Harmonic analysis for affine Hecke algebras

By G.J. Heckman and E.M. Opdam

Summary

Recently the classification of unipotent representations of a simple p-adic group was obtained by Lusztig based on his ideas of character sheaves. An alternative strategy had been proposed by Reeder based on a comparison of formal degrees. We show how the formal degrees can be computed from a residue calculation and thereby turn the approach of Reeder into an efficient route.

1. The affine Hecke algebra

Suppose V is a real vector space of dimension n equipped with a positive definite scalar product $(\cdot,\cdot)$. Let $R \subset V$ be a reduced irreducible root system, and write $R^\vee = \{\alpha^\vee; \alpha \in R\}$ for the dual root system. Here

$$\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$$

is the coroot of a root $\alpha \in R$. A root $\alpha \in R$ will be considered as a linear function on V by $\alpha(\xi) = (\xi,\alpha)$ for $\xi \in V$. Let δ denote the constant function 1 on V. The set R' of affine linear functions on V defined by

$$R' = \{\alpha + m\delta; \alpha \in R, m \in \mathbf{Z}\} \tag{1.1}$$

is called the *affine root system* associated with R.
For each affine root $a \in R'$ let H_a denote the affine hyperplane in V on which a vanishes, and let s_a denote the orthogonal reflection in H_a. Explicitly

$$s_{\alpha+m\delta}(\xi) = \xi - ((\xi,\alpha)+m)\alpha^\vee$$

for $a = \alpha + m\delta \in R'$ and $\xi \in V$. The *affine Weyl group* W' associated with R is the group of euclidean motions of V generated by these reflections. It contains the finite Weyl group W_0 of R (and of $R^\vee$) as the subgroup generated by the reflections s_α for $\alpha \in R$. For each $\alpha \in R$ the product $s_\alpha s_{\alpha+\delta}$ maps $\xi \in V$ to $\xi + \alpha^\vee$, and therefore

$$t(\alpha^\vee) = s_\alpha s_{\alpha+\delta}$$

is translation of V over $\alpha^\vee$. It follows that W' contains the subgroup of translations over elements of the coroot lattice $Q^\vee = \mathbf{Z}R^\vee$ of R. In fact we have a semidirect product decomposition

$$W' = W_0 \ltimes t(Q^\vee). \tag{1.2}$$

For our purpose it will be more convenient to work with the *extended affine Weyl group*

$$W = W_0 \ltimes t(P^\vee) \tag{1.3}$$

with $P^\vee = \{\lambda \in V; (\lambda, \alpha) \in \mathbb{Z}\ \forall \alpha \in R\}$ the coweight lattice of R. It is easy to check that W contains W' as normal subgroup. The quotient W/W' is isomorphic to the finite abelian group $P^\vee/Q^\vee$.

Fix a Weyl chamber V_+ for R. Let R_+ be the set of positive roots relative to V_+, and let $\alpha_1, \ldots, \alpha_n \in R_+$ be the set of simple roots. Let $\theta \in R_+$ be the highest root. The affine roots

$$a_0 = -\theta + \delta,\ a_i = \alpha_i \quad (1 \leq i \leq n) \tag{1.4}$$

form a set of simple roots for R', and the affine Weyl group W' is generated by the reflections $s_i = s_{a_i}$ $(0 \leq i \leq n)$. The alcove

$$C = \{\xi \in V; a_i(\xi) > 0\ (0 \leq i \leq n)\} \tag{1.5}$$

is the unique connected component of $V \setminus \cup H_a$ (union over $a \in R'$) which contains the origin in its closure and is contained in V_+. The affine Weyl group W' permutes the connected components of $V \setminus \cup H_a$ in a simply transitive way, and therefore yields a tessellation of V by congruent simplices. Likewise the extended affine Weyl group W permutes the connected components of $V \setminus \cup H_a$ in a transitive way, and therefore we get a semidirect product decomposition

$$W = \Omega \ltimes W' \tag{1.6}$$

with

$$\Omega = \{w \in W; w(C) = C\} \cong P^\vee/Q^\vee. \tag{1.7}$$

For $w \in W$ the length $l(w) \in \mathbb{N}$ is defined as the number of hyperplanes H_a $(a \in R'_+)$ separating the two alcoves C and wC. Here $R'_+ = \{a \in R'; a(\xi) > 0$ $\forall \xi \in C\}$ is the set of positive affine roots. It is easy to check that

$$R'_+ = R_+ \cup \{R + (\mathbb{N}+1)\delta\}. \tag{1.8}$$

We write

$$\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha \tag{1.9}$$

and denote

$$P^\vee_+ = \{\lambda \in P^\vee ; (\lambda, \alpha) \in \mathbb{N} \quad \forall \alpha \in R_+\} \tag{1.10}$$

for the cone of dominant coweights. Then it is easy to show that

$$l(t(\lambda)) = 2(\lambda, \rho) \qquad \text{for } \lambda \in P^\vee_+. \tag{1.11}$$

Let $q \in \mathbb{C}^\times$ be a nonzero complex parameter. The *extended affine Hecke algebra* $\mathbf{H} = \mathbf{H}(W, q)$ is the associative algebra over $\mathbb{C}$ with a vector space basis T_w indexed by $w \in W$, and multiplication given by

$$(T_i + 1)(T_i - q) = 0 \qquad \text{for } 0 \le i \le n, \tag{1.12}$$

$$T_v T_w = T_{vw} \qquad \text{if } l(v) + l(w) = l(vw). \tag{1.13}$$

Here we abbreviate T_{s_i} by T_i. The subspace of $\mathbf{H}$ with basis T_w $(w \in W')$ is a subalgebra called the *affine Hecke algebra* and denoted $\mathbf{H}' = \mathbf{H}(W', q)$. The subspace of $\mathbf{H}$ with basis T_ω $(\omega \in \Omega)$ is a subalgebra naturally isomorphic to $\mathbb{C}[\Omega]$. The semidirect product (1.6) gives rise to the isomorphism

$$\mathbf{H} \cong \mathbb{C}[\Omega] \otimes \mathbf{H}' \tag{1.14}$$

with $T_v T_\omega = T_\omega T_w$ if $\omega w = v \omega$ $(v, w \in W', \omega \in \Omega)$. The subspace of $\mathbf{H}$ with basis T_w $(w \in W_0)$ is a subalgebra denoted $\mathbf{H}_0 = \mathbf{H}(W_0, q)$. This is the Hecke algebra of the finite Weyl group W_0.

