical inclusions.

Lemma 5.8. *With notation as above set $e = e_2 - e_1$, assume that e is even, and let p be a prime not dividing k. Then $H_*(\mathbb{S}^{\mathcal{J}}[\mathbb{C}\langle \mathbf{e}_1, \mathbf{e}_2\rangle^{\mathrm{gp}}_{\geq 0}];\mathbb{Z}_{(p)})$ is a free module over $H_*(\mathbb{S}^{\mathcal{J}}[\mathbb{C}\langle \mathbf{d}_1, \mathbf{d}_2\rangle^{\mathrm{gp}}_{\geq 0}];\mathbb{Z}_{(p)})$ generated by the images of the canonical generators of $\Sigma^{ie}\mathbb{Z}_{(p)}$ under the homomorphisms*

$$\Sigma^{ie}\mathbb{Z}_{(p)} \cong H_*(\mathbb{S}^{\mathcal{J}}[F^{\mathcal{J}}_{(\mathbf{e}_1,\mathbf{e}_2)^{\sqcup i}}(*)];\mathbb{Z}_{(p)}) \to H_*(\mathbb{S}^{\mathcal{J}}[\mathbb{C}\langle \mathbf{e}_1, \mathbf{e}_2\rangle^{\mathrm{gp}}_{\geq 0}];\mathbb{Z}_{(p)})$$

for $i = 0, \dots, k-1$.

Here the last map is induced by the canonical map $F^{\mathcal{J}}_{(\mathbf{e}_1,\mathbf{e}_2)^{\sqcup i}}(*) \cong F^{\mathcal{J}}_{(\mathbf{e}_1,\mathbf{e}_2)}(*)^{\boxtimes i} \to \mathbb{C}\langle \mathbf{e}_1, \mathbf{e}_2\rangle$.

Proof. Consider the map of $\mathcal{J}$-spaces

$$F^{\mathcal{J}}_{(\mathbf{e}_1,\mathbf{e}_2)^{\sqcup i}}(*) \boxtimes \mathbb{C}\langle \mathbf{d}_1, \mathbf{d}_2\rangle^{\mathrm{gp}}_{\geq 0} \to \mathbb{C}\langle \mathbf{e}_1, \mathbf{e}_2\rangle^{\mathrm{gp}}_{\geq 0} \boxtimes \mathbb{C}\langle \mathbf{e}_1, \mathbf{e}_2\rangle^{\mathrm{gp}}_{\geq 0} \to \mathbb{C}\langle \mathbf{e}_1, \mathbf{e}_2\rangle^{\mathrm{gp}}_{\geq 0}$$

for $i = 0, \dots, k-1$. The statement in the theorem is equivalent to the induced map of symmetric spectra

$$\mathbb{S}^{\mathcal{J}}\left[\textstyle\coprod_{i=0}^{k-1} F^{\mathcal{J}}_{(\mathbf{e}_1,\mathbf{e}_2)^{\sqcup i}}(*) \boxtimes \mathbb{C}\langle \mathbf{d}_1, \mathbf{d}_2\rangle^{\mathrm{gp}}_{\geq 0}\right] \to \mathbb{S}^{\mathcal{J}}[\mathbb{C}\langle \mathbf{e}_1, \mathbf{e}_2\rangle^{\mathrm{gp}}_{\geq 0}]$$

inducing an isomorphism on homology with $\mathbb{Z}_{(p)}$-coefficients. By Proposition 5.2 this is equivalent to the map of spaces

$$\textstyle\coprod_{i=0}^{k-1} \big(F^{\mathcal{J}}_{(\mathbf{e}_1,\mathbf{e}_2)^{\sqcup i}}(*) \boxtimes \mathbb{C}\langle \mathbf{d}_1, \mathbf{d}_2\rangle^{\mathrm{gp}}_{\geq 0}\big)_{h\mathcal{J}_n} \to \Big(\mathbb{C}\langle \mathbf{e}_1, \mathbf{e}_2\rangle^{\mathrm{gp}}_{\geq 0}\Big)_{h\mathcal{J}_n}$$

inducing an isomorphism on homology with $\mathbb{Z}_{(p)}$-coefficients for all n. We can further reduce this to the case $n = 0$ by considering the commutative diagram

$$\begin{tikzcd}
\big(F^{\mathcal{J}}_{(\mathbf{e}_1,\mathbf{e}_2)^{\sqcup i}}(*) \boxtimes \mathbb{C}\langle \mathbf{d}_1, \mathbf{d}_2\rangle^{\mathrm{gp}}\big)_{h\mathcal{J}_{ie+jd}} \arrow[r] & \mathbb{C}\langle \mathbf{e}_1, \mathbf{e}_2\rangle^{\mathrm{gp}}_{h\mathcal{J}_{ie+jd}} \\
F^{\mathcal{J}}_{(\mathbf{e}_1,\mathbf{e}_2)^{\sqcup i}}(*)_{h\mathcal{J}_{ie}} \times \mathbb{C}\langle \mathbf{d}_1, \mathbf{d}_2\rangle^{\mathrm{gp}}_{h\mathcal{J}_{jd}} \arrow[r] \arrow[u, "\sim"] & F^{\mathcal{J}}_{(\mathbf{e}_1,\mathbf{e}_2)^{\sqcup i}}(*)_{h\mathcal{J}_{ie}} \times \mathbb{C}\langle \mathbf{e}_1, \mathbf{e}_2\rangle^{\mathrm{gp}}_{h\mathcal{J}_{jd}} \arrow[u, "\sim"] \\
F^{\mathcal{J}}_{(\mathbf{e}_1,\mathbf{e}_2)^{\sqcup i}}(*)_{h\mathcal{J}_{ie}} \times \mathbb{C}\langle \mathbf{d}_1, \mathbf{d}_2\rangle^{\mathrm{gp}}_{h\mathcal{J}_0} \arrow[r] \arrow[u, "\sim", "\mathrm{id}\times\mathrm{id}_{(\mathbf{d}_1,\mathbf{d}_2)^{\sqcup j}}"'] & F^{\mathcal{J}}_{(\mathbf{e}_1,\mathbf{e}_2)^{\sqcup i}}(*)_{h\mathcal{J}_{ie}} \times \mathbb{C}\langle \mathbf{e}_1, \mathbf{e}_2\rangle^{\mathrm{gp}}_{h\mathcal{J}_0} \arrow[u, "\sim", "\mathrm{id}\times\mathrm{id}_{(\mathbf{e}_1,\mathbf{e}_2)^{\sqcup jk}}"']
\end{tikzcd}$$

where $d = d_2 - d_1 = ke$, and the arrows labeled $\mathrm{id}_{(\mathbf{d}_1,\mathbf{d}_2)^{\sqcup j}}$ and $\mathrm{id}_{(\mathbf{e}_1,\mathbf{e}_2)^{\sqcup jk}}$ are given by left translation by the images of these elements in the group completions. Since the vertical maps are weak homotopy equivalences as indicated, it remains to show that the map $\mathbb{C}\langle \mathbf{d}_1, \mathbf{d}_2\rangle^{\mathrm{gp}}_{h\mathcal{J}_0} \to \mathbb{C}\langle \mathbf{e}_1, \mathbf{e}_2\rangle^{\mathrm{gp}}_{h\mathcal{J}_0}$ induces an isomorphism in homology with $\mathbb{Z}_{(p)}$-coefficients.

Using that $\mathbb{C}\langle \mathbf{n_1}, \mathbf{n_2}\rangle^{\mathrm{gp}}_{h\mathcal{J}} \to \Omega B(\mathbb{C}\langle \mathbf{n_1}, \mathbf{n_2}\rangle^{\mathrm{gp}}_{h\mathcal{J}})$ is a weak equivalence, the next lemma shows that $\mathbb{C}\langle \mathbf{d}_1, \mathbf{d}_2\rangle^{\mathrm{gp}}_{h\mathcal{J}_0} \to \mathbb{C}\langle \mathbf{e}_1, \mathbf{e}_2\rangle^{\mathrm{gp}}_{h\mathcal{J}_0}$ induces multiplication by k on the homotopy groups. Hence the map of homotopy groups becomes an isomorphism after tensoring with $\mathbb{Z}_{(p)}$. This in turn implies that it induces an isomorphism in homology with $\mathbb{Z}_{(p)}$-coefficients (see e.g. [BK72, Proposition V.3.2]). □

Lemma 5.9. *Let k be a positive integer, let $(\mathbf{e_1}, \mathbf{e_2})$ be an object of $\mathcal{J}$, let $(\mathbf{d_1}, \mathbf{d_2}) = (\mathbf{e_1}, \mathbf{e_2})^{\sqcup k}$ and let $\mathbb{C}\langle \mathbf{d}_1, \mathbf{d}_2\rangle \to \mathbb{C}\langle \mathbf{e}_1, \mathbf{e}_2\rangle$ be the map determined by $\mathrm{id}_{(\mathbf{d}_1,\mathbf{d}_2)} \mapsto \mathrm{id}^{\boxtimes k}_{(\mathbf{e}_1,\mathbf{e}_2)}$. Then the induced map $B(\mathbb{C}\langle \mathbf{d}_1, \mathbf{d}_2\rangle^{\mathrm{gp}}_{h\mathcal{J}}) \to B(\mathbb{C}\langle \mathbf{e}_1, \mathbf{e}_2\rangle^{\mathrm{gp}}_{h\mathcal{J}})$ acts as multiplication by k on the homotopy groups.*

Proof. Let $\mathbf{k}$ be the finite set $\mathbf{k} = \{1, \dots, k\}$ and let $\mathbf{k}^{\sqcup}\colon \Sigma \to \Sigma$ be the functor taking $\mathbf{n}$ to the n-fold concatenation $\mathbf{k}^{\sqcup n}$ with its canonical left Σ_n-action. This is a strict symmetric monoidal functor and we use the same notation for the induced functor $\mathbf{k}^{\sqcup}\colon \mathcal{J} \to \mathcal{J}$. It is easy to see that $\mathbf{k}^{\sqcup}\colon \mathcal{J} \to \mathcal{J}$ is naturally isomorphic to the k-fold monoidal product $(\mathbf{n}_1, \mathbf{n}_2) \mapsto (\mathbf{n}_1, \mathbf{n}_2)^{\sqcup k}$. Hence the induced map $B(B\mathcal{J}) \to B(B\mathcal{J})$ acts as multiplication by k on the homotopy groups.

