Effect Algebroids

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Preface

Spending four years on a Ph.D. research project cannot be done alone, so I would like to thank everyone without whom this thesis would not have been possible. In the first place, I wish to express my gratitude to my supervisor Ieke Moerdijk. He helped me finding my way within the research project and shared his deep knowledge of topology, while at the same time giving me freedom to combine it with my other scientific interests.

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Contents

Chapter 1

Introduction

This thesis fits in the tradition of applying tools from category theory and algebraic topology to quantum foundations. Quantum theory governs physical phenomena at very small scales. It has several features that have no counterpart at macroscopic scales, making it fundamentally different from classical physics. An example is the principle of superposition, which states, roughly speaking, that a particle need not have a definite position, but it can be in more than one position at the same time.

While the counterintuitive features are often seen as a hurdle to understanding quantum mechanics, they are also at the heart of the emerging field of quantum computation. This is because some of these features, e.g. contextuality and entanglement, can be used to create quantum algorithms that are more efficient than classical algorithms that solve the same problem, as argued in e.g. [70]. Thus a good comprehension of these features is necessary.

Because quantum phenomena are vastly different from human experience, reasoning intuitively about these can be unreliable. Therefore understanding the quantum world requires a solid mathematical framework. The traditional mathematical machinery used for this, developed by von Neumann in [99], comprises mostly functional analysis and operator algebras.

There is growing evidence that also the mathematical theory of categories is suitable for clarifying certain aspects of quantum theory. Category theory provides an abstract view on mathematical structures, and is often useful as a language to speak about similarities and differences between distinct structures. Since it helps to focus on the most important parts of a structure, ignoring low-level details, category theory is especially good at dealing with complex systems arising in quantum computation.

The main quantum structures that we will be concerned with are effect algebras. These form an algebraic model of quantum logic, and since category theory and topology have already proven to be powerful for dealing with logic (see e.g. [73, 79, 87, 90]), these tools are expected to shed light on many logical aspects of quantum theory.

Our most important mathematical tool for studying effect algebras will be *cyclic cohomology*. Ordinary cohomology theory is suitable for characterizing properties of topological spaces. Cyclic cohomology is a variation applicable to spaces carrying an action of the circle group, or to associative algebras. One of our goals will be to define a generalization of effect algebras which also admits a notion of cyclic cohomology.

Describing cyclic cohomology in categorical language leads to the notions of cyclic sets and abstract circles, which we will use extensively. Abstract circles are an algebraic model for segments on a circle. The definition of an abstract circle is formally similar to that of an effect algebra, even though the examples of both structures and their uses are completely different. To study the similarities between both structures, we will define *effect algebroids*, which form the central new concept in this thesis. Effect algebroids form a common generalization of both effect algebras and abstract circles. It turns out that several results from the theory of effect algebras and the theory of abstract circles generalize to effect algebroids. These generalizations are interesting in their own right, but thay may also provide new insights in other examples of effect algebroids. More specifically, generalizing a known result about effect algebras to algebroids may yield a new fact about abstract circles, and vice versa.

In this thesis, we will discuss several instances of this procedure. For example, each effect algebra has an associated state space. If the effect algebra is thought of as the syntax of a quantum logic, then the state space represents its corresponding semantics. The definition of the state space generalizes effortlessly to effect algebroids, and hence there is a notion of a state space of an abstract circle. This will turn out to be related to measures on the abstract circle.

As a second example, both effect algebras and abstract circles are algebraic structures with a partial operation. For both structures, there is a canonical way to turn the partial operation into a total operation. While seemingly different, these totalizations are both special cases of a totalization procedure on effect algebroids.

The category of effect algebras is both complete and cocomplete, while the category of abstract circles is neither. The construction of limits and colimits of effect algebras can also be performed for effect algebroids, although some additional difficulties emerge. In this way it becomes possible to take limits and colimits of abstract circles, as long as we are willing to work in the larger category of effect algebroids.

For an example application of the theory of abstract circles to effect algebras, we will take a look at embeddings in cyclic sets. There is a full and faithful embedding of the category of abstract circles into the category of cyclic sets, and this extends to a full and faithful embedding of effect algebroids into cyclic sets. It is possible to characterize the essential image of this embedding precisely, using a variation of Segal's characterization of nerves of categories.

This embedding leads to several more novel observations about effect algebras. Each cyclic set, and hence each effect algebroid, has a geometric realization. For abstract circles, these geometric realizations have been studied before, but for effect algebras, it is a new concept, giving a geometric interpretation of effect algebras.

Finally, it is natural to consider cohomology of cyclic sets, which is called cyclic cohomology. This restricts to cyclic cohomology of effect algebroids. The resulting cohomology theory for abstract circles is rather trivial, but for effect algebras we obtain many interesting and natural results. For example, cohomology of effect algebras interacts well with coproducts, products, and unions of effect algebras. These are special properties of cyclic cohomology that do not hold for other possible cohomology theories of effect algebras, such as Hochschild cohomology. This provides evidence that cyclic cohomology of effect algebras, and hence the connections between effect algebras and circles, are indeed very natural subjects to study.

Another reason to examine cohomology of effect algebras is its applicability to the study of contextuality in quantum foundations. The cohomological nature of contextuality was first exhibited in [4], and the role of effect algebras in [117]. Cohomology of effect algebras builds on both results and provides characterizations of contextuality in certain settings.

At some points in this thesis, we will also make use of the theory of partially ordered abelian groups, discussed for example in [55, 53]. Many of the notions occuring in this theory can be given physical interpretations using resource theories, see [27, 49]. The relevance of the theory for effect algebras is that many effect algebras occuring in practice are an interval in some partially ordered group. The relevance for cohomology is that they provide a refinement of the cyclic cohomology of effect algebras. This refinement has consequences for applications to contextuality: in [4], it is shown that if a probabilistic model is non-contextual, then a certain cohomology class associated to the model is zero. However, the converse does not hold, so in some cases false positives arise. The refinement of cyclic cohomology using ordered groups improves this result by making the implication into an equivalence.

Related work

There are already various approaches to using categories in the foundations of quantum physics. These include the use of monoidal categories in [2, 3], the use of topoi in [72, 68], and the use of higher categories in [13]. Category theory has also been applied to effect algebras, for example in [78], where effect algebras are characterized as the Eilenberg–Moore algebras for a certain monad, and in [76], where they are studied from a perspective of categorical logic.

In categorical analysis of quantum structures, topology often plays a major role. This is particularly noticeable in the categorical quantum mechanics based on monoidal categories, since monoidal categories can be represented graphically using string diagrams, see for example [81, 113]. More recently, sheaf theory and cohomology appeared in the study of contextuality. General sheaf theory is discussed in [90], while its applications to contextuality can be found in [1, 4, 5]. Our use of categories and topology is closest to this approach, especially in the description of contextuality using cohomology of effect algebras.

Abstract circles were first defined in [95]. Similar algebraic accounts of cyclically ordered structures can be found in [101, 103, 31, 32]. The connection with effect algebras appears to be new.

The discussion of geometric realizations of effect algebroids in Section 5.4 bears some similarity to the approach to geometric realizations of cyclic sets in [19, 34]. The setting of effect algebroids, and especially the case of effect algebras, is the main new contribution here.

The state space of an effect algebra often plays an important role, especially in Section 6.2, where it is connected to the first cohomology group of the effect algebra. This connection is reminiscent of the connection between convex spaces and base norm spaces, see e.g. [96, 6, 7, 11]. However, we will work in the more general case of states on effect algebras, instead of states on C*-algebras.

Prerequisites

We assume familiarity with the language of category theory, including adjunctions, monads, and Kan extensions. Good introductions to category theory are for example [21, 89].

Since many examples of effect algebras come from quantum mechanics, knowing the basics of Hilbert space theory is useful. For this we refer to [86, 105]. Furthermore, [65] explains the theory with a view towards quantum theory.

Understanding the connections between effect algebras and cyclic sets requires some algebraic topology and homological algebra. The background on algebraic topology can be found in e.g. [64], and a good introduction to the more abstract homological algebra we will need is [122]. The theory of cyclic cohomology is not strictly necessary, since it is explained as needed. Readers who wish to know more about the background may consult [91].

We do not require any knowledge of effect algebras, nor of quantum computation. All preliminary material about effect algebras needed to understand this thesis is outlined in the first two chapters.

Outline

Chapters 2 and 3 contain material that is already known for the most part, but many of the results are spread across the literature. In Chapter 2, we will define effect algebras, and present the main results about them that we will use later on. In particular, we will discuss categorical properties, the connection with partially ordered abelian groups, and how to represent certain effect algebras using tests or Greechie diagrams.

Each effect algebra has an associated state space, which often contains a lot of information about the algebra. The state space is always a convex space. Chapter 3 starts with general facts on convex spaces, and continues by discussing states on effect algebras. We do not aim for an exhaustive coverage of the topic, but only discuss the fragment that is relevant for the results in this thesis.

The original contributions start in Chapter 4, where we define effect algebroids. We start developing the basics of the theory of effect algebroids, by discussing cyclic orders, topology, and states on abstract circles. We will also describe a totalization procedure for effect algebroids, and apply this to prove that the category of effect algebroids is complete and cocomplete.

The development of the theory of effect algebroids is continued in Chapter 5. Here we describe a full and faithful embedding of the category of effect algebroids in the category of cyclic sets. The main result is a characterization of the essential image of this embedding. Furthermore, we treat geometric realizations of effect algebroids, using the embedding in cyclic sets, and provide two equivalent descriptions of this realization.

Finally, in Chapter 6 we introduce cohomology of effect algebroids. We mainly focus on the special case of effect algebras. In fact, we will define two versions of this cohomology: cyclic cohomology, based on the embedding in cyclic sets, and order cohomology, based on the order structure of effect algebras. We will analyze cyclic cohomology extensively, and show that it provides us with versions of the Künneth and Mayer–Vietoris sequences in a new context. The main advantage of cyclic cohomology over order cohomology is that it enables these manipulations using homological algebra. On the other hand, order cohomology will be shown to work better for applications to quantum mechanical no-go theorems. We will give a necessary and sufficient criterion for contextuality using order cohomology, which improves a known result from the literature on cohomology of contextuality.

Chapter 2

Effect algebras

The rules of logical reasoning can be captured by algebraic equations. This fundamental observation due to Boole started the algebraic approach to logic, using structures nowadays called Boolean algebras. These algebraic structures are well suited for classical logic. However, different kinds of reasoning require different logics, and hence different algebraic models. For example, Boolean algebras are inadequate for modeling intuitionistic logic, for which Heyting algebras form a better choice.

Also when reasoning about physical systems, it is possible to form an algebraic model for propositions. If the system under consideration can be described using classical mechanics, then the corresponding logic is classical, so Boolean algebras provide a good formalization. However, Boolean algebras turn out to be unsuitable for reasoning about quantum mechanics. The differences between classical and quantum physics are so fundamental, that they even affect the logic underlying them.

There are two problems that arise when trying to adopt Boolean algebras for quantum logic. Firstly, not all aspects of a quantum system can be measured at the same time. For instance, the position and momentum of a particle cannot both be known accurately at the same time, as witnessed by Heisenberg's uncertainty principle. If P and Q are two propositions about a quantum system that are not simultaneously measurable, then their conjunction $P \wedge Q$ is an untestable proposition. Therefore it is desirable to exclude this proposition from the logic. This suggests that conjunction in quantum logic should be a partial operation, that is only defined whenever two events are jointly measurable.

The second problem is that quantum mechanics is inherently probabilistic. Therefore it is not always possible to assign a definite truth value to a proposition about a quantum mechanical system. The best we can do is assign to each statement a probability that it holds in a certain model, i.e. a number in the unit interval $[0,1] \subset \mathbb{R}$. Since this unit interval does not form a Boolean algebra with its natural ordering, we need a generalization of Boolean algebras to treat the probabilistic aspects of quantum logic properly.

The first attempt to construct a logic governing quantum phenomena was discussed in the famous paper [20], in which Birkhoff and von Neumann proposed to use orthomodular lattices. These form a non-distributive generalization of Boolean algebras. Their relevance lies in the observation that projections on a Hilbert space form an orthomodular lattice, and these are a mathematical model of binary measurements in quantum theory. See [82] for an overview of the theory.

Every orthomodular lattice is a union of Boolean algebras, as shown in [109]. This means that they can be considered as algebraic structures that are locally Boolean. Physically, two elements in an orthomodular lattice represent events that are simultaneously observable if and only if they lie in a common Boolean subalgebra. Hence orthomodular lattices can be made into a structure with partial operations, by stipulating that the conjunction and disjunction of two elements are only defined whenever they lie in the same Boolean subalgebra. In this way orthomodular lattices solve the first issue with Boolean algebras. This view on orthomodular lattices is taken further in the theory of piecewise or partial Boolean algebras, see [84, 59, 33, 67].

Orthomodular lattices are not a completely satisfactory model for quantum logic, since they do not incorporate probabilistic aspects. This is because the set of outcomes of probabilistic experiments is the unit interval $[0,1]$, and this does not form an orthomodular lattice. An overview of quantum mechanical experiments that do not fit in the Birkhoff–von Neumann scheme is given in [111].

Effect algebras form a generalization of orthomodular lattices that model unsharp measurements in quantum mechanics. The unsharpness ensures that they are suitable for multivalued or probabilistic scenarios. Effect algebras were introduced in [44], but the equivalent notions of D-posets and weak orthoalgebras already originated in [85] and [52], respectively. They are also related to MV-algebras introduced by [25] (see also [26, 71, 17]), which can be seen as a model for unsharp measurements in classical mechanics. The book [38] contains an overview of the theory of effect algebras.

2.1 Definition and examples

We shall define effect algebras by abstracting from the algebraic structure of the unit interval $[0,1]$. There is a partial addition on the interval: two numbers can be added, but only if their sum does not exceed 1. Furthermore, the interval has a least element 0 and a greatest element 1. The least element acts as an identity for the partial addition. Finally, the unit interval has complements with respect to the greatest element. That is, for each $a \in [0,1]$ there is a unique $a^\perp \in [0,1]$ such that $a + a^\perp = 1$. We capture this structure in the following notion.

Definition 2.1.1. An *effect algebra* consists of a set A equipped with a partial binary operation $\boxplus$, a unary operation $(-)^\perp$, and constants $0, 1 \in A$, satisfying the following conditions.

- Commutativity: if $a \boxplus b$ is defined, then so is $b \boxplus a$, and $a \boxplus b = b \boxplus a$.
- Associativity: if $a \boxplus b$ and $(a \boxplus b) \boxplus c$ are defined, then so are $b \boxplus c$ and $a \boxplus (b \boxplus c)$, and $(a \boxplus b) \boxplus c = a \boxplus (b \boxplus c)$.
- Zero: $0 \boxplus a$ is always defined and equals a.
- Orthocomplement: for each $a \in A$, $a^\perp$ is the unique element for which $a \boxplus a^\perp = 1$.
- Zero-one law: if $a \boxplus 1$ is defined, then $a = 0$.

Commutativity and associativity ensure that if a sum $a_1 \boxplus a_2 \boxplus \cdots \boxplus a_n$ is defined, then also the sum of any subset of $\{a_1, \ldots, a_n\}$ is defined and independent of the order in which we sum the elements.

Examples 2.1.2.

1. The motivating example is the unit interval $[0,1]$. Addition serves as a partially defined binary operation, and the orthocomplement is given by $a^\perp = 1 - a$.
2. The subset $\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n} = 1\} \subseteq [0,1]$ inherits all effect algebra operations from the unit interval $[0,1]$. An effect algebra of this form is called a finite *linear* effect algebra, or a finite *chain*, and is denoted L_n.
3. Any Boolean algebra B is an effect algebra. The sum $x \boxplus y$ is defined if and only if x and y are disjoint, that is, $x \wedge y = 0$. In that case their sum is defined as $x \vee y$. The orthocomplement is simply the

complement in B. As a special case, any power set $\mathcal{P}(X)$ is an effect algebra in which the partial operation is disjoint union.

This example shows that effect algebras are indeed a generalization of Boolean algebras, and hence of classical logic.

4. Generalizing the previous example, any orthomodular lattice is an effect algebra. A bounded lattice L is called *orthomodular* if there is an operation $(-)^\perp : L \to L$ such that:

   - $x \leq y$ implies $y^\perp \leq x^\perp$.
   - $x^{\perp\perp} = x$.
   - $x \vee x^\perp = 1$.
   - Orthomodular law: if $x \leq y$, then $x \vee (x^\perp \wedge y) = y$.

   Every orthomodular lattice can be turned into an effect algebra where $x \boxplus y$ is defined if and only if $x \leq y^\perp$, and in that case $x \boxplus y = x \vee y$. The elements 0 and 1 exist because we assumed a bounded lattice. The orthocomplement is the operation $(-)^\perp$.

   Because every orthomodular lattice is an effect algebra, the quantum logic of Birkhoff and von Neumann fits in the effect algebra framework. Since projections on a Hilbert space form an orthomodular lattice, they also form an example of an effect algebra that is fundamental in the effect algebraic approach to quantum logic.

5. Projections on a Hilbert space model so-called *sharp* measurements in quantum physics, which satisfy the principle of non-contradiction. Some experiments can only be described using unsharp measurements. These are represented mathematically by effects on a Hilbert space instead of projections. The first treatment of quantum mechanics using effects was in [93].

   Effects are defined in terms of the Löwner order on operators on a Hilbert space. Given two self-adjoint operators A and B on a Hilbert space H, define $A \leq B$ if and only if $\langle x \,|\, Ax \rangle \leq \langle x \,|\, Bx \rangle$ for all x, where $\langle - \mid - \rangle$ denotes the inner product on the Hilbert space. The relation $\leq$ is clearly reflexive and transitive, and it follows from e.g. [86, Lem. 3.9-3] that it is antisymmetric (see also [88, Thm. 1.3.3] for a generalization to C*-algebras), hence it defines a partial order. An *effect* on a Hilbert space is an operator A for which $0 \leq A \leq I$ in the Löwner order. The collection $\mathcal{E}f(H)$ of all effects on H forms an effect algebra with partial addition as binary operation and orthocomplement given by $A^\perp = I - A$. This is one of the motivating examples for the theory of effect algebras, and also the reason for its name.

All effect algebras have certain additional structures. Each effect algebra carries a partial order, defined by $a \leq b$ if and only if there exists an element c such that $a \boxplus c = b$. The axioms for an effect algebra guarantee that $\leq$ is a partial order. We record two commonly used facts connecting the partial order and addition.

Lemma 2.1.3. *Let A be an effect algebra, and $a, b \in A$.*

1. *$a \leq b^\perp$ if and only if $a \boxplus b$ is defined.*
2. *If $a \boxplus b$ is defined and $a' \leq a$, $b' \leq b$, then $a' \boxplus b'$ is also defined.*

Proof.

1. If $a \leq b^\perp$, then $a \boxplus c = b^\perp$ for some c. Hence $a \boxplus c \boxplus b = b^\perp \boxplus b = 1$, so $a \boxplus b$ is defined. Conversely, if $a \boxplus b$ is defined, then $a \boxplus b \boxplus c = 1$ for some c, so $a \boxplus c = b^\perp$ by uniqueness of complements. This proves that $a \leq b^\perp$.
2. Assume that $a \boxplus b$ is defined and $a' \leq a$, $b' \leq b$. Then $a' \leq a \leq b^\perp \leq (b')^\perp$ by part 1 of the lemma, so $a' \boxplus b'$ is defined. □

2.2 The category of effect algebras

We wish to define morphisms of effect algebras and study the properties of the resulting category. Because of the partiality in effect algebras, there are several notions of morphism. We will use two of these notions.

Definition 2.2.1. A function f from A to B is called a *morphism* if:

- f preserves 0, 1, and complements.
- If $a \boxplus b$ is defined, then also $f(a) \boxplus f(b)$ is defined, and $f(a \boxplus b) = f(a) \boxplus f(b)$.

The notation **EA** stands for the category of effect algebras with morphisms in this sense. A *strong* morphism is a morphism for which the condition that $f(a) \boxplus f(b)$ is defined implies that also $a \boxplus b$ is defined.

Sometimes ordinary morphisms are called *weak* to distinguish them from strong morphisms. Most morphisms encountered in the theory of effect algebras are weak, and the category of effect algebras with weak morphisms has better properties than the category with strong morphisms. For example, the category with weak morphisms is complete and cocomplete, whilst the category with strong morphisms is neither. This is why we usually omit the adjective "weak".

Even though weak morphisms have better categorical properties, strong morphisms have certain merits too. We will present two results that require strong morphisms.

Proposition 2.2.2. *Any strong morphism between effect algebras is injective.*

Proof. Suppose that $f : A \to B$ is strong and $f(a) = f(a')$. Then $f(a) \boxplus f((a')^{\perp}) = 1$, so since f is strong, $a \boxplus (a')^{\perp}$ is defined. By part 1 of Lemma 2.1.3, $a \leq a'$. Analogously $a' \leq a$, hence $a = a'$, proving injectivity. □

In the category **EA** of effect algebras with weak morphisms, bijectivity does not suffice to guarantee that a morphism is an isomorphism. Instead we have the following result.

Proposition 2.2.3. *Any bijective strong morphism is an isomorphism in* **EA**.

Proof. If $f : A \to B$ is bijective and strong, then it has a unique set-theoretic inverse $g : B \to A$. To prove that g is a weak morphism, suppose that $b \boxplus b'$ is defined. Then $f(g(b)) = b$ and $f(g(b')) = b'$, so since f is strong, $g(b) \boxplus g(b')$ is defined. Since f is injective and $f(g(b) \boxplus g(b')) = f(g(b \boxplus b'))$, we have $g(b) \boxplus g(b') = g(b \boxplus b')$. Finally, it is easy to see that g preserves 0, 1, and complements. □

The distinction between weak and strong morphisms can be made for all partial algebraic structures. The terminology that we use here comes from [58]. Sometimes strong morphisms are called closed morphisms, following [23]. In the effect algebra literature, one sometimes encounters the term monomorphism. However, we will avoid this term, due to possible confusion with the categorical notion of monomorphism.

Also for defining subobjects in the category of effect algebras we have to be careful with partiality. The following notion of subalgebra is most useful.

Definition 2.2.4. A subset A of an effect algebra B is an *effect subalgebra* if:

- A contains 0 and 1.

- A is closed under complements.
- Whenever $a, a' \in A$ and $a \boxplus a'$ is defined in B, then $a \boxplus a'$ lies in A.

Equivalently, A is a subalgebra of B whenever the inclusion map $A \hookrightarrow B$ is a strong morphism of effect algebras. It is automatically injective by Proposition 2.2.2. We will give an example that illustrates why we require subalgebras to be given by strong injective morphisms.

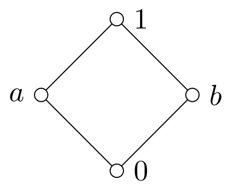
Example 2.2.5. We consider the subsets $A = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ and $B = \{0, \frac{1}{4}, \frac{3}{4}, 1\}$ of $[0,1]$. The subset A is a subalgebra. The set B is an effect algebra with the operations from $[0,1]$, so the inclusion map $B \hookrightarrow [0,1]$ is a weak morphism. However, since $\frac{1}{4} \boxplus \frac{1}{4}$ is defined in $[0,1]$ but not in B, it is not a subalgebra.

Observe that B is isomorphic to the power set effect algebra $\mathcal{P}(2)$, where the singletons in $\mathcal{P}(2)$ correspond to $\frac{1}{4}$ and $\frac{3}{4}$. Hence B does not inherit the order from $[0,1]$: in $[0,1]$, we have $\frac{1}{4} \leq \frac{3}{4}$, while in B, these two elements are incomparable. This is the reason why strong morphisms are more useful for defining subalgebras than weak injective morphisms.

The category **EA** is complete and cocomplete, and possesses a well-behaved tensor product, as proven in [77]. We will regularly need products, coproducts, and tensor products, so we will describe these briefly here. The product of effect algebras is simply the cartesian product with pointwise operations.

To construct the coproduct of A and B, put an equivalence relation $\sim$ on their disjoint union $A \coprod B$ by identifying 0_A with 0_B and 1_A with 1_B. The coproduct $A + B$ is then the quotient $(A \coprod B)/\sim$. Denote the coprojections $A \to A + B$ and $B \to A + B$ by ι_A and ι_B, respectively. Then the sum of two elements $\iota_A(a_1)$ and $\iota_A(a_2)$ is defined if and only if $a_1 \boxplus a_2$ is defined in A, and in that case $\iota_A(a_1) \boxplus \iota_A(a_2) = \iota_A(a_1 \boxplus a_2)$. Likewise one defines the sum of $\iota_B(b_1)$ and $\iota_B(b_2)$. The sum of $\iota_A(a)$ and $\iota_B(b)$ is never defined for $a \neq 0, 1$ and $b \neq 0, 1$. The orthocomplement in $A + B$ is derived from the ones in A and B. Showing that $A + B$ is indeed the coproduct of A and B is straightforward.

Example 2.2.6. Consider the effect algebra $L_2 = \{0, \frac{1}{2}, 1\}$. The coproduct $L_2 + L_2$ is isomorphic to the effect algebra with elements 0, a, b, and 1, where $a \boxplus a = b \boxplus b = 1$ and $a \boxplus b$ is undefined. The underlying partial order of this algebra can be depicted as



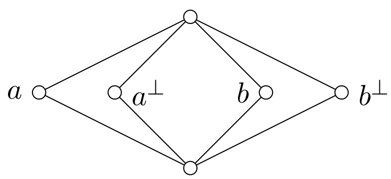
Note that this is the same as the order underlying the power set $\mathcal{P}(2)$, even though $L_2 + L_2$ and $\mathcal{P}(2)$ are not isomorphic as effect algebras.

Definition 2.2.7. A *bimorphism* of effect algebras is a map $f : A \times B \to C$ such that:

- f preserves addition in both variables separately. That is, if $a \boxplus a'$ is defined in A and $b \in B$ is arbitrary, then $f(a,b) \boxplus f(a',b)$ is defined and $f(a,b) \boxplus f(a',b) = f(a \boxplus a', b)$, and similarly for the second variable.
- $f(1,1) = 1$.

In [77] it is shown that any two effect algebras A and B have a tensor product $A \otimes B$, which is constructed in such a way that bimorphisms $A \times B \to C$ correspond bijectively to morphisms $A \otimes B \to C$.

There is also a notion of a free effect algebra for weak morphisms. Formally, this means that the forgetful functor from **EA** to **Sets** has a left adjoint, called $\mathcal{MO} : \mathbf{Sets} \to \mathbf{EA}$. For a set X, the free effect algebra $\mathcal{MO}(X)$ can be described explicitly as follows. Its elements are the formal symbols 0, 1, x and $x^\perp$ for $x \in X$. Complements are defined in the obvious way. Addition is specified by $0 \boxplus z = z = z \boxplus 0$ for $z \in \mathcal{MO}(X)$ and $x \boxplus x^\perp = x^\perp \boxplus x = 1$ for $x \in X$. It is undefined in all cases not covered by these two laws. As an example, the free algebra $\mathcal{MO}(2)$ consists of the elements 0, a, $a^\perp$, b, $b^\perp$, and 1, and has underlying order



Observe that any finite free effect algebra $\mathcal{MO}(n)$ is isomorphic to a coproduct of n copies of the power set $\mathcal{P}(2)$. It is easy to see that the functor $\mathcal{MO}$ is indeed left adjoint to the forgetful functor.

2.3 Totalization

When working with effect algebras, the partial operation forces us to keep track of when a sum is defined. This can make working with effect algebras difficult. Therefore it is sometimes useful to formally add undefined sums, in order to change the effect algebra into a structure with a total operation. The structure obtained by totalization is a *barred commutative monoid*, as shown in [77]. This is defined as a commutative monoid M together with a distinguished element u called the *bar*, satisfying the following requirements:

- Positivity: if $x + y = 0$, then $x = y = 0$.
- Barred cancellativity: if $x + y = x + z = u$, then $y = z$.