From (1.11) and (1.13) it is clear that

$$T_{t(\lambda)} T_{t(\mu)} = T_{t(\lambda + \mu)} \qquad \text{for } \lambda, \mu \in P^\vee_+. \tag{1.15}$$

If now λ is any element of $P^\vee$ then we can write $\lambda = \mu - \nu$ with $\mu, \nu \in P^\vee_+$, and we define

$$\theta_\lambda = q^{-(\lambda, \rho)} T_{t(\mu)} T^{-1}_{t(\nu)}. \tag{1.16}$$

Using (1.15) it is easy to see that this definition is unambiguous, and in addition

$$\theta_\lambda \theta_\mu = \theta_{\lambda + \mu} \qquad \text{for all } \lambda, \mu \in P^\vee. \tag{1.17}$$

Hence the subspace of $\mathbf{H}$ with basis θ_λ $(\lambda \in P^\vee)$ is a subalgebra naturally isomorphic to $\mathbb{C}[P^\vee]$. Next one can show that multiplication in $\mathbf{H}$ defines a vector space isomorphism

$$\mathbf{H} \cong \mathbf{H}_0 \otimes \mathbb{C}[P^\vee]. \tag{1.18}$$

Moreover the algebra structure on $\mathbf{H}$ can be recovered from the algebra structures on $\mathbf{H}_0$ and $\mathbb{C}[P^\vee]$ together with the push relation

$$\theta_\lambda T_i - T_i\theta_{s_i(\lambda)} = (q-1)\frac{\theta_\lambda - \theta_{s_i(\lambda)}}{1-\theta_{-\alpha_i^\vee}} \tag{1.19}$$

for $\lambda \in P^\vee$ and $1 \le i \le n$. Note that the division on the right hand side is possible inside $\mathbb{C}[P^\vee]$. This formula is due to Lusztig [Lu3, Mac1] but on the level of braid groups it was obtained before by van der Lek and Looijenga [Le1,2]. Using (1.19) it is easy to see that the center $\mathbf{Z}$ of $\mathbf{H}$ is equal to

$$\mathbf{Z} = \mathbb{C}[P^\vee]^{W_0} \tag{1.20}$$

with $\mathbb{C}[P^\vee]^{W_0}$ the subalgebra of $\mathbb{C}[P^\vee]$ of invariants for W_0. Here W_0 acts on $\mathbb{C}[P^\vee]$ by $w(\theta_\lambda) = \theta_{w(\lambda)}$ for $w \in W_0$ and $\lambda \in P^\vee$.

Let $P_{W_0}(q) = \sum_{w\in W_0} q^{l(w)}$ be the Poincaré polynomial of W_0. Assume from now on that $P_{W_0}(q) \neq 0$, and let

$$T_0^+ = P_{W_0}(q)^{-1}\left(\sum_{w\in W_0} T_w\right) \in \mathbf{H}_0 \tag{1.21}$$

be the central idempotent corresponding to the trivial representation triv : $\mathbf{H}_0 \to \mathbb{C}$, $\mathrm{triv}(T_w) = q^{l(w)}$. The Satake isomorphism expresses that multiplication by T_0^+ defines an isomorphism of commutative algebras

$$\mathbf{Z} = \mathbb{C}[P^\vee]^{W_0} \xrightarrow{\cong} T_0^+\mathbf{H}T_0^+,\ \varphi \mapsto T_0^+\varphi. \tag{1.22}$$

The spherical subalgebra $T_0^+\mathbf{H}T_0^+$ of $\mathbf{H}$ has the vector space basis $T_0^+\theta_\lambda T_0^+$ with $\lambda \in P_+^\vee$. An explicit inversion for the Satake isomorphism was obtained by Macdonald [Mac2] in the form

$$T_0^+\theta_\lambda T_0^+ = P_{W_0}(q^{-1})^{-1} T_0^+\left(\sum_{w\in W_0} w(c(\cdot,q)\theta_\lambda)\right) \tag{1.23}$$

for $\lambda \in P_+^\vee$, and with the c-function given by

$$c(\cdot,q) = \prod_{\alpha\in R_+} \frac{1-q^{-1}\theta_{-\alpha^\vee}}{1-\theta_{-\alpha^\vee}}. \tag{1.24}$$

Note that although $c(\cdot,q)$ is no longer an element of $\mathbb{C}[P^\vee]$ the averaging over W_0 in the right hand side of (1.23) makes the outcome lie in $\mathbb{C}[P^\vee]^{W_0}$.

The irreducible representations of $\mathbb{C}[P^\vee]$ are parametrized by the complex torus $T = \mathrm{Hom}(P^\vee, \mathbb{C}^\times)$ with character lattice $P^\vee$. For $s \in T$ the induced representation

$$\mathrm{Ind}_{\mathbb{C}[P^\vee]}^{\mathbf{H}}(s) \tag{1.25}$$

is called the *principal series* representation of $\mathbf{H}$ with spectral parameter s. As a module for $\mathbf{H}_0$ it is just the left regular action of $\mathbf{H}_0$ by (1.18). For regular $s \in T$ it is as a module for $\mathbb{C}[P^\vee]$ the direct sum of the one dimensional modules ws $(w \in W_0)$. The principal series (1.25) is irreducible if and only if

$$\text{numerator}(c(s,q)c(s^{-1},q)) \neq 0. \tag{1.26}$$

Each irreducible representation of $\mathbf{H}$ is equivalent to a quotient of some principal series representation.

2. The spherical Plancherel formula

In this section we take $q > 0$ a positive real parameter. The extended affine Hecke algebra $\mathbf{H} = \mathbf{H}(W,q)$ becomes an involutive algebra with respect to

$$(\sum_w c_w T_w)^* = \sum_w \overline{c}_{w^{-1}} T_w. \tag{2.1}$$

The adjoint is an antilinear antiinvolution of $\mathbf{H}$. The canonical trace $\tau : \mathbf{H} \to \mathbb{C}$ is defined by

$$\tau(\sum_w c_w T_w) = c_e, \tag{2.2}$$

and it gives rise to the canonical hermitian inner product

$$(\varphi,\psi) = \tau(\varphi\psi^*) = \sum_w c_w \overline{d}_w q^{l(w)} \tag{2.3}$$

with $\varphi = \sum c_w T_w$, $\psi = \sum d_w T_w \in \mathbf{H}$. Denote by $\mathbf{H}^\wedge$ the set of equivalence classes of irreducible unitary representations of $\mathbf{H}$. The abstract Plancherel theorem [Di] for $\mathbf{H}$ gives the existence of a unique nonnegative measure $\nu_P(\cdot) = \nu_P(\cdot, q)$ on $\mathbf{H}^\wedge$ such that

$$\tau(\varphi) = \int_{\pi \in \mathbf{H}^\wedge} \mathrm{Tr}(\pi(\varphi)) d\nu_P(\pi,q) \quad \forall \varphi \in \mathbf{H}. \tag{2.4}$$

The measure ν_P on $\mathbf{H}^\wedge$ is called the canonical *Plancherel measure* for $\mathbf{H}$.