Let $\tilde{\Sigma}_n$ be the translation category of Σ_n. Its objects are the elements in Σ_n and its morphisms $a\colon b \to c$ are elements in Σ_n such that $ab = c$. We consider the functor $\tilde{\Sigma}_n \to ((\mathbf{d}_1, \mathbf{d}_2)^{\sqcup n} \downarrow \mathcal{J})$ that maps an object a in $\tilde{\Sigma}_n$ to the isomorphism $a_*\colon (\mathbf{d}_1, \mathbf{d}_2)^{\sqcup n} \to (\mathbf{d}_1, \mathbf{d}_2)^{\sqcup n}$ induced by a and the symmetry isomorphism. This defines a weak homotopy equivalence of Σ_n-orbit spaces

$$B\Sigma_n \cong (B\tilde{\Sigma}_n)/\Sigma_n \xrightarrow{\sim} B((\mathbf{d}_1, \mathbf{d}_2)^{\sqcup n} \downarrow \mathcal{J})/\Sigma_n \cong (F^{\mathcal{J}}_{(\mathbf{d}_1,\mathbf{d}_2)}(*)^{\boxtimes n}/\Sigma_n)_{h\mathcal{J}}.$$

Assembling these maps for varying n and forming the corresponding map for $\mathbb{C}\langle \mathbf{e}_1, \mathbf{e}_2\rangle_{h\mathcal{J}}$ defines the horizontal weak equivalences in the following commutative diagram of E_∞ monoids:

$$\begin{tikzcd}
\mathbb{C}\langle \mathbf{d}_1, \mathbf{d}_2\rangle_{h\mathcal{J}} \arrow[d] & B\Sigma \arrow[l, "(\mathbf{d}_1,\mathbf{d}_2)^{\sqcup}"', "\sim"] \arrow[d, "\mathbf{k}^{\sqcup}"] \\
\mathbb{C}\langle \mathbf{e}_1, \mathbf{e}_2\rangle_{h\mathcal{J}} & B\Sigma \arrow[l, "(\mathbf{e}_1,\mathbf{e}_2)^{\sqcup}"', "\sim"]
\end{tikzcd}$$

Since the maps $B(B\Sigma) \to B(B\mathcal{J})$ and $B(\mathbb{C}\langle \mathbf{d}_1, \mathbf{d}_2\rangle_{h\mathcal{J}}) \to B(\mathbb{C}\langle \mathbf{d}_1, \mathbf{d}_2\rangle^{\mathrm{gp}}_{h\mathcal{J}})$ are weak equivalences, the claim follows by the above observation about the map $B(B\mathcal{J}) \to B(B\mathcal{J})$ induced by $\mathbf{k}^{\sqcup}\colon \mathcal{J} \to \mathcal{J}$. □

6. The inclusion of the Adams summand

Let p be an odd prime. It is well-known that there are positive fibrant commutative symmetric ring spectra $ku_{(p)}$ and ℓ modeling respectively the p-local connective complex K-theory spectrum and the corresponding Adams summand. Furthermore, we may assume that the inclusion of the Adams summand $f\colon \ell \to ku_{(p)}$ is realized as a positive fibration of commutative symmetric ring spectra, that the corresponding periodic theories are also represented by positive fibrant commutative symmetric ring spectra $KU_{(p)}$ and L, and that there is a commutative square

$$\begin{tikzcd} \ell \arrow[r] \arrow[d, "f"'] & L \arrow[d] \\ ku_{(p)} \arrow[r] & KU_{(p)} \end{tikzcd} \tag{6.1}$$

in $\mathcal{C}\mathrm{Sp}^{\Sigma}$. This is explained in [Sag14, Section 4.12] and is based on work by Baker and Richter [BR05].

The graded rings of homotopy groups are given by $\pi_*(ku_{(p)}) = \mathbb{Z}_{(p)}[u]$ with $|u| = 2$ and $\pi_*(\ell) = \mathbb{Z}_{(p)}[v]$ with $|v| = 2(p-1)$, and under these isomorphisms f corresponds to the ring homomorphism $\mathbb{Z}_{(p)}[v] \to \mathbb{Z}_{(p)}[u]$ taking v to u^{p-1}. It is proved in [Sag14, Proposition 4.13] that one may choose representatives v in $\Omega^{\mathcal{J}}(\ell)(\mathbf{p-1}, \mathbf{3(p-1)})$ and u in $\Omega^{\mathcal{J}}(ku_{(p)})(\mathbf{1}, \mathbf{3})$ as well as a map $f^{\flat}\colon D(v) \to D(u)$ relating the commutative $\mathcal{J}$-space monoids $D(v)$ and $D(u)$ from Construction 4.2 such that there is a commutative diagram of pre-log ring spectra

$$\begin{tikzcd} (\ell, D(v)) \arrow[r] \arrow[d] & (\ell, j_*\mathrm{GL}_1^{\mathcal{J}}(L)) \arrow[d] \\ (ku_{(p)}, D(u)) \arrow[r] & (ku_{(p)}, j_*\mathrm{GL}_1^{\mathcal{J}}(KU_{(p)})). \end{tikzcd} \tag{6.2}$$

In the diagram, the horizontal maps are induced by the maps $D(v) \to j_*\mathrm{GL}_1^{\mathcal{J}}(L)$ and $D(u) \to j_*\mathrm{GL}_1^{\mathcal{J}}(KU_{(p)})$ from (4.2).

6.1. **Formally log** THH**-étale maps.** In analogy with the notion of *formally* THH-*étale* maps defined in [Rog08, §9.2], we say that a map of pre-log ring spectra $(A, M) \to (B, N)$ is *formally log* THH-*étale* if the induced square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathrm{THH}(A, M) \arrow[d] \\ B \arrow[r] & \mathrm{THH}(B, N) \end{tikzcd}$$

is a homotopy cocartesian diagram of commutative symmetric ring spectra.

Theorem 6.2. *The map of log ring spectra* $(\ell, j_*\mathrm{GL}_1^{\mathcal{J}}(L)) \to (ku_{(p)}, j_*\mathrm{GL}_1^{\mathcal{J}}(KU_{(p)}))$ *is formally* log THH-étale*.*

By Proposition 4.3, the horizontal maps in (6.2) induce stable equivalences when applying THH. This reduces Theorem 6.2 to the next statement.

Theorem 6.3. *The map of pre-log ring spectra* $(\ell, D(v)) \to (ku_{(p)}, D(u))$ *is formally* log THH-étale*.*

Proof. We need to show that the square given by the back face of the following commutative diagram in $\mathcal{C}\mathrm{Sp}^{\Sigma}$ is homotopy cocartesian:

$$\begin{tikzcd}
 & \ell \arrow[rr] \arrow[dd] & & ku_{(p)} \arrow[dd] \\
\mathbb{S}^{\mathcal{J}}[D(v)] \arrow[ur] \arrow[rr] \arrow[dd] & & \mathbb{S}^{\mathcal{J}}[D(u)] \arrow[ur] \arrow[dd] & \\
 & \mathrm{THH}(\ell) \arrow[rr] \arrow[dd] & & \mathrm{THH}(ku_{(p)}) \arrow[dd] \\
\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(D(v))] \arrow[ur] \arrow[rr] \arrow[dd] & & \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(D(u))] \arrow[ur] \arrow[dd] & \\
 & \mathrm{THH}(\ell, D(v)) \arrow[rr] & & \mathrm{THH}(ku_{(p)}, D(u))\,. \\
\mathbb{S}^{\mathcal{J}}[B^{\mathrm{rep}}(D(v))] \arrow[ur] \arrow[rr] & & \mathbb{S}^{\mathcal{J}}[B^{\mathrm{rep}}(D(u))] \arrow[ur] &
\end{tikzcd}$$

The lower left hand face and the lower right hand face are homotopy cocartesian squares by definition. The inner square is homotopy cocartesian since it results from applying THH to the homotopy cocartesian square of Proposition 6.4 below. Hence the bottom face of the diagram is homotopy cocartesian. The top face of the diagram is homotopy cocartesian by Proposition 6.4. Since the back face is already p-local, it is therefore enough to show that the front face becomes a homotopy cocartesian square after p-localization.

Using Proposition 3.2 it suffices to show that $B(D(v)_{h\mathcal{J}}) \to B(D(u)_{h\mathcal{J}})$ induces an isomorphism in homology with $\mathbb{Z}_{(p)}$-coefficients. By Lemma 4.5, this is equivalent to showing that $B(\mathbb{C}\langle v\rangle^{\mathrm{gp}}_{h\mathcal{J}}) \to B(\mathbb{C}\langle u\rangle^{\mathrm{gp}}_{h\mathcal{J}})$ induces an isomorphism in homology with $\mathbb{Z}_{(p)}$-coefficients. Lemma 5.9 shows that the latter map acts as multiplication by $(p-1)$ on the homotopy groups. Hence it induces an isomorphism on homotopy groups after tensoring with $\mathbb{Z}_{(p)}$, which in turn implies that the map in homology with $\mathbb{Z}_{(p)}$-coefficients is an isomorphism. $\square$

The next lemma was used in the proof of Theorem 6.3. The lemma is identical to [Sag14, Proposition 4.15], but we provide a more conceptual argument that is based on the graded Thom isomorphism of Proposition 5.1:

Proposition 6.4. *The commutative square of commutative symmetric ring spectra*

$$\begin{tikzcd}
\mathbb{S}^{\mathcal{J}}[D(v)] \arrow[r] \arrow[d] & \ell \arrow[d] \\
\mathbb{S}^{\mathcal{J}}[D(u)] \arrow[r] & ku_{(p)}
\end{tikzcd}$$

is homotopy cocartesian.

Proof. If we factor $\mathbb{S}^{\mathcal{J}}[D(v)] \to \ell$ as a cofibration $\mathbb{S}^{\mathcal{J}}[D(v)] \to \ell'$ followed by an acyclic fibration $\ell' \to \ell$, then we have to show that the induced map

$$\mathbb{S}^{\mathcal{J}}[D(u)] \wedge_{\mathbb{S}^{\mathcal{J}}[D(v)]} \ell' \to ku_{(p)}$$

is a stable equivalence. These are p-local connective spectra, so it suffices to show that the latter map induces an isomorphism in spectrum homology with $\mathbb{Z}_{(p)}$-coefficients. Consider for each $i = 0, \dots, p-2$ the map of symmetric spectra

$$\mathbb{S}^{\mathcal{J}}[F^{\mathcal{J}}_{(\mathbf{1},\mathbf{3})^{\sqcup i}}(*)] \to \mathbb{S}^{\mathcal{J}}[\mathbb{C}\langle u\rangle] \to ku_{(p)}.$$

The symmetric spectrum $\mathbb{S}^{\mathcal{J}}[F^{\mathcal{J}}_{(\mathbf{1},\mathbf{3})^{\sqcup i}}(*)]$ represents the suspension $\Sigma^{2i}\mathbb{S}$ as an object in the stable homotopy category and the map represents the i-fold product u^i in

$\pi_{2i}(ku_{(p)})$. Hence it follows that the composite map

$$\bigvee_{i=0}^{p-2} \mathbb{S}^{\mathcal{J}}[F^{\mathcal{J}}_{(\mathbf{1},\mathbf{3})^{\sqcup i}}(*)] \wedge \ell' \to ku_{(p)} \wedge ku_{(p)} \to ku_{(p)} \tag{6.3}$$

is a stable equivalence of ℓ'-module spectra.