Any barred commutative monoid gives an effect algebra by restriction to elements below the bar: if (M, u) is a barred commutative monoid, then

$$\{x \in M \mid \text{there exists } y \text{ such that } x + y = u\}$$

is an effect algebra. In the other direction, any effect algebra A can be totalized into a barred commutative monoid. This monoid is obtained by first forming the collection of formal sums of elements of A, and then quotienting out the smallest congruence that identifies $a + b$ with $a \boxplus b$, whenever $a \boxplus b$ is defined, and the empty sum with 0. The bar is the equivalence class of 1.

Different barred commutative monoids may give rise to the same effect algebra, so this construction is not an equivalence of categories. However, it does provide a coreflection between the categories of effect algebras and barred commutative monoids.

Barred commutative monoids are only cancellative below the bar. In general, one cannot say a lot about elements above the bar, and for this reason it may still be difficult to work with the barred monoid constructed from an effect algebra. Many effect algebras admit a group as totalization instead of a monoid. These are called interval effect algebras and have better properties than arbitrary effect algebras, since their totalization is cancellative everywhere.

Before defining interval effect algebras, observe that each monoid can be made into a group. This contruction, called the *Grothendieck construction*, is a generalization of the construction of the integers from the natural numbers: if M is a commutative monoid, then form the set M^2 of pairs (m, n) of elements of M. The interpretation of such a pair is a formal difference $m - n$. Let $\sim$ be the congruence relation on M^2 generated by $(m, n) \sim (m', n')$ if and only if there exists some $k \in M$ for which $m + n' + k = n + m' + k$.

Then $M^2/\sim$ is an abelian group. Abstractly, this gives a functor from commutative monoids to abelian groups, which is left adjoint to the forgetful functor $\mathbf{Ab} \to \mathbf{CMon}$.

Since each effect algebra can be made into a commutative monoid by totalization, the above construction assigns an abelian group to any effect algebra. The resulting group is called the *universal group* in [38]. However, this group may not contain all information about the original effect algebra. If the totalization of the effect algebra is not cancellative above the bar, then the free abelian group generated by the monoid may identify some elements of the monoid, and hence the monoid cannot be reconstructed from it. We will mainly be interested in effect algebras whose enveloping group contains enough information to reconstruct the barred commutative monoid and hence the effect algebra. This requires a partial order on the group.

We will first define a way to obtain effect algebras from groups with partial orders.

Definition 2.3.1. A *partially ordered abelian group* is an abelian group G equipped with a partial order $\leq$, such that the condition $x \leq y$ implies $x + z \leq y + z$. An element $x \in G$ is called *positive* if $x \geq 0$, and *strictly positive* if $x > 0$.

Observe that our definition of positive is not the standard definition for e.g. real numbers. However, it is the usual notion of positivity in the theory of partially ordered groups, and in the theory of operator algebras.

For any partially ordered abelian group G and any positive $u \in G$, the interval $[0, u]_G = \{g \in G \mid 0 \leq g \leq u\}$ forms an effect algebra: the partial addition is given by the group operation $+$, where $g \boxplus h$ is defined if and only if $g + h$ lies below the element u. Complements are given by $g^\perp = u - g$, and the minimal and maximal element are 0 and u. We introduce a name for effect algebras of this form.

Definition 2.3.2. An effect algebra A is called an *interval effect algebra* if there exists a partially ordered abelian group G and a positive element $u \in G$, such that A is isomorphic to $[0, u]_G$.

We can form interval effect algebras using any positive $u \in G$ as a unit, but we cannot say much about the connection between an interval effect algebra and its surrounding group unless u satisfies some extra requirements. It is often desirable that u generates the ordered group G in the following sense.

Definition 2.3.3. An element u in a partially ordered abelian group G is called an *order unit* if for every $g \in G$ there exists a natural number n such that $g \leq n \cdot u$. A partially ordered abelian group together with a chosen order unit is called an *order unit group*. The category of order unit groups is denoted **OUGrp**. Its morphisms are group homomorphisms preserving the order and the order unit.

We will restrict our attention to order unit groups. This does not cause any loss of generality, since any positive u in an ordered group G generates an ordered group

$$\{g \in G \mid -n \cdot u \leq g \leq n \cdot u \text{ for some } n \in \mathbb{N}\}$$

in which u is an order unit. Therefore any interval effect algebra is an interval in some order unit group.

Most of the effect algebras encountered before are interval effect algebras.

Examples 2.3.4.

1. A simple example of a partially ordered abelian group is $\mathbb{R}$ with addition and the usual order. The element 1 is an order unit, and the resulting effect algebra is the unit interval $[0, 1]$.

2. Also the chain $L_n = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\}$ is an interval effect algebra. An enveloping group is the group $\mathbb{Z}$ of integers under addition. Taking $n \in \mathbb{Z}$ as order unit gives the interval $[0, n]_{\mathbb{Z}} = \{0, 1, 2, \ldots, n\}$ whose effect algebra operation is partial addition. This effect algebra is isomorphic to L_n.

3. The Boolean algebra $\mathcal{P}(n)$, viewed as an effect algebra, is an interval effect algebra. The surrounding group is $\mathbb{Z}^n$ with pointwise addition and order, and the order unit is $(1, 1, \ldots, 1)$.

4. Effects on a Hilbert space also form an interval effect algebra. The effect algebra $\mathcal{E}f(H)$ was defined as the unit interval within the set of bounded operators $\mathcal{B}(H)$, but we cannot use $\mathcal{B}(H)$ as enveloping group, since it has no natural order. However, the self-adjoint operators on H do form an ordered abelian group under addition. The order on $\mathcal{SA}(H)$ is given by the Löwner order, in which $A \leq B$ if and only if $\langle x \mid Ax \rangle \leq \langle x \mid Bx \rangle$ for all vectors $x \in H$. The identity $I \in \mathcal{SA}(H)$ is an order unit, and the unit interval $[0, I]_{\mathcal{SA}(H)}$ is precisely $\mathcal{E}f(H)$.

5. For an example of a non-interval effect algebra, we consider the following algebra from [107]. The algebra has elements 0, a, $2a$, b,

$2b$, $3b$, c, $2c$, 1, and the addition is determined by $3a = 4b = 3c = a \boxplus b \boxplus c = 1$. Suppose that this algebra is the unit interval in some order unit group (G, u). Then in G we have $12a + 12b + 12c = 4(3a) + 3(4b) + 4(3c) = 4u + 3u + 4u = 11u$, but also $12a + 12b + 12c = 12(a + b + c) = 12u$, hence $11u = 12u$. This cannot happen in a non-trivial order unit group, so the algebra is not an interval. The same argument shows that this is an example of an effect algebra whose totalization as a monoid is not cancellative above the bar.

We will now present some results that help showing that certain effect algebras are interval effect algebras. The following states that any subalgebra of an interval effect algebra is again an interval effect algebra.

Proposition 2.3.5. *Let $A = [0, u]_G$ be an interval effect algebra, and $B \subseteq A$ an effect subalgebra of A. Then $B = [0, u]_H$ for some subgroup H of G containing u.*

Proof. This proof is adapted from [38, Cor. 1.4.5], see also [18]. Since B is a subset of G, it generates a subgroup H of G. We will show that $B = [0, u]_H$. It is clear that $B \subseteq [0, u]_H$. For the reverse inclusion, take an $h \in H$ for which $0 \leq h \leq u$. Since B generates H, h can be written as $h = b_1 + \cdots + b_n$ where each b_i lies either in B or in $-B$. We will prove that $h \in B$, by induction on n. If $n = 1$ then $h = b_1$ for $b_1 \in B \cup -B$. In this case b_1 cannot lie in $-B$ (unless $b_1 = 0$), since h is positive. Therefore $h = b_1 \in B$. For the induction step, suppose that for all positive $h' = b_1 + \cdots + b_{n-1}$, if each b_i lies in $B \cup -B$, then $h' \in B$. Since $h = b_1 + \cdots + b_{n-1} + b_n$ is positive, either $b_1 + \cdots + b_{n-1}$ is positive or b_n is positive. In the first case, apply the induction hypothesis and use that $-b_n \leq b_1 + \cdots + b_{n-1}$. In the second case, use that $-(b_1 + \cdots + b_{n-1}) \leq b_n$. □

Example 2.3.6. Projections on a Hilbert space H form an orthomodular lattice, and hence an effect algebra. This is a subalgebra of the effects on H. Since $\mathcal{E}f(H)$ is an interval effect algebra, $\mathcal{P}roj(H)$ is an interval effect algebra as well. Effects are the unit interval in the group $\mathcal{SA}(H)$, so according to the proof of the above proposition, $\mathcal{P}roj(H)$ is the unit interval in the subgroup of $\mathcal{SA}(H)$ generated by $\mathcal{P}roj(H)$. Concretely, this group is given by $\{\pm P_1 \pm \cdots \pm P_n \mid n \in \mathbb{N},\ P_i \in \mathcal{P}roj(H)\}$.

In [45] it has been shown that interval effect algebras are also closed under taking products, coproducts, and tensor products. Before we can prove this, we need several facts on partially ordered abelian groups. The following fact guarantees that certain quotients of partially ordered abelian groups exist.

Lemma 2.3.7. *Let U be a subgroup of a partially ordered abelian group G. Assume that 0 is the only positive element in U. Then G/U acquires the structure of a partially ordered abelian group if we set $[g] \leq [h]$ if and only if there exists an element $u \in U$ such that $g + u \leq h$ in G.*

Proof. It is clear that G/U is an abelian group, and that the order is reflexive and transitive. It remains to prove anti-symmetry. Suppose that $[g] \leq [h]$ and $[h] \leq [g]$. Then $g + u \leq h$ and $h + v \leq g$ for certain $u, v \in U$, so $g + u + v \leq g$, whence $-(u + v) \geq 0$. Since $-(u + v)$ lies in U, the hypothesis gives $u + v = 0$. Then in the chain $g = g + u + v \leq h + v \leq g$, all inequalities must be equalities, hence $g + u + v = h + v$. It follows that $h - g \in U$, which gives $[g] = [h]$, as desired. □

We also need tensor products of partially ordered abelian groups. Given two such groups G and H, let $G \otimes H$ be their tensor product as abelian groups. We say that an element in $G \otimes H$ is positive if it can be written as a sum of tensors $g \otimes h$, where $g \geq 0$ in G and $h \geq 0$ in H. Then the order on $G \otimes H$ is defined by stipulating that $x \leq y$ for $x, y \in G \otimes H$ if and only if $y - x$ is positive. Explicitly, this means that $y = x + \sum_i g_i \otimes h_i$ for certain positive elements $g_i \in G$, $h_i \in H$.

The closure properties of interval effect algebras are as follows.

Proposition 2.3.8. *Let $A = [0, u]_G$ and $B = [0, v]_H$ be interval effect algebras.*

1. *The product $A \times B$ is the interval $[0, (u, v)]_{G \times H}$.*
2. *Let $U \subseteq G \times H$ be the subgroup generated by $(u, -v)$, and let K be the quotient $(G \times H)/U$. Then $A + B \cong [0, (u, 0)]_K$.*
3. *The tensor product $A \otimes B$ is the interval $[0, u \otimes v]_{G \otimes H}$.*

Proof. Since the proofs of 1 and 3 are straightforward, we will only prove 2. Lemma 2.3.7 applies, so K is indeed an ordered abelian group. We will use the description of $A + B$ from Section 2.2. To simplify notation, we assume that $A \cap B = \{0, 1\}$ and write a for an element of the form $\iota_A(a)$, and b for an element of the form $\iota_B(b)$.

Define a map $\varphi : A + B \to [0, (u, 0)]_K$ by $\varphi(a) = [(a, 0)]$ and $\varphi(b) = [(0, b)]$. It is clear that $\varphi(a)$ lies in the interval $[0, (u, 0)]$, and $\varphi(b)$ lies in the interval because $[(u, 0)] = [(0, v)]$ in the quotient.

By Propositions 2.2.2 and 2.2.3 it suffices to show that φ is strong and surjective. To show that φ is strong, suppose first that $\varphi(a) \boxplus \varphi(a')$ is defined for certain $a, a' \in A$. Then $(a + a' + nu, -nv) \leq (u, 0)$ for some

$n \in \mathbb{Z}$. All strictly positive multiples of v are greater than or equal to v, and all negative multiples of v are less than zero. Therefore, since $-nv \leq 0$, we obtain $n \geq 0$. Hence $a + a' \leq (1 - n)u \leq u$, which means that $a \boxplus a'$ is defined. Similarly the condition that $\varphi(b) \boxplus \varphi(b')$ is defined implies that $b \boxplus b'$ is defined. Now consider the situation where $\varphi(a) \boxplus \varphi(b)$ is defined. Then there is an integer n such that $a + nu \leq u$ and $b - nv \leq 0$. Since a lies in the interval $[0, u]$ and b in $[0, v]$, we get $n \leq 1$ and $n \geq 0$, so $n = 0$ or $n = 1$. In the first case $b = 0$, and in the second case $a = 0$, so in both cases $a \boxplus b$ is defined. This finishes the proof of strength.

To prove surjectivity, suppose that $[(g, h)] \geq 0$ and $[(g, h)] \leq [(u, 0)]$. Then there are integers n, m such that $(g + nu, h - nv) \geq 0$ and $(g + mu, h - mv) \leq (u, 0)$. In other words, $-nu \leq g \leq (1 - m)u$ and $nv \leq h \leq mv$. Define $a = g + nu$ and $b = h - nv$. Rewriting the inequalities for g and h gives $0 \leq a \leq (1 - (m - n))u$ and $0 \leq b \leq (m - n)v$. In particular, $m - n \geq 0$ and $1 - (m - n) \geq 0$, so $m - n$ is either 0 or 1. If it is 0, then $b = 0$ and $a \in [0, u]_G$, so we can apply the map φ to a. This gives $\varphi(a) = [(g + nu, 0)] = [(g + nu, b)] = [(g + nu, h - nv)] = [(g, h)]$. Similarly, if $m - n = 1$, then $b \in [0, v]_H$ and $\varphi(b) = [(g, h)]$. □

Example 2.3.9. The free effect algebra $\mathcal{MO}(2)$ is an interval effect algebra, because it is the coproduct of two copies of $\mathcal{P}(2)$. The enveloping group of $\mathcal{P}(2)$ is $\mathbb{Z}^2$ with unit element $(1, 1)$. Hence the enveloping group of $\mathcal{MO}(2)$ is the quotient of $\mathbb{Z}^2 \times \mathbb{Z}^2 \cong \mathbb{Z}^4$ by the subgroup U generated by $(1, 1, -1, -1)$. As unit element in $\mathbb{Z}^4/U$ we take $(1, 1, 0, 0)$, which ensures that $[0, (1, 1, 0, 0)]_{\mathbb{Z}^4/U} \cong \mathcal{MO}(2)$.

There is a more explicit way to describe this group. Let A be the abelian group $\{(a, b, c, d) \in \mathbb{Z}^4 \mid a + d = b + c\}$, with order inherited from $\mathbb{Z}^4$. The map from $\mathbb{Z}^4/U$ to A sending a congruence class $(a, b, c, d) + U$ to $(a + c, a + d, b + c, b + d)$ is a well-defined isomorphism $\mathbb{Z}^4/U \cong A$. Since it maps the unit $(1, 1, 0, 0)$ of $\mathbb{Z}^4/U$ to $(1, 1, 1, 1)$ in A, the interval $[0, (1, 1, 0, 0)]$ in $\mathbb{Z}^4/U$ corresponds to the interval $[0, (1, 1, 1, 1)]$ in A. The elements in this interval are

$$(0, 0, 0, 0),\ (1, 1, 0, 0),\ (0, 0, 1, 1),\ (1, 0, 1, 0),\ (0, 1, 0, 1),\ (1, 1, 1, 1),$$

and it is easy to see that this is precisely $\mathcal{MO}(2)$ with the addition from A.

Despite the many similarities between interval effect algebras and partially ordered abelian groups, the study of interval effect algebras cannot be reduced to the study of ordered groups. This is because the two structures are not categorically equivalent. This will be shown in Example 3.4.6.

In certain effect algebras, it is possible to multiply elements with scalars from the unit interval $[0, 1]$. This leads to a commonly used subcategory of **EA**.

Definition 2.3.10. An *effect module* is an effect algebra A endowed with a scalar multiplication $\cdot : [0, 1] \times A \to A$, such that for all $r, s \in [0, 1]$ and $a, b \in A$ we have:

- $r \cdot (s \cdot a) = (rs) \cdot a$.
- If $r + s \leq 1$, then $r \cdot a \boxplus s \cdot a$ is defined and $(r + s) \cdot a = r \cdot a \boxplus s \cdot a$.
- If $a \boxplus b$ is defined, then $r \cdot a \boxplus r \cdot b$ is defined and $r \cdot (a \boxplus b) = r \cdot a \boxplus r \cdot b$.
- $1 \cdot a = a$.

A morphism of effect modules is a morphism of effect algebras that additionally preserves the scalar multiplication. This results in a category denoted **EMod**.

Effect modules were introduced in [62] under the name convex effect algebras. Here it is also proven that every effect module is an interval effect algebra. Thus we obtain a large class of effect algebras for which it is easy to check that they are intervals.

2.4 Orthoalgebras

Effect algebras model unsharp measurements in quantum mechanics. Unsharpness means that measurements need not satisfy the principle of non-contradiction, that is, it may happen that $a \wedge a^{\perp}$ is non-zero. Since not all effect algebras form a lattice, we have to be careful if we want to phrase this condition formally.

Definition 2.4.1. An element a of an effect algebra A is called *sharp* if, whenever $b \leq a$ and $b \leq a^{\perp}$, then $b = 0$.

In other words, a is sharp if and only if $a \wedge a^{\perp}$ exists and equals zero. To gain more insight into unsharpness in effect algebras, it is interesting to characterize the class of sharp effect algebras inside it. The following definition comes from [48, 46].

Definition 2.4.2. An *orthoalgebra* is an effect algebra in which $a \boxplus a$ is never defined, unless $a = 0$.

Proposition 2.4.3. *An effect algebra is an orthoalgebra if and only if all its elements are sharp.*

Proof. Suppose that A is an orthoalgebra, and that $b \leq a$ and $b \leq a^{\perp}$. The sum $a \boxplus a^{\perp}$ is defined, hence by part 2 of Lemma 2.1.3, $b \boxplus b$ is defined too. Since A is an orthoalgebra, $b = 0$.

For the converse, suppose that a is sharp and $a \boxplus a$ is defined. Then $a \leq a^{\perp}$ by part 1 of Lemma 2.1.3. Of course we also have $a \leq a$, so $a = 0$, as desired. □

Boolean algebras are among the simplest examples of orthoalgebras. General orthoalgebras can be obtained by gluing several Boolean algebras together. Hence orthoalgebras become easier to analyze if we understand their constituent Boolean algebras, and the gluing construction, so we will take a look at this construction here.

We will frequently use the notion of a test on an effect algebra.

Definition 2.4.4. An *n-test* on A consists of n elements $a_1, \ldots, a_n$ such that $a_1 \boxplus \cdots \boxplus a_n$ is defined and equals 1.

We introduce the following notation for tests:

$$\mathcal{T}_n(A) = \{(a_0, \ldots, a_n) \mid a_0 \boxplus \cdots \boxplus a_n = 1\}$$

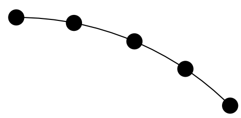
Note that $\mathcal{T}_n(A)$ contains the $(n+1)$-tests; this convention will turn out to be beneficial when defining cohomology of effect algebras. If all but one elements of a test are known, then the final one is fixed since orthocomplements in an effect algebra are unique. Therefore $\mathcal{T}_n(A)$ is isomorphic to the set

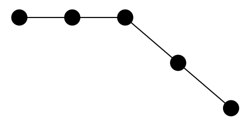
$$\{(a_1, \ldots, a_n) \mid a_1 \boxplus \cdots \boxplus a_n \text{ is defined}\}.$$

All information contained in a finite orthoalgebra can be conveniently organized into a *Greechie diagram*. Our description of Greechie diagrams follows [69]. More background on the topic can be found in [82, 119]. We need a generalization of the notion of a graph, called a hypergraph. A graph consists of points and a set of two-element subsets of the points, representing the edges. A hypergraph generalizes this by dropping the requirement that the subsets have two points.

Definition 2.4.5. A *hypergraph* comprises a set P of points and a set $H \subseteq \mathcal{P}(P)$, elements of which are called *hyperedges* or *lines*, such that $\bigcup H = P$ and $\varnothing \notin H$.

Each hypergraph can be represented pictorially. To do this, we simply draw a point for each point in the hypergraph. A hyperedge is drawn as a smooth curve connecting all points in the corresponding hyperedge. For example, consider the following two diagrams.

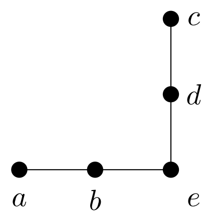




The diagram on the left represents a hypergraph with 5 points and one single hyperedge containing all of those points. The diagram on the right represents a hypergraph with 5 points and two hyperedges of 3 points, because it has a corner.

To any finite orthoalgebra A we can assign a hypergraph, called its Greechie diagram. A non-zero element a of an effect algebra is called an *atom* if the only element lying below a is 0. A test on A consisting of only atoms is called a *maximal test*, since it has no refinements without zeroes. The Greechie diagram of an orthoalgebra A is a hypergraph with a point for each atom, and a hyperedge for each maximal test.

Example 2.4.6. The Greechie diagram



represents an orthoalgebra with 5 atoms a, b, c, d, e, in such a way that $\{a,b,e\}$ and $\{c,d,e\}$ are maximal tests. This means that $a \boxplus b \boxplus e = 1$, $c \boxplus d \boxplus e = 1$, and that the sum of an atom in $\{a,b\}$ and an atom in $\{c,d\}$ is undefined. The condition $a \boxplus b \boxplus e = c \boxplus d \boxplus e$ implies that $a \boxplus b = c \boxplus d$. Thus the orthoalgebra consists of 12 elements 0, a, b, c, d, e, $a \boxplus b = c \boxplus d$, $a \boxplus e$, $b \boxplus e$, $c \boxplus e$, $d \boxplus e$, 1, with partial addition determined by the maximal tests.

Note that the Greechie diagram is more concise than a description of the full orthoalgebra. This is the reason why finite orthoalgebras are often defined in terms of their Greechie diagrams.

The construction of an orthoalgebra from a Greechie diagram is made more precise using the framework of test spaces, see for example [47, 43]. It can also be interpreted as pasting Boolean subalgebras together, as discussed in [63, 98]. Each maximal test in an orthoalgebra generates a maximal Boolean subalgebra, elements of which are sums of its atoms. A maximal Boolean subalgebra is called a *block*. Conversely, the atoms of any block form a maximal test. Thus there is a one-to-one correspondence between blocks and maximal tests in any finite orthoalgebra. Since each finite orthoalgebra is completely determined by its atoms and maximal tests, it is the union of its blocks.

When gluing blocks in an orthoalgebra together, it is often desirable to know how tests on a union relate to tests on the constituents. The following result gives such a relation.

Proposition 2.4.7. *Let A and B be subalgebras of an effect algebra E, such that $E = A \cup B$. Any test on E is a test on A or a test on B.*

Proof. Suppose that $(t_0, \dots, t_n)$ is a test on E. Assume towards a contradiction that it is neither a test on A, nor a test on B. Then there are i and j such that $t_i \notin A$ and $t_j \notin B$. Since $(t_0, \dots, t_n)$ is a test on the union, we have $t_i \in B \setminus A$ and $t_j \in A \setminus B$, and $t_i \boxplus t_j$ is defined in $E = A \cup B$. Without loss of generality, assume that $t_i \boxplus t_j \in A$. Then $t_i \boxplus t_j \boxplus a = 1$ for some $a \in A$, so t_i is the orthocomplement of $t_j \boxplus a$. The sum $t_j \boxplus a$ is defined in E, and both t_j and a lie in A. Since A is a subalgebra of E, the sum $t_j \boxplus a$ is also defined in A. Therefore $t_i = (t_j \boxplus a)^\perp$ also lies in A, which is a contradiction. □

Chapter 3

Convexity

Effects on a physical system represent observables, or binary measurements, on the system. Another fundamental notion associated to the system is a state, which can be thought of as the result of a preparation procedure in an experiment. States and effects together determine the statistics of the experiment. The reason for this is that when a state and a measurement on a system are given, then one can compute a probability distribution over the possible outcomes of the measurement.

Quantum physics is inherently probabilistic. Therefore a good mathematical model for quantum states should also incorporate probabilistic aspects. An important aspect of probabilistic states is that we can form mixtures of those states. This means that, whenever $\sigma_1, \sigma_2, \ldots, \sigma_n$ are states and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are non-negative real numbers whose sum is 1, then there should be a state corresponding to the convex combination $\lambda_1\sigma_1 + \lambda_2\sigma_2 + \cdots + \lambda_n\sigma_n$. The algebraic structure that enables forming convex combinations in a coherent way will be called a convex set. In Sections 3.1 through 3.3 we will study properties of convex sets in general. In Section 3.4 we will look at the interplay between effects and their corresponding convex set of states.

3.1 Convex sets

Roughly speaking, a convex set is a set equipped with enough algebraic structure to speak of forming convex combinations, or probabilistic mixtures. There are various notions of convex sets, see for example [118, 61], or [60] for their relevance in quantum foundations. We will start with a very simple one, based on the so-called distribution monad on the category **Sets**, like in [74]. Later, we will gradually add more structure to obtain more refined notions. Define a functor $\mathcal{D} : \mathbf{Sets} \to \mathbf{Sets}$ sending a set X to the collection of maps $\varphi : X \to [0,1]$ for which the support $\mathrm{supp}(\varphi) = \{x \in X \mid \varphi(x) \neq 0\}$ is finite and $\sum_{x\in X} \varphi(x) = 1$. Maps like this can be thought of as finite convex combinations, or probability distributions, on X. An element φ of $\mathcal{D}(X)$ can also be written as a formal sum $\lambda_1 x_1 + \cdots + \lambda_n x_n$, where $\{x_1, \ldots, x_n\} = \mathrm{supp}(\varphi)$ and $\lambda_i = \varphi(x_i)$. On a morphism $f : X \to Y$, the functor $\mathcal{D}$ is defined as

$$\mathcal{D}(f)(\lambda_1 x_1 + \cdots + \lambda_n x_n) = \lambda_1 f(x_1) + \cdots + \lambda_n f(x_n).$$

The functor $\mathcal{D}$ is a monad with unit and multiplication given by

$$\begin{array}{ccc} X \xrightarrow{\eta} \mathcal{D}(X) & \qquad & \mathcal{D}(\mathcal{D}(X)) \xrightarrow{\mu} \mathcal{D}(X) \\ x \longmapsto 1x & \qquad & \sum_i \lambda_i \left(\sum_j \mu_{ij} x_{ij}\right) \longmapsto \sum_{i,j} \lambda_i \mu_{ij} x_{ij} \end{array}$$

The monad $\mathcal{D}$ is called the *distribution monad.*

Definition 3.1.1. A *convex set* is an Eilenberg–Moore algebra for the distribution monad. The category of convex sets is denoted **Conv**.

In a convex set X, we can assign to each convex combination $\sum_i \lambda_i = 1$ a function $X^n \to X$ denoted $(x_1, \ldots, x_n) \mapsto \sum_i \lambda_i x_i$. A morphism of convex sets preserves all convex combinations and is called an *affine* map.

Observe that if we know the functions $\mathrm{cc}_\lambda : (x, y) \mapsto \lambda x + (1 - \lambda) y$ for all $\lambda \in [0, 1]$, then all convex combination maps $(x_1, \ldots, x_n) \mapsto \sum_i \lambda_i x_i$ are known by iterating the binary convex combinations with suitable coefficients. Therefore a convex set can also be defined as a set equipped with maps cc_λ satisfying certain conditions. This is the approach taken in [118] and the first known definition of convex sets.

Examples 3.1.2.