Let $\mathbf{C} = \mathbf{C}(W,q)$ be the formal completion of $\mathbf{H}$ with respect to the basis T_w $(w \in W)$. Hence $\varphi \in \mathbf{C}$ means that $\varphi = \sum c_w T_w$ is a formal infinite sum with complex coefficients. Clearly for $\varphi \in \mathbf{C}$ and $\psi \in \mathbf{H}$ the product $\varphi\psi \in \mathbf{C}$ and the hermitian pairing $(\varphi,\psi) \in \mathbb{C}$ are well defined. The subspace $T_0^+ \mathbf{C} T_0^+$ is the space of all spherical functions in $\mathbf{C}$. Now suppose in relation (2.4) that $\varphi \in T_0^+ \mathbf{H} T_0^+$. Clearly $\mathrm{Tr}(\pi(\varphi)) = 0$ in the right hand side of (2.4) unless $\pi \in \mathbf{H}^\wedge_{\mathrm{sph}}$ with

$$\mathbf{H}^\wedge_{\mathrm{sph}} = \{\pi \in \mathbf{H}^\wedge ; [\pi|_{\mathbf{H}_0} : \mathrm{triv}] = 1\}. \tag{2.5}$$

For $\pi \in \mathbf{H}^{\wedge}_{\mathrm{sph}}$ the *elementary spherical function* $\varphi_\pi \in T_0^+\mathbf{C}T_0^+$ is uniquely characterized by

$$(\varphi_\pi, \varphi^*) = \mathrm{Tr}(\pi(\varphi)) \qquad \forall \varphi \in T_0^+\mathbf{H}T_0^+. \tag{2.6}$$

It is easy to check that

$$\varphi\varphi_\pi = (\varphi_\pi, \varphi^*)\varphi_\pi \quad \forall \varphi \in T_0^+\mathbf{H}T_0^+. \tag{2.7}$$

The scalar (2.6) is called the *spherical Fourier transform* of φ (evaluated at π), and is also denotes $\widehat{\varphi}(\pi)$. The *spherical Plancherel formula* states that for all $\varphi, \psi \in T_0^+\mathbf{H}T_0^+$

$$(\varphi, \psi) = \int_{\pi \in \mathbf{H}^{\wedge}_{\mathrm{sph}}} \widehat{\varphi}(\pi)(\varphi_\pi, \psi) d\nu_P^+(\pi, q) \tag{2.8}$$

with ν_P^+ the restriction of the Plancherel measure ν_P to $\mathbf{H}^{\wedge}_{\mathrm{sph}}$.

For $s \in T = \mathrm{Hom}(P^\vee, \mathbb{C}^\times)$ let π_s be the unique spherical subquotient of the principal series representation (1.25) with spectral parameter s. An immediate consequence of (1.23) is the explicit formula of Macdonald [Mac2, Theorem 4.1.2]

$$(\varphi_s, \theta_\lambda^*) = P_{W_0}(q^{-1})^{-1} \sum_{w \in W_0} c(ws, q) ws(\theta_\lambda) \tag{2.9}$$

for $\lambda \in P_+^\vee$, $s \in T$ regular for W_0, and φ_s short for φ_{π_s}. In case $q > 1$ the spherical Plancherel measure ν_P^+ has also been determined by Macdonald [Mac2, Theorem 5.1.2] in the form

$$d\nu_P^+(s, q) = \frac{q^{-N} ds}{|W_0| c(s, q) c(s^{-1}, q)} \tag{2.10}$$

with $N = \# R_+$ and ds the normalized Haar measure on the compact form T_c of T. The difference by a factor $P_{W_0}(q)$ between [Mac2, Theorem 5.1.2] and (2.10) comes from a different normalization of Haar measure. Substituting $\psi = \theta_\lambda^*$ in (2.8) and using (2.9) and (2.10) yields (still for $q > 1$)

$$\begin{aligned}
(\varphi, \theta_\lambda^*) &= \int_{T_c} \widehat{\varphi}(s)(\varphi_s, \theta_\lambda^*) d\nu_P^+(s, q) \\
&= P_{W_0}(q)^{-1} \int_{T_c} \widehat{\varphi}(s) s(\theta_\lambda) \frac{ds}{c(s^{-1}, q)}
\end{aligned}$$

for $\lambda \in P_+^\vee$ and $\widehat{\varphi}(s) = \widehat{\varphi}(\pi_s)$. The integrand is meromorphic on T with simple poles along submanifolds of the form $\{t \in T; t(\theta_{\alpha^\vee}) = q\}$ for $\alpha \in R_+$. Therefore the contour of integration T_c can be shifted in the direction of the negative chamber

$$(\varphi, \theta_\lambda^*) = P_{W_0}(q)^{-1} \int_{s_0 T_c} \widehat{\varphi}(s) s(\theta_\lambda) \frac{ds}{c(s^{-1}, q)}, \tag{2.11}$$

or more precisely as long as $s_0 \in T_v = \{t \in T; t(\theta_\lambda) > 0 \ \forall \lambda \in P^\vee\}$ satisfies

$$s_0(\theta_{\alpha^\vee}) < q \qquad \text{for all } \alpha \in R_+. \tag{2.12}$$

Since both sides of (2.11) are analytic in q formula (2.11) therefore remains valid for all $q > 0$ as long as s_0 satisfies (2.12). In order to give (2.11) an L^2-interpretation for $0 < q < 1$ the contour $s_0 T_c$ should be shifted back at the cost of picking up residues. Grouping residues together according to orbits of W_0 enables one to compare the residue contributions in (2.11) with the ones obtained by a contour shift in the integral

$$\int_{s_0 T_c} \widehat{\varphi}(s)(\varphi_s, \theta_\lambda^*) d\nu_P^+(s,q). \tag{2.13}$$

The term $\widehat{\varphi}(s)(\varphi_s, \theta_\lambda^*)$ is holomorphic as function of s on all of T, and therefore the residues all come from the poles in the spherical Plancherel measure (2.10).

A connected algebraic subgroup of T is called a subtorus, and we will use the phrase "affine subtorus" (by analogy with the concept affine subspace of a vector space) for any subvariety S of T for which $s^{-1}S$ ($s \in S$) is a subtorus of T.

DEFINITION 2.1. *If for $S \subset T$ an affine subtorus we write*

$$z(S) = \#\{\alpha \in R; s(\theta_{\alpha^\vee}) = 1 \quad \forall s \in S\} \tag{2.14}$$

$$p(S) = \#\{\alpha \in R; s(\theta_{\alpha^\vee}) = q \quad \forall s \in S\} \tag{2.15}$$

for the number of zeros and poles along S in the analytic continuation of the spherical Plancherel measure (2.10) then S is called residual if

$$p(S) \geq z(S) + \operatorname{codim}(S). \tag{2.16}$$

THEOREM 2.2. *For S a residual affine subtorus of T one always has the equality*

$$p(S) = z(S) + \operatorname{codim}(S). \tag{2.17}$$

The proof of this theorem reduces immediately to the case that the affine subtorus S is just a point of T. Moreover one can classify the finite list of residual points of T for each root system case by case, and thereby verify (2.17) by inspection of the list. However the deeper reason for (2.17) is that it explains why under the contour shift in (2.13) measures are picked up rather than just distributions (as should in accordance with the abstract Plancherel theorem).

THEOREM 2.3. *For $S \subset T$ a residual affine subtorus let*

$$S_c = \{s \in S; \text{ distance from } s \text{ to } T_c \text{ is minimal}\}. \tag{2.18}$$

Here the distance is taken with respect to a translation invariant metric on T, which is also invariant under W_0. Then S_c is a compact real form of S, and let μ_S denote the normalized invariant measure on S_c. For $0 < q < 1$ the spherical Plancherel measure $d\nu_P^+(s,q)$ is the measure on T given by

$$d\nu_P^+(s,q) = \sum_S c_S q^{-N} \frac{\Pi'|1 - s(\theta_{\alpha^\vee})|}{\Pi'|1 - q^{-1}s(\theta_{\alpha^\vee})|} d\mu_S(s) \tag{2.19}$$

with the sum over all residual affine subtori S of T. Here c_S is a nonnegative rational number with $c_{wS} = c_S$ for $w \in W_0$, and Π' denotes the product over all nonzero factors.