Using the $\mathcal{J}$-equivalences $D(v) \to \mathbb{C}\langle v\rangle^{\mathrm{gp}}_{\geq 0}$ and $D(u) \to \mathbb{C}\langle u\rangle^{\mathrm{gp}}_{\geq 0}$ from Lemma 4.5, and Lemma 5.8, it follows that $H_*(\mathbb{S}^{\mathcal{J}}[D(u)];\mathbb{Z}_{(p)})$ is a free $H_*(\mathbb{S}^{\mathcal{J}}[D(v)];\mathbb{Z}_{(p)})$-module. Inspecting Lemma 5.8, we see that it is generated by the images of the canonical generators of $\Sigma^{2i}\mathbb{Z}_{(p)}$ under the homomorphisms

$$\Sigma^{2i}\mathbb{Z}_{(p)} \cong H_*(\mathbb{S}^{\mathcal{J}}[F^{\mathcal{J}}_{(\mathbf{1},\mathbf{3})^{\sqcup i}}(*)];\mathbb{Z}_{(p)}) \to H_*(\mathbb{S}^{\mathcal{J}}[\mathbb{C}\langle u\rangle];\mathbb{Z}_{(p)}) \to H_*(\mathbb{S}^{\mathcal{J}}[D(u)];\mathbb{Z}_{(p)})$$

for $i = 0, \ldots, p-2$. Hence the Tor spectral sequence [EKMM97, Theorem IV.4.1]

$$\mathrm{Tor}^{E_*(\mathbb{S}^{\mathcal{J}}[D(v)])}_{s,t}(E_*(\mathbb{S}^{\mathcal{J}}[D(u)]), E_*(\ell')) \Longrightarrow E_{s+t}(\mathbb{S}^{\mathcal{J}}[D(u)] \wedge_{\mathbb{S}^{\mathcal{J}}[D(v)]} \ell')$$

with $E = H\mathbb{Z}_{(p)}$ collapses. The result follows from this since the specified generators are compatible with the stable equivalence in (6.3). □

7. Logarithmic THH of the Adams summand

We will now demonstrate that the current definition of logarithmic topological Hochschild homology, $\mathrm{THH}(A, M)$, in terms of $\mathrm{THH}(A)$ and the cyclic and replete bar constructions, lends itself to nontrivial explicit computations, going beyond the cases of discrete rings previously studied by Hesselholt–Madsen [HM03, §2]. In particular, we will realize the program to compute $V(1)_*\,\mathrm{THH}(ku) \cong V(1)_*\,\mathrm{THH}(ku_{(p)})$ outlined by Ausoni in [Aus05, §10], using $\mathrm{THH}(\ell, D(v))$ and $\mathrm{THH}(ku_{(p)}, D(u))$ in place of the then-hypothetical constructions $\mathrm{THH}(\ell|L)$ and $\mathrm{THH}(ku_{(p)}|KU_{(p)})$. In Theorem 8.5 we use this to recover the full algebra structure on $V(1)_*\,\mathrm{THH}(ku)$, for $p \geq 5$. The subsequent construction of $\mathrm{THH}(\ell|L)$ and $\mathrm{THH}(ku_{(p)}|KU_{(p)})$ by Blumberg–Mandell [BM11], using simplicially enriched Waldhausen categories, is not known to lend itself to such explicit calculations. On the other hand, their models are known to admit good trace maps from algebraic K-theory, so one may hope to prove that the two constructions are equivalent, in the cases where both are defined.

Notation 7.1. For any prime p, let $H = H\mathbb{F}_p$ be the mod p Eilenberg–Mac Lane spectrum and write $H_*X = \pi_*(H \wedge X)$ for the mod p homology groups of X. For $p \geq 3$ let

$$V(1) = \mathrm{cone}(v_1\colon \Sigma^{2p-2}S/p \to S/p) \simeq S \cup_p e^1 \cup_{\alpha_1} e^{2p-1} \cup_p e^{2p}$$

be the Smith–Toda complex of type 2. When $p \geq 5$ it admits the structure of a homotopy commutative ring spectrum [Oka79], and $V(1) \to H$ is a map of homotopy commutative ring spectra. Write $V(1)_*X = \pi_*(V(1) \wedge X)$ for the $V(1)$-homotopy groups of X.

When X is a ring spectrum and $p \geq 5$, $V(1)_*X$ becomes a graded $\mathbb{F}_p$-algebra, and $V(1)_*X \to H_*X$ is an algebra homomorphism. When X has a circle action $S^1_+ \wedge X \to X$, $V(1)_*X$ inherits a suspension operator

$$\sigma\colon V(1)_*X \to V(1)_{*+1}(S^1_+ \wedge X) \to V(1)_{*+1}X,$$

compatible with the operator $\sigma\colon H_*X \to H_{*+1}(S^1_+ \wedge X) \to H_{*+1}X$. When the adjoint circle action $X \to F(S^1_+, X)$ is a ring spectrum map, both suspension operators are derivations [AR05, Proposition 5.10].

There is an equivalence $V(1) \wedge \ell \simeq H$ of homotopy commutative ℓ-algebras. When X is an ℓ-module or commutative ℓ-algebra there is an equivalence $V(1) \wedge X \simeq$

$H \wedge_\ell X$ of spectra or homotopy commutative ring spectra, respectively, and a corresponding isomorphism $V(1)_*X \cong \pi_*(H \wedge_\ell X)$. (Smash products are understood as left derived smash products here.) In particular, the natural map $V(1)_*X \to H_*X$ is split injective. When $p = 3$ and X is a commutative ℓ-algebra we give $V(1)_*X$ the algebra structure from $\pi_*(H \wedge_\ell X)$.

Let (A, M) be a pre-log ring spectrum with $M = M_{\geq 0}$ concentrated in non-negative degrees. Suppose, without loss of generality, that (A, M) is cofibrant, so that the maps $\mathbb{S}^{\mathcal{J}}[M] \to A$ and $\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)] \to \mathrm{THH}(A)$ are cofibrations of commutative symmetric ring spectra. This ensures that the smash product

$$\mathrm{THH}(A, M) = \mathrm{THH}(A) \wedge_{\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)]} \mathbb{S}^{\mathcal{J}}[B^{\mathrm{rep}}(M)]$$

captures the correct homotopy type.

In order to determine the structure of a Künneth spectral sequence associated to this smash product, we shall use a natural chain of $\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)]$-module maps

$$\mathbb{S}^{\mathcal{J}}[B^{\mathrm{rep}}(M)] \longrightarrow \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}_{\{0\}}(M^{\mathrm{gp}})] \longleftarrow \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M_{\{0\}})]\,.$$

The left hand map is defined by first projecting onto $\mathbb{S}^{\mathcal{J}}[B^{\mathrm{rep}}_{\{0\}}(M)]$ as in [RSS15, Definition 6.9] and then composing with the canonical map to $\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}_{\{0\}}(M^{\mathrm{gp}})]$, while the right hand map uses the identification $B^{\mathrm{cy}}(M_{\{0\}}) = B^{\mathrm{cy}}_{\{0\}}(M)$ and the group completion map $M \to M^{\mathrm{gp}}$. Hence there is a chain of $\mathrm{THH}(A)$-module maps

$$\mathrm{THH}(A, M) \longrightarrow \mathrm{THH}(A) \wedge_{\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)]} \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}_{\{0\}}(M^{\mathrm{gp}})] \longleftarrow \mathrm{THH}(A/(M_{>0}))\,, \tag{7.1}$$

where $A/(M_{>0}) = A \wedge_{\mathbb{S}^{\mathcal{J}}[M]} \mathbb{S}^{\mathcal{J}}[M_{\{0\}}]$.

We shall use a corresponding chain of Künneth spectral sequences to transport information about the spectral sequence for $\mathrm{THH}(A/(M_{>0}))$ to the spectral sequence for $\mathrm{THH}(A, M)$. This chain has the following E^2-terms:

$$\begin{aligned}
&\mathrm{Tor}^{H_*\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)]}_{**}(V(1)_*\,\mathrm{THH}(A), H_*\mathbb{S}^{\mathcal{J}}[B^{\mathrm{rep}}(M)])\\
&\qquad\longrightarrow \mathrm{Tor}^{H_*\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)]}_{**}(V(1)_*\,\mathrm{THH}(A), H_*\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}_{\{0\}}(M^{\mathrm{gp}})])\\
&\qquad\qquad\longleftarrow \mathrm{Tor}^{H_*\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)]}_{**}(V(1)_*\,\mathrm{THH}(A), H_*\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M_{\{0\}})])
\end{aligned}$$

and converges to the $V(1)$-homotopy of the chain of $\mathrm{THH}(A)$-modules displayed above. It will be constructed in the proof of Theorem 7.3. The reader may want to compare the following calculations with those in [RSS15, Section 5], where we handled the case of a discrete pre-log structure.

Our assumption that (A, M) is cofibrant implies that A is a cofibrant commutative symmetric ring spectrum. Now suppose in addition that A is augmented over $H\mathbb{F}_p$, and let $H \to H\mathbb{F}_p$ be a cofibrant replacement in commutative A-algebras. If X is an A-module, we write $H^A_*(X) = \pi_*(H \wedge_A X)$. Using the isomorphism

$$\mathrm{THH}(A) \wedge_{\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)]} \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M_{\{0\}})] \cong \mathrm{THH}(A/(M_{>0}))$$

and the maps $\mathbb{S} \to A \to H$ we get a pushout square of commutative H-algebras

$$\begin{array}{ccc}
H \wedge \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)] & \longrightarrow & H \wedge \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M_{\{0\}})]\\
\downarrow & & \downarrow\\
H \wedge_A \mathrm{THH}(A) & \longrightarrow & H \wedge_A \mathrm{THH}(A/(M_{>0}))\,.
\end{array}$$

which is homotopy cocartesian by our cofibrancy assumptions. We first study the associated Tor spectral sequence

$$\begin{aligned}
E^2_{**} &= \mathrm{Tor}^{H_*\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)]}_{**}(H^A_*\,\mathrm{THH}(A), H_*\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M_{\{0\}})])\\
&\Longrightarrow H^A_*\,\mathrm{THH}(A/(M_{>0}))\,.
\end{aligned} \tag{7.2}$$

We shall use the notation $d^r(x) \doteq y$ to indicate that $d^r(x)$ equals a unit in $\mathbb{F}_p$ times y.