1. The standard simplex $\mathcal{D}(n) = \{(x_1, \ldots, x_n) \mid x_i \in [0,1],\ \sum_i x_i = 1\}$ is a convex set. In fact, $\mathcal{D}(n)$ is the free convex set generated by n points, in the sense that each point in $\mathcal{D}(n)$ can be written as a convex combination of its vertices in a unique way. Categorically speaking, this follows from the fact that the distribution functor $\mathcal{D}$ considered as a functor from **Sets** to **Conv** is left adjoint to the forgetful functor **Conv** $\to$ **Sets**.

   Since there are multiple possible notations for the standard simplex, a remark about our choice of notation is in place. We denote the convex set $\{(x_1, \ldots, x_n) \mid x_i \in [0,1],\ \sum_i x_i = 1\}$ by $\mathcal{D}(n)$, but this set is also often written as Δ_{n-1}, i.e. the index denotes the dimension of the simplex instead of the number of generators. The notation $\mathcal{D}(n)$ is mostly used in the context of convexity theory, while the notation Δ_{n-1} is more common in topology. Therefore we adapt the following convention: when we wish to speak about the simplex equipped with its natural convex structure, we write it as $\mathcal{D}(n)$, and when we consider the simplex as a bare topological space, we write it as Δ_{n-1}.

This convention has the minor drawback that it requires two distinct notations for the same object, but it also has several advantages. Later we will define coproducts and tensor products of convex sets. The virtue of denoting the free convex space as $\mathcal{D}(n)$ is that this notation behaves better with respect to these constructions, because we get $\mathcal{D}(n) + \mathcal{D}(m) \cong \mathcal{D}(n+m)$ and $\mathcal{D}(n) \otimes \mathcal{D}(m) \cong \mathcal{D}(nm)$. The asset of the notation Δ_{n-1} is that it is traditional in topology, and that it works well in discussions about simplicial sets and cyclic sets, encountered in Chapter 5. Another advantage of distinguishing two notations is that it clarifies when we take the convex structure on the simplex into account, and when we ignore it.

2. Let H be a Hilbert space. A density matrix on H is a positive trace-class operator $\rho : H \to H$ for which $\mathrm{tr}(\rho) = 1$. The collection of all density matrices is denoted $\mathcal{DM}(H)$ and forms a convex set. This is because a convex combination of positive trace-class operators is again a positive trace-class operator, and the trace preserves convex combinations. This example is important in quantum mechanics, since density matrices represent mixed states.

3. Let L be a join-semilattice. Then L forms a convex set with binary convex combinations defined in the following way:

$$\lambda x + (1-\lambda) y = \begin{cases} x & \text{if } \lambda = 1 \\ x \vee y & \text{if } \lambda \in (0,1) \\ y & \text{if } \lambda = 0 \end{cases}$$

A common way to obtain a convex set is by taking a subset of an $\mathbb{R}$-vector space that is closed under convex combinations. In the above example, $\mathcal{D}(n)$ and $\mathcal{DM}(H)$ arise in this way, since $\mathcal{D}(n)$ can be embedded in $\mathbb{R}^n$, and $\mathcal{DM}(H)$ is a convex subset of the space of self-adjoint trace-class operators. The final example of a semilattice is not a convex subspace of a vector space, since convex combinations $\lambda x + (1-\lambda)y$ are independent of the choice of λ, except in the extreme cases where $\lambda \in \{0, 1\}$. This phenomenon cannot occur in a vector space. We wish to use convex sets to model probabilistic scenarios. These are better behaved if we exclude examples like the last one, and focus on those convex sets that are subsets of a vector space. The following characterization due to [100] is often helpful.

Proposition 3.1.3. *A convex set X can be embedded in an $\mathbb{R}$-vector space if and only if it is* cancellative*. This means that if $\lambda x+(1-\lambda)y = \lambda x'+(1-\lambda)y$ for some $\lambda \in (0,1)$, then $x = x'$.*

Proof. Since each vector space is cancellative, so is each of its convex subspaces. Conversely, suppose that X is a cancellative convex set. We will construct an enveloping vector space V in two steps. First we will construct a module over the semiring $\mathbb{R}_{\geq 0}$. Let $\sim$ be the equivalence relation on $\mathbb{R}_{\geq 0} \times X$ that identifies each $(0, x)$ with each $(0, y)$. Then the quotient $M := \mathbb{R}_{\geq 0} \times X/\sim$ can be made into an $\mathbb{R}_{\geq 0}$-module with addition defined by

$$(r, x) + (s, y) = \left(r + s, \frac{r}{r+s}x + \frac{s}{r+s}y\right),$$

and scalar multiplication by $r \cdot (s, x) = (rs, x)$. There is an embedding $X \hookrightarrow M$ sending x to $(1, x)$. The $\mathbb{R}_{\geq 0}$-module M is cancellative in the sense that $a + b = a' + b$ implies that $a = a'$. This follows from cancellativity of the convex set X. Thus we have proven that X embeds in a cancellative $\mathbb{R}_{\geq 0}$-module.

The second step is to construct an $\mathbb{R}$-vector space out of the module M. Applying the Grothendieck construction to M gives an abelian group $V := M \times M/\sim$, where $(m, n) \sim (m', n')$ if and only if there exists $k \in M$ such that $m+n'+k = m'+n+k$. Since M is cancellative, this is equivalent to $m+n' = m'+n$. Define multiplication by scalars in $\mathbb{R}$ via $r\cdot(m, n) = (rm, rn)$ for $r \geq 0$, and $r \cdot (m, n) = (-rn, -rm)$ for $r < 0$. This makes V into an $\mathbb{R}$-vector space. The map $M \to V$ given by $m \mapsto (m, 0)$ is injective since $(m, 0) \sim (m', 0)$ if and only if $m = m'$. Note that cancellativity is essential here. Hence the composition $X \hookrightarrow M \hookrightarrow V$ is also injective, proving that X embeds in the vector space V. □

The above proof can be conveniently summarized using categories. Since every $\mathbb{R}$-vector space is in particular a cancellative convex set, there is a forgetful functor from $\mathbf{Vect}_{\mathbb{R}}$ to the category of cancellative convex sets. This functor has a left adjoint, which is precisely the construction of a vector space out of a convex set in the proof.

3.2 Compact convex spaces

In order to describe probabilistic systems using convex sets, we need cancellative convex sets, or equivalently, convex sets that can be embedded in a vector space. For many purposes, it is not enough to work with subsets of an arbitrary vector space. For example, to work with extreme points in convex sets, or to develop a duality theory, we need more structure on the surrounding vector space. We shall require that it carries a topology that interacts well with the convex structure.

Definition 3.2.1. A topological vector space is said to be *locally convex* if its topology has a basis of convex sets.

There are several equivalent definitions of local convexity; the above one comes from [105].

We shall mostly be concerned with compact convex subspaces of locally convex vector spaces, since these form a rich and well-behaved class of convex spaces. Many vector spaces encountered in practice are locally convex, so this condition is not very restrictive. For an example of a convex subspace of a vector space that does not lie in a locally convex space, see [108]. We will introduce some terminology to simplify talking about the convex spaces under consideration, following [115].

Definition 3.2.2. We call a compact convex subspace of a locally convex topological vector space simply a *compact convex space*. Make these into a category **KConv** whose objects are pairs (X, V), where V is a locally convex space and X a compact convex subspace of V. A morphism from (X, V) to (Y, W) is simply a continuous affine map $X \to Y$. The surrounding vector space is ignored to ensure that properties of convex sets do not depend on a particular embedding.

An advantage of compact convex spaces defined in this way is that they have sufficiently many extreme points. Therefore many problems about a compact convex space can be solved by first analyzing the problem on extreme points.

Definition 3.2.3. An *extreme point* of a convex set X is a point $x \in X$ such that if $x = \lambda y + (1 - \lambda)z$ for some $\lambda \in (0, 1)$, then $x = y = z$.

In other words, an extreme point cannot be written as a non-trivial convex combination of other points. For example, in the convex set $\mathcal{D}(X)$, the extreme points are $1x$ for $x \in X$. In the set $\mathcal{DM}(H)$ of density matrices on a Hilbert space, the extreme points are the pure states $\rho_x : H \to H$ for $x \in H$, defined by $\rho_x(y) = \langle x \mid y \rangle x$.

The Krein–Milman Theorem roughly states that any compact convex space is generated by its extreme points. If X is a convex set without topology and $A \subseteq X$ is any subset, then the smallest convex subset of X containing A is called the *convex hull* of A. It is obtained by taking all possible convex combinations of points in A, yielding the set

$$\{\sum_i \lambda_i a_i \mid \lambda_i \in [0,1],\ \sum_i \lambda_i = 1,\ a_i \in A\}.$$

For compact convex spaces, we also need to take the topology into account. The smallest compact convex space surrounding a subspace is the closure of the convex hull in this case. Thus the Krein–Milman Theorem states that any point in a compact convex space can be obtained by taking a limit of convex combinations of extreme points. For a proof, see e.g. [105, Thm. 2.5.4].

Theorem 3.2.4 (Krein–Milman)**.** *Let X_{ext} be the set of extreme points in a compact convex space X. Then the convex hull of X_{ext} is dense in X.*

In the previous section, we characterized convex sets as algebras for the distribution monad $\mathcal{D}$. There is a similar categorical characterization of compact convex spaces, discussed in [14, 114, 115, 120, 50, 94]. The basic idea is that a compact convex space is a topological version of a convex set. Convex sets are algebras for the distribution monad $\mathcal{D}$, which represents discrete probability distributions on a set. The continuous analogue of a probability distribution is a probability measure. If we replace the monad $\mathcal{D}$ by a monad describing measures, we will obtain the desired continuous generalization.

Any compact Hausdorff space X gives rise to a Borel σ-algebra Σ_X of measurable sets. A probability measure on X is a map $\mu : \Sigma_X \to [0,1]$ such that:

- For all pairwise disjoint measurable $M_1, M_2, \ldots$, we have

$$\mu\left(\bigcup_{i=1}^{\infty} M_i\right) = \sum_{i=1}^{\infty} \mu(M_i).$$

- $\mu(X) = 1$.

A probability measure μ will be called a *Radon measure* if

$$\mu(M) = \sup_{K \subseteq M} \mu(K)$$

for all measurable M, where the supremum runs over all compact subsets of M. Informally, this says that the measure is determined by its values on compact sets. The set of Radon measures on X forms a compact convex space, denoted by $\mathcal{R}(X)$.

The significance of Radon measures lies in their tight connection to integration. If μ is a Radon measure on X and $f : X \to [0,1]$ is a continuous function, then we can integrate f along μ to obtain $\int_X f \,\mathrm{d}\mu \in [0,1]$. Thus the measure μ determines an integration operator $\int(-)\,\mathrm{d}\mu : C(X,[0,1]) \to [0,1]$. Since integration preserves sums and scalar multiplication, this is a morphism of effect modules. The content of the famous Riesz Representation Theorem is that all integration operators are obtained in this way. This can be considered as an equivalence between measures and integration. To state the Riesz Representation Theorem, observe that the collection of effect module morphisms $C(X,[0,1]) \to [0,1]$ forms a compact convex space, in which convex combinations are calculated pointwise.

Theorem 3.2.5 (Riesz Representation Theorem)**.** *For any compact Hausdorff space X, the map $\mathcal{R}(X) \to \mathbf{EMod}(C(X,[0,1]),[0,1])$ given by $\mu \mapsto \int_X(-)\,\mathrm{d}\mu$ is an isomorphism of compact convex spaces.*

A proof can e.g. be found in [105, Thm. 6.3.4].

The formation of Radon measures can be made functorial, as noticed in [115]. Denote the category of compact Hausdorff spaces and continuous maps by **KHaus**. Then there is a *Radon functor* $\mathcal{R} : \mathbf{KHaus} \to \mathbf{KConv}$, whose object part gives the Radon measures on a space. On a morphism $f : X \to Y$ in **KHaus**, the Radon functor is defined by $\mathcal{R}(f)(\mu)(N) = \mu(f^{-1}[N])$ for $\mu \in \mathcal{R}(X)$ and measurable $N \subseteq Y$.

Proposition 3.2.6. *The Radon functor $\mathcal{R}$ is left adjoint to the forgetful functor* **KConv** $\to$ **KHaus**.

This means that the collection of Radon measures $\mathcal{R}(X)$ can be viewed as the free compact convex space generated by X. It is indeed a generalization of the free convex set $\mathcal{D}(X)$ for finite X, since $\mathcal{R}(X) \cong \mathcal{D}(X)$ for all finite spaces X. All ingredients for the proof are in [14]; see also [115] for the categorical version.

Proof. Let X be a compact Hausdorff space, and Y a compact convex space. We have to establish a bijective correspondence between continuous maps $X \to Y$ and affine continuous maps $\mathcal{R}(X) \to Y$. If $f : \mathcal{R}(X) \to Y$ is affine and continuous, then it restricts to a continuous map $X \to Y$ given by $x \mapsto f(\delta_x)$. Here δ_x is the Dirac measure

$$\delta_x(M) = \begin{cases} 1 & \text{if } x \in M \\ 0 & \text{otherwise} \end{cases}$$

In the other direction, if $g : X \to Y$ is any continuous map, then it extends to an affine map $\mathcal{D}(X) \to Y$ sending $\sum_i \lambda_i x_i$ to $\sum_i \lambda_i g(x_i)$. Every finite probability distribution in $\mathcal{D}(X)$ can be considered as a probability measure in $\mathcal{R}(X)$, so there is an embedding $\mathcal{D}(X) \hookrightarrow \mathcal{R}(X)$. Explicitly this embedding maps $\sum_i \lambda_i x_i$ to $\sum_i \lambda_i \delta_{x_i}$. By the Krein–Milman Theorem, this embedding is dense, so the map $\mathcal{D}(X) \to Y$ extends uniquely to an affine continuous map $\mathcal{R}(X) \to Y$.

We will now check that both constructions are mutual inverses. Extending $g : X \to Y$ gives a map $\bar{g} : \mathcal{R}(X) \to Y$, and restriction of $\bar{g}$ gives the map $x \mapsto \bar{g}(\delta_x)$. Since δ_x lies in the image of the embedding $\mathcal{D}(X) \hookrightarrow \mathcal{R}(X)$, the extension $\bar{g}$ maps δ_x to $g(x)$, as desired.

For the other direction, let $f : \mathcal{R}(X) \to Y$ be an affine continuous map, and let $g : X \to Y$ be its restriction $g(x) = f(\delta_x)$. Then g extends to a map $\bar{g} : \mathcal{R}(X) \to Y$. On the subset $\mathcal{D}(X)$ of $\mathcal{R}(X)$, the action of $\bar{g}$ is given by $\bar{g}(\sum_i \lambda_i x_i) = \sum_i \lambda_i f(\delta_{x_i})$. Since f is affine, this is equal to $f\left(\sum_i \lambda_i \delta_{x_i}\right)$. Hence the map f coincides with $\bar{g}$ on $\mathcal{D}(X)$. But since f and $\bar{g}$ are continuous and $\mathcal{D}(X)$ is dense in $\mathcal{R}(X)$, they coincide everywhere, which is what we wanted to show. □

Now we are ready to explain why the Radon functor $\mathcal{R}$ is indeed a continuous version of the distribution functor $\mathcal{D}$, and that compact convex spaces are indeed a continuous version of convex sets. The adjunction

$$\mathbf{KConv} \underset{\mathcal{R}}{\overset{\dashv}{\leftrightarrows}} \mathbf{KHaus}$$

gives rise to a monad on **KHaus**, also denoted $\mathcal{R}$. Then the following result from [120] (see also [50]) gives a solid categorical foundation for the theory of compact convex spaces.

Theorem 3.2.7. *The category of Eilenberg–Moore algebras for the Radon monad $\mathcal{R}$ is equivalent to the category* **KConv**.

Hence compact convex spaces can be regarded as compact spaces in which each probability measure can be "integrated" to obtain an element from the space. This is analogous to the characterization of convex sets as algebras for the distribution monad, with finite probability distributions replaced by probability measures. This provides a more intrinsic characterization of compact convex spaces than our original definition, since it does not require an ambient locally convex vector space. Therefore we will often speak about a convex space X instead of a convex space (X, V).

3.3 Constructions of convex spaces

We will take a look at the categorical properties of compact convex spaces. In [115] it has been shown that the category **KConv** is complete and cocomplete, and that it carries a tensor product characterizing bimorphisms. We will frequently need products, coproducts, and tensor products of compact convex spaces, so we will describe their constructions here.

The product of two compact convex spaces X and Y is simply their cartesian product $X \times Y$ with pointwise operations. If X embeds in the locally convex space V and Y embeds in the locally convex space W, then the product $X \times Y$ embeds in $V \oplus W$.

The coproduct of two convex spaces can be described geometrically, using the embedding in a locally convex vector space. The following description is a slight modification of the construction in [115]. Suppose that (X, V) and (Y, W) are compact convex spaces. Then the coproduct $X + Y$ can be embedded in the vector space $V \oplus W \oplus \mathbb{R}$. To construct this coproduct, embed X in this larger vector space via the inclusion $x \mapsto (x, 0, 1)$, and embed Y via the inclusion $y \mapsto (0, y, 0)$. The convex hull of the disjoint union of X and Y is the coproduct of X and Y. This is made precise in the following.

Proposition 3.3.1. *If (X, V) and (Y, W) are two objects in the category* **KConv***, then their coproduct is*

$$X + Y = \{(rx, (1-r)y, r) \mid r \in [0, 1],\ x \in X,\ y \in Y\} \subseteq V \oplus W \oplus \mathbb{R}.$$

Proof. Define embeddings $i_X : X \to X + Y$ and $i_Y : Y \to X + Y$ via $i_X(x) = (x, 0, 1)$ and $i_Y(y) = (0, y, 0)$. Given affine maps $f : X \to Z$ and $g : Y \to Z$, define $h : X + Y \to Z$ by

$$h(rx, (1-r)y, r) = rf(x) + (1-r)g(y).$$

Then $h \circ i_X = f$ and $h \circ i_Y = g$, so it remains to be shown that h is the unique map with this property. Suppose that $h' : X + Y \to Z$ is an affine map for which $h' \circ i_X = f$ and $h' \circ i_Y = g$. Then

$$h'(rx, (1-r)y, r) = h'\left(r(x, 0, 1) + (1-r)(0, y, 0)\right) = rf(x) + (1-r)g(y),$$

which proves uniqueness. □

Example 3.3.2. Denote the one-point convex space by 1. The coproduct $1 + \cdots + 1$ of n copies of this space is the convex hull of n points, embedded in $\mathbb{R}^{n-1}$ in such a way that they are all affinely independent. Therefore this coproduct is the standard simplex $\mathcal{D}(n)$.

We continue with a discussion of the tensor product of compact convex spaces. There are various notions of tensor product, several of which are compared in [97]. We will use the tensor product characterizing bimorphisms of convex spaces.

If X, Y, and Z are compact convex spaces, then a map $X \times Y \to Z$ is called *bi-affine* is it is affine in both variables separately. A *tensor product* of X and Y is a compact convex space $X \otimes Y$ equipped with a bi-affine map $\otimes : X \times Y \to X \otimes Y$ such that for every compact convex space Z and every bi-affine $f : X \times Y \to Z$ there exists a unique affine map $g : X \otimes Y \to Z$ such that $g \circ \otimes = f$. Semadeni proves in [115] that any two compact convex spaces admit a tensor product, and that it is unique up to isomorphism.

There are several ways to construct the tensor product $X \otimes Y$. A standard way is by starting with the free compact convex space generated by $X \times Y$, which is $\mathcal{R}(X \times Y)$. The quotient by a suitable congruence relation gives the tensor product $X \otimes Y$. Since this is a very common way to form tensor products, we will not provide the details here.

Another construction of the tensor product uses a double dualization procedure. Let $\mathrm{BiAff}(X, Y; [0,1])$ denote the collection of bi-affine morphisms $X \times Y \to [0,1]$. This collection forms an effect module with pointwise operations. Then the effect module morphisms $\mathrm{BiAff}(X, Y; [0,1]) \to [0,1]$ form a compact convex space that serves as a tensor product of X and Y.

The above tensor product enjoys many good properties. The one-point convex space 1 acts as a unit for the tensor. Furthermore, the tensor product distributes over coproducts. From these two facts, together with the isomorphism $\mathcal{D}(n) \cong 1+\cdots+1$, it can be deduced that the tensor product of standard simplices is $\mathcal{D}(n) \otimes \mathcal{D}(m) \cong \mathcal{D}(nm)$. More generally, the tensor product of two free compact convex spaces is $\mathcal{R}(X) \otimes \mathcal{R}(Y) \cong \mathcal{R}(X \times Y)$.

3.4 State spaces

Definition 3.4.1. A *state* on an effect algebra A is a morphism from A to $[0,1]$. The collection of all states on A is called the *state space* and denoted $\mathrm{St}(A)$.

The state space of an effect algebra can be equipped with a topology. It is always a subset of the product $[0,1]^A$, and hence $\mathrm{St}(A)$ inherits a subspace topology from the product topology on $[0,1]^A$.

Proposition 3.4.2. *The state space of an effect algebra A is always a compact convex space.*

Proof. We have to show that $\mathrm{St}(A)$ is a compact convex subspace of a locally convex vector space. If σ and τ are two states on A and $\lambda \in [0,1]$, then $\lambda\sigma + (1-\lambda)\tau$ is again a morphism of effect algebras. This means that the state space is a convex set.

The state space $\mathrm{St}(A)$ is a subspace of the vector space $\mathbb{R}^A$, which is locally convex in the product topology. So it remains to prove that $\mathrm{St}(A)$ is compact. The space $[0,1]^A$ is compact by Tychonoff's Theorem, and since $\mathrm{St}(A) = \{\sigma \in [0,1]^A \mid \sigma(a \boxplus b) = \sigma(a) + \sigma(b),\ \sigma(1) = 1\}$ it is a closed subset of $[0,1]^A$, hence compact. □

Examples 3.4.3.

1. The identity function is obviously a state on the effect algebra $[0,1]$. We will show that it is the only state, so that the state space of $[0,1]$ is a singleton convex space. Suppose that $\sigma : [0,1] \to [0,1]$ is a state. Then $\sigma(0) = 0$ and $\sigma(1) = 1$. Since σ preserves addition, we have $n\sigma(\frac{1}{n}) = 1$, hence $\sigma(\frac{1}{n}) = \frac{1}{n}$. It follows that σ acts as the identity on all rational numbers. Now if r is any real number in $[0,1]$, there is an increasing sequence p_n of rationals converging to r, and also a decreasing sequence q_n of rationals converging to r. Then $p_n = \sigma(p_n) \leq \sigma(r) \leq \sigma(q_n) = q_n$ for all n, hence $\sigma(r) = r$.

2. The state space of the power set algebra $\mathcal{P}(n)$ is the standard simplex $\mathcal{D}(n)$. To see this, observe that a morphism $\mathcal{P}(n) \to [0,1]$ is determined by its values on singletons, since it maps disjoint unions to sums. Since a state preserves the top element, the sum of all values of singletons must be 1. Thus a state can be identified with a sequence of numbers in $[0,1]$ summing to 1, in other words, an element of $\mathcal{D}(n)$.

3. If σ is a state on the effect algebra $C(X,[0,1])$ of continuous functions on X, then it additionally preserves scalar multiplications. This is the case because it preserves multiplication by rational scalars, and every real number can be approximated by rationals. Hence the state space of $C(X,[0,1])$ is the space $\mathcal{R}(X)$ of Radon measures by the Riesz Representation Theorem 3.2.5.

4. Let H be a Hilbert space. The state space of $\mathcal{E}f(H)$ is the space of density matrices $\mathcal{DM}(H)$. The isomorphism $\mathcal{DM}(H) \to \mathrm{St}(\mathcal{E}f(H))$ is given by $\rho \mapsto \mathrm{tr}(\rho(-))$. This gives the well-known connection between mixed states and effects in quantum mechanics; see e.g. [24] for a proof or [65] for an extensive discussion. Also, if $\dim H \geq 3$, then the state space of $\mathcal{P}roj(H)$ is $\mathcal{DM}(H)$. This is the content of Gleason's Theorem [54, 36].

Many effect algebras encountered in applications are interval effect algebras. Recall that an interval effect algebra is an interval $[0,u]_G$ in some partially ordered abelian group G. We may always assume that (G,u) is an order unit group. Each morphism of order unit groups $(G,u) \to (\mathbb{R},1)$ restricts to a state on the unit interval $[0,u]_G$. A morphism $(G,u) \to (\mathbb{R},1)$ is called a *state* on G. Be aware that not all states on $[0,u]_G$ extend to states on G, so these two notions of state are not equivalent. Nevertheless, states on the enveloping group may provide valuable information about the states on the effect algebra, so we will study these states now. The theory of states on ordered groups is elaborated in [55], which we will follow here. Most results discussed here trace back to [56, 57, 37].

We will start with a result that is similar to the Hahn–Banach Theorem in functional analysis. It tells that any state on a subgroup of an order unit group extends to a state on the larger group.

Theorem 3.4.4. *Let G be an order unit group, and $H \subseteq G$ an order unit subgroup. If $\sigma : H \to \mathbb{R}$ is a state on H, then there exists a state $\tau : G \to \mathbb{R}$ on G whose restriction to H is σ.*

We will prove the above theorem by extending the map σ step by step. Suppose that x is an element of G that does not lie in H. Then we wish to define the value of σ on x. This amounts to the same as extending σ to a map $H + \mathbb{Z}x \to \mathbb{R}$. By repeating this procedure for all elements in $G \setminus H$ using Zorn's Lemma, we obtain an extension to the full group G. This proof sketch indicates that we need the following lemma.

Lemma 3.4.5. *Let G be an order unit group with order unit subgroup $H \subseteq G$, and let $\sigma : H \to \mathbb{R}$ be a state. For any $x \in G \setminus H$, there exists a state $\tau : H + \mathbb{Z}x \to \mathbb{R}$ for which $\tau|_H = \sigma$.*

Proof. Define the real numbers

$$p = \sup\left\{ \frac{\sigma(y)}{m} \;\middle|\; y \in H,\ m \in \mathbb{N},\ y \leq mx \right\},$$

$$r = \inf\left\{ \frac{\sigma(z)}{n} \;\middle|\; z \in H,\ n \in \mathbb{N},\ nx \leq z \right\}.$$

Intuitively, p is obtained by approximating x from below and applying the function σ to it, while r is obtained by approximating x from above.

We first have to show that the supremum and infimum really exist. Since G has an order unit u, there is a natural number n such that $x \leq nu$. Hence, if $y \leq mx$, then $\frac{\sigma(y)}{m} \leq n\sigma(u) = n$. Therefore $\frac{\sigma(y)}{m}$ is bounded above by n, so the supremum in the definition of p exists. Similarly the infimum in the definition of r exists.

Secondly, we show that $p \leq r$. To this end, suppose that $y \leq mx$ and $nx \leq z$. Then $ny \leq mnx \leq mz$, so $\frac{\sigma(y)}{m} \leq \frac{\sigma(z)}{n}$. Since y, z, m, and n were arbitrary, $p \leq r$.

Take any $q \in [p, r]$ and define τ by $\tau(h) = \sigma(h)$ for $h \in H$, and $\tau(x) = q$. We have to show that τ is well-defined. Suppose that $h + nx = h' + n'x$. In order to prove that $\sigma(h) + nq = \sigma(h') + n'q$, assume without loss of generality that $n \leq n'$. Then $h - h' = (n' - n)x$ where $n' - n$ is a natural number, so $\frac{\sigma(h-h')}{n'-n} \leq p \leq q$ and $\frac{\sigma(h-h')}{n'-n} \geq r \geq q$. Therefore $\frac{\sigma(h-h')}{n'-n} = q$, thus $\sigma(h) + nq = \sigma(h') + n'q$.