Remark 2.4. Note that $\mathbb{R}_{>0}$ acts on $T = T_v T_c$ by homotheties in T_v. The dependence of a residual affine subtorus S of T on q is simply a scale factor. The rationality of c_S therefore implies that c_S is independent of $q \in (0,1)$. All complications of the residue calculation are captured in the rational constants c_S. The actual computation of c_S can be difficult with complexity being exponential in $z(S)$. For $z(S) = 0$ it is easy and for $z(S) = 2$ it is manageable. By our method it is equally hard to decide whether $c_S > 0$ or $c_S = 0$, and therefore to conclude whether S_c really lies in the tempered spherical spectrum or not. However in the next section we will show using the work of Kazhdan and Lusztig [KL] that in fact

$$c_S > 0 \tag{2.20}$$

for each residual affine subtorus S of T.

Remark 2.5. For reasons of minimizing technicalities we have restricted ourselves to Hecke algebras $\mathbf{H}(W,q)$ with a single parameter q. However one can relax the quadratic relation (1.12) by requiring

$$(T_i + 1)(T_i - q^{k_i}) = 0 \tag{2.21}$$

instead. Here k_i are natural numbers satisfying $k_i = k_j$ if s_i and s_j are conjugated inside W. These multilabel Hecke algebras $\mathbf{H}(W, q^{\underline{k}})$ are important for the representation theory of reductive groups over a p-adic field. Virtually without change the results of this section go through for these Hecke algebras $\mathbf{H}(W, q^{\underline{k}})$. However in case of multilabel Hecke algebras it may happen in contrast with (2.20) that for $0 < q < 1$ and some $\underline{k} = (k_i)$

$$c_S = 0 \tag{2.22}$$

for some residual affine subtori S of T [HO1, Proposition 4.16 or Table 4.18].

Remark 2.6. It is quite likely that the method discussed in this section can be extended (by working with arbitrary matrix coefficients rather than just spherical ones) to recover the full Plancherel formula in more explicit terms.

An additional complication in comparison with the spherical case is that the residue contributions need no longer be single irreducible representations but finite collections (called packets) of these. All members of the same tempered packet should differ in Plancherel measure only by numbers independent of q.

Remark 2.7. The idea of picking up discrete spectrum from a residue calculation in the analytic continuation of the spectral measure of the continuous spectrum is familiar, and can be traced back to H. Weyl [We] in his treatment of the spectral problem for the hypergeometric function. It has been used by Selberg [Se] to find residual discrete spectrum in the meromorphic continuation of Eisenstein series. In several variables this method was used by Macdonald [Mac2, Section 5.2] and Matsumoto [Mat] in particular cases, and in full generality by Langlands [La] in the context of automorphic forms. A detailed exposition of this work of Langlands has been given by Moeglin and Waldspurger [MW].

3. The classification of Kazhdan and Lusztig

The classification of the irreducible representations of the unilabel extended affine Hecke algebras $\mathbf{H}(W,q)$ is quite well understood from the work of Kazhdan and Lusztig [KL]. The complex torus

$$T = \mathrm{Hom}(P^\vee, \mathbb{C}^\times) \tag{3.1}$$

parametrizing principal series representations (1.25) of $\mathbf{H}$ can be viewed as a maximal torus in the connected simply connected almost simple complex group G with root system $R^\vee$. The group G is the dual group in the sense of Langlands whose geometry of conjugacy classes ties up with the representation theory of the Hecke algebra $\mathbf{H}$.

The set $\mathbf{H}^\sqcap$ of equivalence classes of irreducible representations of $\mathbf{H}$ is partitioned into finite *packets* $\Pi(s,u)$ indexed by conjugacy classes of pairs (s,u) with $s \in G$ semisimple, $u \in G$ unipotent and satisfying the relation

$$sus^{-1} = u^q. \tag{3.2}$$

All members of the packet $\Pi(s,u)$ have the same central character s, and therefore are all subquotients of the principal series with spectral parameter s. Let

$$A(s,u) = Z(s,u)/Z(s,u)^0$$

be the component group of the simultaneous centralizer of s and u in G, and let $B(s,u)$ be the variety of Borel subgroups of G containing both s and u. Now the members of the packet $\Pi(s,u)$ are parametrized by the irreducible representations $\rho \in A(s,u)^\wedge$ which occur in the homology of the variety $B(s,u)$. The set of these $\rho \in A(s,u)^\wedge$ will be denoted by $A(s,u)^\wedge_{\mathrm{geom}}$. Denote by

$$\pi(s,u,\rho) \in \Pi(s,u) \tag{3.3}$$

the irreducible representation of $\mathbf{H}$ corresponding to $\rho \in A(s,u)^\wedge_{\mathrm{geom}}$. It can happen that $A(s,u)^\wedge_{\mathrm{geom}}$ is strictly smaller than $A(s,u)^\wedge$.However $A(s,u)^\wedge_{\mathrm{geom}}$ is never empty since the trivial representation $\rho = 1$ always does occur in $A(s,u)^\wedge_{\mathrm{geom}}$. This is the classification of $\mathbf{H}^\sqcap$ as obtained by Kazhdan and Lusztig [KL]. The parametrization of the packets by pairs (s,u) as above is a special case of more general conjectures by Deligne and Langlands. The parametrization of the packet $\Pi(s,u)$ by $A(s,u)^\wedge_{\mathrm{geom}}$ resembles the Springer classification of Weyl group representations [Sp].

Let $T(s,u)$ be a maximal torus of the reductive group $Z(s,u)$, and choose the maximal torus T of G such that both s and $T(s,u)$ lie in T. It follows from the classification of unipotent classes by Bala and Carter [Ca, Ch 5] that

$$S = sT(s,u) \tag{3.4}$$

is a residual affine subtorus of T in the sense of Definition 2.1. Moreover the map

$$(s,u) \mapsto (s,S) \tag{3.5}$$

is a bijection between pairs (s,u) with $s,u \in G$ semisimple and unipotent respectively satisfying (3.2) upto conjugation and pairs (s,S) with $s \in S$ and $S \subset T$ a residual affine subtorus upto action of W_0. Therefore we can write $\Pi(s,S), A(s,S), \ldots$ instead of $\Pi(s,u), A(s,u), \ldots$. The following result is due to Kazhdan and Lusztig [KL, Theorem 8.3] and (for the last statement to) Lusztig [Lu5, Proposition 9.1].

THEOREM 3.1. *Suppose $q > 1$. The representation $\pi(s,S,\rho) \in \Pi(s,S)$ for $\rho \in A(s,S)^\wedge_{\mathrm{geom}}$ is square integrable if and only if all members of $\Pi(s,S)$ are square integrable, and this happens precisely in case $S = \{s\}$ is a residual point. Moreover in this case the representation*

$$\pi(s, S = \{s\}, \rho = 1) \tag{3.6}$$

is the unique antispherical representation of $\mathbf{H}$ with central character s.