Proposition 7.2. *Consider the case $A = \ell$, $M = D(v)$ and $A/(M_{>0}) \simeq H\mathbb{Z}_{(p)}$ of the spectral sequence* (7.2) *above. It is an algebra spectral sequence*

$$\begin{aligned} E^2_{**} &= \mathrm{Tor}_{**}^{H_\circledast(B^{\mathrm{cy}}(D(v))_{h\mathcal{J}})}(V(1)_* \mathrm{THH}(\ell), H_\circledast(B^{\mathrm{cy}}(D(v))_{h\mathcal{J}_0})) \\ &\Longrightarrow V(1)_* \mathrm{THH}(\mathbb{Z}_{(p)}) \,. \end{aligned}$$

With $C_* = H_*(B^{\mathrm{cy}}(D(v))_{h\mathcal{J}_0})$ as in (5.2) *we have*

$$\begin{aligned} H_\circledast(B^{\mathrm{cy}}(D(v))_{h\mathcal{J}}) &\cong P(v) \otimes E(dv) \otimes C_* \\ V(1)_* \mathrm{THH}(\ell) &\cong E(\lambda_1, \lambda_2) \otimes P(\mu_2) \\ V(1)_* \mathrm{THH}(\mathbb{Z}_{(p)}) &\cong E(\epsilon_1, \lambda_1) \otimes P(\mu_1) \,, \end{aligned}$$

with $|v| = 2p-2$, $|dv| = 2p-1$, $|\lambda_1| = 2p-1$, $|\lambda_2| = 2p^2-1$, $|\mu_2| = 2p^2$, $|\epsilon_1| = 2p-1$ and $|\mu_1| = 2p$. Here

$$E^2_{**} \cong E(\lambda_1, \lambda_2) \otimes P(\mu_2) \otimes E([v]) \otimes \Gamma([dv])$$

where $[v]$ has bidegree $(1, 2p-2)$ and $[dv]$ has bidegree $(1, 2p-1)$. There are nontrivial d^p-differentials

$$d^p(\gamma_k[dv]) \doteq \lambda_2 \cdot \gamma_{k-p}[dv]$$

for all $k \geq p$, leaving

$$E^\infty_{**} \cong E(\lambda_1) \otimes P(\mu_2) \otimes E([v]) \otimes P_p([dv]) \,.$$

Hence $[v]$ represents ϵ_1 (modulo λ_1), $[dv]$ represents μ_1 and μ_2 represents μ_1^p (up to units in $\mathbb{F}_p$) in the abutment, and there is a multiplicative extension $[dv]^p \doteq \mu_2$.

Proof. Building on the graded Thom isomorphism, Proposition 5.5 provides isomorphisms $H_*\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}_{\{0\}}(D(v))] \cong H_*B^{\mathrm{cy}}(D(v)_{\{0\}})_{h\mathcal{J}} = C_*$ and $H_*\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(D(v))] \cong H_\circledast(B^{\mathrm{cy}}(D(v))_{h\mathcal{J}}) \cong P(v) \otimes E(dv) \otimes C_*$.

Bökstedt computed $\pi_*(S/p \wedge \mathrm{THH}(\mathbb{Z})) \cong \pi_*(S/p \wedge \mathrm{THH}(\mathbb{Z}_{(p)})) = E(\lambda_1) \otimes P(\mu_1)$, so we have $V(1)_* \mathrm{THH}(\mathbb{Z}_{(p)}) = E(\epsilon_1, \lambda_1) \otimes P(\mu_1)$, where ϵ_1 is a mod v_1 Bockstein element in degree $2p-1$. McClure and Staffeldt [MS93] computed that $V(1)_* \mathrm{THH}(\ell) = E(\lambda_1, \lambda_2) \otimes P(\mu_2)$. See [AR12, §3, §4] for further details.

This leads to the E^2-term

$$\begin{aligned} E^2_{**} &= \mathrm{Tor}_{**}^{P(v)\otimes E(dv)\otimes C_*}(E(\lambda_1, \lambda_2) \otimes P(\mu_2), C_*) \\ &\cong \mathrm{Tor}_{**}^{P(v)\otimes E(dv)}(E(\lambda_1, \lambda_2) \otimes P(\mu_2), \mathbb{F}_p) \\ &\cong E(\lambda_1, \lambda_2) \otimes P(\mu_2) \otimes E([v]) \otimes \Gamma([dv]) \,, \end{aligned}$$

where we have used change-of-rings and the fact that $P(v) \otimes E(dv)$ acts trivially on $E(\lambda_1, \lambda_2) \otimes P(\mu_2)$. To verify the last assertion, we use the factorization

$$H_*\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(D(v))] \to H_* \mathrm{THH}(\ell) \to H^\ell_*(\mathrm{THH}(\ell)) \cong V(1)_* \mathrm{THH}(\ell) \,.$$

The first homomorphism extends $H_*\mathbb{S}^{\mathcal{J}}[D(v)] \to H_*\ell$, hence takes v to 0 since $v \in \pi_{2p-2}(\ell)$ has Adams filtration 1, and takes dv to 0 by compatibility with the suspension operator coming from the circle action.

The algebra generators λ_1, λ_2, μ_2, $[v]$ and $[dv]$ must be infinite cycles for filtration reasons. To determine the differentials on the remaining algebra generators, namely the divided powers $\gamma_{p^i}[dv]$ for $i \geq 1$, we note that the abutment $E(\epsilon_1, \lambda_1) \otimes P(\mu_1)$ has at most two generators in each degree. In total degree $2p^2-1$ the E^2-term is generated by the three classes λ_2, $\lambda_1 \cdot \gamma_{p-1}[dv]$ and $[v] \cdot \gamma_{p-1}[dv]$. Hence one of these must be a boundary, and for filtration reasons the only possibility is $d^p(\gamma_p[dv]) \doteq \lambda_2$.

This implies that λ_1, $[v]$, $[dv]$ and μ_2 survive to the E^∞-term, where they must represent λ_1, ϵ_1 (modulo λ_1), μ_1 and μ_1^p, respectively (up to units in $\mathbb{F}_p$). It follows that each generating monomial in $E(\lambda_1) \otimes P(\mu_2) \otimes E([v]) \otimes P_p([dv])$ is an infinite cycle that represents a nonzero product in the abutment, so these classes cannot be boundaries.

Now consider total degree $2p^3 - 1$. After the differential on $\gamma_p[dv]$, only the three generators

$$\lambda_1 \cdot \mu_2^{p-1} \cdot \gamma_{p-1}[dv] \quad , \quad \mu_2^{p-1} \cdot [v] \cdot \gamma_{p-1}[dv] \quad \text{and} \quad \lambda_2 \cdot \gamma_{p^2-p}[dv]$$

remain. As we have just noticed, the first two monomials cannot be hit by differentials. Since only two generators can survive in this degree, and the only possible source (or target) of a differential is $\gamma_{p^2}[dv]$, we must have $d^p(\gamma_{p^2}[dv]) \doteq \lambda_2 \cdot \gamma_{p^2-p}[dv]$. By induction, the corresponding argument in degree $2p^{i+1} - 1$ establishes the nontrivial differential on $\gamma_{p^i}[dv]$, for each $i \geq 2$. $\square$

Analogously to the homotopy cocartesian square leading to (7.2), the smash product defining $\mathrm{THH}(A, M)$ gives rise to a homotopy cocartesian square

$$\begin{tikzcd} H \wedge \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)] \arrow[r] \arrow[d] & H \wedge \mathbb{S}^{\mathcal{J}}[B^{\mathrm{rep}}(M)] \arrow[d] \\ H \wedge_A \mathrm{THH}(A) \arrow[r] & H \wedge_A \mathrm{THH}(A, M) \end{tikzcd}$$

of commutative H-algebras, and an associated Tor spectral sequence

$$\begin{aligned} E^2_{**} &= \mathrm{Tor}_{**}^{H_*\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)]}(H^A_* \mathrm{THH}(A), H_*\mathbb{S}^{\mathcal{J}}[B^{\mathrm{rep}}(M)]) \\ &\Longrightarrow H^A_* \mathrm{THH}(A, M)\,. \end{aligned} \tag{7.3}$$

Theorem 7.3. *Consider the case $A = \ell$ and $M = D(v)$ of the spectral sequence* (7.3) *above. It is an algebra spectral sequence*

$$\begin{aligned} E^2_{**} &= \mathrm{Tor}_{**}^{H_\circledast(B^{\mathrm{cy}}(D(v))_{h\mathcal{J}})}(V(1)_* \mathrm{THH}(\ell), H_\circledast(B^{\mathrm{rep}}(D(v))_{h\mathcal{J}})) \\ &\Longrightarrow V(1)_* \mathrm{THH}(\ell, D(v))\,, \end{aligned}$$

where

$$\begin{aligned} H_\circledast(B^{\mathrm{cy}}(D(v))_{h\mathcal{J}}) &= P(v) \otimes E(dv) \otimes C_* \\ H_\circledast(B^{\mathrm{rep}}(D(v))_{h\mathcal{J}}) &= P(v) \otimes E(d\log v) \otimes C_* \\ V(1)_* \mathrm{THH}(\ell) &= E(\lambda_1, \lambda_2) \otimes P(\mu_2)\,, \end{aligned}$$

with $|d\log v| = 1$ and the remaining degrees as above. Here

$$E^2_{**} = E(\lambda_1, \lambda_2) \otimes P(\mu_2) \otimes E(d\log v) \otimes \Gamma([dv])$$

where $[dv]$ has bidegree $(1, 2p-1)$. There are nontrivial differentials

$$d^p(\gamma_k[dv]) \doteq \lambda_2 \cdot \gamma_{k-p}[dv]$$

for all $k \geq p$, leaving

$$E^\infty_{**} = E(\lambda_1) \otimes P(\mu_2) \otimes E(d\log v) \otimes P_p([dv])\,.$$

There is a multiplicative extension $[dv]^p \doteq \mu_2$, so the abutment is

$$V(1)_* \mathrm{THH}(\ell, D(v)) = E(\lambda_1, d\log v) \otimes P(\kappa_1)\,,$$

where κ_1 is represented by $[dv]$ in degree $2p$.

Together with the stable equivalence $\mathrm{THH}(\ell, D(v)) \to \mathrm{THH}(\ell, j_*\mathrm{GL}_1^{\mathcal{J}}(L))$ from Proposition 4.3, and Proposition 8.9 below, the previous theorem implies Theorem 1.3 from the introduction. Note that $\kappa_1 \in V(1)_{2p} \mathrm{THH}(\ell, D(v))$ is only defined modulo $\lambda_1 \cdot d\log v$.