The map τ is a group homomorphism by construction. To show that it is order-preserving, it suffices to show that it maps positive elements to positive elements. Thus assume that $h + nx \in H + \mathbb{Z}x$ is positive. Then $-h \leq nx$, so $-\frac{\sigma(h)}{n} \leq p \leq q$, hence $\tau(h + nq) = \sigma(h) + nq \geq 0$. Finally, τ preserves the order unit because σ does, proving that τ is a morphism in the category **OUGrp**. □

Proof of Theorem 3.4.4. Let P be the collection of all pairs $(K, \tau : K \to \mathbb{R})$, where $H \subseteq K \subseteq G$ is an intermediate subgroup and τ is a state on K. Put an order on P by $(K, \tau) \leq (K', \tau')$ if and only if $K \subseteq K'$ and $\tau'|_K = \tau$. By Zorn's Lemma, P possesses a maximal element (K, τ). We are done if we can prove that $K = G$. Assume for contradiction that there is an element $x \in G \setminus K$. Then, by Lemma 3.4.5, the state τ extends to a state $\tau' : K + \mathbb{Z}x \to \mathbb{R}$. But then $(K + \mathbb{Z}x, \tau')$ is an element of P that is strictly larger than (K, τ), contradicting maximality. □

The analogous result for interval effect algebras is false. This is one of the reasons why it is sometimes necessary to look at ambient groups.

Example 3.4.6. Consider the effect algebra $\mathcal{MO}(3)$ and denote its elements by $0, p, p^\perp, q, q^\perp, r, r^\perp, 1$. Furthermore, let X be the set $\{a, b, c, d, e, f\}$. There is an embedding $i : \mathcal{MO}(3) \hookrightarrow \mathcal{P}(X)$ determined by

$$i(p) = \{a, b, c\}$$
$$i(q) = \{a, b, d\}$$
$$i(r) = \{c, d, e\}$$

We will show that there is a state σ on $\mathcal{MO}(3)$ that does not extend to a state on $\mathcal{P}(X)$. Define σ by $\sigma(p) = \frac{1}{2}$, $\sigma(q) = \sigma(r) = \frac{1}{6}$. Suppose that there is a state τ on $\mathcal{P}(X)$ that restricts to σ. Then

$$\tau(a) + \tau(b) + \tau(c) = \tfrac{1}{2}$$
$$\tau(a) + \tau(b) + \tau(d) = \tfrac{1}{6}$$
$$\tau(c) + \tau(d) + \tau(e) = \tfrac{1}{6}$$

Adding the last two equations gives $\tau(a) + \tau(b) + \tau(c) + 2\tau(d) + \tau(e) = \frac{1}{3}$. Since τ maps into positive numbers, $\tau(a)+\tau(b)+\tau(c) \leq \frac{1}{3}$. This contradicts the first equation.

Note that $\mathcal{MO}(3)$ and $\mathcal{P}(X)$ are both interval effect algebras, by Proposition 2.3.8. Hence every state on the enveloping group of $\mathcal{MO}(3)$ can be extended to a state on the larger group. We are forced to conclude that the state space of an order unit group differs from the state space of its unit interval, since σ defined above cannot arise from a state on the surrounding group. It also follows that interval effect algebras are not categorically equivalent to order unit groups, since for an order unit group G morphisms $G \to \mathbb{R}$ do not correspond to morphisms $[0, u]_G \to [0, 1]$.

We would like to know to what extent the state space of an effect algebra determines the effect algebra. The state space contains a maximal amount of information when the states separate elements from the effect algebra. We will now present conditions under which this happens.

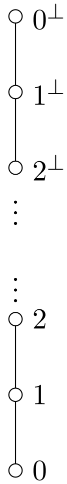
Definition 3.4.7. The state space of an effect algebra A is called *order-determining* whenever the following condition holds: if $\sigma(a) \leq \sigma(b)$ for all states σ on A, then $a \leq b$.

There is a similar notion for order unit groups. We will search for an equivalent condition, that is easier to check. The following example is instructive for finding a necessary condition.

Example 3.4.8. Let A be the effect algebra with underlying set $\mathbb{N} \cup \{n^\perp \mid n \in \mathbb{N}\}$. Here $n^\perp$ is a formal symbol denoting the complement of n, so that the underlying set actually consists of two disjoint copies of $\mathbb{N}$. Addition is defined according to the following rules:

- $n \boxplus m$ is always defined and equals $n + m$.
- $n \boxplus m^\perp$ is defined if and only if $n \leq m$, and in that case $n \boxplus m^\perp = (m - n)^\perp$. The case $n^\perp \boxplus m$ is treated symmetrically.
- $n^\perp \boxplus m^\perp$ is never defined.

The order underlying A can be visualized as a copy of $\mathbb{N}$ with another copy of $\mathbb{N}$ put upside down on top:



If $\sigma : A \to [0, 1]$ is a state, then σ is determined by its value on 1, because all other elements of A are obtained by addition and complementation. Suppose that $\sigma(1) \neq 0$. Then $\sigma(1) > \frac{1}{n}$ for some $n \in \mathbb{N}$, hence $\sigma(n) > 1$, which is impossible for a state. Therefore A has only one state, namely the one that maps all elements $n \in \mathbb{N}$ to 0 and all elements $n^\perp$ to 1. Thus the state space of A is not order-determining. The problem here is that the element 1 in A is infinitesimal: it is not possible to get above all elements of the effect algebra by adding it to itself.

This suggests that only effect algebras or order unit groups without infinitesimal elements can have an order-determining state space. This will be called the Archimedean property.

Definition 3.4.9. An order unit group is called *Archimedean* if it satisfies the following property: whenever $nx \leq y$ for all $n \in \mathbb{N}$, then $x \leq 0$. An interval effect algebra is called Archimedean if its totalization as a group is Archimedean.

Lemma 3.4.10. *Let G be an Archimedean order unit group, $m \in \mathbb{N}$, $x \in G$. If $mx \geq 0$, then $x \geq 0$.*

Proof. Write u for the order unit of G. Then $-nu \leq x$ for some $n \in \mathbb{N}$, hence $-knu \leq kx$ for $k = 0, 1, \ldots, m-1$. It follows that $-mnu \leq kx$ for these values of k. We will show that $-mnu \leq \ell x$ for all natural numbers ℓ. Write ℓ as $pm + k$ where $p \in \mathbb{N}$ and $0 \leq k \leq m-1$. Since $mx \geq 0$, we also have $pmx \geq 0$. Combining this with $-mnu \leq kx$ gives $-mnu \leq (pm+k)x = \ell x$. Hence $\ell(-x) \leq mnu$ for all ℓ. From the Archimedean property it follows that $-x \leq 0$, thus x is positive. □

An ordered abelian group satisfying the property in the above lemma is often called *unperforated*. We are now ready to prove our characterization of order-determining state spaces.

Theorem 3.4.11. *An order unit group has an order-determining state space if and only if it is Archimedean.*

Proof. Suppose the order unit group (G, u) is Archimedean. To show that its state space is order-determining, it suffices to show that $x \geq 0$ whenever $\sigma(x) \geq 0$ for all σ. Assume that $\sigma(x)$ is positive for all states σ. The subgroup $\mathbb{Z}u$ of G has a unique state $nu \mapsto n$. By Lemma 3.4.5, this state extends to a state $\tau : \mathbb{Z}u + \mathbb{Z}x \to \mathbb{R}$. The proof of the same lemma shows that we can in fact choose τ in such a way that

$$\tau(x) = p := \left\{ \frac{k}{m} \;\middle|\; k \in \mathbb{Z},\ m \in \mathbb{N},\ ku \leq mx \right\}.$$

The state τ extends to a state σ on all of G by Theorem 3.4.4, and by assumption, $\sigma(x) \geq 0$. Therefore $p = \tau(x)$ is also positive. Since p is defined as a supremum of real numbers, this means that for any $n \in \mathbb{N}$, there exist numbers $k \in \mathbb{Z}$ and $m \in \mathbb{N}$ such that $ku \leq mx$ and $\frac{k}{m} > -\frac{1}{n}$. Then $kn > -m$, so $mnx \geq knu > -mu$, hence $m(nx + u) > 0$. Applying Lemma 3.4.10 gives $nx + u > 0$, or equivalently $n(-x) < u$. Since n was arbitrary, the Archimedean property now shows that x is positive. Thus we have shown that if G is Archimedean, then its state space is order-determining.

For the converse, suppose that $\mathrm{St}(G)$ is order-determining and that $nx \leq y$ for all $n \in \mathbb{N}$. Then $n\sigma(x) = \sigma(nx) \leq \sigma(y)$ for all states $\sigma : G \to \mathbb{R}$, so since $\mathbb{R}$ is Archimedean, we get that $\sigma(x) \leq 0$ for all σ. Since the state space is order-determining, it follows that $x \leq 0$, establishing the Archimedean property. ∎

Corollary 3.4.12. *An effect algebra has an order-determining state space if and only if it is an Archimedean interval effect algebra.*

Proof. Suppose that A is an Archimedean interval effect algebra. Then A is the interval $[0, u]_G$ in some Archimedean order unit group G. If $\sigma(a) \leq \sigma(b)$ for all states σ on A, then in particular $\sigma(a) \leq \sigma(b)$ for all states σ on G, because every state on G restricts to a state on A. Now Theorem 3.4.11 gives $a \leq b$.

Conversely, suppose that A has an order-determining state space. We first have to show that it is an interval effect algebra. The set $[0, 1]^{\mathrm{St}(A)}$ of functions from the state space to $[0, 1]$ forms an effect algebra with pointwise operations. There is an embedding $\eta : A \to [0,1]^{\mathrm{St}(A)}$ given by $\eta(a)(\sigma) = \sigma(a)$. We shall prove that this map identifies A with a subalgebra of $[0,1]^{\mathrm{St}(A)}$ by showing that η is strong, hence also injective by Proposition 2.2.2. To prove strength, suppose that $\eta(a) \boxplus \eta(b)$ is defined. Then $\sigma(a) \boxplus \sigma(b)$ is defined for all σ, hence $\sigma(a) \leq \sigma(b)^\perp = \sigma(b^\perp)$ by part 1 of Lemma 2.1.3. Since the state space is order-determining, we obtain $a \leq b^\perp$, whence $a \boxplus b$ is defined by the same lemma. We conclude that A is a subalgebra of $[0,1]^{\mathrm{St}(A)}$. The larger algebra $[0,1]^{\mathrm{St}(A)}$ is clearly an interval effect algebra, since it embeds in $\mathbb{R}^{\mathrm{St}(A)}$. Hence, by Proposition 2.3.5, A is an interval effect algebra as well. Any subalgebra of an Archimedean effect algebra is again Archimedean, and $[0,1]^{\mathrm{St}(A)}$ is Archimedean, so A is an Archimedean interval effect algebra. □

Chapter 4

Effect algebroids

Effect algebras are an abstract generalization of the unit interval $[0, 1]$. Similarly, the note [95] introduced abstract circles as a generalization of the unit circle $\mathbb{S}^1$, see also [103, 19, 34, 31, 32] for related approaches. There are many similarities between these two structures: both involve a partial addition and complements with respect to a maximal element. Here we will make these similarities explicit by defining a common generalization of effect algebras and abstract circles, called effect algebroids.

Effect algebroids will be defined as a category-like structure: it has points and arrows between the points, and a partial composition operation on the arrows. Effect algebras can be recovered as the one-point effect algebroids, and abstract circles as algebroids with many points and very few arrows. Hence, if we view effect algebroids as analogous to categories, then effect algebras are analogous to monoids and abstract circles to posets. Thus effect algebroids can be considered as a categorification of the notion of an effect algebra. Several other well-known structures arising in mathematical descriptions of quantum mechanics also have interesting categorifications. For example, 2-Hilbert spaces are a categorification of Hilbert spaces, and C*-categories of C*-algebras; see [12, 51, 66] for more information about these approaches. Therefore the study of effect algebroids fits in the research line of categorifications of quantum structures.

4.1 Effect algebroids

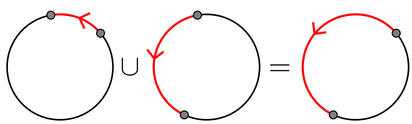
Effect algebroids generalize effect algebras and abstract circles. Effect algebras have already been discussed thoroughly in Chapter 2. Abstract circles are defined as follows in [95].

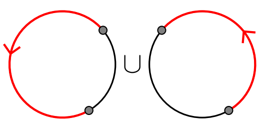
Definition 4.1.1. An *abstract circle* consists of a set P of points, and for each two points x, y a set $\mathrm{Hom}(x, y)$ of segments from x to y. Furthermore, there are partial functions $\cup : \mathrm{Hom}(x, y) \times \mathrm{Hom}(y, z) \rightharpoonup \mathrm{Hom}(x, z)$, functions $(-)^\perp : \mathrm{Hom}(x, y) \to \mathrm{Hom}(y, x)$, and segments $0_x, 1_x \in \mathrm{Hom}(x, x)$ for each x. These are subject to the following requirements:

- Associativity: If $a \cup b$ and $(a \cup b) \cup c$ are defined, then so are $b \cup c$ and $a \cup (b \cup c)$, and $(a \cup b) \cup c = a \cup (b \cup c)$.
- Zero: For each $a \in \mathrm{Hom}(x, y)$, $0_x \cup a = a$.
- Cyclic complement law: $a \cup b = c$ if and only if $c^\perp \cup a = b^\perp$.
- Double complement law: $a^{\perp\perp} = a$.
- Zero complement law: $0_x^\perp = 1_x$.
- Positivity: If $a \cup b = 0_x$, then $a = 0_x$.
- Totality: For all $a \in \mathrm{Hom}(x, y)$ and $b \in \mathrm{Hom}(y, z)$, at least one of $a \cup b$ and $b^\perp \cup a^\perp$ exists.
- Trivial automorphisms: $\mathrm{Hom}(x, x) = \{0_x, 1_x\}$.

We will often write 0 and 1 instead of 0_x and 1_x, when the domain is clear from context.

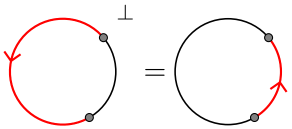
Any subset of the unit circle $\mathbb{S}^1$ provides an important example of an abstract circle. For $x \neq y$, the set $\mathrm{Hom}(x, y)$ is a singleton, whose element represents the circle segment from x to y, counterclockwise. The homset $\mathrm{Hom}(x, x)$ has two elements 0_x and 1_x, where 0_x represents the segment consisting of the single point x, and 1_x represents a full circle. The composition of the segment from x to y and the segment from y to z is given by gluing the segments, which is defined whenever the segments together do not exceed the circle. This can be visualized as follows:





is undefined

The complement of the unique segment in $\mathrm{Hom}(x, y)$ is the unique segment in $\mathrm{Hom}(y, x)$:



The connection between effect algebras and abstract circles looks a bit like the connection between groups and groupoids: an effect algebra is “almost” an abstract circle with one point. However, the condition on trivial automorphisms means that we can get only trivial effect algebras as one-point circles. Therefore we will now introduce a generalization of the notion of an abstract circle. We call our generalization an effect algebroid, since effect algebras will turn out to be effect algebroids with one point.

Definition 4.1.2. An *effect algebroid* A consists of a set P of points, and for each two points x, y a set $\mathrm{Hom}(x, y)$ of segments from x to y. Furthermore, there are partial functions $\cup : \mathrm{Hom}(x, y) \times \mathrm{Hom}(y, z) \rightharpoonup \mathrm{Hom}(x, z)$, functions $(-)^{\perp} : \mathrm{Hom}(x, y) \to \mathrm{Hom}(y, x)$, and segments $0_x, 1_x \in \mathrm{Hom}(x, x)$ for each x. These are subject to the following requirements:

- Associativity: If $a \cup b$ and $(a \cup b) \cup c$ are defined, then so are $b \cup c$ and $a \cup (b \cup c)$, and $(a \cup b) \cup c = a \cup (b \cup c)$.
- Zero: For each $a \in \mathrm{Hom}(x, y)$, $0_x \cup a = a = a \cup 0_y$.
- Orthocomplement: For all $a \in \mathrm{Hom}(x, y)$ and $b \in \mathrm{Hom}(y, x)$, we have

$$a \cup b = 1_x \iff a = b^{\perp} \iff b = a^{\perp}$$

- Zero-one law: For any $a \in \mathrm{Hom}(x, y)$, if $a \cup 1_y$ is defined, then $a = 0_y$. Also, if $1_x \cup a$ is defined, then $a = 0_x$.

A morphism $F : A \to B$ between effect algebroids consists of a function from the points of A to the points of B and functions $\mathrm{Hom}(x, y) \to \mathrm{Hom}(F(x), F(y))$, such that F preserves 0_x, 1_x, and the complement, and subject to the following functoriality condition: whenever $a \cup b$ is defined, then also $F(a) \cup F(b)$ is defined, and $F(a \cup b) = F(a) \cup F(b)$. The category of effect algebroids is denoted **EAd**.

We will look at some classes of examples of effect algebroids. It is easy to see which effect algebroids correspond to effect algebras.

Proposition 4.1.3. *Commutative effect algebroids with one object are precisely the effect algebras.*

Example 4.1.4. We give an example of an effect algebroid with one object, which is a "non-commutative effect algebra". Let A be the set of order-isomorphisms $f : \mathbb{R} \to \mathbb{R}$ that satisfy $x \leq f(x) \leq x+1$ and $f(x+1) = f(x)+1$. Function composition is a partial operation on this set, where $g \circ f$ is defined if and only if $(g \circ f)(x)$ always lies between x and $x+1$. The assignment $f^{\perp}(x) = f^{-1}(x+1)$ defines a two-sided complement, because of the condition $f(x+1) = f(x)+1$. Zero and one are provided by the functions id and $x \mapsto x+1$. This structure satisfies all requirements for an effect algebra, except for commutativity.

Non-commutative effect algebras are also studied in [39] under the name pseudo-effect algebras. The difference with our one-object effect algebroids is that left and right complements in a pseudo-effect algebra need not coincide.

We can also characterize abstract circles as a special case of effect algebroids.

Proposition 4.1.5. *Let P be an effect algebroid satisfying 'Totality' and 'Trivial automorphisms' from Definition 4.1.1. Then P is an abstract circle.*

Proof. The associativity and zero laws follow from the definition of an effect algebroid. The double complement law follows readily from the orthocomplement law. Using the fact that we can cancel double complements, the cyclic complement law follows from applying the orthocomplement law twice:

$$a \cup b = c \iff c^{\perp} \cup a \cup b = 1 \iff c^{\perp} \cup a = b^{\perp}$$

The zero complement law follows immediately from $0 \cup 1 = 1$. For positivity, suppose that $a \cup b = 0$. Then $a \cup b \cup 1 = 1$, so in particular $a \cup b \cup 1$ is defined. By associativity, $b \cup 1$ is defined, so by the zero-one law, $b = 0$. Hence $a = a \cup b = 0$. □

Summarizing, we have shown that both effect algebras and abstract circles are extreme cases of effect algebroids: effect algebras are effect algebroids with only one object, while abstract circles are effect algebroids with "very few" morphisms. More precisely, in an abstract circle, $\mathrm{Hom}(x, y)$ is a singleton for $x \neq y$, and $\mathrm{Hom}(x, x)$ always consists of two elements.

There are also effect algebroids that are neither effect algebras nor abstract circles. The following construction gives one example of such an effect algebroid, similar to an action groupoid. It will be generalized in the next section.

Example 4.1.6. Let X be a set carrying an action of the circle group $\mathbb{S}^1 = \{e^{i\theta} \mid \theta \in [0, 2\pi)\}$. We will define an effect algebroid $X//\mathbb{S}^1$, called the *action algebroid*. The set of points is simply X. A segment from x to y is a number $\theta \in [0, 2\pi]$ for which $e^{i\theta} \cdot x = y$. Observe that we include 2π in the set of segments, to ensure that we have a maximal segment on each point. Composition of segments $\theta \in \mathrm{Hom}(x, y)$ and $\varphi \in \mathrm{Hom}(y, z)$ is defined whenever $\theta + \varphi \leq 2\pi$, in which case the composition is $\theta + \varphi$. The zero and one segments on x are 0 and 2π, respectively, and both act indeed trivially on x. Complements are given by $\theta^\perp = 2\pi - \theta$. It is not hard to check that this yields an effect algebroid.

Another new class of examples is obtained by generalizing the construction of free effect algebras from Section 2.2. Every effect algebroid has an underlying directed graph. Denote the category of directed graphs by **Graph**, then the forgetful functor **EAd** $\to$ **Graph** has a left adjoint. Applying this left adjoint to a graph $\mathbf{\Gamma}$ gives an effect algebroid A, which we will call the free effect algebroid on $\mathbf{\Gamma}$. The points of A are simply the points of $\mathbf{\Gamma}$. If x and y are distinct points, then a segment from x to y is either an edge from x to y in $\mathbf{\Gamma}$, or it is of the form $a^\perp$ for an edge a from y to x, where $^\perp$ is a formal symbol. Segments from x to itself are the formal symbols 0_x, 1_x, a, and $a^\perp$ for self-edges a in $\mathbf{\Gamma}$. Then composition in the free effect algebroid is determined by the following laws:

- $0 \cup z = z \cup 0 = z$ for each segment z in A.
- $a \cup a^\perp = 1$ and $a^\perp \cup a = 1$ for each edge a in $\mathbf{\Gamma}$.
- Composition is undefined in all cases not covered by the first two laws.

Complements are defined in the obvious way.

4.2 Order and topology

On any effect algebra, we can define a partial order by $a \leq b$ if and only if $a \boxplus c = b$ for some c. Similarly, each abstract circle carries a cyclic order. See [101, 102] for an introduction to cyclic orders.

Definition 4.2.1. A *partial cyclic order* on a set X is a ternary relation $[-, -, -]$ satisfying:

- Cyclicity: If $[x, y, z]$, then $[y, z, x]$.
- Asymmetry: If $[x, y, z]$, then not $[z, y, x]$.

- Transitivity: If $[w, x, y]$ and $[w, y, z]$, then $[w, x, z]$.

A *cyclic order* is a partial cyclic order that additionally satisfies:

- Totality: For all distinct x, y, z, either $[x, y, z]$ or $[z, y, x]$.

The above definition is analogous to a strict order, since the condition $[x, y, z]$ can only hold if x, y, z are all distinct. For examples of cyclic orders, think of collections of points arranged on a circle, and say that $[x, y, z]$ holds if and only if, when walking along the circle counterclockwise starting in x, point y is encountered before point z.

Lemma 4.2.2. *Let x, y be distinct points on an abstract circle P. Then $\mathrm{Hom}(x, y)$ contains at most one segment.*

Proof. Take two segments $a, b \in \mathrm{Hom}(x, y)$. Since abstract circles have trivial automorphisms, $a \cup b^{\perp}$ is either 0_x or 1_x. It can't be 0_x by positivity, so $a \cup b^{\perp} = 1_x$, and hence $a = b$. □

We will denote the unique segment from x to y by $(x \to y)$.

Proposition 4.2.3. *For an abstract circle P, define a ternary relation $[-, -, -]$ on P by*

$$[x, y, z] \iff x,\ y,\ z \text{ are distinct and } (x \to y) \cup (y \to z) \text{ is defined.}$$

This relation is a cyclic order.

Proof. We start with cyclicity. If $(x \to y) \cup (y \to z)$ is defined, then it is equal to $(x \to z)$, so by the cyclic complement law, $(z \to x) \cup (x \to y) = (z \to y)$, hence $[z, x, y]$.

For asymmetry, suppose that $(x \to y) \cup (y \to z) = (x \to z)$ and $(z \to y) \cup (y \to x) = (z \to x)$ for distinct x, y, z. Then

$$\begin{aligned} 1_x &= (x \to z) \cup (z \to x) \\ &= (x \to y) \cup (y \to z) \cup (z \to y) \cup (y \to x) \\ &= (x \to y) \cup 1_y \cup (y \to x). \end{aligned}$$

Therefore $(x \to y) \cup 1_y = (x \to y)$, so $(x \to y) = 0_y$ according to the zero-one law. But then $x = y$, which is a contradiction.

Transitivity of the relation follows directly from associativity, and totality of the relation from totality of the abstract circle. □

In fact, abstract circles are the same as cyclic orders. However, this correspondence does not extend to morphisms, hence it does not give a categorical equivalence.

Proposition 4.2.4. *For a cyclically ordered set X, define an abstract circle structure on X as follows: for $x \neq y$, put* $\mathrm{Hom}(x, y) = 1$, *and put* $\mathrm{Hom}(x, x) = \{0_x, 1_x\}$. *For distinct x, y, z, we say that $(x \to y) \cup (y \to z)$ is defined iff $[x, y, z]$, and in that case it equals $x \to z$. All other cases of $\cup$ and the orthocomplement are defined in the obvious way. This makes X into an abstract circle.*

Proof. Associativity follows from cyclicity and transitivity. The orthocomplement and zero-one laws hold by definition. Totality of the circle follows from totality of the cyclic order, and it is immediate that the automorphisms are trivial. □

Partial cyclic orders can be used to find some new classes of examples of effect algebroids. Any partial cyclic order is itself an effect algebroid, via essentially the same construction as in Proposition 4.2.4.

Another construction associates an effect algebroid to any cyclically ordered group acting on a set. This is a generalization of Example 4.1.6. A *cyclically ordered group* is a group G that carries a partial cyclic order for which $[x, y, z]$ implies $[gx, gy, gz]$ and $[xg, yg, zg]$. Examples include finite cyclic groups, and the circle group $\mathbb{S}^1$. Suppose that the cyclically ordered group G acts on a set X. Define the action algebroid $X//G$ with set of points X and segments

$$\mathrm{Hom}(x, y) = \begin{cases} \{g \in G \mid g \cdot x = y\} & \text{if } x \neq y \\ \{g \in G \mid g \cdot x = x\} \cup \{u_x\} & \text{if } x = y \end{cases}$$

where the u_x are elements not lying in G. They are written as u if no confusion is possible. The element u_x will act as the maximal segment connecting x to itself. To define composition in $X//G$, let $x \xrightarrow{g} y \xrightarrow{h} z$ be a sequence of two segments. Distinguish the following cases:

- If at least one of g and h is the identity, then it is clear how to define the composition.
- Otherwise, if $gh = e$, then x has to be equal to z. In this situation, $g \cup h$ is defined and equals u_x.
- Otherwise, if $[e, g, hg]$, then $g \cup h$ is defined and equals hg.
- In all other cases, $g \cup h$ is undefined.

The zero and one segments on a point x are given by e and u_x, respectively. Orthocomplements are defined by $e^\perp = u$, $u^\perp = e$, and $g^\perp = g^{-1}$ for all other g. In the special case where G is the circle group $\mathbb{S}^1$, Example 4.1.6 gives a concrete description of this effect algebroid.

Proposition 4.2.5. *The action algebroid $X//G$ is an effect algebroid.*

Proof. All axioms are easy to check except associativity. Suppose that $g \cup h$ and $(g \cup h) \cup k$ are defined. We may assume that none of the elements g, h and k is the identity. Then also none of them is u, so we are left with the case in which $[e, g, gh]$ and $[e, gh, ghk]$. Since the group operation is compatible with the cyclic order, it follows that $[g^{-1}, e, h]$ and $[g^{-1}, h, hk]$. Cyclicity gives $[h, g^{-1}, e]$ and $[h, hk, g^{-1}]$, hence $[h, hk, e]$ by transitivity. Again applying cyclicity shows that $h \cup k$ is defined. To see that $g \cup (h \cup k)$ is defined, apply transitivity to the assumptions $[e, g, gh]$ and $[e, gh, ghk]$. The two compositions are equal since the group operation is associative. $\square$

There are several connections between cyclic orders and linear orders. If X is a set that is linearly ordered by $<$, then

$$[x, y, z] \iff x < y < z \text{ or } y < z < x \text{ or } z < x < y$$

defines a cyclic order on X. This construction is called *rolling* the linear order X. Each cyclic order arises by rolling a linear order, but there may be more than one linear order that gives the same cyclic order. A *cut* on a cyclically ordered set X is a linear order on X for which the condition $x < y < z$ implies $[x, y, z]$. Cuts on a cyclic order are exactly the linear orders that give rise to the same cyclic order by rolling.

Another way to construct linear orders out of cyclic orders involves intervals. If x and y are distinct points in X, then the interval $(x, y) := \{z \mid [x, z, y]\}$ is linearly ordered via $z < z'$ if and only if $[x, z, z']$. Similarly, for $x \in X$, the set $X \backslash \{x\}$ can be linearly ordered via $z < z'$ if and only if $[x, z, z']$. Informally, $X \backslash \{x\}$ is the open interval from x to x running around the circle.