Here $\pi \in \mathbf{H}^\sqcap$ is called *antispherical* if $[\pi|_{\mathbf{H}_0} : \mathrm{sign}] = 1$ with $\mathrm{sign} : \mathbf{H}_0 \to \mathbb{C}$, $\mathrm{sign}(T_w) = (-1)^{l(w)}$. The next conjecture is in accordance with the general Langlands philosophy about formation of L-packets, and goes back in this precise form to Reeder [Re1, Conjecture 7.2].

Conjecture 3.2. Suppose $q > 1$, and let $S = \{s\}$ be a residual point of T. The formal degree (or Plancherel measure) of the square integrable representation $\pi(s,s,\rho) \in \Pi(s,s)$ is given by

$$f\deg(\pi(s,s,\rho)) = \dim(\rho) f\deg(\pi(s,s,1)). \tag{3.7}$$

In particular all members of a square integrable packet differ in formal degree only by absolute (i.e. independent of q) constants. Of course this supports the hope expressed in Remark 2.6.

The map $T_i \mapsto -q^{-1}T_i$ extends to a unique isomorphism

$$\mathbf{H}(W,q) \xrightarrow{\cong} \mathbf{H}(W,q^{-1}) \tag{3.8}$$

preserving the adjoint and the canonical trace τ. On the level of representations this gives a bijection

$$\mathbf{H}(W,q)^{\wedge}_{\text{antisph}} \longleftrightarrow \mathbf{H}(W,q^{-1})^{\wedge}_{\text{sph}} \tag{3.9}$$

preserving canonical Plancherel measures. If ν_P^- denotes the restriction of the Plancherel measure ν_P on $\mathbf{H}(W,q)^\wedge$ to the antispherical unitary dual $\mathbf{H}(W, q)^{\wedge}_{\text{antisph}}$ then we get

$$d\nu_P^-(s,q) = d\nu_P^+(s,q^{-1}). \tag{3.10}$$

Using (2.19) we can compute (3.10) for $q > 1$ upto absolute constants, and thereby get an explicit formula for the right hand side of (3.7) upto absolute constants.

Remark 3.3. The classification of Kazhdan and Lusztig breaks down for multilabel extended affine Hecke algebras, whereas the method of Section 2 essentially should go through as mentioned in Remarks 2.5 and 2.6. It suggests that packets ought to be parametrized by pairs (s,S) upto action of W_0 with $s \in S$ and S a residual affine subtorus of T (once this notion is properly defined in the multilabel context [HO1,2]). However the actual parametrization of a packet in geometric terms remains unclear in the multilabel context.

4. Affine Hecke algebras as intertwining algebras

In this section we will explain the role of affine Hecke algebras for the representation theory of reductive groups over a p-adic field. These results are due to Lusztig [Lu6] and Morris [Mo] generalizing earlier work of Iwahori and Matsumoto [IM] and Howlett and Lehrer [HL].

Let $\mathcal{G}$ be a tame separable locally compact unimodular group with Haar measure $\mu_{\mathcal{G}}$. Let $\mathcal{G}^\wedge$ denote the unitary dual of $\mathcal{G}$, and let $L(\mathcal{G})$ be the vector space of continuous complex valued functions on $\mathcal{G}$ with compact support. For

$\rho \in \mathcal{G}^\wedge$ with representation space V_ρ the Fourier transform $\rho(\varphi)$ of $\varphi \in L(G)$ is defined by

$$\rho(\varphi) = \int_{\mathcal{G}} \varphi(x)\rho(x)d\mu_{\mathcal{G}}(x). \tag{4.1}$$

It turns out that $\rho(\varphi)$ is of trace class, and the Plancherel formula [Di] gives the existence of a unique nonnegative measure μ_P on $\mathcal{G}^\wedge$ such that

$$\varphi(e) = \int_{G^\wedge} \mathrm{Tr}_{V_\rho}(\rho(\varphi))d\mu_P(\rho) \quad \forall \varphi \in L(\mathcal{G}). \tag{4.2}$$

Suppose $K < \mathcal{G}$ is a compact open subgroup and let $\sigma \in K^\wedge$ with representation Hilbert space V_σ.

DEFINITION 4.1. *The Hecke algebra* $\mathbf{H}(\sigma) = \mathbf{H}(\mathcal{G}, K, \sigma)$ *is the space*

$$\mathbf{H}(\sigma) = \{\varphi \in L(\mathcal{G}) \otimes End(V_\sigma); \varphi(k_1 x k_2) = \sigma(k_1)\varphi(x)\sigma(k_2) \forall x \in \mathcal{G}, \forall k_1, k_2 \in K\}$$

viewed as an associative algebra under convolution by

$$\varphi * \psi(x) = \int_{\mathcal{G}} \varphi(xy^{-1})\psi(y)d\mu_{\mathcal{G}}(y) \quad \textit{for } \varphi, \psi \in \mathbf{H}(\sigma). \tag{4.3}$$

We suppress the notation $*$ for convolution and simply write $\varphi\psi = \varphi * \psi$. The adjoint of $\varphi \in \mathbf{H}(\sigma)$ is defined by $\varphi^*(x) = \varphi(x^{-1})^*$ with the second $*$ the usual adjoint on $End(V_\sigma)$. The hermitian inner product on $\mathbf{H}(\sigma)$ is defined by

$$(\varphi, \psi) = \mathrm{Tr}_{V_\sigma}(\varphi\psi^*(e)) \quad \textit{for } \varphi, \psi \in \mathbf{H}(\sigma). \tag{4.4}$$

Note that the unit element T_e of $\mathbf{H}(\sigma)$ is given by

$$T_e(x) = \begin{cases} vol(K)^{-1}\sigma(x) & \textit{if } x \in K \\ 0 & \textit{if } x \in \mathcal{G} \backslash K. \end{cases} \tag{4.5}$$

The induced representation $\mathrm{Ind}_K^{\mathcal{G}}(\sigma)$ is defined on the vector space

$$\{\psi \in L(\mathcal{G}) \otimes V_\sigma; \psi(kx) = \sigma(k)\psi(x) \forall x \in \mathcal{G}, \forall k \in K\}. \tag{4.6}$$

Clearly $\mathbf{H}(\sigma)$ acts on (4.6) by left multiplication and $\mathcal{G}$ acts by right multiplication. Moreover these two actions commute and in fact are mutually centralizing. In turn this yields on the algebraic level a bijection

$$\mathcal{G}(\sigma)^\sqcap \ni \rho \longleftrightarrow \pi \in \mathbf{H}(\sigma)^\sqcap \tag{4.7}$$

between

$$\begin{aligned} \mathcal{G}(\sigma)^\sqcap = \{&\text{equivalence classes of admissible} \\ &\text{irreducible representations } \rho \text{ of } \mathcal{G} \text{ with } [\rho|_K : \sigma] \geq 1\} \end{aligned} \tag{4.8}$$

and

$$\mathbf{H}(\sigma)^{\sqcap} = \{\text{equivalence classes of finite dimensional irreducible representations of } \mathbf{H}(\sigma)\}. \tag{4.9}$$