Proof of Theorem 7.3. Recall the chain (7.1). We apply the same cobase changes as before, and get a chain of three Tor spectral sequences with E^2-terms

$$\begin{aligned}&\mathrm{Tor}_{**}^{P(v)\otimes E(dv)\otimes C_*}(E(\lambda_1,\lambda_2)\otimes P(\mu_2), P(v)\otimes E(d\log v)\otimes C_*)\\ &\qquad\longrightarrow \mathrm{Tor}_{**}^{P(v)\otimes E(dv)\otimes C_*}(E(\lambda_1,\lambda_2)\otimes P(\mu_2), E(d\log v)\otimes C_*)\\ &\qquad\qquad\longleftarrow \mathrm{Tor}_{**}^{P(v)\otimes E(dv)\otimes C_*}(E(\lambda_1,\lambda_2)\otimes P(\mu_2), C_*)\,,\end{aligned}$$

which by change-of-rings is isomorphic to the chain

$$\begin{aligned}&\mathrm{Tor}_{**}^{E(dv)}(E(\lambda_1,\lambda_2)\otimes P(\mu_2), E(d\log v))\\ &\qquad\longrightarrow \mathrm{Tor}_{**}^{P(v)\otimes E(dv)}(E(\lambda_1,\lambda_2)\otimes P(\mu_2), E(d\log v))\\ &\qquad\qquad\longleftarrow \mathrm{Tor}_{**}^{P(v)\otimes E(dv)}(E(\lambda_1,\lambda_2)\otimes P(\mu_2), \mathbb{F}_p)\,,\end{aligned}$$

hence takes the form

$$\begin{aligned}&E(\lambda_1,\lambda_2)\otimes P(\mu_2)\otimes E(d\log v)\otimes \Gamma([dv])\\ &\qquad\longrightarrow E(\lambda_1,\lambda_2)\otimes P(\mu_2)\otimes E(d\log v)\otimes E([v])\otimes\Gamma([dv])\\ &\qquad\qquad\longleftarrow E(\lambda_1,\lambda_2)\otimes P(\mu_2)\otimes E([v])\otimes\Gamma([dv])\,,\end{aligned}$$

and converges to the chain

$$\begin{aligned}&V(1)_*\,\mathrm{THH}(\ell, D(v))\\ &\qquad\longrightarrow V(1)_*(\mathrm{THH}(\ell)\wedge_{\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(D(v))]}\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}_{\{0\}}(D(v)^{\mathrm{gp}})])\\ &\qquad\qquad\longleftarrow V(1)_*\,\mathrm{THH}(\mathbb{Z}_{(p)})\,.\end{aligned}$$

The known differentials $d^p(\gamma_k[dv]) \doteq \lambda_2\cdot\gamma_{k-p}[dv]$ in the right hand spectral sequence (which is that of Proposition 7.2) remain nonzero in the middle spectral sequence, since the right hand arrow of E^2-terms is injective and $d\log v$ is an infinite cycle for filtration reasons. Likewise the multiplicative extension $[dv]^p \doteq \mu_2$ carries over to the middle. Furthermore, since the left hand arrow of E^2-terms is also injective, it follows that the differentials and multiplicative extensions lift to the left hand spectral sequence.

This implies that the asserted d^p-differentials are the first nonzero differentials in the left hand spectral sequence, which is the spectral sequence in the statement of the theorem. This leaves the E^{p+1}-term

$$E^{p+1}_{**} = E(\lambda_1)\otimes P(\mu_2)\otimes E(d\log v)\otimes P_p([dv])\,,$$

which must be equal to the E^∞-term for filtration reasons. Letting κ_1 in degree $2p$ be a class in the abutment represented by $[dv]$ in bidegree $(1, 2p-1)$, we find that $\kappa_1^p \doteq \mu_2$, leading to the asserted algebraic structure of the abutment. □

By Theorem 4.4 we have a homotopy cofiber sequence

$$\mathrm{THH}(\ell)\xrightarrow{\rho}\mathrm{THH}(\ell, D(v))\xrightarrow{\partial}\Sigma\,\mathrm{THH}(\mathbb{Z}_{(p)})$$

of $\mathrm{THH}(\ell)$-modules, where ρ is a map of commutative symmetric ring spectra. Let $\tau\colon \mathrm{THH}(\mathbb{Z}_{(p)})\to\mathrm{THH}(\ell)$ denote the homotopy fiber map of ρ, which we like to think of as a kind of transfer map.

Lemma 7.4. *There is a long exact sequence*

$$\begin{aligned}&\dots\longrightarrow V(1)_*\,\mathrm{THH}(\mathbb{Z}_{(p)})\xrightarrow{\tau_*} V(1)_*\,\mathrm{THH}(\ell)\\ &\qquad\xrightarrow{\rho_*} V(1)_*\,\mathrm{THH}(\ell, D(v))\xrightarrow{\partial_*} V(1)_{*-1}\,\mathrm{THH}(\mathbb{Z}_{(p)})\longrightarrow\dots\end{aligned}$$

of $V(1)_*\,\mathrm{THH}(\ell)$-modules, where ρ_* is an algebra homomorphism, and

(1) $\tau_*(\epsilon_1\mu_1^{p-1}) \doteq \lambda_2$,

(2) $\rho_*(\lambda_1) = \lambda_1$,
(3) $\rho_*(\mu_2) \doteq \kappa_1^p$,
(4) $\partial_*(d\log v \cdot \kappa_1^k) \doteq \mu_1^k$ *for* $k \geq 0$ *and*
(5) $\partial_*(\kappa_1^k) \doteq \epsilon_1 \mu_1^{k-1} \pmod{\lambda_1 \mu_1^{k-1}}$ *for* $p \nmid k \geq 1$.

Proof. The repletion homomorphism

$$\rho_* \colon E(\lambda_1, \lambda_2) \otimes P(\mu_2) \longrightarrow E(\lambda_1, d\log v) \otimes P(\kappa_1)$$

is induced by the canonical H-algebra map

$$H \wedge_\ell \mathrm{THH}(\ell) \longrightarrow H \wedge_\ell \mathrm{THH}(\ell, D(v)),$$

and therefore factors through the edge homomorphism of the spectral sequence in Theorem 7.3. Hence it is an algebra homomorphism satisfying $\rho_*(\lambda_1) = \lambda_1$, $\rho_*(\lambda_2) = 0$ and $\rho_*(\mu_2) \doteq \kappa_1^p$.

It follows that

$$\ker(\rho_*) = \mathrm{im}(\tau_*) = E(\lambda_1) \otimes P(\mu_2)\{\lambda_2\}$$

is the free $E(\lambda_1) \otimes P(\mu_2)$-submodule of $V(1)_* \mathrm{THH}(\ell)$ generated by λ_2. Only (a unit in $\mathbb{F}_p$ times) $\epsilon_1 \lambda_1 \mu_1^{p-1}$ can map under τ_* to $\lambda_1 \lambda_2$, so $\tau_*(\epsilon_1 \mu_1^{p-1}) \doteq \lambda_2$. Likewise

$$\mathrm{im}(\rho_*) = \ker(\partial_*) = E(\lambda_1) \otimes P(\kappa_1^p)$$

is the free $E(\lambda_1) \otimes P(\mu_2)$-submodule of $V(1)_* \mathrm{THH}(\ell, D(v))$ generated by 1. The classes $d\log v \cdot \kappa_1^k$ can only map non-trivially under ∂_* to (units in $\mathbb{F}_p$ times) μ_1^k, so $\partial_*(d\log v \cdot \kappa_1^k) \doteq \mu_1^k$ for $k \geq 0$. The classes κ_1^k with $p \nmid k \geq 1$ must map to classes in the span of $\epsilon_1 \mu_1^{k-1}$ and $\lambda_1 \mu_1^{k-1}$ that are linearly independent of

$$\partial_*(\lambda_1 \cdot d\log v \cdot \kappa_1^{k-1}) \doteq \lambda_1 \mu_1^{k-1},$$

hence must map to (a unit in $\mathbb{F}_p$ times) $\epsilon_1 \mu_1^{k-1}$ modulo a multiple of $\lambda_1 \mu_1^{k-1}$. □

Lemma 7.5. *The adjoint pre-log structure map*

$$\bar\alpha \colon \mathbb{S}^{\mathcal{J}}[B^{\mathrm{rep}} D(v)] \longrightarrow \mathrm{THH}(\ell, D(v))$$

induces an algebra homomorphism

$$\bar\alpha_* \colon H_{\circledast}(B^{\mathrm{rep}}(D(v))_{h\mathcal{J}}) \longrightarrow V(1)_* \mathrm{THH}(\ell, D(v))$$

satisfying $\bar\alpha_*(v) = 0$ *and* $\bar\alpha_*(d\log v) = d\log v$.

Proof. The adjoint pre-log structure homomorphism

$$\bar\alpha_* \colon P(v) \otimes E(d\log v) \otimes C_* \longrightarrow E(\lambda_1, d\log v) \otimes P(\kappa_1)$$

is induced by the canonical H-algebra map

$$H \wedge \mathbb{S}^{\mathcal{J}}[B^{\mathrm{rep}} D(v)] \longrightarrow H \wedge_\ell \mathrm{THH}(\ell, D(v)),$$

and therefore factors through the edge homomorphism of the spectral sequence in Theorem 7.3. Hence it is an algebra homomorphism satisfying $\bar\alpha_*(v) = 0$ and $\bar\alpha_*(d\log v) = d\log v$. □

8. Logarithmic THH of the connective complex K-theory spectrum

We now return to the setup of Section 6 and consider the tamely ramified map $f \colon \ell \to ku_{(p)}$, inducing $f_* \colon \mathbb{Z}_{(p)}[v] \to \mathbb{Z}_{(p)}[u]$ with $f_*(v) = u^{p-1}$ in homotopy, and its log THH-étale extension

$$(f, f^\flat) \colon (\ell, D(v), \alpha) \to (ku_{(p)}, D(u), \beta)$$

where $\alpha\colon D(v) \to \Omega^{\mathcal{J}}(\ell)$ and $\beta\colon D(u) \to \Omega^{\mathcal{J}}(ku_{(p)})$ denote the Adams and Bott pre-log structures on ℓ and $ku_{(p)}$, respectively, and $f^\flat(v) = u^{p-1}$. We note that the induced map of residue ring spectra

$$f/(f^\flat_{>0})\colon \ell/(D(v)_{>0}) \xrightarrow{\sim} ku_{(p)}/(D(u)_{>0})$$

is a stable equivalence, with both sides equivalent to $H\mathbb{Z}_{(p)}$. Therefore Theorem 4.4 provides a diagram of homotopy cofiber sequences

$$\begin{array}{ccccc} \mathrm{THH}(\ell) & \xrightarrow{\rho} & \mathrm{THH}(\ell, D(v)) & \xrightarrow{\partial} & \Sigma\,\mathrm{THH}(\mathbb{Z}_{(p)}) \\ {\scriptstyle f}\downarrow & & {\scriptstyle (f,f^\flat)}\downarrow & & \| \\ \mathrm{THH}(ku_{(p)}) & \xrightarrow{\rho'} & \mathrm{THH}(ku_{(p)}, D(u)) & \xrightarrow{\partial'} & \Sigma\,\mathrm{THH}(\mathbb{Z}_{(p)}) \end{array} \tag{8.1}$$

where the left hand square is strictly commutative and the right hand square is homotopy commutative. We have proved in Theorem 6.3 that the induced map

$$ku_{(p)} \wedge_\ell \mathrm{THH}(\ell, D(v)) \xrightarrow{\sim} \mathrm{THH}(ku_{(p)}, D(u)) \tag{8.2}$$

is a stable equivalence, where here the smash product over ℓ should be understood as a left derived smash product.

The stable equivalence $\mathrm{THH}(ku_{(p)}, D(u)) \to \mathrm{THH}(ku_{(p)}, j_*\mathrm{GL}_1^{\mathcal{J}}(KU_{(p)}))$ from Proposition 4.3, Proposition 8.9 below, and the following theorem imply Theorem 1.7 from the introduction.