Using intervals, we can define a natural topology on any abstract circle. If X is an abstract circle, viewed as a cyclically ordered set, then the basic opens for the topology on X are the intervals (x, y) for distinct $x, y \in X$, and $X \backslash \{x\}$ for $x \in X$. This topology is called the *cyclic topology*. We will now present some properties of this topology.

Lemma 4.2.6. *The cyclic topology on a cyclically ordered set X is Hausdorff.*

Proof. Let x and y be two distinct points in X. Distinguish the following cases:

1. The intervals (x, y) and (y, x) are both empty. Because a cyclic order is total, x and y are the only two points of X. Hence $\{x\} = X \backslash \{y\}$ and $\{y\} = X \backslash \{x\}$ are both open and disjoint.

2. Exactly one of (x, y) and (y, x) is empty. Say that (x, y) is empty and $z \in (y, x)$. In this case, let $U = (z, y)$ and $V = (x, z)$. We claim that U and V are disjoint. For if $p \in U \cap V$, then $[z, p, y]$ and $[x, p, z]$, so $[p, y, z]$ and $[p, z, x]$ by cyclicity. Transitivity gives $[p, y, x]$, so $[x, p, y]$, hence $p \in (x, y) = \varnothing$. Thus $U \cap V = \varnothing$, so we are finished if we can show that $x \in U$ and $y \in V$. Since $(x, y) = \varnothing$, the relation $[x, z, y]$ does not hold, so $[y, z, x]$ by totality. From cyclicity we get $x \in U$ and $y \in V$.

3. Both (x, y) and (y, x) are non-empty. Pick $w \in (x, y)$ and $z \in (y, x)$ and put $U = (z, w)$ and $V = (w, z)$. By asymmetry, U and V are disjoint. Furthermore, since $[x, w, y]$ and $[x, y, z]$, it follows that $[x, w, z]$, hence $x \in U$. Similarly $y \in V$, as desired. □

The following result relates the topological property of compactness to order-theoretic completeness properties. As usual, we write $[x, y]$ for the closed interval $\{x\} \cup (x, y) \cup \{y\}$.

Proposition 4.2.7. *Let X be a cyclically ordered set. The following are equivalent:*

1. *X is compact in the cyclic topology.*

2. *For any cut $<$ on X, X has a least or a greatest element w.r.t. $<$.*

3. *Every closed interval $[x, y] \subset X$ is complete (i.e. every subset of $[x, y]$ has an infimum and supremum in $[x, y]$).*

4. *Every closed interval $[x, y] \subset X$ is sup-complete (i.e. every subset has a supremum).*

5. *Every closed interval $[x, y] \subset X$ is inf-complete.*

Proof. The equivalence of 2 and 3 is proven in [104]. We will now show that 1 and 3 are equivalent. If X is compact, then the interval $[x, y]$ is also compact since it is a closed subset of X. But $[x, y]$ carries an order topology, so it is complete. Now assume that each closed interval in X is complete. Then each closed interval is also compact. Because $X = [x, y] \cup [y, x]$, it follows that X is itself compact.

The next step is to show that 4 implies 5 (and hence 3). Suppose that each $[x, y]$ is sup-complete. Let $(p_i)_{i \in I}$ be a decreasing sequence in $[x, y]$; we have to show that its infimum exists. Intuitively, the infimum of the sequence is the limit point when walking along the sequence on the circle counterclockwise. We will construct this infimum by walking along the circle clockwise instead, and taking the supremum. This will give the same limit point because the circle is round. For the precise proof, we may assume that I has at least two elements and that (p_i) is strictly decreasing. Let p_0 and p_1 be two elements in the sequence such that $p_1 < p_0$ in $[x, y]$. We will seek for an infimum in the closed interval $[p_0, p_1]$. Define $p = \sup\{q \in [p_0, p_1] \mid q \leq p_i \text{ for all } p_i \in (p_0, p_1]\}$, which exists since each closed interval is sup-complete. We claim that p is the infimum of the sequence p_i. From transitivity and cyclicity of the cyclic order it follows that p lies in $[x, y]$. It is easy to see that p is actually the greatest lower bound for (p_i).

By duality, 3 is also equivalent to 5, so the proof of the proposition is finished. □

4.3 Measures on abstract circles

We have seen that the state space is an interesting invariant of an effect algebra. This space consists of effect algebra morphisms into the unit interval $[0, 1]$. The unit interval can be considered as a one-point effect algebroid, so we can generalize the notion of state space to an arbitrary effect algebroid A by defining $\mathrm{St}(A)$ to be the maps $A \to [0, 1]$ of effect algebroids. In this section we will investigate state spaces of abstract circles. If P is an abstract circle and $S = \coprod_{x,y} \mathrm{Hom}(x, y)$ is its collection of segments, a morphism $f : P \to [0, 1]$ amounts to a function $f : S \to [0, 1]$ for which $f(a \cup b) = f(a) + f(b)$ whenever $a \cup b$ exists, $f(0_x) = 0$ and $f(1_x) = 1$. These maps are closely related to measures on the circle P, as we will show now.

Measures are assumed to preserve countable unions of chains. If we want to connect states and measures, we need a similar requirement on states. The proper categorical analogue of a union is a colimit, so we will define colimits in effect algebroids. For this we have to compare compositions in categories and compositions in effect algebroids. In effect algebroids, the composition of $(x \xrightarrow{f} y \xrightarrow{g} z)$ is written as $f \cup g$, while in categories, this composition is usually written as $g \circ f$. The different directions make comparison between the two structures less transparant, so we will often denote composition in categories as $f\,;g$ instead.

Definition 4.3.1. Let $\mathbf{C}$ be a category and A an effect algebroid. A *diagram* of shape $\mathbf{C}$ in A consists of an assignment $D : \mathrm{Obj}(\mathbf{C}) \to \mathrm{Pt}(A)$ and functorial assignments $D : \mathrm{Hom}(x, y) \to \mathrm{Hom}(D(x), D(y))$. Functoriality means that $D(\mathrm{id}_x) = 0_{D(x)}$, and whenever $f : x \to y$ and $g : y \to z$ are composable morphisms in $\mathbf{C}$, the composition $D(f) \cup D(g)$ exists and equals $D(f\,;g)$.

A *cocone* of a diagram D consists of a point x in A together with segments $a_c : D(c) \to x$ for $c \in \mathbf{C}$ such that, for each $f : c \to d$ in $\mathbf{C}$, the composition $D(f) \cup a_d$ is always defined and equals a_c. This cocone is called a *colimit* if it is universal, which means that for any cocone $(y, (b_c)_{c \in \mathbf{C}})$ there exists a unique segment $f : x \to y$ such that $D(a_c) \cup f$ is always defined and equals $D(b_c)$.

A *chain* in A is a diagram whose shape category is a linear order. An effect algebroid is *chain-cocomplete* if all chains have a colimit, and a morphism of effect algebroids is *upper semicontinuous* if it preserves colimits of chains.

For abstract circles, chain-cocompleteness can be expressed in terms of the linear order on intervals: an abstract circle P is chain-cocomplete if for all $x, y \in P$ and all chains C in the interval $(x, y]$, the supremum of C exists in $(x, y]$. A map $P \to [0, 1]$ is upper semicontinuous if and only if, for all chains C in (x, y) with supremum y, $f(x \to y) = \sup_{c \in C} f(x \to c)$.

Observe that upper semicontinuous maps into $[0, 1]$ are closed under taking convex combinations, so they form a convex set. Before stating the next theorem, we start with a quick recap on measures and valuations. If X is a topological space, then a *valuation* on X is a map $v : \mathcal{O}(X) \to [0, 1]$ satisfying

- $v(\varnothing) = 0$, $v(X) = 1$.
- Monotonicity: if $U \subseteq V$, then $v(U) \leq v(V)$.
- Modularity: $v(U) + v(V) = v(U \cup V) + v(U \cap V)$.

The valuation is called *continuous* if for each chain (U_i), we have $v(\bigcup_i U_i) = \sup_i U_i$. Each measure on the Borel σ-algebra of X restricts to a valuation. This valuation need not be continuous, but it is countably continuous in the sense that it satisfies the continuity condition for all countable chains. If X is a compact Hausdorff space, then the converse is also true: every continuous valuation can be extended to a Borel measure. In fact, this holds for a larger class of spaces, but we will only be concerned with compact Hausdorff spaces here. See for example [8, 9, 83] for proofs and generalizations of these assertions. For second countable topological spaces, continuity of a valuation is equivalent to countable continuity. Therefore there is a one-to-one correspondence between (countably) continuous valuations and measures on a second countable compact Hausdorff space. This will be used in the proof of the following result.

Theorem 4.3.2. *Suppose that P is an abstract circle which is compact and second countable in the cyclic topology. Then the convex set of upper semicontinuous maps $P \to [0,1]$ is isomorphic to the convex set of Radon measures on P.*

Proof. Let $\mu : \Sigma_P \to [0,1]$ be a Radon measure on P. Define $f : P \to [0,1]$ by $f(x \to y) = \mu([x,y))$ and $f(0_x) = 0$, $f(1_x) = 1$. Observe that all singletons in P are closed and hence measurable, since P is Hausdorff. Therefore each half-open interval $[x,y) = \{x\} \cup (x,y)$ is measurable, so f is well-defined. It is easy to see that f is a map of effect algebroids. To show that it is upper semicontinuous, it suffices to prove that it preserves colimits of countable chains, since P is second countable. Let (y_i) be a countable chain in (x,y) with supremum y. Then

$$\sup_i f(x \to y_i) = \sup_i \mu([x,y_i)) = \mu\left(\bigcup_i [x,y_i)\right) = \mu([x,y)) = f(x \to y).$$

Conversely, suppose that $f : P \to [0,1]$ is upper semicontinuous. In order to define a measure on P, we will first define a countably continuous valuation v on the open sets of P. If x and y are distinct points in P, define the valuation on the open interval (x,y) by

$$v((x,y)) = \sup\{f(z \to y) \mid [x,z,y]\}.$$

Furthermore, for $x \in P$, define

$$v(P \backslash \{x\}) = \sup\{f(z \to x) \mid z \neq x\}.$$

The idea behind this definition is as follows: when extending the valuation to a measure μ, we would like that $\mu([x,y)) = f(x \to y)$. By approximating x from above (i.e. by elements z for which $[x,z,y]$), we obtain the value of the measure at the open interval (x,y). Since the topology on P is generated by the open intervals (x,y) and $P\backslash\{x\}$, the definition extends to a map defined on all open sets. We will show that v is a countably continuous valuation. To simplify notation, we will write both cases in the definition as

$$v((x,y)) = \sup_{z>x} f(z \to y),$$

where we think of the set $P\backslash\{x\}$ as the open interval from x to x, running around the circle. It suffices to prove the requirements for a valuation on the basis for the topology.

- For monotonicity, suppose that $(x, y) \subseteq (x', y')$. Then

$$\begin{aligned} v((x,y)) &= \sup_{z>x} f(z \to y) \\ &\leq \sup_{z>x} f(z \to y') \\ &\leq \sup_{z>x'} f(z \to y') \\ &= v((x', y')). \end{aligned}$$

- To prove modularity, let (x, y) and (x', y') be intervals. Assume that the points are cyclically arranged in the order x, x', y, y'; the other cases are either similar or trivial. Then

$$\begin{aligned} &v((x,y)) + v((x',y')) \\ &\quad = \sup_{z>x} f(z \to y) + \sup_{z>x'} f(z \to y') \\ &\quad = \sup_{z>x} f(z \to x') + f(x' \to y) + \sup_{z>x'} f(z \to y) + f(y \to y') \end{aligned}$$

and

$$\begin{aligned} &v((x,y) \cup (x',y')) + v((x,y) \cap (x',y')) \\ &\quad = v((x,y')) + v((x',y)) \\ &\quad = \sup_{z>x} f(z \to y') + \sup_{z>x'} f(z \to y) \\ &\quad = \sup_{z>x} f(z \to x') + f(x' \to y) + f(y \to y') + \sup_{z>x'} f(z \to y), \end{aligned}$$

which gives modularity.

- Finally we will prove that v is countably continuous. Let $(x_0, y_0) \subseteq (x_1, y_1) \subseteq \cdots$ be a chain of intervals in P. Then

$$\begin{aligned} v\left(\bigcup_{i\in I}(x_i,y_i)\right) &= v((\textstyle\bigwedge_i x_i, \bigvee_j y_j)) \\ &= \sup_{z>\bigwedge x_i} f(z \to \textstyle\bigvee_j y_j) \\ &= \sup_i \sup_{z>x_i} f(z \to \textstyle\bigvee_j y_j) \\ &= \sup_i \sup_{z>x_i} \sup_j f(z \to y_j) \qquad \text{(upper semicontinuity)} \\ &= \sup_i \sup_{z>x_i} f(z \to y_i) \\ &= \sup_i v((x_i, y_i)) \end{aligned}$$

The map v is a countably continuous valuation, and the space P is second countable, compact, and Hausdorff, hence v gives a unique measure μ for which $\mu((x,y)) = v((x,y))$.

To show that μ is Radon, we will prove the stronger statement that every measure on a space with the cyclic topology is Radon. Let x and y be points in P, then to prove the Radon property it suffices to show that

$$\mu((x,y)) = \sup\{\mu(K) \mid K \subseteq (x,y) \text{ compact}\}.$$

Assume that there exist sequences $x_0 > x_1 > \cdots$ and $y_0 < y_1 < \cdots$ in (x,y) with $\inf_i x_i = x$ and $\sup_i y_i = y$. (If such sequences do not exist, then an easy modification of the following argument will work.) Then

$$\begin{aligned}\mu((x,y)) &= \mu(\textstyle\bigcup_i [x_i, y_i]) \\ &= \sup_i \mu([x_i, y_i]) \\ &\leq \sup\{\mu(K) \mid K \subseteq (x,y) \text{ compact}\} \\ &\leq \mu((x,y))\end{aligned}$$

All inequality signs must be equalities, hence the Radon property holds.

We will finish by showing that the construction of the measure is the inverse of the construction of the upper semicontinuous map. If μ is a measure and $f(x \to y) = \mu([x,y))$, then the measure μ_f corresponding to f is determined by

$$\mu_f((x,y)) = \sup_{z>x} f(z \to y) = \sup_{z>x} \mu([z,y)) = \mu((z,y)).$$

The measures μ and μ_f agree on open intervals, hence on the entire Borel σ-algebra. Now if f is upper semicontinuous and μ is the corresponding measure, then μ gives an upper semicontinuous map f_μ for which

$$\begin{aligned}f_\mu(x \to y) &= \mu([x,y)) \\ &= \mu(\{x\}) + \mu((x,y)) \\ &= 1 - \mu(P \backslash \{x\}) + \mu((x,y)) \\ &= 1 - \sup_{z>x} f(z \to x) + \sup_{z>x} f(z \to y) \\ &= 1 - \left(\sup_{z>x} f(z \to y) + f(y \to x)\right) + \sup_{z>x} f(z \to y) \\ &= 1 - f(y \to x) \\ &= f(x \to y)\end{aligned}$$

It is clear that the inverse constructions preserve convex combinations. □

Dually, we can also obtain a correspondence between lower semicontinuous maps and Radon measures via $f(x \to y) = \mu((x,y])$.

4.4 Totalization

It can be difficult to work with effect algebroids because of the partial operation. Therefore it is sometimes useful to formally add undefined compositions, in order to change the effect algebroid into a structure with a total operation. For effect algebras and abstract circles these totalizations are already known, and the construction for effect algebroids will be a common generalization of both. The totalization of effect algebras has been discussed in Section 2.3. Here we will start by reviewing the totalization of abstract circles.

The totalization of an abstract circle is called an *Archimedean set* and originated in [31]. It consists of a linearly ordered set X together with an automorphism Θ, such that $x < \Theta(x)$ for each x, and for all $x, y \in X$ there is a natural number n such that $y \leq \Theta^n(x)$. A morphism of Archimedean sets is an equivalence class of monotone maps $f : (X, \Theta) \to (X', \Theta')$ such that $f \circ \Theta = \Theta' \circ f$, modulo the equivalence relation defined by $f \sim g$ if and only if $g = f \circ \Theta^n$ for some $n \in \mathbb{Z}$.

Each Archimedean set gives an abstract circle. Define an equivalence relation $\sim$ on an Archimedean set X by putting $x \sim y$ whenever there exists $n \in \mathbb{Z}$ such that $\Theta^n(x) = y$. Then the quotient $X/\sim$ is the set of points in an abstract circle. To define the segments of this circle, we say that there is a unique segment between any two different points x and y, denoted $(x \to y)$, and there are two segments from any point to itself. Composition $(x \to y) \cup (y \to z)$ is defined if and only if $x \leq y \leq z \leq \Theta(x)$.

To construct an Archimedean set from an abstract circle P, pick a point $x \in P$. Then the set S of segments starting in x is linearly ordered. Let X be the quotient of $\mathbb{Z} \times S$ by the equivalence relation that identifies $(n, 1_x)$ with $(n+1, 0_x)$, and endow it with the lexicographic order. Then X forms an Archimedean set when equipped with the automorphism $\Theta(n, a) = (n+1, a)$. These two constructions establish a categorical equivalence between abstract circles and Archimedean sets.

The totalization of an arbitrary effect algebroid gives a certain kind of category. It is a generalization of a barred commutative monoid. Again we will write composition in the category from left to right, that is, the composition of $f : x \to y$ and $g : y \to z$ is written as $f\,;g$.

Definition 4.4.1. A *barred category* is a category $\mathbf{C}$ together with a natural transformation $u : \mathrm{id}_{\mathbf{C}} \Rightarrow \mathrm{id}_{\mathbf{C}}$ called the *bar* such that:

- Positivity: If $f\,;g = \mathrm{id}_x$, then $f = g = \mathrm{id}_x$.
- Barred cancellativity: If $f\,;g = f\,;h = u_x$, then $g = h$.

- Cyclicity: For morphisms $f : x \to y$ and $g : y \to x$, if $f \,;\, g = u_x$, then $g \,;\, f = u_y$.

A *barred functor* from $(\mathbf{C}, u)$ to $(\mathbf{D}, v)$ is a functor $F : \mathbf{C} \to \mathbf{D}$ such that $F(u_x) = v_{F(x)}$ for all objects x in $\mathbf{C}$. The category of barred categories and barred functors is denoted **BarCat**.

The cyclicity condition implies that whenever $f_1 \,;\, \cdots \,;\, f_n$ is equal to some u_x, then any cyclic permutation of the maps f_i has composition u_y for some y.

The natural transformation in a barred category represents one single loop. Therefore we can associate an effect algebroid to any barred category by restricting to morphisms that are "smaller" than the natural transformation, i.e. those that can be extended to a full loop. Call a morphism $f : x \to y$ in a barred category *enclosed* if $a \,;\, f \,;\, b = u_w$ for certain morphisms $a : w \to x$, $b : y \to w$. The conditions for a barred category are mainly about enclosed morphisms, and ensure that they form an effect algebroid.

Lemma 4.4.2. *The enclosed morphisms in a barred category form an effect algebroid.*

Proof. Let $\mathbf{C}$ be a barred category. For two enclosed morphisms f and g in $\mathbf{C}$, we say that $f \cup g$ is defined whenever $f \,;\, g$ is defined and enclosed. The zero morphisms are the identities, and 1_x is defined to be u_x. To define an orthocomplement for an enclosed morphism f, let a, b be such that $a \,;\, f \,;\, b = u$ and set $f^\perp = b \,;\, a$. To show that this is well-defined, suppose that both $a \,;\, f \,;\, b = u$ and $a' \,;\, f \,;\, b' = u$. By cyclicity $f \,;\, b \,;\, a = f \,;\, b' \,;\, a' = u$, hence $b \,;\, a = b' \,;\, a'$. The morphism $f^\perp$ is enclosed because of cyclicity.

We will show that the above assignment yields an effect algebroid. The orthocomplement law has been established above. It is clear that $(f;g);h = f;(g;h)$ whenever both sides are defined, so to prove associativity, it suffices to show that $g \,;\, h$ is enclosed whenever $f \,;\, g$ and $(f \,;\, g) \,;\, h$ are. Determine a, b for which $a \,;\, f \,;\, g \,;\, h \,;\, b = u$. Then it is immediate that $g \,;\, h$ is enclosed.

For the zero-one law, suppose that $f \,;\, u_y$ is enclosed. Then $a;f;u;b = u$ for certain a, b, so $u;b;a;f = u;\mathrm{id}$, hence $b;a;f = \mathrm{id}$ by barred cancellativity. Positivity gives $f = \mathrm{id}$. □

If $F : (\mathbf{C}, u) \to (\mathbf{D}, v)$ is a barred functor and f is an enclosed morphism in $\mathbf{C}$, then $F(f)$ is enclosed in $\mathbf{D}$. Hence taking enclosed morphisms defines a functor $\mathcal{E} : \mathbf{BarCat} \to \mathbf{EAd}$.

To construct a barred category from an effect algebroid, take sequences of adjacent segments, where we identify two sequences whenever the algebroid forces them to be equal. This will form a barred category with concatenation of sequences as composition. We shall describe this construction in more detail.

Let A be an effect algebroid. A *trajectory* in A is a finite sequence of points and segments

$$\left(x_0 \xrightarrow{a_1} x_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} x_n\right)$$

Note that the composition of two adjacent segments in a trajectory need not be defined. Trajectories in A are the morphisms in a category, whose objects are the points of A. Composition of two trajectories is given by concatenation. The identity on x is given by the singleton trajectory (x). The resulting category is denoted A^*.

We can obtain the desired barred category by quotienting out an equivalence relation on A^*. Let $\sim$ be the smallest equivalence relation for which:

- $\left(x \xrightarrow{a} y \xrightarrow{b} z\right) \sim \left(x \xrightarrow{a \cup b} z\right)$ whenever $a \cup b$ is defined.
- $(x) \sim \left(x \xrightarrow{0_x} x\right)$.
- $\left(x \xrightarrow{a} y \xrightarrow{1_y} y\right) \sim \left(x \xrightarrow{1_x} x \xrightarrow{a} y\right)$.
- If $\varphi \sim \psi$ for trajectories φ and ψ, and χ is an arbitrary trajectory, then $\varphi\chi \sim \psi\chi$ and $\chi\varphi \sim \chi\psi$ whenever both sides are defined, where juxtaposition indicates concatenation of trajectories.

Lemma 4.4.3. *Let $a, a_1, \ldots, a_n$ be segments in A for which*

$$\left(x_0 \xrightarrow{a_1} \cdots \xrightarrow{a_n} x_n\right) \sim \left(x_0 \xrightarrow{a} x_n\right).$$

Then the composition $a_1 \cup \cdots \cup a_n$ is defined and equal to a.

Proof. Define a new equivalence relation $\approx$ on A^* by declaring two trajectories $x_0 \xrightarrow{a_1} x_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} x_n$ and $x_0 = y_0 \xrightarrow{b_1} y_1 \xrightarrow{b_2} \cdots \xrightarrow{b_m} y_m = x_n$ to be equivalent if either both compositions $a_1 \cup \cdots \cup a_n$ and $b_1 \cup \cdots \cup b_m$ are defined and equal, or both are undefined. We will show that $\sim \subseteq \approx$. It is not hard to check that $\approx$ satisfies all conditions from the definition of $\sim$. Since $\sim$ is the smallest equivalence relation with these properties, it is contained in $\approx$. Combining this with the assumption of the lemma gives $\left(x_0 \xrightarrow{a_1} \cdots \xrightarrow{a_n} x_n\right) \approx \left(x_0 \xrightarrow{a} x_n\right)$. But the composition of the right-hand side is certainly defined, so $a_1 \cup \cdots \cup a_n$ is defined and equal to a. □

Proposition 4.4.4. *For any effect algebroid A, $A^*/\sim$ is a barred category whose algebroid of enclosed morphisms is isomorphic to A.*

Proof. Define the transformation u by letting u_x be the trajectory consisting of only the segment 1_x. This transformation is natural by the third condition for $\sim$. To show that $A^*/\sim$ is a barred category, we start with positivity. Suppose that $x_0 \xrightarrow{a_1} \cdots \xrightarrow{a_n} x_n$ and $x_n = y_0 \xrightarrow{b_1} \cdots \xrightarrow{b_m} y_m$ are trajectories whose concatenation is $\sim$-equivalent with the identity on x_0. Then it is also equivalent with $x_0 \xrightarrow{0} x_0$, so $a_1 \cup \cdots \cup a_n \cup b_1 \cup \cdots \cup b_m = 0$ by Lemma 4.4.3. Using positivity in the effect algebroid repeatedly proves that all a_i and b_j are zero, hence they give identities in the barred category.

To prove barred cancellativity, suppose that

$$\left(x_0 \xrightarrow{a_1} \cdots \xrightarrow{a_n} x_n = y_0 \xrightarrow{b_1} \cdots \xrightarrow{b_m} y_m = x_0\right) \sim \left(x_0 \xrightarrow{1} x_0\right)$$

and

$$\left(x_0 \xrightarrow{a_1} \cdots \xrightarrow{a_n} x_n = z_0 \xrightarrow{c_1} \cdots \xrightarrow{c_k} z_k = x_0\right) \sim \left(x_0 \xrightarrow{1} x_0\right)$$

Again use the lemma to conclude that $a_1 \cup \cdots \cup a_n \cup b_1 \cup \cdots \cup b_m = a_1 \cup \cdots \cup a_n \cup c_1 \cup \cdots \cup c_k = 1$. From this it follows that $b_1 \cup \cdots \cup b_m = c_1 \cup \cdots \cup c_k$, hence the corresponding trajectories are equivalent.

Cyclicity is proven in a similar way: if a concatenation of two trajectories is equivalent to the bar, then the lemma shows that their composition is equal in the effect algebroid. Cyclicity of the effect algebroid then gives the desired result.

Finally we have to show that the algebroid of enclosed morphisms is isomorphic to A. Clearly each segment a in A gives a trajectory of length one in $A^*/\sim$. Conversely, if a trajectory $x_0 \xrightarrow{a_1} \cdots \xrightarrow{a_n} x_n$ is enclosed, then Lemma 4.4.3 gives $b_1, \ldots, b_m$ and $c_1, \ldots, c_k$ such that $b_1 \cup \cdots \cup b_m \cup a_1 \cup \cdots \cup a_n \cup c_1 \cup \cdots \cup c_k = 1$. Hence the composition $a_1 \cup \cdots \cup a_n$ is defined, so the trajectory of these segments is equivalent to a segment of length one. It is straightforward to see that this provides an isomorphism of effect algebroids. □

The barred category $A^*/\sim$ associated to an effect algebroid A will be called the *totalization* of A and denoted $\mathcal{T}(A)$.

Examples 4.4.5.

1. The totalization of $[0, 1]$ is the barred category $\mathbb{R}_{\geq 0}$ as a one-object category. More generally, for any interval effect algebra A, we get the positive part of its enveloping ordered group.

2. The totalization of $\mathbb{S}^1$ is a circle wrapping around itself. This means that the set of objects is simply $\mathbb{S}^1$, and a morphism from $e^{i\theta}$ to $e^{i\varphi}$ is a natural number, representing the number of times that one walks around the circle. This intuition helps to describe composition. Let n be a morphism from $e^{i\theta}$ to $e^{i\varphi}$, and m a morphism from $e^{i\varphi}$ to $e^{i\psi}$. If at least two of θ, φ, and ψ are equal, then the composition $n \cup m$ is $n+m$. Also if $[\theta, \varphi, \psi]$ in the cyclic order, then $n \cup m$ is $n+m$. In all other cases the composition of the two "walks" on the circle gives an additional winding, so $n \cup m = n + m + 1$.