On the analytic level the bijection (4.7) restricts after completion to a bijection

$$\mathcal{G}(\sigma)^{\wedge}_{\text{temp}} \ni \rho \longleftrightarrow \pi \in \mathbf{H}(\sigma)^{\wedge}_{\text{temp}} \tag{4.10}$$

for the corresponding tempered (or restricted) unitary duals.
Moreover the Plancherel measures are preserved by this correspondence in the sense that

$$d\mu_P(\rho) = d\mu_P(\pi). \tag{4.11}$$

Here by abuse of notation μ_P also denotes the Plancherel measure for $\mathbf{H}(\sigma)$. It is the unique nonnegative measure on $\mathbf{H}(\sigma)^{\wedge}$ such that

$$\mathrm{Tr}_{V_\sigma}(\varphi(e)) = \int_{\mathbf{H}(\sigma)^{\wedge}} \mathrm{Tr}_{V_\pi}(\pi(\varphi)) d\mu_P(\pi) \quad \forall \varphi \in \mathbf{H}(\sigma). \tag{4.12}$$

Now the strategy of understanding the restriction of the Plancherel measure for $\mathcal{G}$ to $\mathcal{G}(\sigma)^{\wedge}$ from the Plancherel measure for $\mathbf{H}(\sigma)$ instead is only successful if the Hecke algebra $\mathbf{H}(\sigma)$ allows an alternative description in more elementary terms. The following situation illustrates in a nice way the power of such an approach.

Let F be a nonarchimedean local field with finite residue field $\mathbb{F}_q$. Let $\mathcal{G}$ be the group of F-rational points of a simple algebraic group defined over F. Suppose P is a parahoric subgroup of $\mathcal{G}$ with prounipotent radical U. Then the quotient M of P by U is the group of $\mathbb{F}_q$-rational points of a reductive algebraic group defined over $\mathbb{F}_q$. Suppose σ is the inflation from M to P of a cuspidal unipotent representation of M. The following result is due to Lusztig [Lu6] and Morris [Mo].

THEOREM 4.2. *With the above notations we have an isomorphism of associative algebras*

$$\mathbf{H}(\mathcal{G}, P, \sigma) \cong \mathbf{H}(W, q^{\underline{k}}) \tag{4.13}$$

with $\mathbf{H}(W, q^{\underline{k}})$ the (possibly multilabel, possible extended) affine Hecke algebra associated with a root system R.

The rank n of the root system R is equal to the difference of the split ranks of $\mathcal{G}$ and M. The group W is called the relative extended affine Weyl group of the pair $(\mathcal{G}, P)$. For the explicit determination of the relative root system R and the labels $\underline{k} = (k_i)$ from the pair $(\mathcal{G}, P)$ we refer to [Mo] and [Lu4].

Remark 4.3. Taking adjoints is preserved under the isomorphism (4.13). However the hermitian inner products (4.4) of $\mathbf{H}(\mathcal{G}, P, \sigma)$ and (2.3) on $\mathbf{H}(W, q^{\underline{k}})$ need not coincide. Indeed from (4.5) we get

$$(T_e, T_e) = \mathrm{vol}(P)^{-1} \dim(\sigma) = f \deg(\sigma)$$

in $\mathbf{H}(\mathcal{G}, P, \sigma)$ while

$$(T_e, T_e) = 1$$

for the canonical hermitian inner product on $\mathbf{H}(W, q^{\underline{k}})$. In turn this implies

$$\mu_P = f \deg(\sigma) \nu_P \tag{4.14}$$

with μ_P the Plancherel measure for $\mathbf{H}(\mathcal{G}, P, \sigma)$ and ν_P the canonical Plancherel measure for $\mathbf{H}(W, q^{\underline{k}})$.

5. Formation of L-packets of square integrable unipotent representations

Let $\mathcal{G}$ be the group of F-rational points of a simple algebraic group defined over the nonarchimedean local field F. Let K be a good maximal compact open subgroup of $\mathcal{G}$, and we normalize the Haar measure $\mu_{\mathcal{G}}$ on $\mathcal{G}$ by $\mathrm{vol}(K) = 1$. The following definition is due to Lusztig [Lu1].

DEFINITION 5.1. *An irreducible admissible representation ρ of $\mathcal{G}$ is called unipotent if there is a parahoric subgroup P of $\mathcal{G}$ and a unipotent cuspidal representation σ of P (or more precisely the inflation from M to P of such a representation with M the reductive quotient of P over $\mathbb{F}_q$) such that $[\rho|_P : \sigma] \geq 1$.*

In particular the unipotent unitary dual $\mathcal{G}^\wedge_{\mathrm{unip}}$ of $\mathcal{G}$ is given by

$$\mathcal{G}^\wedge_{\mathrm{unip}} = \bigcup_{(P,\sigma)} \mathcal{G}(\sigma)^\wedge \tag{5.1}$$

with P a parahoric subgroup of $\mathcal{G}$, σ a unipotent cuspidal representation of P and the union taken over pairs (P, σ) upto conjugation. Restriction to the discrete series representations yields

$$\mathcal{G}^\wedge_{\mathrm{unip,ds}} = \bigcup_{(P,\sigma)} \mathcal{G}(\sigma)^\wedge_{\mathrm{ds}} \tag{5.2}$$

The next corollary is immediate from (4.14) keeping in mind the normalization of Haar measure on $\mathcal{G}$.

COROLLARY 5.2. *Combination of (4.10) and (4.13) yields a bijection*

$$\mathcal{G}(\sigma)^\wedge_{\mathrm{ds}} \ni \rho \longleftrightarrow \pi \in \mathbf{H}(W, q^{\underline{k}})^\wedge_{\mathrm{ds}} \tag{5.3}$$

and the relation between the formal degrees becomes

$$f\deg(\rho) = P_{W_K}(q)P_{W_P}(q)^{-1}\dim(\sigma)f\deg(\pi) \tag{5.4}$$

with P_{W_K} and P_{W_P} the Poincaré polynomials of the Weyl groups associated with K and P respectively. Moreover $f\deg(\pi)$ is the canonical formal degree of π as introduced in Section 2.

It follows from Theorem 3.1 that for $q > 1$ the collection

$$\mathbf{H}(W, q)^{\wedge}_{\mathrm{antisph,ds}} \tag{5.5}$$

is parametrized by the set of residual points in T upto action of W_0. Moreover it follows from (2.19) and (3.10) how to compute their canonical formal degrees (at least upto absolute constants).

Conjecture 5.3. For multilabel extended affine Hecke algebras $\mathbf{H}(W, q^{\underline{k}})$ the collection

$$\mathbf{H}(W, q^{\underline{k}})^{\wedge}_{\mathrm{ds}} \tag{5.6}$$

is partitioned into nonempty packets $\Pi(s, s)$ which are parametrized by residual points $s \in T$ upto the action of W_0. The parameter s is the common central character of all representations in $\Pi(s, s)$. Moreover all members of the packet $\Pi(s, s)$ have canonical formal degrees which only differ by absolute constants and which are given by (the analogous expression for multilabel Hecke algebras of) the term on the right hand side of (2.19) for $S = \{s\}$ the residual point of T.