Theorem 8.1. *There is an algebra isomorphism*

$$V(1)_*\,\mathrm{THH}(ku_{(p)}, D(u)) \cong P_{p-1}(u) \otimes E(\lambda_1, d\log u) \otimes P(\kappa_1)$$

with $|u| = 2$, $|\lambda_1| = 2p-1$, $|d\log u| = 1$ *and* $|\kappa_1| = 2p$. *The cobase change equivalence* (8.2) *induces the isomorphism*

$$P_{p-1}(u) \otimes E(\lambda_1, d\log v) \otimes P(\kappa_1) \xrightarrow{\cong} P_{p-1}(u) \otimes E(\lambda_1, d\log u) \otimes P(\kappa_1)$$

that maps $d\log v$ *to* $-d\log u$ *and preserves the other terms. The adjoint pre-log structure map*

$$\bar\beta\colon \mathbb{S}^{\mathcal{J}}[B^{\mathrm{rep}}(D(u))] \longrightarrow \mathrm{THH}(ku_{(p)}, D(u))$$

induces an algebra homomorphism

$$\bar\beta_*\colon H_\circledast(B^{\mathrm{rep}}(D(u))_{h\mathcal{J}}) \longrightarrow V(1)_*\,\mathrm{THH}(ku_{(p)}, D(u))$$

satisfying $\bar\beta_*(u) = u$ *and* $\bar\beta_*(d\log u) = d\log u$. *The suspension operator satisfies* $\sigma(u) = u \cdot d\log u$ *and* $\sigma(d\log u) = 0$.

Proof. The Tor spectral sequence

$$\begin{aligned} E^2_{**} &= \mathrm{Tor}^{\pi_*\ell}_{**}(\pi_* ku_{(p)}, V(1)_*\,\mathrm{THH}(\ell, D(v))) \\ &\Longrightarrow V(1)_*\,\mathrm{THH}(ku_{(p)}, D(u)) \end{aligned}$$

collapses at the E^2-term in filtration 0, since $\pi_* ku_{(p)}$ is a free $\pi_*\ell$-module and $V(1)_*\,\mathrm{THH}(\ell, D(v))$ is a trivial $\pi_*\ell$-module. The term $P_{p-1}(u)$ arises as

$$\pi_* ku_{(p)} \otimes_{\pi_*\ell} \mathbb{F}_p \cong P(u) \otimes_{P(v)} \mathbb{F}_p\,.$$

Chasing the class u in $\mathcal{J}$-degree 2 around the commutative square

$$\begin{array}{ccc} H_\circledast(D(u)_{h\mathcal{J}}) & \longrightarrow & V(1)_* ku_{(p)} \\ \downarrow & & \downarrow \\ H_\circledast(B^{\mathrm{rep}}(D(u)_{h\mathcal{J}})) & \xrightarrow{\bar\beta_*} & V(1)_*\,\mathrm{THH}(ku_{(p)}, D(u)) \end{array}$$

we see that $\bar\beta_*(u) = u$.

Chasing the class $d\log v$ in $\mathcal{J}$-degree 0 around the commutative square

$$\begin{tikzcd} H_{\circledast}(B^{\mathrm{rep}}(D(v))_{h\mathcal{J}}) \arrow[r, "\bar\alpha_*"] \arrow[d, "f^\flat_\circledast"'] & V(1)_* \operatorname{THH}(\ell, D(v)) \arrow[d, "f_*"] \\ H_{\circledast}(B^{\mathrm{rep}}(D(u))_{h\mathcal{J}}) \arrow[r, "\bar\beta_*"] & V(1)_* \operatorname{THH}(ku_{(p)}, D(u)) \end{tikzcd}$$

we find that $f^\flat_\circledast(d\log v) = (p-1)d\log u = -d\log u$ maps under $\bar\beta_*$ to the image of $\bar\alpha_*(d\log v) = d\log v$ under f_*. Hence we can trade $d\log u$ for $d\log v$ as a generator in $V(1)_* \operatorname{THH}(ku_{(p)}, D(u))$, giving the asserted formulas.

The $(2p-3)$-connected map $V(1) \to H$ induces a commutative diagram

$$\begin{tikzcd} H_{\circledast}(B^{\mathrm{cy}}(D(u))_{h\mathcal{J}}) \arrow[d, "\rho_\circledast"'] & V(1)_*\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(D(u))] \arrow[l] \arrow[r, "\bar\beta_*"] \arrow[d, "\rho_*"'] & V(1)_* \operatorname{THH}(ku_{(p)}) \arrow[d, "\rho'_*"] \\ H_{\circledast}(B^{\mathrm{rep}}(D(u))_{h\mathcal{J}}) & V(1)_*\mathbb{S}^{\mathcal{J}}[B^{\mathrm{rep}}(D(u))] \arrow[l] \arrow[r, "\bar\beta_*"] & V(1)_* \operatorname{THH}(ku_{(p)}, D(u)) \,, \end{tikzcd}$$

where the left hand horizontal arrows are $(2p-3)$-connected. By Proposition 5.5 we have $\rho_\circledast(du) = u \cdot d\log u = \sigma(u)$ at the left hand side. By the connectivity estimate we have $\rho_*(du) = u \cdot d\log u = \sigma(u)$ in the middle (modulo $\alpha_1 \in \pi_3 V(1)$ for $p = 3$), which implies that $\rho'_*(du) = u \cdot d\log u = \sigma(u)$ at the right hand side. By Proposition 5.5 we also have $\sigma(d\log u) = 0$ at the left hand side, so by the same connectivity estimate we have $\sigma(d\log u) = 0$ in the middle and at the right hand side. □

In the next lemma we consider the square obtained by applying the left derived cobase change along $\ell \to ku_{(p)}$ to the right hand square in (8.1). We write

$$\chi\colon ku_{(p)} \wedge_\ell \Sigma \operatorname{THH}(\mathbb{Z}_{(p)}) \to \Sigma \operatorname{THH}(\mathbb{Z}_{(p)})$$

for the induced map of $ku_{(p)}$-modules.

Lemma 8.2. *The homotopy commutative square*

$$\begin{tikzcd} ku_{(p)} \wedge_\ell \operatorname{THH}(\ell, D(v)) \arrow[r, "1\wedge\partial"] \arrow[d, "\sim"'] & ku_{(p)} \wedge_\ell \Sigma \operatorname{THH}(\mathbb{Z}_{(p)}) \arrow[d, "\chi"] \\ \operatorname{THH}(ku_{(p)}, D(u)) \arrow[r, "\partial'"] & \Sigma \operatorname{THH}(\mathbb{Z}_{(p)}) \end{tikzcd}$$

induces a commutative square

$$\begin{tikzcd} P_{p-1}(u) \otimes V(1)_* \operatorname{THH}(\ell, D(v)) \arrow[r, "1\otimes\partial_*"] \arrow[d, "\cong"', "\cdot"] & P_{p-1}(u) \otimes V(1)_{*-1} \operatorname{THH}(\mathbb{Z}_{(p)}) \arrow[d, "\chi_*"] \\ V(1)_* \operatorname{THH}(ku_{(p)}, D(u)) \arrow[r, "\partial'_*"] & V(1)_{*-1} \operatorname{THH}(\mathbb{Z}_{(p)}) \end{tikzcd}$$

of $P_{p-1}(u) \otimes V(1)_* \operatorname{THH}(\ell)$-modules, where $\chi_*(1 \otimes x) = x$ and $\chi_*(u^k \otimes x) = 0$ for all $x \in V(1)_{*-1} \operatorname{THH}(\mathbb{Z}_{(p)})$ and $k \geq 1$. Hence $\partial'_*(1 \cdot y) = \partial_*(y)$ and $\partial'_*(u^k \cdot y) = 0$ for all $y \in V(1)_* \operatorname{THH}(\ell, D(v))$ and $k \geq 1$.

Proof. It follows from Proposition 6.4 and [RSS15, Proposition 6.11] that the map χ may be obtained from the analogous map

$$\mathbb{S}^{\mathcal{J}}[D(u)] \wedge_{\mathbb{S}^{\mathcal{J}}[D(v)]} \Sigma\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}_{\{0\}}(D(v))] \longrightarrow \Sigma\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}_{\{0\}}(D(u))] \,,$$

by cobase change along

$$\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(D(v))] \to \operatorname{THH}(\ell) \quad \text{and} \quad \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(D(u))] \to \operatorname{THH}(ku_{(p)})$$

(compare to the proof of Theorem 6.3). Clearly the $\mathbb{S}^{\mathcal{J}}[D(u)]$-module structure on $\Sigma\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}_{\{0\}}(D(u))]$ factors through the projection $\mathbb{S}^{\mathcal{J}}[D(u)] \to \mathbb{S}^{\mathcal{J}}[D(u)_{\{0\}}]$ to the

$\mathcal{J}$-degree 0 part which implies that $\chi_*(u^k \otimes x) = 0$ for all $k \geq 1$. The remaining claims follow by naturality. $\square$

Let $\tau' \colon \mathrm{THH}(\mathbb{Z}_{(p)}) \to \mathrm{THH}(ku_{(p)})$ denote the homotopy fiber map of the map ρ' in (8.1). As usual, we write $(u) = \mathbb{F}_p\{u^k \mid 1 \leq k \leq p-2\}$ for the ideal in $P_{p-1}(u)$ generated by u.

Lemma 8.3. *There is a long exact sequence*

$$\begin{aligned} \dots \longrightarrow V(1)_* \mathrm{THH}(\mathbb{Z}_{(p)}) &\xrightarrow{\tau'_*} V(1)_* \mathrm{THH}(ku_{(p)}) \\ &\xrightarrow{\rho'_*} V(1)_* \mathrm{THH}(ku_{(p)}, D(u)) \xrightarrow{\partial'_*} V(1)_{*-1} \mathrm{THH}(\mathbb{Z}_{(p)}) \longrightarrow \dots \end{aligned}$$

of $V(1)_* \mathrm{THH}(ku_{(p)})$*-modules, where* ρ'_* *is an algebra homomorphism. The resulting short exact sequence*

$$0 \to \ker(\rho'_*) \longrightarrow V(1)_* \mathrm{THH}(ku_{(p)}) \longrightarrow \mathrm{im}(\rho'_*) \to 0$$

is a square-zero extension of $P_{p-1}(u) \otimes V(1)_* \mathrm{THH}(\ell)$*-algebras, where*

$$\ker(\rho'_*) \cong E(\lambda_1) \otimes P(\mu_2)\{\lambda_2\}$$

and

$$\mathrm{im}(\rho'_*) = E(\lambda_1) \otimes P(\kappa_1^p) \ \oplus \ (u) \otimes E(\lambda_1, d\log u) \otimes P(\kappa_1)\,.$$

Proof. By exactness $\mathrm{im}(\rho'_*) = \ker(\partial'_*)$, which by Lemma 7.4 and Lemma 8.2 is the direct sum of $\ker(\partial_*) = E(\lambda_1) \otimes P(\kappa_1^p)$ and $(u) \otimes E(\lambda_1, d\log u) \otimes P(\kappa_1)$, inside

$$V(1)_* \mathrm{THH}(ku_{(p)}, D(u)) = P_{p-1}(u) \otimes E(\lambda_1, d\log u) \otimes P(\kappa_1).$$

Similarly $\ker(\rho'_*) \cong \mathrm{cok}(\partial'_*) = \mathrm{cok}(\partial_*) \cong \ker(\rho_*)$ equals $E(\lambda_1) \otimes P(\mu_2)\{\lambda_2\}$. This is a square-zero ideal inside $V(1)_* \mathrm{THH}(\ell) = E(\lambda_1, \lambda_2) \otimes P(\mu_2)$, hence is also a square-zero ideal inside $V(1)_* \mathrm{THH}(ku_{(p)})$. $\square$

The following is essentially copied from [Aus05, Definition 9.13].