Totalization can be made functorial. If $F : A \to B$ is a morphism of effect algebroids, then F extends to a mapping $F^* : A^* \to B^*$ by letting $F^*\left(x_0 \xrightarrow{a_1} \cdots \xrightarrow{a_n} x_n\right) = \left(F(x_0) \xrightarrow{F(a_1)} \cdots \xrightarrow{F(a_n)} F(x_n)\right)$. The map F^* respects the defining properties of the equivalence relation $\sim$, hence it gives a barred functor $A^*/\sim \to B^*/\sim$. This shows that totalization is a functor $\mathcal{T} : \mathbf{EAd} \to \mathbf{BarCat}$ The above result shows that $\mathcal{E} \circ \mathcal{T}$ is naturally isomorphic to the identity on $\mathbf{EAd}$. Since the composition $\mathcal{T} \circ \mathcal{E}$ is not isomorphic to id, the functors do not constitute a categorical equivalence. They do however give a coreflection.

Proposition 4.4.6. *The functor $\mathcal{T} : \mathbf{EAd} \to \mathbf{BarCat}$ is left adjoint to $\mathcal{E}$, and makes $\mathbf{EAd}$ into a coreflective subcategory of $\mathbf{BarCat}$.*

Proof. Each barred functor $\mathcal{T}(A) \to \mathbf{C}$ restricts to a map $A \to \mathcal{E}(\mathbf{C})$, since barred functors preserve enclosed morphisms. Also, each morphism of effect algebroids $F : A \to \mathcal{E}(\mathbf{C})$ extends to a barred functor $\mathcal{T}(A) \to \mathbf{C}$, given by $\left(x_0 \xrightarrow{a_1} \cdots \xrightarrow{a_n} x_n\right) \mapsto F(a_1) ; \cdots ; F(a_n)$. We obtain an adjunction since these constructions are natural and mutually inverse. The coreflection property follows because the unit map $\mathrm{id} \Rightarrow \mathcal{E} \circ \mathcal{T}$ is an isomorphism by Proposition 4.4.4. □

Interval effect algebras form a nice class of examples of effect algebras, because they have a totalization that is a group rather than just a monoid. Similarly there is a class of effect algebroids that has an ordered groupoid as totalization, rather than just a barred category. The following definition of an ordered groupoid is a generalization of the oriented groupoids from [32], using partial orders instead of total orders.

Definition 4.4.7. An *ordered groupoid* is a groupoid $\mathbf{G}$ in which each set $\mathrm{Hom}(x, \_) = \bigcup_{y \in \mathbf{G}} \mathrm{Hom}(x, y)$ carries a partial order $\leq$, in such a way that $g \leq h$ implies $x ; g \leq x ; h$ and $g ; y \leq h ; y$. A *strong unit* in an ordered groupoid is a natural transformation $u : \mathrm{id}_{\mathbf{G}} \to \mathrm{id}_{\mathbf{G}}$ such that for each morphism $g : x \to y$ there is a natural number n such that $g \leq u_x^n$. A morphism of ordered groupoids with order unit is a functor that preserves the order on homsets and the unit. The resulting category of ordered groupoids with a strong unit is denoted **OGrpdU**.

A morphism $g : x \to y$ in an ordered groupoid is called *positive* if $\mathrm{id}_x \leq g$. Positive morphisms in an ordered groupoid with order unit form a barred category, and hence the enclosed morphisms give an effect algebroid. The following examples are also adapted from [32].

Examples 4.4.8.

1. Let X be a poset, and let G be a group acting freely on X by order automorphisms. The orbits of this action give rise to an ordered groupoid $\mathbf{G}$. The set of objects is $X/\sim$, where $\sim$ is the equivalence relation that identifies x with each $g \cdot x$. The set of all morphisms is $(X \times X)/\sim$, where $(x, y) \sim (g \cdot x, g \cdot y)$ for all $x, y \in X$ and $g \in G$. The domain and codomain maps of the groupoid are the first and second projections $(X \times X)/\sim \to X/\sim$, and the identity is the diagonal. The homsets inherit the order from X, making $\mathbf{G}$ into an ordered groupoid.

2. Every Archimedean set gives an ordered groupoid. If (X, Θ) is an Archimedean set, then the group $\mathbb{Z}$ acts on X by powers of Θ. The action is free since $x < \Theta(x)$ for all x, hence the previous example gives an ordered groupoid $\mathbf{G}$. Objects of $\mathbf{G}$ are orbits in X, which are of the form $[x] = \{\Theta^n(x) \mid x \in X\}$ for some $x \in X$. A morphism from $[x]$ to $[y]$ is an orbit $\{(\Theta^n(x'), \Theta^n(y')) \mid n \in \mathbb{Z}\}$ in $X \times X$ that contains (x, y). We can identify the set $\mathrm{Hom}([x], [y])$ with $\mathbb{Z}$, since no element of X is equal to its image under a power of Θ.

   Define a natural transformation $u : \mathrm{id}_{\mathbf{G}} \to \mathrm{id}_{\mathbf{G}}$ by letting u_x be the equivalence class of $(x, \Theta(x))$. Concretely, $u_x = \{(\Theta^n(x), \Theta^{n+1}(x)) \mid n \in \mathbb{Z}\}$. This definition endows $\mathbf{G}$ with the structure of an ordered groupoid with a strong unit. Restricting to enclosed morphisms in $\mathbf{G}$ gives an effect algebroid, whose objects are orbits of the $\mathbb{Z}$-action, and for which there is exactly one morphism between any two different objects. Hence this gives precisely the abstract circle associated to the original Archimedean set.

   Summarizing the above discussion, the construction of an abstract circle from an Archimedean set factorizes as

$$\mathbf{ArchSet} \to \mathbf{OGrpdU} \to \mathbf{BarCat} \to \mathbf{EAd}.$$

Both the category of Archimedean sets and the category of barred commutative monoids can be embedded in the category of barred categories, in such a way that restricting to enclosed morphisms gives the right effect algebroid. Therefore a barred category is indeed a well-behaved notion of totalization of an effect algebroid.

4.5 The category of effect algebroids

In this section we will study the categorical properties of effect algebroids. In particular we will show that the category **EAd** is complete and cocomplete.

It is easy to see that the category of effect algebroids is complete, by constructing products and equalizers. The product of effect algebroids is just their cartesian product with pointwise operations, and the equalizer of two morphisms $F, G : A \to B$ is given by the subalgebroid of A with points $x \in \mathrm{Pt}(A)$ for which $F(x) = G(x)$, and segments $a \in \mathrm{Sg}(A)$ for which $F(a) = G(a)$.

It is also straightforward to construct coproducts, by taking the disjoint union of both points and segments. Note however that the coproduct of two effect algebras does not coincide with their coproduct as one-point algebroids. This is because the coproduct of two effect algebras still has only one point, and the zeroes and ones of both algebras are identified, while their coproduct as algebroids has two points.

The only difficulty in proving that the category **EAd** is cocomplete is the construction of coequalizers. Coequalizers of effect algebras are discussed in [77], and coequalizers of categories in [15, 21]. Both constructions are rather subtle, and coequalizers in effect algebroids combine the issues involved in both cases. We will start by reviewing coequalizers of categories as described in [15], continue by showing how these adapt to barred categories, and then show how they descend to the level of effect algebroids.

Coequalizers of categories are obtained by quotienting out a certain congruence relation. We have to be careful with the precise definition of congruence, since it does not suffice to simply use a congruence relation on objects and morphisms. Instead, we have to look at generalized congruences, which are congruences defined on sequences of morphisms rather than single morphisms. We will simply call these congruences. Given a category $\mathbf{C}$, denote by $\mathbf{C}^+$ the set of non-empty sequences $\mathbf{f} = (f_1, \ldots, f_n)$ of (not necessarily composable) morphisms. We define the domain and codomain of such a sequence by $\operatorname{dom} \mathbf{f} = \operatorname{dom} f_1$ and $\operatorname{cod} \mathbf{f} = \operatorname{cod} f_n$. Given two sequences $\mathbf{f} = (f_1, \ldots, f_n)$ and $\mathbf{g} = (g_1, \ldots, g_m)$, write $\mathbf{fg}$ for their concatenation $(f_1, \ldots, f_n, g_1, \ldots, g_m)$. Recall that a partial equivalence relation is a symmetric and transitive relation.

Definition 4.5.1. A *congruence* on a category $\mathbf{C}$ consists of an equivalence relation $\simeq$ on $\mathrm{Obj}(\mathbf{C})$ and a partial equivalence relation $\simeq$ on $\mathbf{C}^+$ such that the following properties hold:

1. If $\mathbf{fg} \simeq \mathbf{h}$, then $\mathrm{cod}\,\mathbf{f} \simeq \mathrm{dom}\,\mathbf{g}$.
2. If $\mathbf{f} \simeq \mathbf{g}$, then $\mathrm{dom}\,\mathbf{f} \simeq \mathrm{dom}\,\mathbf{g}$ and $\mathrm{cod}\,\mathbf{f} \simeq \mathrm{cod}\,\mathbf{g}$.
3. If $x \simeq y$, then $(\mathrm{id}_x) \simeq (\mathrm{id}_y)$.
4. If $\mathbf{f} \simeq \mathbf{f}'$ and $\mathbf{g} \simeq \mathbf{g}'$ and $\mathrm{cod}\,\mathbf{f} \simeq \mathrm{dom}\,\mathbf{g}$, then $\mathbf{fg} \simeq \mathbf{f}'\mathbf{g}'$.
5. If $f : x \to y$ and $g : y \to z$, then $(f\,;g) \simeq (f, g)$.

If $\simeq$ is a congruence on a category $\mathbf{C}$, then we can form a quotient category $\mathbf{C}/\simeq$. The objects of the quotient are equivalence classes of objects in $\mathbf{C}$. A morphism from $[x]$ to $[y]$ is an equivalence class of sequences $\mathbf{f} \in \mathbf{C}^+$ for which $\mathbf{f} \simeq \mathbf{f}$ and $\mathrm{dom}\,\mathbf{f} \simeq x$, $\mathrm{cod}\,\mathbf{f} \simeq y$. Composition is given by concatenating sequences, and the equivalence classes of identities in $\mathbf{C}$ provide identities in $\mathbf{C}/\simeq$.

Lemma 4.5.2. *For a congruence $\simeq$ on a category $\mathbf{C}$, the quotient $\mathbf{C}/\simeq$ forms a category.*

Proof. Morphisms have well-defined domains and codomains by Axiom 2 in Definition 4.5.1. From Axioms 1 and 4 it follows that composition $[\mathbf{f}]\,;[\mathbf{g}]$ is defined if and only if $\mathrm{cod}\,f \simeq \mathrm{dom}\,g$, and Axiom 4 also shows that it is independent of the choice of representatives. The identity is well-defined by Axiom 3. To show that it behaves like an identity, take a sequence $\mathbf{f} = (f_1, \ldots, f_n)$ with $\mathrm{dom}\,\mathbf{f} = x$, and assume that $x \simeq x'$. Then $(\mathrm{id}_{x'}, f_1, \ldots, f_n) \simeq (\mathrm{id}_x, f_1, \ldots, f_n)$ by Axioms 3 and 4, and the latter is equivalent to $((\mathrm{id}_x\,; f_1), f_2, \ldots, f_n) = (f_1, \ldots, f_n)$ by Axiom 5. It follows that $[\mathrm{id}_x]$ is the identity on x in $\mathbf{C}/\simeq$. Finally, associativity simply follows from associativity of concatenation. □

Now let $F, G : \mathbf{C} \to \mathbf{D}$ be two parallel functors. To construct their coequalizer, let $\simeq_{F,G}$ be the smallest congruence on $\mathbf{D}$ for which $F(c) \simeq_{F,G} G(c)$ for all objects c in $\mathbf{C}$, and $(F(f)) \simeq_{F,G} (G(f))$ for all morphisms f.

Proposition 4.5.3. *The coequalizer of two functors $F, G : \mathbf{C} \to \mathbf{D}$ in $\mathbf{Cat}$ is $\mathbf{D}/\simeq_{F,G}$.*

This is proven in [15]; we will prove a variation valid for barred categories.

Definition 4.5.4. A congruence $\simeq$ on a barred category $(\mathbf{C}, u)$ is called a *bar congruence* if it satisfies the following requirements:

1. If $x \simeq y$, then $(u_x) \simeq (u_y)$.
2. If $\mathbf{fg} \simeq (\mathrm{id}_x)$, then $\mathbf{f} \simeq \mathbf{g} \simeq (\mathrm{id}_x)$.
3. If $\mathbf{fg} \simeq \mathbf{fh} \simeq (u_x)$, then $\mathbf{g} \simeq \mathbf{h}$.
4. If $\mathbf{fg} \simeq (u_{\mathrm{dom}\,\mathbf{f}})$, then $\mathbf{gf} \simeq (u_{\mathrm{dom}\,\mathbf{g}})$.

Example 4.5.5. Any barred functor $F : (\mathbf{C}, u) \to (\mathbf{D}, v)$ induces a bar congruence $\simeq_F$ on $(\mathbf{C}, u)$ in the following way. We declare two objects x and y in $\mathbf{C}$ to be equivalent if and only if $F(x) = F(y)$. For sequences $\mathbf{f} = (f_1, \dots, f_n)$ and $\mathbf{g} = (g_1, \dots, g_m)$ in $\mathbf{C}^+$, put $\mathbf{f} \simeq_F \mathbf{g}$ if and only if $F(f_1) \,;\cdots;\, F(f_n)$ and $F(g_1) \,;\cdots;\, F(g_m)$ are both defined and equal. Checking the conditions from Definitions 4.5.1 and 4.5.4 is straightforward, so $\simeq_F$ is indeed a bar congruence.

Lemma 4.5.6. *For a bar congruence $\simeq$ on a barred category $(\mathbf{C}, u)$, the quotient $\mathbf{C}/\simeq$ forms a barred category with bar defined by $[u]_{[x]} = [u_x]$ for $x \in \mathbf{C}$.*

Proof. The bar is well-defined by Axiom 1 from Definition 4.5.4. Since each bar congruence is in particular a congruence on the underlying category, $\mathbf{C}/\simeq$ forms a category by Lemma 4.5.2. The requirements for a barred category follow readily from the defining properties of a bar congruence. □

Now we have the necessary preparation to construct coequalizers of barred categories. Let $F, G : (\mathbf{C}, u) \to (\mathbf{D}, v)$ be two parallel barred functors. Let $\simeq_{F,G}$ be the smallest bar congruence on $\mathbf{D}$ for which $F(c) \simeq_{F,G} G(c)$ for all objects c in $\mathbf{C}$, and $(F(f)) \simeq_{F,G} (G(f))$ for all morphisms f. This smallest congruence exists, since bar congruences are closed under intersections.

Proposition 4.5.7. *The coequalizer of two barred functors $F, G : (\mathbf{C}, u) \to (\mathbf{D}, v)$ is the quotient $\mathbf{D}/\simeq_{F,G}$.*

Proof. To simplify notation, write $\simeq$ for $\simeq_{F,G}$ Let $Q : \mathbf{D} \to \mathbf{D}/\simeq$ be the canonical projection functor. It satisfies $QF = QG$ because $F(c) \simeq G(c)$ and $F(f) \simeq G(f)$. Suppose that $R : (\mathbf{D}, v) \to (\mathbf{E}, w)$ also satisfies $RF = RG$. We have to show that there is a unique $S : \mathbf{D}/\simeq \to \mathbf{E}$ such that $SQ = R$. For an object $[d]$ in $\mathbf{D}/\simeq$, define $S([d]) = R(d)$; and for a morphism $[\mathbf{f}] = [(f_1, \dots, f_n)]$, define $S([\mathbf{f}]) = R(f_1) \,;\cdots;\, R(f_n)$.

Before we prove that S is well-defined, we will first show that $\simeq \subseteq \simeq_R$, where $\simeq_R$ is the congruence induced by R, as defined in Example 4.5.5. To this end, remark that $\simeq_R$ is a bar congruence for which $F(c) \simeq_R G(c)$ and $(F(f)) \simeq_R (G(f))$. Since $\simeq$ is the smallest such congruence, the assertion $\simeq \subseteq \simeq_R$ follows.

To show that S is well-defined, suppose first that $d \simeq d'$. Since $\simeq \subseteq \simeq_R$, we get $d \simeq_R d'$, hence $R(d) = R(d')$. Now suppose that $\mathbf{f} \simeq \mathbf{g}$. Then $\mathbf{f} \simeq_R \mathbf{g}$, hence $R(f_1) ; \cdots ; R(f_n)$ and $R(g_1) ; \cdots ; R(g_m)$ are both defined and equal.

The mapping S is easily seen to be a barred functor making the diagram commute. It is also the unique barred functor with this property, which establishes the claim. □

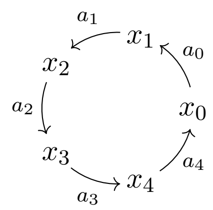
Theorem 4.5.8. *The category of effect algebroids is cocomplete.*

Proof. Since the category **EAd** clearly has coproducts and an initial object, it suffices to show that it has coequalizers. By Proposition 4.4.6, it is a coreflective subcategory of **BarCat**, which has coequalizers. From general category theory we know that any coreflective subcategory inherits all colimits from the larger category (this is e.g. the dual of [21, Prop. 3.5.3]), so **EAd** has coequalizers as well. Explicitly, to compute the coequalizer of two morphisms $F, G : A \to B$, first determine the coequalizer $\mathbf{C}$ of $\mathcal{T}(F)$ and $\mathcal{T}(G)$. Then the coequalizer of F and G is $\mathcal{E}(\mathbf{C})$. It follows that **EAd** has all colimits. □

Chapter 5

Cyclic sets

We will investigate the interplay between an effect algebroid and its cycles. Intuitively, a cycle in an effect algebroid is a sequence of points and segments arranged in a circle



where we require that the composition $a_0 \cup \cdots \cup a_n$ of the segments is 1. The cycles will turn out to contain all the information about the effect algebroid.

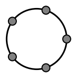
Cycles can be organized into a structure called a cyclic set, which is a variation of a simplicial set. Simplicial sets can be thought of as combinatorial models for topological spaces. Analogously, cyclic sets are combinatorial models for topological spaces carrying an action of the circle group. Since the cycles of an effect algebroid form a cyclic set, it is possible to apply some of the theory of cyclic sets to effect algebroids, which is what we will do in this chapter.

The connection between effect algebroids and cyclic sets is analogous to the connection between categories and simplicial sets. Therefore most of the results in this chapter can be viewed as translations of corresponding results about categories and simplicial sets. For example, the embedding of effect algebroids into cyclic sets behaves in the same way as the nerve of a category, and there are corresponding Segal conditions. Furthermore, we will define geometric realizations of effect algebroids in such a way that they are analogous to geometric realizations (or classifying spaces) of categories.

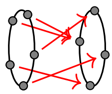
5.1 Cycles in effect algebroids

Before we can define cycles in an effect algebroid, we will take a look at a more abstract version of cycles using Connes' cycle category $\mathbf{\Lambda}$ from [28, 30, 91], see also [42] for a picture from a more general point of view. Let **FinAbsCirc** be the full subcategory of **EAd** consisting of the finite abstract circles. The objects of **FinAbsCirc** are configurations of a finite number of points on the unit circle $\mathbb{S}^1$. We wish to work with a fixed skeleton $\mathbf{\Lambda}$ of the category **FinAbsCirc**. Two finite configurations of points on the circle are isomorphic if and only if they have the same number of points, so fixing a skeleton means picking one configuration of n points for each natural number n. Given n, let Λ_n be the abstract circle $\{e^{2\pi ik/(n+1)} \mid k = 0, \ldots, n\}$, called the *$n$-cycle*. Keep in mind that the n-cycle has $n + 1$ points; this is customary in the theory of cyclic sets. The collection of all Λ_n gives a skeleton of **FinAbsCirc** denoted $\mathbf{\Lambda}$. An object of $\mathbf{\Lambda}$ is called a *cycle*, and the category itself is called the *cycle category*.

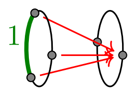
Manipulations with cycles in effect algebroids often boil down to calculations with morphisms in the category $\mathbf{\Lambda}$. To simplify these calculations, we will use a graphical notation for the objects and morphisms in $\mathbf{\Lambda}$. The object Λ_n is depicted as a circle with $n+1$ points on it, in the configuration we fixed before, for instance:

$\Lambda_4 =$ 

A morphism is represented by its action on points:

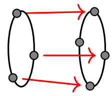


For most morphisms in $\mathbf{\Lambda}$, there is only one possible definition for the action on segments when the action on points is known, so a diagram of this form suffices to describe it. The only exceptions are the constant morphisms. If $\varphi : \Lambda_m \to \Lambda_n$ is constant with value x, then exactly one segment in Λ_m is sent to 1_x in Λ_n, and all other segments are sent to 0_x. In such a case, the segment that is sent to 1_x is indicated by a 1 next to the segment.

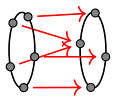


The cycle category is generated by morphisms of the following forms:

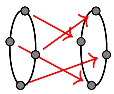
- Face maps $\delta_i^n : \Lambda_{n-1} \to \Lambda_n$ for $i = 0, \ldots, n$, where δ_i^n is the injection that skips the i^{th} point in Λ_n. As an example, $\delta_2^3 : \Lambda_2 \to \Lambda_3$ is the map



- Degeneracy maps $\sigma_i^n : \Lambda_{n+1} \to \Lambda_n$ for $i = 0, \ldots, n$, where σ_i^n is the surjection that hits the i^{th} point twice. Thus $\sigma_2^3 : \Lambda_4 \to \Lambda_3$ is for example:



- Cyclic permutations $\tau^n : \Lambda_n \to \Lambda_n$ that map each point to the next point counterclockwise. For example, for $n = 3$ this is the map



The generating maps $\delta_i^n, \sigma_i^n, \tau^n$ will often be written as δ_i, σ_i, τ when no confusion is possible. These generators are subject to the following relations, yielding a presentation of the category $\mathbf{\Lambda}$:

$$\delta_j^{n+1} \circ \delta_i^n = \delta_i^{n+1} \circ \delta_{j-1}^n \text{ for } i < j.$$

$$\sigma_j^n \circ \sigma_i^{n+1} = \sigma_i^n \circ \sigma_{j+1}^{n+1} \text{ for } i \leq j.$$

$$\sigma_j^n \circ \delta_i^{n+1} = \begin{cases} \delta_i^n \circ \sigma_{j-1}^{n-1} & \text{for } i < j \\ \text{id} & \text{for } i = j \text{ or } i = j+1 \\ \delta_{i-1}^n \circ \sigma_j^{n-1} & \text{for } i > j+1 \end{cases}$$

$$\tau^n \circ \delta_i^n = \begin{cases} \delta_{i+1}^n \circ \tau^{n-1} & \text{for } i < n \\ \delta_0^n & \text{for } i = n \end{cases}$$

$$\tau^n \circ \sigma_i^n = \begin{cases} \sigma_{i+1}^n \circ \tau^{n+1} & \text{for } i < n \\ \sigma_0^n \circ \left(\tau^{n+1}\right)^2 & \text{for } i = n \end{cases}$$

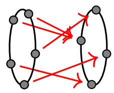
$$(\tau^n)^{n+1} = \text{id}$$

Since these equations offer less insight than the graphical notation introduced earlier, we will often refrain from using the presentation, except when absolutely necessary.

An object in $\mathbf{\Lambda}$ can be seen as an abstract cycle. A concrete n-cycle in an effect algebroid A is defined as a morphism $\Lambda_n \to A$ of effect algebroids.

Example 5.1.1. Let A be an effect algebra, viewed as a one-object effect algebroid. An n-cycle in A can be identified with a sequence of elements $a_0, \dots, a_n$ of A for which $a_0 \boxplus \cdots \boxplus a_n = 1$. This is usually called an $(n+1)$-test on the effect algebra. Thus cycles can be seen as a generalization of tests. Note that our convention to denote denote the set of $(n+1)$-tests by $\mathcal{T}_n(A)$ coincides with the cycle indexing convention.

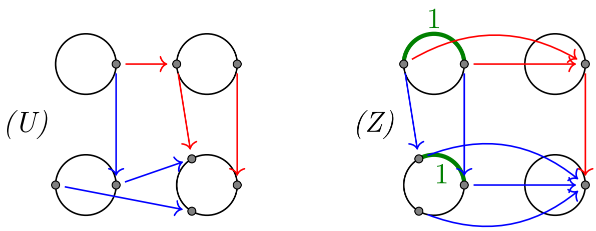
A *cyclic set* is a presheaf on $\mathbf{\Lambda}$. If X is a cyclic set, then we will write X_n for $X(\Lambda_n)$. The category $\mathbf{\Lambda}$ acts from the right on X, and we will denote the action by $x \cdot f = X(f)(x)$. This notation for the action can be combined with the graphical notation for morphisms in $\mathbf{\Lambda}$. Since the morphisms are written from left to right in the graphical notation, we have to use a left action. For example, we can write

 $\cdot\, x$

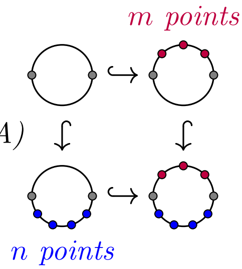
for $x \in X_3$.

Cycles in an effect algebroid can be organized into a cyclic set. To be precise, there is a functor $\mathcal{C} : \mathbf{EAd} \to \mathbf{cSets}$ given by $\mathcal{C}(A)(\Lambda_n) = \mathbf{EAd}(\Lambda_n, A)$. We wish to characterize the essential image of $\mathcal{C}$. This is similar to characterizing the simplicial sets that are nerves of categories. In [112], Segal describes a characterization of nerves of categories as simplicial sets that send certain pushouts to pullbacks, attributed to unpublished work by Grothendieck. We aim for a similar characterization of cycles in effect algebroids, so we will first take a look at pushouts in the category $\mathbf{\Lambda}$.

Lemma 5.1.2. *The diagrams*

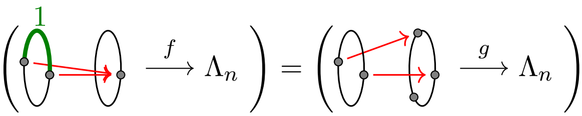


are pushouts in $\mathbf{\Lambda}$. *Also, any diagram of the form*

(A) 

is a pushout in $\mathbf{\Lambda}$*. The names* (U)*,* (Z)*, and* (A) *stand for 'Uniqueness', 'Zero-one law', and 'Associativity', respectively.*

Proof. We will prove this for the diagram (Z). Let $f : \Lambda_0 \to \Lambda_n$ and $g : \Lambda_2 \to \Lambda_n$ be morphisms for which



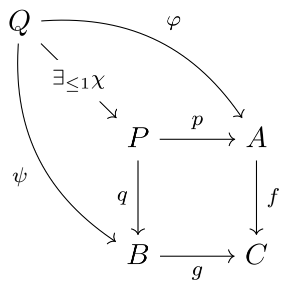
Then g maps the "first" segment of Λ_2 to a full circle in Λ_n. But that is only possible if g maps both other segments of Λ_2 to the zero segment in Λ_n. This implies that there is a point $x \in \Lambda_n$ such that g maps all points of Λ_2 to x, and f maps the single point of Λ_0 to x. Define $h : \Lambda_0 \to \Lambda_n$ by sending the point in Λ_0 to x. Then h makes the appropriate triangles commute, and it is easily seen to be unique. □

When reconstructing a category from its nerve, the composition comes from a certain mediating morphism in a pullback. If we want to reconstruct an effect algebroid, the composition is only partially defined. Therefore we will need a weakening of the concept of pullback, in which the mediating morphism does not always exist. However, we do require it to be unique whenever it is defined.

Definition 5.1.3. A commutative square

$$\begin{array}{ccc} P & \xrightarrow{p} & A \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[g]{} & C \end{array}$$

is called a *subpullback* if for every two morphisms $\varphi : Q \to A$ and $\psi : Q \to B$, there is at most one $\chi : Q \to P$ making the diagram



commute.

Theorem 5.1.4. *The cycle functor* $\mathcal{C} : \mathbf{EAd} \to \mathbf{cSets}$ *is full and faithful. Its essential image consists of those cyclic sets* $X : \mathbf{\Lambda}^{\mathrm{op}} \to \mathbf{Sets}$ *for which:*

- X *sends the pushout (U) of Lemma 5.1.2 to a subpullback;*
- X *sends the pushout (Z) to a pullback;*
- X *sends the family (A) of pushouts to pullbacks.*

The conditions in the above theorem will be called the *partial Segal conditions*. The proof of the theorem will occupy the next two sections. Section 5.2 will set the stage by showing how to recover an effect algebroid from a cyclic set satisfying the partial Segal conditions. After this preparatory material, we will tackle the claims of Theorem 5.1.4 one by one in Section 5.3.