This conjecture essentially sums up what was said before in Remarks 2.5 and 2.6. As stated there the technique of Section 2 of a contour shift and a residue computation should suffice for the proof.

Assuming the validity of this conjecture we can now discuss the formation of L-packets of square integrable unipotent representations of $\mathcal{G}$. *Two square integrable unipotent representations of $\mathcal{G}$ should lie in the same L-packet if their formal degrees only differ by absolute constants.* This is the strategy as proposed by Reeder [Re1]. We have checked that this formation of square integrable unipotent L-packets for all exceptional groups coincides with the classification of unipotent representations of $\mathcal{G}$ as obtained by Lusztig [Lu4]. In the next section we will discuss the example that $\mathcal{G}$ is of type E_8.

Remark 5.4. Let U be the prounipotent radical of the good maximal compact open subgroup K of $\mathcal{G}$. One might expect that everything said so far for unipotent representations of $\mathcal{G}$ can be generalized for U-spherical representations of $\mathcal{G}$ as well. Indeed in this more general context the results of Morris [Mo] remain valid, and therefore the computation of formal degrees of square integrable U-spherical representation of $\mathcal{G}$ can be transferred to corresponding questions about representation theory of affine Hecke algebras.

6. The example E_8

In this final section we discuss the example that the group $\mathcal{G}$ is split of type E_8. There are upto conjugation 5 parahoric subgroups P of $\mathcal{G}$ for which the associated reductive quotient M over $\mathbb{F}_q$ has a unipotent cuspidal representation σ. The pairs (P,σ) are given in Table 6.1. The unipotent cuspidal representations were classified by Lusztig [Lu2]. Throughout this section we will use his notation and results as presented in [Ca].

The packets $\Pi(s,u)$ of square integrable representations of the Hecke algebra $\mathbf{H}(W(\widetilde{E}_8),q)$ are parametrized by residual points $s \in T$ upto action of W_0. Write s in polar decomposition

$$s = s_v s_c$$

with $s_v \in T_v$ and $s_c \in T_c$. Now s is residual if an only if

$$R(s_c) = \{\alpha \in R; s_c(\theta_{\alpha^\vee}) = 1\} \subset R(E_8)$$

is a root subsystem of rank 8, and s_v is residual with respect to $R(s_c)$. The possible s_v upto action of $W_0(s_c) = \{w \in W_0; w(s_c) = s_c\}$ are in bijection with the distinguished unipotent classes in the group $Z(s_c) = \{g \in G; gs_c = s_c g\}$. In Table 6.2 we have listed the 31 possibilities by the pairs $(R(s_c), m)$ with m an index for the distinguished unipotent class in $Z(s_c)$ (ordered such that $z(s)$ in (2.14) increases with m, cf [Ca, p.174-177]). In this table we have also given the formal degrees of the corresponding representations. The multiplication of the canonical formal degree by $P_{W(E_8)}(q)$ is in accordance with (5.4). The actual calculations were performed using maple. These square integrable packets are sometimes called *generalized special*. The Steinberg representation is one of them and corresponds to $(E_8, 1)$.

On the other extreme unipotent cuspidal representations of the group $G(E_8, \mathbb{F}_q)$ lifted to K and induced upto $\mathcal{G}$ yield irreducible *supercuspidal* representations of $\mathcal{G}$. Their formal degree equals the dimension of the original unipotent cuspidal representation σ that one starts with. Comparison of [Ca, p.488] with Table 6.2 forces to which of the 31 generalized special packets these supercuspidal representations should be attached. Likewise for the intermediate cases with (P,σ) of type D_4, E_6 or E_7 the matching of formal degrees based on (5.4) forces how the various packets should be linked. The outcome has been tabulated in Table 6.3. The L-packets of unipotent square integrable representations of $\mathcal{G}$ are obtained by grouping packets in horizontal lines together.

The fact that a generalized special packet $\Pi(s,u)$ does not fill up its L-packet is connected with the fact that

$$A(s,u)^{\wedge}_{\mathrm{geom}} \subsetneq A(s,u)^{\wedge}.$$

For example for $(\widetilde{E}_8, E_8, 11)$ we have $A(s,u) \cong S_5$ and $A(s,u)^{\wedge} \backslash A(s,u)^{\wedge}_{\mathrm{geom}} = \{\mathrm{sign}\}$. The additional supercuspidal representation accounts for the missing sign representation. It is expected that the full L-packet should be parametrized by $A(s,u)^{\wedge}$.

Table 6.1. Relative root systems

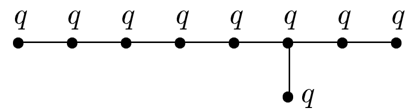
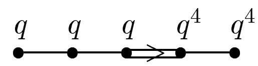
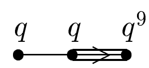
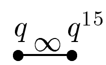
W_P	σ	relative affine root system with labels
$\emptyset$	triv	
D_4	D_4	
E_6	$E_6[\theta^j]$	
E_7	$E_7[\pm\xi]$	
E_8	$E_8[\zeta^i], E_8[\theta^j], E_8[-\theta^j],$ $E_8[\pm i], E_8[-1], E_8^I[1], E_8^{II}[1]$	$\emptyset$

Table 6.2. Formal degrees (up to rational constants) for $H(W(\widetilde{E}_8), q)$