Definition 8.4. Assume $p \geq 3$, and let Θ_* be the graded-commutative unital $P_{p-1}(u) \otimes P(\mu_2)$-algebra with generators

$$\begin{cases} a_i & 0 \leq i \leq p-1\,, \\ b_j & 1 \leq j \leq p-1\,, \end{cases}$$

and relations

$$\begin{cases} u^{p-2} a_i = 0 & 0 \leq i \leq p-2\,, \\ u^{p-2} b_j = 0 & 1 \leq j \leq p-1\,, \\ b_i b_j = u b_{i+j} & i+j \leq p-1\,, \\ a_i b_j = u a_{i+j} & i+j \leq p-1\,, \\ b_i b_j = u b_{i+j-p} \mu_2 & i+j \geq p\,, \\ a_i b_j = u a_{i+j-p} \mu_2 & i+j \geq p\,, \\ a_i a_j = 0 & 0 \leq i,j \leq p-1\,. \end{cases}$$

By convention $b_0 = u$. The degrees of the generators are $|a_i| = 2pi + 3$ and $|b_j| = 2pj + 2$.

Theorem 8.5. [Aus05, Theorem 9.15] *Let* $p \geq 3$*. There is an isomorphism*

$$V(1)_* \mathrm{THH}(ku_{(p)}) \cong E(\lambda_1) \otimes \Theta_*$$

of $P_{p-1}(u) \otimes E(\lambda_1) \otimes P(\mu_2)$*-algebras. Under this identification,* $\rho'_*(a_i) = u \cdot d\log u \cdot \kappa_1^i$ *for* $0 \leq i \leq p-1$*,* $\rho'_*(b_j) = u\kappa_1^j$ *for* $1 \leq j \leq p-1$ *and* $\rho'_*(\mu_2) = \kappa_1^p$*, all in* $\mathrm{im}(\rho'_*)$*, and* λ_2 *in* $\ker(\rho'_*)$ *maps to a unit in* $\mathbb{F}_p$ *times* $u^{p-2} a_{p-1}$*.*

Definition 8.6. The assignments

$$\begin{aligned} \lambda_1 &\longmapsto \lambda_1 \\ a_i &\longmapsto u \cdot d\log u \cdot \kappa_1^i \\ b_j &\longmapsto u\kappa_1^j \\ \mu_2 &\longmapsto \kappa_1^p \end{aligned}$$

define a surjective algebra homomorphism $\bar{\theta}\colon E(\lambda_1) \otimes \Theta_* \to \operatorname{im}(\rho'_*)$, with kernel $E(\lambda_1) \otimes P(\mu_2)\{u^{p-2}a_{p-1}\}$. For $p \geq 5$ it lifts uniquely to an algebra homomorphism

$$\theta\colon E(\lambda_1) \otimes \Theta_* \longrightarrow V(1)_* \operatorname{THH}(ku_{(p)}),$$

because $\ker(\rho'_*) = 0$ in the degrees of the algebra generators and relations of Θ_*. (This is not true for $p = 3$.) For brevity, let $z = u^{p-2}\kappa_1^{p-1}$.

In the remainder of this section we will give a new proof of Ausoni's theorem, in the cases $p \geq 5$. The only obstruction to carrying out the same proof for $p = 3$ is the need to check that $\bar{\theta}$ admits a multiplicative lift θ, i.e., that the relations in degree $|\lambda_2| = 2p^2 - 1 = 17$ and $|\lambda_1\lambda_2| = 2p^2 + 2p - 2 = 22$ in Θ_* also hold in $V(1)_* \operatorname{THH}(ku_{(p)})$. We do not carry out this check. A similar complication occurs in Ausoni's original calculation for $p = 3$, cf. [Aus05, p. 1305].

Lemma 8.7. *For $p \geq 5$, the homomorphism θ is an isomorphism in degrees $* < |\lambda_2| = 2p^2 - 1$, and maps $\ker(\bar{\theta}) = E(\lambda_1) \otimes P(\mu_2)\{u^{p-2}a_{p-1}\}$ to $\ker(\rho'_*) \cong E(\lambda_1) \otimes P(\mu_2)\{\lambda_2\}$. In degree $2p^2 - 1$ it maps $u^{p-2}a_{p-1}$ to the product $du \cdot z$, which is a multiple of λ_2. It is an isomorphism in all degrees if and only if this multiple is nonzero.*

Proof. The lift θ is an isomorphism if and only if it maps $\ker(\bar{\theta})$ isomorphically to $\ker(\rho'_*)$, which happens if and only if θ maps $u^{p-2}a_{p-1} = a_0 \cdot u^{p-3}b_{p-1}$ to a nonzero multiple of λ_2. Since a_0 maps to $u \cdot d\log u = \rho'_*(du)$ and $u^{p-3}b_{p-1}$ maps to $z = u^{p-2}\kappa_1^{p-1}$, this happens if and only if $du \cdot z \doteq \lambda_2$ in $V(1)_* \operatorname{THH}(ku_{(p)})$. The only alternative is that $du \cdot z = 0$. □

Proposition 8.8. *Consider the case $A = ku_{(p)}$ and $M = D(u)$ of the spectral sequence* (7.3)*. It is an algebra spectral sequence*

$$\begin{aligned} E^2_{**} &= \operatorname{Tor}_{**}^{H_\circledast(B^{\mathrm{cy}}(D(u))_{h\mathcal{J}})}(V(1)_* \operatorname{THH}(ku_{(p)}), H_\circledast(B^{\mathrm{rep}}(D(u))_{h\mathcal{J}})) \\ &\Longrightarrow V(1)_* \operatorname{THH}(ku_{(p)}, D(u)), \end{aligned}$$

where

$$\begin{aligned} H_\circledast(B^{\mathrm{cy}}(D(u))_{h\mathcal{J}}) &= P(u) \otimes E(du) \otimes C_* \\ H_\circledast(B^{\mathrm{rep}}(D(u))_{h\mathcal{J}}) &= P(u) \otimes E(d\log u) \otimes C_* \\ V(1)_* \operatorname{THH}(ku_{(p)}) &\cong E(\lambda_1) \otimes \Theta_* \\ V(1)_* \operatorname{THH}(ku_{(p)}, D(u)) &= P_{p-1}(u) \otimes E(\lambda_1, d\log u) \otimes P(\kappa_1). \end{aligned}$$

Here

$$E^2_{**} = \operatorname{Tor}_{**}^{E(du)}(E(\lambda_1) \otimes \Theta_*, E(d\log u))$$

is the tensor product of $E(\lambda_1, d\log u) \otimes P(\mu_2)$ with

$$\begin{aligned} \mathbb{F}_p\{1, u^{p-3}b_{p-1}, u^i b_j \mid 0 \leq i \leq p-4, 0 \leq j \leq p-1\} \\ \oplus\ \Gamma([du])\{u^{p-3}b_{j-1}, a_j \mid 1 \leq j \leq p-1\} \end{aligned}$$

where $b_0 = u$. There are nontrivial differentials

$$d^2(\gamma_k[du] \cdot u^{p-3}b_{j-1}) \doteq \gamma_{k-2}[du] \cdot a_j$$

modulo $(\lambda_1, d\log u)$, for all $k \geq 2$ and $1 \leq j \leq p-1$, leaving $E^3_{**} = E^\infty_{**}$ equal to the tensor product of $E(\lambda_1, d\log u) \otimes P(\mu_2)$ with

$$\mathbb{F}_p\{1, u^{p-3}b_{p-1}, u^i b_j \mid 0 \leq i \leq p-4, 0 \leq j \leq p-1\} \\ \oplus\ E([du])\{u^{p-3}b_{j-1} \mid 1 \leq j \leq p-1\}\,.$$

Here u represents u, $[du]u^{p-3}b_{j-1}$ represents κ_1^j up to a unit in $\mathbb{F}_p$, for $1 \leq j \leq p-1$, and there are multiplicative extensions $u \cdot [du]u^{p-3}b_{j-1} \doteq b_j$ and $[du]u^{p-3}b_{j-1} \cdot [du]u^{p-3}b_{k-1} \doteq \mu_2$ for $1 \leq j, k \leq p-1$ with $j + k = p$.

Proof. We first rewrite the E^2-term to clarify its dependence on $V(1)_* \operatorname{THH}(ku_{(p)})$, using change-of-rings:

$$\begin{aligned} E^2_{**} &= \operatorname{Tor}^{P(u)\otimes E(du)\otimes C_*}_{**}(V(1)_* \operatorname{THH}(ku_{(p)}), P(u) \otimes E(d\log u) \otimes C_*) \\ &\cong \operatorname{Tor}^{E(du)}_{**}(V(1)_* \operatorname{THH}(ku_{(p)}), E(d\log u))\,. \end{aligned}$$

Here $E(du)$ acts trivially on $E(d\log u)$, so only the $E(du)$-module structure on $V(1)_* \operatorname{THH}(ku_{(p)})$ is relevant. We know that θ is an isomorphism in degrees $* < |\lambda_2| = 2p^2 - 1 < |\mu_2| = 2p^2$, so in this range of degrees $V(1)_* \operatorname{THH}(ku_{(p)})$ is additively isomorphic to the tensor product of $E(\lambda_1)$ and the $E(du)$-module

$$E(du)\{1, u^{p-3}b_{p-1}, u^i b_j \mid 0 \leq i \leq p-4, 0 \leq j \leq p-1\} \\ \oplus\ \mathbb{F}_p\{u^{p-3}b_{j-1}, a_j \mid 1 \leq j \leq p-1\}\,,$$

with $u^{p-3}b_{p-1}$ corresponding to z in degree $2p^2 - 4$.

It follows that the E^2-term is as stated in the proposition in bidegrees (s, t) with $t < 2p^2 - 1$. In particular, it is isomorphic to a free module over $E(\lambda_1, d\log u)$ in this range of degrees. Since the abutment is a free module over $E(\lambda_1, d\log u)$ on the monomial generators of $P_{p-1}(u) \otimes P(\kappa_1)$, which are concentrated in even degrees, it follows that the E^2-classes $\gamma_i[du] \cdot a_j$ in (odd) total degrees less than $2p^2 - 1$ cannot survive to the E^∞-term. By induction over increasing total degrees $s + t$, and over decreasing filtration degrees s within each total degree, it follows that there must be nonzero d^2-differentials as stated in the proposition, cancelling the $E(\lambda_1, d\log u)$-module generators $\gamma_k[du] \cdot u^{p-3}b_{j-1}$ and $\gamma_{k-2}[du] \cdot a_j$ for $k \geq 2$ and $1 \leq j \leq p-1$, in total degrees $s + t < 2p^2 - 1$.