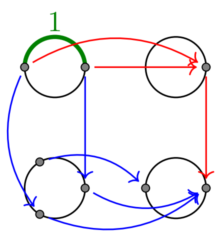
5.2 Reconstruction of effect algebroid

We will prove Theorem 5.1.4 in several steps. First we will show that the partial Segal conditions imply that one other pushout is also sent to a pullback.

Lemma 5.2.1.

1. *The following diagram is a pushout in* $\mathbf{\Lambda}$.

(O)



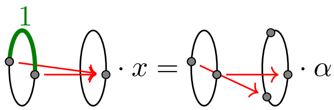
The name (O) stands for 'Orthocomplement'.

2. *If a cyclic set satisfies the partial Segal conditions, then it sends this pushout to a pullback.*

Proof.

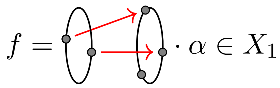
1. Straightforward.

2. Let X be a cyclic set satisfying the partial Segal conditions. To show that it sends the diagram (O) to a pullback, take $x \in X_0$ and $\alpha \in X_2$ such that

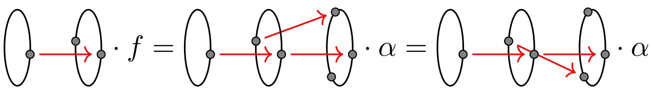


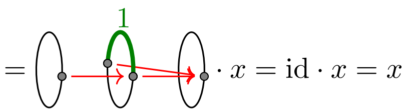
We have to find a unique $f \in X_1$ such that

Take

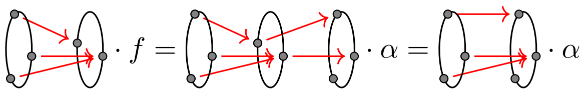


Then the first equation for f is easy to verify:

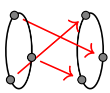




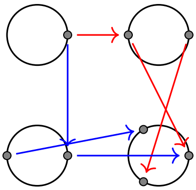
To check the second equation, first note that



Call the right-hand side β; we have to show that it is equal to α. We will accomplish this by showing that both morphisms are mediating morphisms in the same subpullback. By the partial Segal conditions, X sends the pushout (U) of Lemma 5.1.2 to a subpullback. Since the cyclic permutation

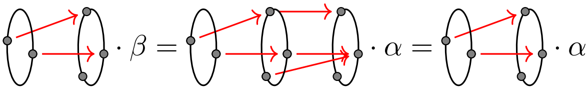


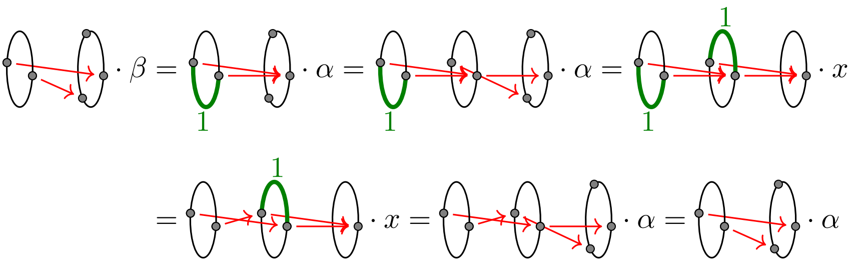
is an automorphism of Λ_2, also the diagram



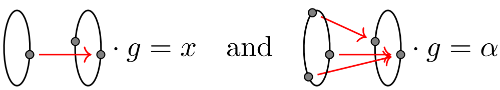
is sent to a subpullback. This means that two elements of X_2 are equal whenever the morphisms

act on them in the same way. Thus the following computations show that $\alpha = \beta$:

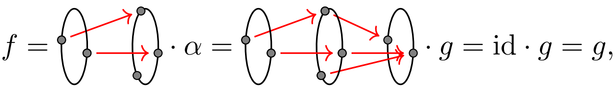




Now we have to show that f is the unique element of X_1 with these properties. Suppose that also $g \in X_1$ satisfies



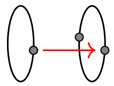
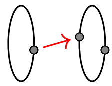
Then



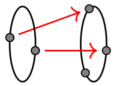
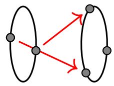
which finishes the proof that the diagram is a pullback. □

The second step of the proof is showing how to construct an effect algebroid $\mathcal{A}(X)$ from a cyclic set X satisfying the partial Segal conditions. Let X_0 be the collection of points, and X_1 the collection of all segments of

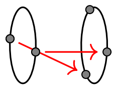
$\mathcal{A}(X)$. We say that $f \in X_1$ is a segment from x to y whenever $\mathrm{dom} \cdot f = x$ and $\mathrm{cod} \cdot f = y$, where the domain and codomain functions are defined by

dom =  and cod = 

To define the partial composition, suppose that f and g are segments with $\mathrm{cod} \cdot f = \mathrm{dom} \cdot g$. We say that $f \cup g$ is defined if and only if there exists a “bound” $\alpha \in X_2$ for which

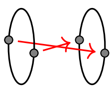
 $\cdot\, \alpha = f$ and  $\cdot\, \alpha = g$

If such a bound exists, then it is unique by the first partial Segal condition. In this case, the composition $f \cup g$ is defined as

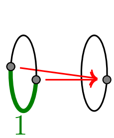
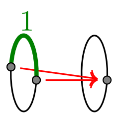
 $\cdot\, \alpha$

Intuitively, α can be thought of as a 2-cycle in which the first segment represents f and the second segment represents g. The composition is obtained by taking these two segments together. Using bounds is a common technique for constructing partial operations, also used in e.g. [10, 75, 76, 117].

The orthocomplement in $\mathcal{A}(X)$ is obtained by swapping two segments:

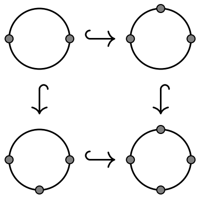
$f^{\perp} =$  $\cdot\, f$

Finally, the zero and one segments on a point $x \in X_0$ are defined by

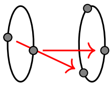
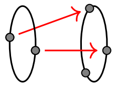
$0_x =$  $\cdot\, x$ and $1_x =$  $\cdot\, x$

Lemma 5.2.2. *If X is a cyclic set satisfying the partial Segal conditions, then $\mathcal{A}(X)$ is an effect algebroid.*

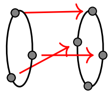
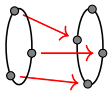
Proof. We start by proving associativity. Suppose that $f \cup g$ and $(f \cup g) \cup h$ are defined. Let α be a bound for f and g, and let β be a bound for $(f \cup g)$ and h. By assumption, the cyclic set X sends the diagram



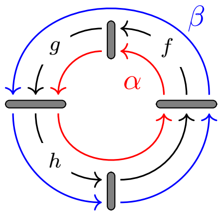
to a pullback in **Sets**. Therefore, since

 $\cdot\, \alpha = f \cup g =$  $\cdot\, \beta,$

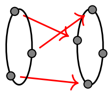
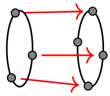
there exists a unique $\Theta \in X_3$ for which

 $\cdot\, \Theta = \alpha$ and  $\cdot\, \Theta = \beta.$

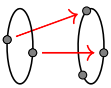
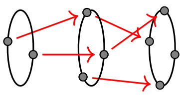
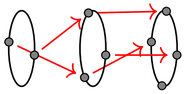
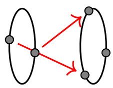
Think of Θ as a 3-cycle representing the following information:

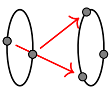
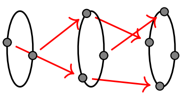
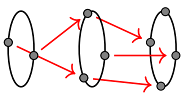
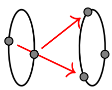


We can find bounds for g, h and $f, g \cup h$ by extracting suitable 2-cycles from Θ. Define

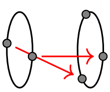
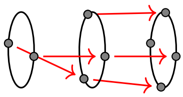
$\gamma =$  $\cdot\, \Theta$ and $\delta =$  $\cdot\, \Theta.$

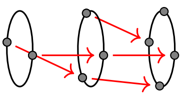
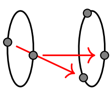
Then γ is a bound for g and h:

 $\cdot\, \gamma =$  $\cdot\, \Theta =$  $\cdot\, \Theta =$  $\cdot\, \alpha = g$

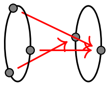
 $\cdot\, \gamma =$  $\cdot\, \Theta =$  $\cdot\, \Theta =$  $\cdot\, \beta = h$

Similarly δ is a bound for f and $g \cup h$. The next calculation finishes the associativity proof:

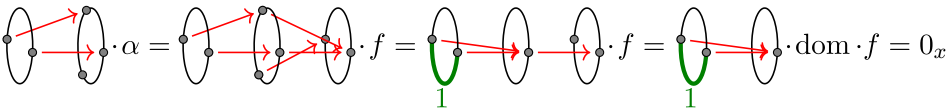
$f \cup (g \cup h) =$  $\cdot\, \delta =$  $\cdot\, \Theta$

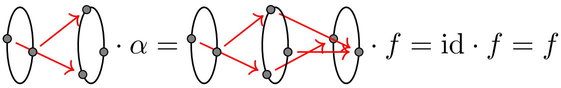
$=$  $\cdot\, \Theta =$  $\cdot\, \beta = (f \cup g) \cup h$

The next step is to show that 0_x is a neutral element. Take any $f \in \mathrm{Hom}(x, y)$; we will prove that $0_x \cup f$ is defined and equals f. As a bound for the composition, take

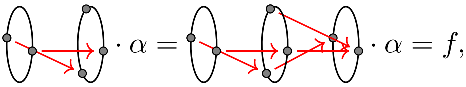
$\alpha =$  $\cdot f.$

This is indeed a bound because:



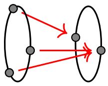


The composition is

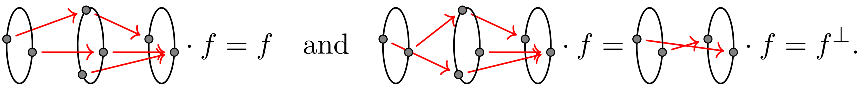


so $0_x \cup f = f$. Analogously $f \cup 0_y = f$.

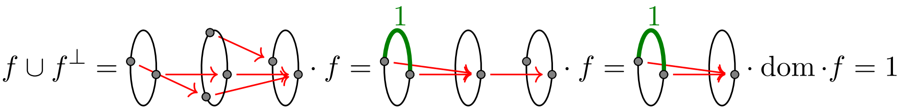
We continue with the orthocomplement law. We will prove that $f \cup f^\perp = 1$ and that $f \cup g = 1$ implies $g = f^\perp$; the other aspects of the law follow from cyclic permutations of these arguments. To show that $f \cup f^\perp = 1$, use the bound

$\alpha =$  $\cdot f.$

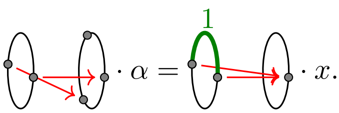
Then α bounds f and $f^\perp$ because



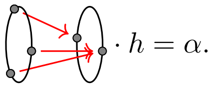
It gives the composition



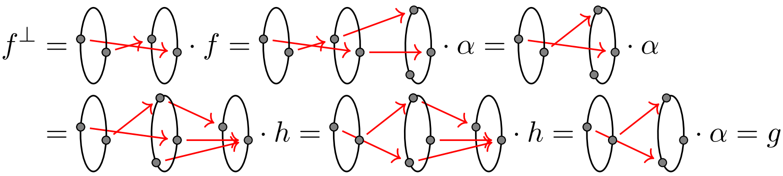
Now assume that $f \cup g = 1_x$ via a bound α. Then



Therefore, since the diagram (O) from Lemma 5.2.1 is sent to a pullback, there exists $h \in X_1$ for which

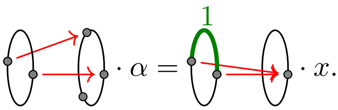


This gives

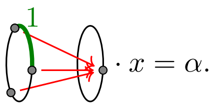


Thus the orthocomplement law holds.

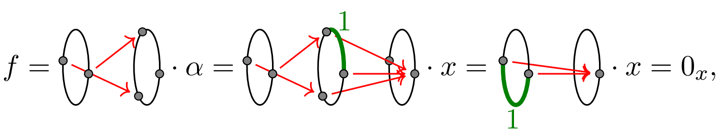
Finally we have to check the zero-one law. Suppose that $1_x \cup f$ is defined, for $f \in \mathrm{Hom}(x, y)$. Determine a bound α for 1_x and f, then



The diagram (Z) in Lemma 5.1.2 is sent to a pullback, so x satisfies



It follows that



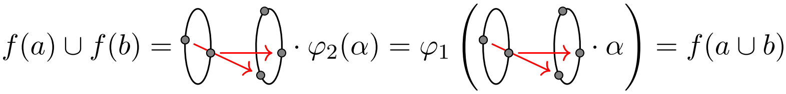
as required. □

5.3 Proof of Theorem 5.1.4

With the above preparations, we are now ready to prove the first assertion in Theorem 5.1.4.

Claim 1. *The cycle functor $\mathcal{C}$ is full and faithful.*

Proof. Let $\varphi : \mathcal{C}(A) \to \mathcal{C}(B)$ be any map of cyclic sets. Since 0-cycles in an effect algebroid correspond to points, and 1-cycles correspond to segments, we can simply define a morphism of effect algebroids $f : A \to B$ on points as φ_0 and on segments as φ_1. Then f preserves 0, 1 and complements because φ is natural. To prove the functoriality condition, suppose that $a \cup b$ is defined in A. Consider a and b as elements of $X_1 \cong \mathcal{C}(\mathcal{A}(X))_1$, and let $\alpha \in X_2$ be a bound. Then $\varphi_2(\alpha)$ is a bound for $f(a)$ and $f(b)$. Hence $f(a) \cup f(b)$ is defined, and it equals $f(a \cup b)$ because:

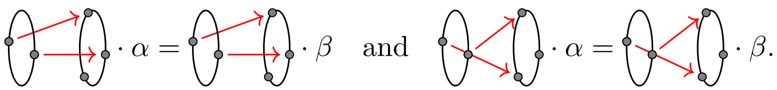


The morphism f satisfies $\mathcal{C}(f) = \varphi$ because φ is natural, and f is unique since points are 0-cycles and segments are 1-cycles. □

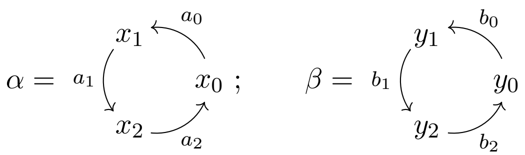
For the second assertion of the theorem, it suffices to show that cyclic sets in the image of the cycle functor satisfy the partial Segal conditions, and that $\mathcal{C}(\mathcal{A}(X))$ is naturally isomorphic to X if X satisfies the partial Segal conditions. We start with the former claim.

Claim 2. *If A is an effect algebroid, then $\mathcal{C}(A)$ satisfies the partial Segal conditions.*

Proof. To show that the diagram (U) is sent to a subpullback, suppose that $\alpha, \beta \in \mathcal{C}(A)_2$ satisfy



We have to show that $\alpha = \beta$. Identify α and β with the following 2-cycles in A:

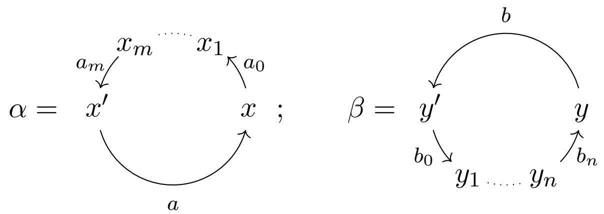


The equations for α and β give $a_0 = b_0$ and $a_1 = b_1$. Therefore, also all points in the 2-cycles are the same. Furthermore, $a_2 = (a_0 \cup a_1)^\perp = (b_0 \cup b_1)^\perp = b_2$, which shows that $\alpha = \beta$.

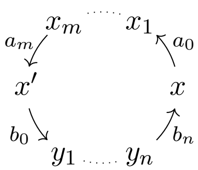
We also have to show that $\mathcal{C}(A)$ sends the pushout (Z) to a pullback. For this, assume that we have a point x in A and a 2-cycle α, whose first segment a_0 is equal to 1_x. We are done if we can show that the other segments a_1

and a_2 are both zero. The composition $a_0 \cup a_1 \cup a_2 = 1_x \cup a_1 \cup a_2$ is equal to 1_x, so by the zero-one law $a_1 \cup a_2 = 0_x$, and hence $a_1 = a_2 = 0_x$.

The next step is to show that the class (A) of diagrams in Lemma 5.1.2 is sent to pullbacks. Take cycles of the following forms in A:



Assume that the 1-cycles $(\Lambda_1 \hookrightarrow \Lambda_{1+m}) \cdot \alpha$ and $(\Lambda_1 \hookrightarrow \Lambda_{1+n}) \cdot \beta$ are equal. Then $x = y$ and $x' = y'$, and furthermore $a_0 \cup \cdots \cup a_m = b$ and $b_0 \cup \cdots \cup b_n = a$. Hence the cycle



reduces to α and β under the relevant inclusions, and it is clearly the unique cycle with this property. □

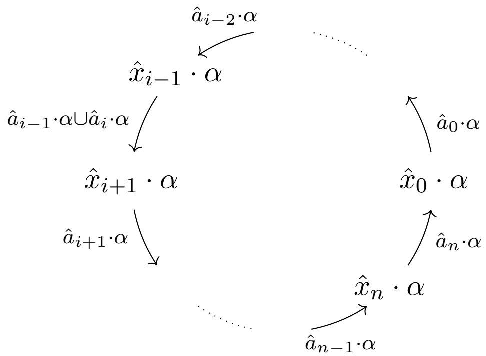
Now let X be any cyclic set satisfying the partial Segal conditions. We will show that it lies in the essential image of the cycle functor by providing an isomorphism $\Phi : X \to \mathcal{C}(\mathcal{A}(X))$. Take any $\alpha \in X_n$, then we have to define an n-cycle $\Phi_n(\alpha)$ in $\mathcal{A}(X)$. For $i = 0, \ldots, n$, let $\hat{x}_i : \Lambda_0 \to \Lambda_n$ be the map that sends the single point in Λ_0 to the i^{th} point in Λ_n, and let $\hat{a}_i : \Lambda_1 \to \Lambda_n$ send the first segment of Λ_1 to the i^{th} segment in Λ_n. Then define $\Phi_n(\alpha)$ to be the n-cycle with points $\hat{x}_0 \cdot \alpha, \ldots, \hat{x}_n \cdot \alpha$, and segments $\hat{a}_0 \cdot \alpha, \ldots, \hat{a}_n \cdot \alpha$.

Claim 3. *The map Φ is a natural transformation.*

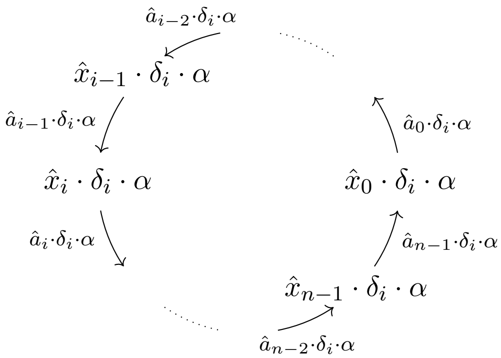
Proof. This can be checked by proving that Φ preserves the generators of $\mathbf{\Lambda}$. We will perform the computations for the generator $\delta_i : \Lambda_{n-1} \to \Lambda_n$. For this we have to show that the diagram

$$\begin{array}{ccc} X_n & \xrightarrow{\Phi_n} & \mathcal{C}(\mathcal{A}(X))_n \\ {\scriptstyle \delta_i}\downarrow & & \downarrow{\scriptstyle \delta_i} \\ X_{n-1} & \xrightarrow[\Phi_{n-1}]{} & \mathcal{C}(\mathcal{A}(X))_{n-1} \end{array}$$

commutes. Take any $\alpha \in X_n$, then the upper right route in the diagram maps α to the $(n-1)$-cycle



and the lower left route takes it to



We need a few computations to show that these two cycles are equal. First we check that they have the same points. Each point in either of the two cycles is obtained by applying a map $\Lambda_0 \to \Lambda_n$ to α, so it suffices to show that each of the maps involved sends the single point x_0 of Λ_0 to the same point in Λ_n. We have $\hat{x}_j(x_0) = x_j$ for each j, and

$$\delta_i(x_j) = \begin{cases} x_j & \text{for } j < i \\ x_{j+1} & \text{for } j \geq i. \end{cases}$$

It follows that $\delta_i \circ \hat{x}_j = \hat{x}_j$ for $j < i$, and $\delta_i \circ \hat{x}_j = \hat{x}_{j+1}$ for $j \geq i$. Therefore all points in both cycles are equal.

We continue by verifying that the cycles have the same segments, by showing that the corresponding morphisms $\Lambda_1 \to \Lambda_n$ have the same values on $x_0, x_1 \in \Lambda_1$. For $j < i-1$ and $k \in \{0,1\}$, we get $\delta_i(\hat{a}_j)(x_k) = \delta_i(x_{j+k}) = x_{j+k} = \hat{a}_j(x_k)$, so $\hat{a}_j \cdot \delta_i \cdot \alpha = \hat{a}_j \cdot \alpha$. If $j = i-1$, then $\delta_i(\hat{a}_j)(x_0) = \delta_i(x_{i-1}) =$

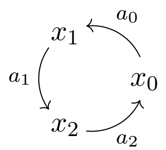
x_{i-1} and $\delta_i(\hat{a}_j)(x_1) = \delta_i(x_i) = x_{i+1}$, whence $\hat{a}_{i-1} \cdot \delta_i \cdot \alpha = \hat{a}_{i-1} \cdot \alpha \cup \hat{a}_i \cdot \alpha$. Finally, in the case where $j > i - 1$, we have $\delta_i(\hat{a}_j(x_k)) = \delta_i(x_{j+k}) = x_{j+k+1} = \hat{a}_{j+1}(x_k)$, from which it follows that all segments of both cycles are equal.

The verification that Φ preserves the generators σ_j and τ is analogous. Since Φ commutes with all generators of the category $\mathbf{\Lambda}$, it commutes with all morphisms, hence it is natural. □

Our proof that $X \cong \mathcal{C}(\mathcal{A}(X))$ will be finished if we can show that Φ is a natural isomorphism.

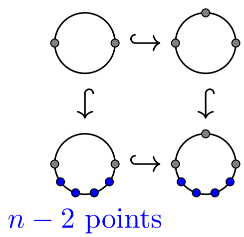
Claim 4. *The map Φ is an isomorphism at each level.*

Proof. For Φ_0 and Φ_1 this follows because points in X correspond to 0-cycles in $\mathcal{A}(X)$ and segments correspond to 1-cycles. If



is a 2-cycle in $\mathcal{A}(X)$, then the composition $a_0 \cup a_1$ is defined. That means that there exists a unique $\alpha \in X_2$ for which $\hat{a}_i \cdot \alpha = a_i$ for all i. But then $\Phi_2(\alpha)$ is the above 2-cycle, so Φ_2 is an isomorphism.

We treat the cases of Φ_n with $n \geq 3$ inductively. Both functors X and $\mathcal{C}(\mathcal{A}(X))$ send the diagram



to a pullback. Hence $X_n \cong X_{n-1} \times_{X_1} X_2$, which is isomorphic to

$$\mathcal{C}(\mathcal{A}(X))_{n-1} \times_{\mathcal{C}(\mathcal{A}(X))_1} \mathcal{C}(\mathcal{A}(X))_2$$

by the induction hypothesis. This implies that $X_n \cong \mathcal{C}(\mathcal{A}(X))_n$, and therefore Φ is a natural isomorphism. □

5.4 Geometric realization

Every simplicial set has a geometric realization. This realization can be seen as the space represented by the combinatorial complex. For cyclic sets there is a similar realization functor, which always gives a topological space with an action of the circle group $\mathbb{S}^1$. It was originally defined in [80], but see also [40, 19]. Since the cycles in any effect algebroid form a cyclic set, one can also assign a geometric realization to an effect algebroid. We will compute some examples of realizations of effect algebroids, and look at alternative descriptions of the realization using the topology on an effect algebroid.

Denote the category of topological spaces with an action of the circle group by $\mathbb{S}^1$-**Top**. There is a functor $|-|$ from $\mathbf{\Lambda}$ to $\mathbb{S}^1$-**Top**, that sends Λ_n to the space $\mathbb{S}^1 \times \Delta_n$. The circle group acts by multiplication on $\mathbb{S}^1$ and trivially on Δ_n. To make this assignment into a functor, we will first propose some notational conveniences. We will identify the circle $\mathbb{S}^1$ with the quotient $\mathbb{R}/\mathbb{Z}$. Elements of the standard simplex Δ_n will be written as $p = (p_0, \ldots, p_n)$, where the entries p_i are positive and sum to 1. To define the functor $|-| : \mathbf{\Lambda} \to \mathbb{S}^1\text{-}\mathbf{Top}$ on morphisms, we specify how it acts on the generators of $\mathbf{\Lambda}$:

$$\begin{aligned} \mathbb{S}^1 \times \Delta_{n-1} &\xrightarrow{|\delta_i|} \mathbb{S}^1 \times \Delta_n \\ (\theta, (p_0, \ldots, p_{n-1})) &\longmapsto (\theta, (p_0, \ldots, p_{i-1}, 0, p_i, \ldots, p_{n-1})) \end{aligned}$$

$$\begin{aligned} \mathbb{S}^1 \times \Delta_{n+1} &\xrightarrow{|\sigma_i|} \mathbb{S}^1 \times \Delta_n \\ (\theta, (p_0, \ldots, p_{n+1})) &\longmapsto (\theta, (p_0, \ldots, p_{i-1}, p_i + p_{i+1}, p_{i+2} \ldots, p_{n+1})) \end{aligned}$$

$$\begin{aligned} \mathbb{S}^1 \times \Delta_n &\xrightarrow{|\tau|} \mathbb{S}^1 \times \Delta_n \\ (\theta, (p_0, \ldots, p_n)) &\longmapsto (\theta - p_n, (p_n, p_0, \ldots, p_{n-1})) \end{aligned}$$

Lemma 5.4.1. *The assignment $|-| : \mathbf{\Lambda} \to \mathbb{S}^1$-**Top** yields a well-defined functor.*

Proof. The maps $|\delta_i|$ and $|\sigma_i|$ are equivariant because they leave the $\mathbb{S}^1$-component unchanged. The map $|\tau|$ is equivariant because

$$|\tau|(\varphi + \theta, p) = (\varphi + \theta - p_n, (p_n, p_0, \ldots, p_{n-1})) = \varphi \cdot |\tau|(\theta, p).$$

To prove that $|-|$ is functorial, it suffices to show that it respects the defining relations of the category $\mathbf{\Lambda}$. The equation $|\delta_j| \circ |\delta_i| = |\delta_i| \circ |\delta_{j-1}|$ holds for $i < j$, since both sides map (θ, p) to

$$(\theta, (p_0, \ldots, p_{i-1}, 0, p_i, \ldots, p_{j-2}, 0, p_{j-1}, \ldots, p_n)).$$

To prove that $|\sigma_j| \circ |\sigma_i| = |\sigma_i| \circ |\sigma_{j+1}|$ for $i \leq j$, note that both sides yield

$$(\theta, (p_0, \ldots, p_i + p_{i+1}, \ldots, p_{j+1} + p_{j+2}, \ldots, p_{n+1}))$$

for $i < j$, and

$$(\theta, (p_0, \ldots, p_i + p_{i+1} + p_{i+2}, \ldots, p_{n+1}))$$

for $i = j$.