s	$P_{W(E_8)}(q) f \deg(\pi_s)$
A_8	$q^8 \ \Phi_1^8\Phi_2^8\Phi_4^4\Phi_5^2\Phi_7\Phi_8^2\Phi_9^2\Phi_{10}^2\Phi_{14}\Phi_{18}\Phi_{20}\Phi_{27}$
$E_8, 1$	$\Phi_1^8\Phi_7\Phi_{11}\Phi_{13}\Phi_{17}\Phi_{19}\Phi_{23}\Phi_{29}$
$E_8, 2$	$q \ \Phi_1^8\Phi_7^2\Phi_9\Phi_{11}\Phi_{13}\Phi_{14}\Phi_{17}\Phi_{19}\Phi_{23}\Phi_{30}$
$E_8, 3$	$q^2 \ \Phi_1^8\Phi_7^2\Phi_8^2\Phi_{11}^2\Phi_{13}\Phi_{14}\Phi_{17}\Phi_{19}\Phi_{24}\Phi_{30}$
$E_8, 4$	$q^3 \ \Phi_1^8\Phi_4^4\Phi_7\Phi_8^2\Phi_9\Phi_{11}\Phi_{12}\Phi_{13}^2\Phi_{17}\Phi_{20}\Phi_{24}\Phi_{30}$
$E_8, 5$	$q^4 \ \Phi_1^8\Phi_7^4\Phi_9^2\Phi_{11}^2\Phi_{13}\Phi_{14}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$
$E_8, 6$	$q^5 \ \Phi_1^8\Phi_5^5\Phi_7\Phi_8^2\Phi_{10}^2\Phi_{11}\Phi_{13}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$
$E_8, 7$	$q^6 \ \Phi_1^8\Phi_5^5\Phi_7^2\Phi_9\Phi_{10}^2\Phi_{11}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$
$E_8, 8$	$q^7 \ \Phi_1^8\Phi_4^4\Phi_7^4\Phi_8^2\Phi_{11}\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$
$E_8, 9$	$q^8 \ \Phi_1^8\Phi_3^9\Phi_6^4\Phi_7\Phi_9^2\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$
$E_8, 10$	$q^{10} \ \Phi_1^8\Phi_5^5\Phi_7^2\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$
$E_8, 11$	$q^{16} \ \Phi_1^8\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$
$A_1E_7, 1$	$q^3 \ \Phi_1^8\Phi_3^2\Phi_5^2\Phi_7\Phi_8^2\Phi_9\Phi_{11}\Phi_{13}\Phi_{15}\Phi_{16}\Phi_{17}\Phi_{24}\Phi_{26}$
$A_1E_7, 2$	$q^5 \ \Phi_1^8\Phi_3^2\Phi_4^4\Phi_5^3\Phi_7\Phi_9\Phi_{10}^2\Phi_{11}\Phi_{12}\Phi_{13}\Phi_{15}\Phi_{20}^2\Phi_{30}$
$A_1E_7, 3$	$q^7 \ \Phi_1^8\Phi_3^2\Phi_5^2\Phi_7^3\Phi_8^2\Phi_9\Phi_{11}\Phi_{14}^2\Phi_{15}\Phi_{16}\Phi_{24}\Phi_{30}$
$A_1E_7, 4$	$q^9 \ \Phi_1^8\Phi_3^2\Phi_5^3\Phi_7^3\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{14}\Phi_{15}\Phi_{20}\Phi_{24}\Phi_{30}$
$A_1E_7, 5$	$q^{13} \ \Phi_1^8\Phi_3^2\Phi_5^4\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$
$A_1E_7, 6$	$q^{16} \ \Phi_1^8\Phi_3^2\Phi_4^4\Phi_5^2\Phi_7\Phi_9\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$
$A_2E_6, 1$	$q^7 \ \Phi_1^8\Phi_2^6\Phi_4^4\Phi_5^2\Phi_7^2\Phi_8^2\Phi_{10}^2\Phi_{11}\Phi_{12}\Phi_{14}\Phi_{20}\Phi_{21}\Phi_{24}$
$A_2E_6, 2$	$q^{10} \ \Phi_1^8\Phi_2^6\Phi_4^4\Phi_5^3\Phi_7^2\Phi_8^2\Phi_{10}^2\Phi_{14}\Phi_{15}^2\Phi_{20}\Phi_{30}$
$A_2E_6, 3$	$q^{16} \ \Phi_1^8\Phi_2^6\Phi_4^4\Phi_5^2\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{14}\Phi_{20}\Phi_{24}\Phi_{30}$
$A_3D_5, 1$	$q^{12} \ \Phi_1^8\Phi_3^4\Phi_4^4\Phi_5^2\Phi_7^2\Phi_8^2\Phi_9\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{20}\Phi_{24}$
$A_3D_5, 2$	$q^{16} \ \Phi_1^8\Phi_2^8\Phi_3^4\Phi_5^2\Phi_6^4\Phi_7\Phi_9\Phi_{10}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{30}$
A_4A_4	$q^{16} \ \Phi_1^8\Phi_2^8\Phi_3^4\Phi_4^4\Phi_6^4\Phi_7\Phi_8^2\Phi_9\Phi_{12}^2\Phi_{14}\Phi_{18}\Phi_{24}$
$A_1A_2A_5$	$q^{16} \ \Phi_1^8\Phi_2^6\Phi_3^2\Phi_4^4\Phi_5^2\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{20}$
A_1A_7	$q^{11} \ \Phi_1^8\Phi_3^4\Phi_4^5\Phi_5^2\Phi_7\Phi_8^3\Phi_9\Phi_{12}^2\Phi_{15}\Phi_{16}\Phi_{20}\Phi_{24}$
$D_8, 1$	$q^4 \ \Phi_1^8\Phi_3^4\Phi_5^2\Phi_7^2\Phi_9^2\Phi_{11}\Phi_{13}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{22}\Phi_{28}$
$D_8, 2$	$q^6 \ \Phi_1^8\Phi_3^4\Phi_5^3\Phi_7^2\Phi_9^2\Phi_{10}^2\Phi_{11}\Phi_{14}\Phi_{15}\Phi_{20}^2\Phi_{30}$
$D_8, 3$	$q^8 \ \Phi_1^8\Phi_3^7\Phi_5^2\Phi_6^4\Phi_7^2\Phi_9^2\Phi_{12}^3\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$
$D_8, 4$	$q^{10} \ \Phi_1^8\Phi_3^4\Phi_5^4\Phi_7^2\Phi_9\Phi_{10}^3\Phi_{14}\Phi_{15}\Phi_{20}\Phi_{24}\Phi_{30}$
$D_8, 5$	$q^{16} \ \Phi_1^8\Phi_3^4\Phi_5^2\Phi_7\Phi_8^2\Phi_9\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$

Table 6.3. Formation of L-packets

$W_P = \emptyset$	$W_P = W(D_4)$	$W_P = W(E_6)$	$W_P = W(E_7)$	$W_P = W(E_8)$
$(\widetilde{E}_8, A_8)$		$(\widetilde{G}_2, A_2)$		
$(\widetilde{E}_8, E_8, 1)$				
2				
$\vdots$				
10				
11				$(\emptyset, E_8^{II}[1])$
$(\widetilde{E}_8, A_1E_7, 1)$	$(\widetilde{F}_4, F_4, 1)$			
2	2			
$\vdots$	$\vdots$			
5	5			
6	6			$(\emptyset, E_8[-1])$
$(\widetilde{E}_8, A_2E_6, 1)$		$(\widetilde{G}_2, G_2, 1)$		
2		2		
3		3		$(\emptyset, E_8[\theta^j])$
$(\widetilde{E}_8, A_3D_5, 1)$	$(\widetilde{F}_4, A_1B_3, 1)$		$(\widetilde{A}_1, A_1)$	
2	2			$(\emptyset, E_8[\pm i])$
$(\widetilde{E}_8, A_4A_4)$				$(\emptyset, E_8[\zeta^i])$
$(\widetilde{E}_8, A_1A_2A_5)$	$(\widetilde{F}_4, A_2A_2)$	$(\widetilde{G}_2, A_1A_1)$		$(\emptyset, E_8[-\theta^j])$
$(\widetilde{E}_8, A_1A_7)$	$(\widetilde{F}_4, A_1A_3)$		$(\widetilde{A}_1, A_1)$	
$(\widetilde{E}_8, D_8, 1)$	$(\widetilde{F}_4, B_4, 1)$			
2	2			
3	3			
4	4			
5	5			$(\emptyset, E_8^{I}[1])$

Department of Mathematics, University of Nijmegen, Toernooiveld, 6525 ED Nijmegen, The Netherlands (email: heckman@sci.kun.nl)

Department of Mathematics, University of Leiden, P.O. Box 9512, 2300 RA Leiden, The Netherlands (email: opdam@wi.leidenuniv.nl)

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