If $du \cdot z \doteq \lambda_2$, so that θ is an isomorphism, the same inductive argument continues to cover all total degrees, extending linearly over $P(\mu_2)$. The E^∞-term is then concentrated in filtration degrees $0 \leq s \leq 1$, and the final claims of the proposition follow directly from a comparison with the known abutment.

It remains to exclude the alternative, namely that $du \cdot z = 0$. In that hypothetical case, $V(1)_* \operatorname{THH}(ku_{(p)})$ would be isomorphic in degrees $* \leq 2p^2 - 1$ to the $E(du)$-module

$$E(du)\{1, u^i b_j \mid 0 \leq i \leq p-4, 0 \leq j \leq p-1\} \\ \oplus\ \mathbb{F}_p\{z, \lambda_2, u^{p-3}b_{j-1}, a_j \mid 1 \leq j \leq p-1\}\,.$$

This would lead to a modified E^2-term, where the summand $\mathbb{F}_p\{u^{p-3}b_{p-1}\}$ is replaced by $\Gamma([du])\{z, \lambda_2\}$, at least in internal degrees $t \leq 2p^2 - 1$.

By our initial analysis for $t < 2p^2 - 1$, all E^2-generators in total degree $s + t = 2p^2 - 2$ support linearly independent d^2-differentials. Hence all generators in total degree $2p^2 - 1$ must be d^2-cycles.

Under this assumption, the E^2-generators in total degree $s + t = 2p^2 - 1$ would be λ_2 in filtration $s = 0$, the $m = (p-1)/2$ classes

$$\{\gamma_{pi-1}[du] \cdot a_{p-2i} \mid 1 \leq i \leq m\}\,,$$

and some λ_1- or $d\log u$-multiples of classes in lower total degrees.

The E^2-generators in total degree $s+t=2p^2$ and filtration degree $s \geq 2$ would be the m classes
$$\{\gamma_{pi+1}[du] \cdot u^{p-3} b_{p-2i-1} \mid 1 \leq i \leq m\}\,,$$
and some λ_1- or $d\log u$-multiples of classes in lower total degrees.

By our previous analysis, the λ_1- or $d\log u$-multiples in total degree $2p^2$ and filtration degree ≥ 2 support d^2-differentials that kill all but one of the λ_1- or $d\log u$-multiples in total degree $2p^2-1$, leaving only $\lambda_1[du]u^{p-3}b_{p-2}$ in filtration degree $s=1$. This is λ_1 times the permanent cycle $[du]u^{p-3}b_{p-2}$ representing κ_1^{p-1}.

The only remaining d^r-differentials affecting total degree $2p^2-1$, for $r \geq 2$, are those mapping from the m classes $\gamma_{pi+1}[du] \cdot u^{p-3} b_{p-2i-1}$ to the $m+2$ classes $\gamma_{pi-1}[du] \cdot a_{p-2i}$, λ_2 and $\lambda_1[du]u^{p-3}b_{p-2}$. It follows that at least $(m+2)-m=2$ linearly independent classes are left at the E^∞-term in total degree $2p^2-1$. This contradicts the fact that the abutment in degree $2p^2-1$ is generated by the single class $\lambda_1\kappa_1^{p-1}$.

This contradiction eliminates the possibility that $du \cdot z = 0$. Hence θ is an isomorphism in all degrees, and the structure of the spectral sequence is as asserted. □

Proof of Theorem 8.5 for $p \geq 5$. In view of Lemma 8.7, and the conclusion from the proof of the previous proposition that $du \cdot z \doteq \lambda_2$, the claims of the theorem have now been verified for $p \geq 5$. □

Proposition 8.9. *(i) In $V(1)_* \operatorname{THH}(\ell) \cong E(\lambda_1, \lambda_2) \otimes P(\mu_2)$ the suspension operator satisfies $\sigma(\lambda_1)=0$, $\sigma(\lambda_2)=0$ and $\sigma(\mu_2)=0$.*

(ii) In $V(1)_* \operatorname{THH}(\ell, D(v)) \cong E(\lambda_1, d\log v) \otimes P(\kappa_1)$ the suspension operator satisfies $\sigma(\lambda_1)=0$, $\sigma(d\log v)=0$ and $\sigma(\kappa_1)=\kappa_1 \cdot d\log v$.

(iii) In $V(1)_* \operatorname{THH}(ku_{(p)}) \cong E(\lambda_1) \otimes \Theta_*$ the suspension operator satisfies $\sigma(\lambda_1)=0$, $\sigma(a_i)=0$, $\sigma(b_j)=(1-j)a_j$ and $\sigma(\mu_2)=0$.

(iv) In $V(1)_* \operatorname{THH}(ku_{(p)}, D(u)) \cong P_{p-1}(u) \otimes E(\lambda_1, d\log u) \otimes P(\kappa_1)$ the suspension operator satisfies $\sigma(u)=u \cdot d\log u$, $\sigma(\lambda_1)=0$, $\sigma(d\log u)=0$ and $\sigma(\kappa_1)=-\kappa_1 \cdot d\log u$.

Proof. *(i)* The Hurewicz image of λ_2 in $H_*(V(1) \wedge \operatorname{THH}(\ell))$ is $\sigma\bar{\xi}_2$, hence $\sigma(\lambda_2)=0$. The classes $\sigma(\lambda_1)$ and $\sigma(\mu_2)$ are zero because they lie in trivial groups.

(iv) We saw that $\sigma(u)=u \cdot d\log u$ and $\sigma(d\log u)=0$ in Theorem 8.1. The class $\sigma(\lambda_1)$ is zero by case *(i)*, via naturality with respect to $\operatorname{THH}(\ell) \to \operatorname{THH}(ku_{(p)}, D(u))$. Under the trace map, Ausoni's class $b \in V(1)_{2p+2}K(ku)$ is mapped to the class $b_1 \in V(1)_{2p+2} \operatorname{THH}(ku_{(p)})$. Hence $\sigma(b_1)=0$. This can also be deduced from the formula for Connes' B-operator; compare [Aus05, Remark 3.4] and [Aus10, Lemma 6.3]. Since $\rho'_*(b_1)=u\kappa_1$ it follows that $0=u \cdot d\log u \cdot \kappa_1 + u \cdot \sigma(\kappa_1)$ and $\sigma(\kappa_1)=-\kappa_1 \cdot d\log u$.

(ii) This follows from case *(iv)* by naturality with respect to the morphism $(f, f^\flat)\colon (\ell, D(v)) \to (ku_{(p)}, D(u))$.

(iii) This follows from case *(iv)* by naturality with respect to the morphism $\rho'\colon \operatorname{THH}(ku_{(p)}) \to \operatorname{THH}(ku_{(p)}, D(u))$. □

References

[AR05] V. Angeltveit and J. Rognes, *Hopf algebra structure on topological Hochschild homology*, Algebr. Geom. Topol. **5** (2005), 1223–1290 (electronic). MR2171809 (2007b:55007)

[AR12] C. Ausoni and J. Rognes, *Algebraic K-theory of the first Morava K-theory*, J. Eur. Math. Soc. (JEMS) **14** (2012), no. 4, 1041–1079. MR2928844

[Aus05] C. Ausoni, *Topological Hochschild homology of connective complex K-theory*, Amer. J. Math. **127** (2005), no. 6, 1261–1313. MR2183525 (2006k:55016)

[Aus10] ———, *On the algebraic K-theory of the complex K-theory spectrum*, Invent. Math. **180** (2010), no. 3, 611–668. MR2609252 (2011g:19009)

[BK72] A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Mathematics, Vol. 304, Springer-Verlag, Berlin-New York, 1972. MR0365573 (51 #1825)

[BM11] A. J. Blumberg and M. A. Mandell, *Localization for THH(ku) and the topological Hochschild and cyclic homology of Waldhausen categories*, 2011. `arXiv:1111.4003`.

[BR05] A. Baker and B. Richter, *On the Γ-cohomology of rings of numerical polynomials and E_∞ structures on K-theory*, Comment. Math. Helv. **80** (2005), no. 4, 691–723. MR2182697 (2006i:55012)

[EKMM97] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May, *Rings, modules, and algebras in stable homotopy theory*, Mathematical Surveys and Monographs, vol. 47, American Mathematical Society, Providence, RI, 1997. With an appendix by M. Cole. MR1417719 (97h:55006)

[Goo85] T. G. Goodwillie, *Cyclic homology, derivations, and the free loopspace*, Topology **24** (1985), no. 2, 187–215. MR793184 (87c:18009)

[HM03] L. Hesselholt and I. Madsen, *On the K-theory of local fields*, Ann. of Math. (2) **158** (2003), no. 1, 1–113. MR1998478 (2004k:19003)

[HV92] J. Hollender and R. M. Vogt, *Modules of topological spaces, applications to homotopy limits and E_∞ structures*, Arch. Math. (Basel) **59** (1992), no. 2, 115–129. MR1170635 (93e:55015)

[MS93] J. E. McClure and R. E. Staffeldt, *On the topological Hochschild homology of* bu. *I*, Amer. J. Math. **115** (1993), no. 1, 1–45. MR1209233 (94d:55020)

[Oka79] S. Oka, *Ring spectra with few cells*, Japan. J. Math. (N.S.) **5** (1979), no. 1, 81–100. MR614695 (82i:55009)

[Rog08] J. Rognes, *Galois extensions of structured ring spectra. Stably dualizable groups*, Mem. Amer. Math. Soc. **192** (2008), no. 898, viii+137. MR2387923 (2009c:55007)

[Rog09] ———, *Topological logarithmic structures*, New Topological Contexts for Galois Theory and Algebraic Geometry (BIRS 2008), 2009, pp. 401–544.

[RSS15] J. Rognes, S. Sagave, and C. Schlichtkrull, *Localization sequences for logarithmic topological Hochschild homology*, Math. Ann. **363** (2015), no. 3-4, 1349–1398. MR3412362

[Sag14] S. Sagave, *Logarithmic structures on topological K-theory spectra*, Geom. Topol. **18** (2014), 447–490 (electronic). MR3159166

[Sag15] ———, *Spectra of units for periodic ring spectra and group completion of graded E_∞ spaces*, 2015. `arXiv:1111.6731v3`, to appear in Algebr. Geom. Topol.

[SS12] S. Sagave and C. Schlichtkrull, *Diagram spaces and symmetric spectra*, Adv. Math. **231** (2012), no. 3-4, 2116–2193. MR2964635

[SS14] ———, *Virtual vector bundles and graded Thom spectra*, 2014. `arXiv:1410.4492`.

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