We now turn our attention to the interaction between face and degeneracy maps. If $i < j$, then

$$\begin{aligned}
|\sigma_j|(|\delta_i|(\theta, (p_0, \ldots, p_n))) &= |\sigma_j|(\theta, (p_0, \ldots, p_{i-1}, 0, p_i, \ldots, p_n)) \\
&= (\theta, (p_0, \ldots, p_{i-1}, 0, p_i, \ldots, p_{j-1} + p_j, \ldots, p_n)) \\
&= |\delta_i|(\theta, (p_0, \ldots, p_{j-1} + p_j, \ldots, p_n)) \\
&= |\delta_i|(|\sigma_j|(\theta, (p_0, \ldots, p_n))).
\end{aligned}$$

Analogous computations show that $|\sigma_j| \circ |\delta_i| = \mathrm{id}$ for $i = j$ or $i = j+1$, and that $|\sigma_j| \circ |\delta_i| = |\delta_{i-1}| \circ |\sigma_j|$ for $i > j+1$.

Now consider the interaction between cyclic permutations and face maps. For $i < n$ we have

$$\begin{aligned}
|\tau|(|\delta_i|(\theta, (p_0, \ldots, p_{n-1}))) &= |\tau|(\theta, (p_0, \ldots, p_{i-1}, 0, p_i, \ldots, p_{n-1})) \\
&= (\theta - p_{n-1}, (p_{n-1}, p_0, \ldots, p_{i-1}, 0, p_i, \ldots, p_{n-2})) \\
&= |\delta_{i+1}|(\theta - p_{n-1}, (p_{n-1}, p_0, \ldots, p_{n-2})) \\
&= |\delta_{i+1}|(|\tau|(\theta, (p_0, \ldots, p_{n-1})))
\end{aligned}$$

and we have a similar result for $i = n$.

Combining cyclic permutations and degeneracy maps gives

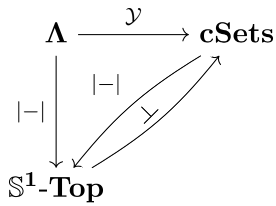
$$\begin{aligned}
|\tau|(|\sigma_i|(\theta, (p_0, \ldots, p_{n+1}))) &= |\tau|(\theta, (p_0, \ldots, p_i + p_{i+1}, \ldots, p_{n+1})) \\
&= (\theta - p_{n+1}, (p_{n+1}, p_0, \ldots, p_i + p_{i+1}, \ldots, p_n)) \\
&= |\sigma_{i+1}|(\theta - p_{n+1}, (p_{n+1}, p_0, \ldots, p_n)) \\
&= |\sigma_{i+1}|(|\tau|(\theta, (p_0, \ldots, p_{n+1})))
\end{aligned}$$

for $i < n$, and

$$\begin{aligned}
|\tau|(|\sigma_n|(\theta, (p_0, \ldots, p_{n+1}))) &= (\theta - (p_n + p_{n+1}), (p_n + p_{n+1}, p_0, \ldots, p_{n-1})) \\
&= |\sigma_0|(\theta - p_{n+1} - p_n, (p_n, p_{n+1}, p_0, \ldots, p_{n-1})) \\
&= |\sigma_0|(|\tau|(\theta - p_{n+1}, (p_{n+1}, p_0, \ldots, p_n))) \\
&= |\sigma_0|(|\tau|^2(\theta, (p_0, \ldots, p_{n+1}))).
\end{aligned}$$

Finally, the equation $\tau^{n+1} = \mathrm{id}$ holds because $p_0 + \cdots + p_n = 1$ for $(p_0, \ldots, p_n) \in \Delta_n$, and cycling this element $n+1$ times gives back the same thing. $\square$

Since the category $\mathbb{S}^1$-**Top** is cocomplete, the functor $|-|$ has a left Kan extension along the Yoneda embedding. The extension is called *geometric realization* and also denoted $|-|$:



Explicitly, the geometric realization of a cyclic set can be described via the colimit

$$|X| = \operatorname{colim}\left(\int_{\mathbf{\Lambda}} X \longrightarrow \mathbf{\Lambda} \xrightarrow{|-|} \mathbb{S}^1\text{-}\mathbf{Top}\right).$$

The geometric realization of an effect algebroid A is defined as the realization of the cyclic set $\mathcal{C}(A)$. The colimit in the definition of the realization can be difficult to calculate in general. However, for the case of finite effect algebroids the next result provides an easier way to determine the geometric realization.

Proposition 5.4.2. *Let A be a finite effect algebroid. The geometric realization of A is isomorphic (as an $\mathbb{S}^1$-space) to the space of upper semicontinuous effect algebroid morphisms from $\mathbb{S}^1$ to A.*

We will denote the space of upper semicontinuous maps from $\mathbb{S}^1$ to A by $\mathrm{Hom}(\mathbb{S}^1, A)$. The $\mathbb{S}^1$-action on this set is given by $(\theta \cdot f)(\varphi) = f(\varphi - \theta)$. To endow it with a topology, we will first define a metric. Any set of numbers $\Theta = \{\theta_0 < \theta_1 < \cdots < \theta_n = \theta_0 + 1\} \subseteq \mathbb{R}$ can be considered as a cyclically arranged set of points on the circle, using the isomorphism $\mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$. For any such Θ and $f, g \in \mathrm{Hom}(\mathbb{S}^1, A)$, define

$$d_\Theta(f, g) = \sum_{i=1}^{n} (\theta_i - \theta_{i-1}) d_{\mathrm{Disc}}(f(\theta_{i-1} \to \theta_i), g(\theta_{i-1} \to \theta_i)),$$

where d_{Disc} represents the discrete metric on segments of A. Using this definition, let $d(f, g)$ be the supremum over all such Θ of $d_\Theta(f, g)$. This supremum is well-defined, since $d_\Theta(f, g) \leq \sum_{i=1}^{n}(\theta_i - \theta_{i-1}) = 1$. It makes

$\mathrm{Hom}(\mathbb{S}^1, A)$ into a metric space, hence also into a Hausdorff topological space.

With this description of the $\mathbb{S}^1$-action and topology on the set of upper semicontinuous morphisms, we can start proving Proposition 5.4.2. We will first prove the assertion for the special case where A is the abstract circle Λ_n.

Lemma 5.4.3. *The* $\mathbb{S}^1$*-spaces* $\mathbb{S}^1 \times \Delta_n$ *and* $\mathrm{Hom}(\mathbb{S}^1, \Lambda_n)$ *are isomorphic.*

Proof. During this proof, we will pick representatives for elements of $\mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$ in the half-open interval $(0, 1]$ whenever possible.

If $\Phi : \mathbb{S}^1 \times \Delta_n \to \mathrm{Hom}(\mathbb{S}^1, \Lambda_n)$ is any equivariant map, then it satisfies $\Phi(\theta, p) = \theta \cdot \Phi(0, p)$. Therefore, to define the isomorphism Φ, it suffices to define $\Phi(0, p)$ and extend it via the action. Let $i \in \{0, \ldots, n\}$ be the index for which θ lies in the half-open interval

$$\left(\sum_{k=0}^{i-1} p_k, \sum_{k=0}^{i} p_k \right],$$

and then define $\Phi(0, (p_0, \ldots, p_n))(\theta)$ to be the i^{th} point of Λ_n.

An easier way to describe this: call the points of Λ_n $x_0, \ldots, x_n$. If θ lies in the interval $(0, p_0]$, then we map it to x_0. If it lies in $(p_0, p_0 + p_1]$, then we map it to x_1, and so on, until the case where θ lies in $(p_0 + \cdots + p_{n-1}, 1]$, in which case it is mapped to x_n.

If $\Phi(0, p)$ is not constant on the points of $\mathbb{S}^1$, then its action on segments is fixed by its action on points. If the map is constant, then the probability distribution $(p_0, \ldots, p_n)$ is degenerate, in the sense that $p_i = 1$ for a certain i. In this case, define the action on segments by $\Phi(\theta, (p_0, \ldots, p_n))(a) = 1$ if θ lies in the interior of the segment a, and 0 otherwise.

The mapping $\Phi(0, p)$ is upper semicontinuous, since we used half-open intervals that are closed on the right.

To define an inverse Ψ for Φ, take any $f \in \mathrm{Hom}(\mathbb{S}^1, \Lambda_n)$. We distinguish two cases:

- If f is not constant, then let $\theta \in \mathbb{S}^1$ be the limit (in $\mathbb{S}^1$) of $f^{-1}(x_0)$. Set $\Psi(f) = (\theta, (\mu(f^{-1}(x_0)), \ldots, \mu(f^{-1}(x_n))))$, where μ is the Lebesgue measure on the circle normalized to 1.

- Assume that f is constant with value x_i. Let Θ be the collection

  $$\{\theta \in \mathbb{S}^1 \mid f(a) = 1_{x_i} \text{ for all segments } a \text{ containing } \theta\}$$

  Here we say that a *contains* θ whenever θ lies in the interior of a. We will show that Θ is a singleton. To prove that Θ contains at most one point, suppose that θ and θ' are two distinct points in Θ. Pick disjoint segments a and a' such that θ lies in a, θ' lies in a', $f(a) = f(a') = 1_{x_i}$ and $\mathrm{cod}(a) = \mathrm{dom}(a')$. Then $a \cup a'$ is defined, but $f(a) \cup f(a')$ isn't, contradicting the fact that f is a morphism of effect algebroids.

We also have to show that Θ contains at least one point. Assume towards a contradiction that Θ is empty. Then for each $\theta \in \mathbb{S}^1$ there exists a non-zero segment a_θ around θ such that $f(a_\theta) = 0$. Call the endpoints of a_θ y_θ and z_θ, and let I_θ be the open interval (y_θ, z_θ). Then the collection of all I_θ covers the circle, so since $\mathbb{S}^1$ is compact, there is a finite subcover $I_{\theta_1}, \dots, I_{\theta_m}$. Order the endpoints $y_{\theta_1}, z_{\theta_1}, \dots, y_{\theta_m}, z_{\theta_m}$ of the intervals cyclically, that is, let $w_1, \dots, w_k$ be a cyclic arrangement of points on the circle for which $\{w_1, \dots, w_k\} = \{y_{\theta_1}, z_{\theta_1}, \dots, y_{\theta_m}, z_{\theta_m}\}$. Then each circle segment connecting two adjacent points $(w_i \to w_{i+1})$ (or $(w_k \to w_1)$) is a subsegment of some $(y_j \to z_j)$, hence it is mapped to 0 by f. However, the concatenation of all segments $(w_i \to w_{i+1})$ is the full circle, so f also maps the full circle to 0, which is a contradiction. We conclude that Θ consists of exactly one point.

We will use the fact that Θ is always a singleton to define $\Psi(f)$. Recall that x_i was our name for the constant value of f. Define $\Psi(f) = (\theta, p)$ where θ is the unique element of Θ, and p is the degenerate probability distribution $p_i = 1$, $p_j = 0$ for $j \neq i$.

We continue by proving that $\Phi \circ \Psi = \mathrm{id}$. Again we distinguish the cases of a constant and a non-constant function $f : \mathbb{S}^1 \to \Lambda_n$. First let f be a non-constant function. We have to check that $\Phi\Psi(f)(\varphi) = f(\varphi)$. For this we again consider several cases, corresponding to the cases in the definition of Φ. Suppose that $\varphi - \lim f^{-1}(x_0)$ lies in the interval $(0, \mu(f^{-1}(x_0))]$; in that case $\Phi\Psi(f)(\varphi) = x_0$. Since $f^{-1}(x_0)$ is an interval, this happens if and only if $\varphi \in (\lim f^{-1}(x_0), \lim f^{-1}(x_0) + \mu(f^{-1}(x_0))]$. This is in turn equivalent with $f(\varphi) = x_0$, because f is upper semicontinuous. In the case where $\varphi - \lim f^{-1}(x_0)$ does not lie in the interval $(0, \mu(f^{-1}(x_0))])$, it must lie in an interval of the form $(\mu(f^{-1}(x_0)) + \cdots + \mu(f^{-1}(x_{i-1})), \mu(f^{-1}(x_0)) + \cdots + \mu(f^{-1}(x_i))]$. Then $\varphi \in (\lim f^{-1}(x_i), \mathrm{colim}\, f^{-1}(x_i)]$, so because f is upper semicontinuous we again get that $f(\varphi) = x_i$.

The other case is when f is a constant function, say with value x_i. Since the second component of $\Psi(f)$ is a degenerate probability distribution, $\Phi\Psi(f)(\varphi)$ is always equal to x_i. It remains to be checked that $\Phi\Psi(f)$ and f coincide on segments. This is the case because $\Phi\Psi(f)(a) = 1$ if and only if θ lies in a, where θ is the unique element of Θ defined above. This is equivalent with $f(a) = 1$.

Subsequently we have to show that $\Psi \circ \Phi = \mathrm{id}$. Let $p \in \Delta_n$, and assume that $\Phi(0,p)$ is not constant. Then

$$\Psi\Phi(0,p) = \left(\lim f^{-1}(x_0), \left(\mu(f^{-1}(x_i))\right)_{i=1}^{n}\right),$$

where $f = \Phi(0,p)$. Since $f^{-1}(x_i) = (\sum_{k=0}^{i-1} p_k, \sum_{k=0}^{i} p_k]$, it follows that $\mu(f^{-1}(x_i)) = p_i$, which implies that $\Psi\Phi(0,p) = (0,p)$.

Now consider the case in which $\Phi(0,p)$ is constant. Then p is of the form $p_i = 1$, $p_j = 0$ for $j \neq i$. It follows that $\Psi\Phi(0,p) = (\theta, p)$, where θ is the unique element of $\mathbb{S}^1$ for which $\Phi(0,p)(a) = 1$ if and only if θ lies in a. But then θ has to be zero by definition of $\Phi(0,p)$.

To finish the proof, we have to show that Φ and Ψ are continuous. We will show that Φ is continuous using the metrics on $\mathbb{S}^1 \times \Delta_n$ and $\mathrm{Hom}(\mathbb{S}^1, \Lambda_n)$. Suppose that $(\theta, p), (\theta', p') \in \mathbb{S}^1 \times \Delta_n$ are such that $|\theta - \theta'| < \delta$ and $|p_i - p'_i| < \delta$ for each i. Then $|\sum_{k=0}^{i} p_i - \sum_{k=0}^{i} p'_i| < (i+1)\delta$ for each i. Since $\Phi(\theta,p)(\varphi) = x_i$ for the index i such that $\varphi - \theta \in (\sum_{k=0}^{i-1} p_i, \sum_{k=0}^{i} p_i]$ and similarly for $\Phi(\theta',p')(\varphi)$, and $|\theta - \theta'| < \delta$, the set of points φ where $\Phi(\theta,p)(\varphi)$ and $\Phi(\theta',p')(\varphi)$ differ has size less than $\delta + 2\delta + \cdots + (n+2)\delta$. Hence the distance between $\Phi(\theta,p)$ and $\Phi(\theta',p')$ is less than $(1+2+\cdots+(n+2))\delta$, from which continuity follows.

Since Φ is a continuous bijection from the compact space $\mathbb{S}^1 \times \Delta_n$ into the Hausdorff space $\mathrm{Hom}(\mathbb{S}^1, \Lambda_n)$, it is a homeomorphism. □

Proof of Proposition 5.4.2. We have to prove that $|\mathcal{C}(A)| \cong \mathrm{Hom}(\mathbb{S}^1, A)$. This will be achieved by factorizing a map $\mathbb{S}^1 \to A$ through some Λ_n, which enables us to apply the previous lemma.

The cycle functor is full and faithful by Theorem 5.1.4, so

$$|\mathcal{C}(A)| \cong \operatorname*{colim}_{\mathcal{C}(\Lambda_n) \to \mathcal{C}(A)} |\mathcal{C}(\Lambda_n)| \cong \operatorname*{colim}_{\Lambda_n \to A} \mathrm{Hom}(\mathbb{S}^1, \Lambda_n)$$

where we used Lemma 5.4.3 for the second isomorphism. Therefore, to prove the result, it suffices to show that $\mathrm{Hom}(\mathbb{S}^1, A)$ is the colimit of $\mathrm{Hom}(\mathbb{S}^1, \Lambda_n)$, taken over the comma category $(\mathbf{\Lambda} \downarrow A)$.

Given maps $\Lambda_n \to A$ and $\mathbb{S}^1 \to \Lambda_n$, composition provides a map in $\mathrm{Hom}(\mathbb{S}^1, A)$. This map is upper semicontinuous because every map $\Lambda_n \to A$ is upper semicontinuous. The collection of all compositions gives a cocone to $\mathrm{Hom}(\mathbb{S}^1, A)$. To show that this is universal, let X be an arbitrary $\mathbb{S}^1$-space equipped with a cocone of maps $\varphi_\alpha : \mathrm{Hom}(\mathbb{S}^1, \Lambda_n) \to X$, indexed by cycles $\alpha : \Lambda_n \to A$. In order to define $\psi : \mathrm{Hom}(\mathbb{S}^1, A) \to X$, take any $f \in \mathrm{Hom}(\mathbb{S}^1, A)$. We will show that this f factors trough some finite cycle Λ_n.

Recall that we can write segments in $\mathbb{S}^1$ as $(\theta \to \varphi)$, where θ and φ are angles in $[0, 2\pi]$. Let S be the set $\{f(0 \to \theta) \mid \theta \in (0, 2\pi]\}$. This is a set of segments in A, hence it is finite. We will equip S with a linear order. Given a segment a in S, let θ_a denote the largest number in $(0, 2\pi]$ for which $f(0 \to \theta_a) = a$. This number exists since f is upper semicontinuous. For segments a, b in S, put $a \leq b$ if and only if $\theta_a \leq \theta_b$ in $(0, 2\pi]$. Rolling the resulting linear order on S gives a cyclic order, so S is isomorphic to some Λ_n. Our goal is now to factorize f as $\mathbb{S}^1 \xrightarrow{g} S \xrightarrow{h} A$.

Define $g : \mathbb{S}^1 \to S$ by $g(\theta) = f(0 \to \theta)$. Here we take θ to be an angle in $(0, 2\pi]$ and we consider $(0 \to 2\pi)$ as the full circle segment on 0. To show that g is a map of effect algebroids, we will prove that g maps the *linear* order on $(0, 2\pi]$ into the linear order on S. It will follow that g preserves the cyclic order and hence it is a morphism of effect algebroids. Suppose that $\theta \leq \varphi$ in $(0, 2\pi]$. Let θ' be the largest number for which $f(0 \to \theta') = f(0 \to \theta)$, and define φ' similarly. In order to show that $g(\theta) \leq g(\varphi)$, we have to show that $\theta' \leq \varphi'$. Assume towards a contradiction that $\varphi' < \theta'$. Since $\theta \leq \varphi$, $\theta \leq \theta'$, and $\varphi \leq \varphi'$, we must have $\theta \leq \varphi \leq \varphi' < \theta'$. Then

$$f(0 \to \theta) = f(0 \to \theta') = f(0 \to \theta) \cup f(\theta \to \theta'),$$

so $f(\theta \to \theta') = 0$ by the cancellation property of effect algebroids. But then also all of $f(\theta \to \varphi)$, $f(\varphi \to \varphi')$, and $f(\varphi' \to \theta')$ have to be 0, because of positivity and the arrangement of the angles. Therefore

$$f(0 \to \varphi) = f(0 \to \varphi) \cup f(\varphi \to \theta') = f(0 \to \theta').$$

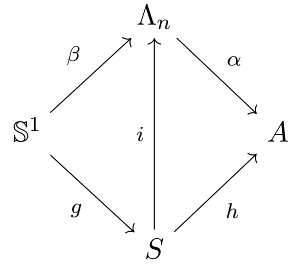
But φ' was the largest number for which $f(0 \to \varphi) = f(0 \to \varphi')$, so this contradicts our assumption that $\varphi' < \theta'$. We may conclude that g preserves the linear order and hence the effect algebroid structure.

For the second half of the factorization, define $h(a) = f(\theta_a)$. This is a map of effect algebroids because f is a map of effect algebroids. Furthermore $h(g(\theta)) = h(f(0 \to \theta)) = f(\theta')$, where θ' is again the largest number with $f(0 \to \theta) = f(0 \to \theta')$. As we computed above, $f(\theta \to \theta') = 0$, hence $f(\theta) = f(\theta')$, thus $h(g(\theta)) = f(\theta)$.

Having shown that f factors as $\mathbb{S}^1 \xrightarrow{g} \Lambda_n \xrightarrow{h} A$ for some n, we return to defining the mediating map ψ for the cocone φ_α. We can define $\psi(f) = \varphi_h(g)$, since h is an n-cycle in A, and g lies in $\mathrm{Hom}(\mathbb{S}^1, \Lambda_n)$. To show that this defines a mediating morphism, we have to prove $\psi(\alpha \circ \beta) = \varphi_\alpha(\beta)$ for $\beta : \mathbb{S}^1 \to \Lambda_n$ and $\alpha : \Lambda_n \to A$. Let $\mathbb{S}^1 \xrightarrow{g} S \xrightarrow{h} A$ be the factorization of $\alpha \circ \beta$ defined above. Note that this need not coincide with the factorization $\mathbb{S}^1 \xrightarrow{\beta} \Lambda_n \xrightarrow{\alpha} A$, since S may differ from Λ_n. However, since

$$S = \{\alpha\beta(0 \to \theta) \mid \theta \in (0, 2\pi]\} \subseteq \{\alpha(\beta(0) \to x_i) \mid x_i \in \Lambda_n\} \cong \Lambda_n,$$

there is always an inclusion map $i : S \to \Lambda_n$. We will show that the inclusion i makes the diagram



commute. We start with the triangle on the left. Because the isomorphism $\{\alpha(\beta(0) \to x_i) \mid x_i \in \Lambda_n\} \to \Lambda_n$ maps a segment $\alpha(\beta(0) \to x_i)$ to x_i, it follows that

$$i(g(\theta)) = i(\alpha\beta(0 \to \theta)) = i(\alpha(\beta(0) \to \beta(\theta))) = \beta(\theta).$$

The following computation shows that the right triangle commutes:

$$\alpha i(a) = \alpha i(f(0 \to \theta_a)) = \alpha i g(\theta_a) = \alpha\beta(\theta_a) = h(a)$$

The maps φ_α form a cocone and $\alpha \circ i = h$, so $\varphi_\alpha(g \circ i) = \varphi_h(g)$. Using the left triangle we conclude that $\psi(\alpha \circ \beta) = \varphi_h(g) = \varphi_\alpha(g \circ i) = \varphi_\alpha(\beta)$. Thus ψ satisfies the requirement for a mediating morphism. It is also the unique map with this property, since every $f : \mathbb{S}^1 \to A$ factors through some finite cycle. □

Examples 5.4.4.

1. The geometric realization of a finite abstract circle Λ_n is isomorphic to $\mathbb{S}^1 \times \Delta_n$. From the characterization in Proposition 5.4.2, together with the isomorphism $\mathrm{Hom}(\mathbb{S}^1, A \times B) \cong \mathrm{Hom}(\mathbb{S}^1, A) \times \mathrm{Hom}(\mathbb{S}^1, B)$, it follows that the geometric realization functor preserves products of finite effect algebroids. Hence the realization of a k-dimensional torus $\Lambda_{n_1} \times \cdots \times \Lambda_{n_k}$ is $(\mathbb{S}^1)^k \times \Delta_{n_1} \times \cdots \times \Delta_{n_k}$. As a special case, note that Λ_0 is the effect algebra $\{0, 1\}$. Hence the geometric realization of the power set algebra $\mathcal{P}(k) \cong \{0,1\}^k \cong (\Lambda_0)^k$ is $(\mathbb{S}^1)^k$.

2. We will compute the geometric realization of the finite effect algebra $L_n = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n} = 1\}$ using Proposition 5.4.2. The symmetric group S_n acts on $(\mathbb{S}^1)^n$ by permutations. Let $(\mathbb{S}^1)^n/S_n$ be the space of orbits. An element of $(\mathbb{S}^1)^n/S_n$ can be thought of as an unordered

list of n points in $\mathbb{S}^1$. That is, we take the multiplicity of the points into account, but not the order. We will show that $(\mathbb{S}^1)^n/S_n$ is the geometric realization of L_n.

Define a map $\Phi : (\mathbb{S}^1)^n/S_n \to \mathrm{Hom}(\mathbb{S}^1, L_n)$ by letting

$$\Phi(x_1,\ldots,x_n)(\theta \to \varphi) = \tfrac{1}{n}\big(\#\{i \mid x_i \in (\theta,\varphi]\}\big).$$

This map is well-defined since it is invariant under the S_n-action. To define an inverse Ψ for Φ, take any $f \in \mathrm{Hom}(\mathbb{S}^1, L_n)$. Let Θ be the collection

$$\{\theta \in \mathbb{S}^1 \mid f(\varphi \to \theta) \neq 0 \text{ for all } \varphi \neq \theta\}.$$

We claim that Θ contains at most n points. Suppose that $\theta_1,\ldots,\theta_{n+1}$ are distinct points in Θ. Without loss of generality we may assume that they are cyclically arranged in the order $\theta_1,\ldots,\theta_{n+1}$. By definition of Θ, each $f(\theta_i \to \theta_{i+1})$ is non-zero, so it is at least $\frac{1}{n}$. In the same way $f(\theta_{n+1} \to \theta_1)$ is at least $\frac{1}{n}$. Since the points $\theta_1,\ldots,\theta_{n+1}$ are arranged cyclically in this order, the composition $f(\theta_1 \to \theta_2) \cup \cdots \cup f(\theta_n \to \theta_{n+1}) \cup f(\theta_{n+1} \to \theta_1)$ is defined in L_n. But this composition is at least a sum of $n+1$ copies of $\frac{1}{n}$, so it is more than one, which is impossible. We are forced to conclude that Θ has no more than n elements.

Since Θ has at most n elements, it is in particular finite, so we can write $\Theta = \{\theta_1,\ldots,\theta_m\}$ where $m \le n$. Again we assume that the points are numbered in such a way that $\theta_1,\ldots,\theta_m$ are cyclically ordered. Define the *multiplicity* of a point θ_i as $m(\theta_i) = nf(\theta_{i-1} \to \theta_i)$ for $i \neq 1$, and $m(\theta_1) = nf(\theta_m \to \theta_1)$. Define $\Psi(f)$ to be the set of points $\theta_1,\ldots,\theta_m$, where the point θ_i occurs with multiplicity $m(\theta_i)$. Since $\sum_i m(\theta_i) = nf(1) = n$, this indeed gives an unordered list of n points.

We will now prove that $\Phi \circ \Psi = \mathrm{id}$. Let f be any function, and let $\theta_1,\ldots,\theta_m$ be the cyclically ordered set of points of Θ. Then $\Phi(\Psi(f))(\varphi \to \psi)$ is the number of i for which θ_i lies in the interval $(\varphi,\psi]$, counted with multiplicity, and then divided by n. Let $\theta_j, \theta_{j+1},\ldots,\theta_k$ be the list of points that lie in $(\varphi,\psi]$. Note that k may be lower than j, because the points are arranged in a circle. Then

$$\begin{aligned} &f(\varphi \to \psi) \\ &\quad = f(\varphi \to \theta_j) \cup f(\theta_j \to \theta_{j+1}) \cup \cdots \cup f(\theta_{k-1} \to \theta_k) \cup f(\theta_k \to \psi) \\ &\quad = f(\varphi \to \theta_j) + \frac{m(\theta_{j+1})}{n} + \cdots + \frac{m(\theta_k)}{n} + f(\theta_k \to \psi). \end{aligned}$$

Now we claim that $f(\varphi \to \theta_j) = \frac{m(\theta_j)}{n}$. Since $\varphi \notin \Theta$, there exists a φ' in the interval $[\theta_{j-1}, \varphi)$ for which $f(\varphi' \to \varphi) = 0$. Take for

φ' the lowest number in $[\theta_{j-1}, \varphi)$ with this property. Suppose that $\varphi' \neq \theta_{j-1}$. Then for all φ'' between θ_{j-1} and φ', we have $f(\varphi'' \to \varphi') \neq 0$, since otherwise $f(\varphi'' \to \varphi)$ would be zero, contradicting minimality of φ'. But this implies that $f(\varphi'' \to \varphi') \neq 0$ for all $\varphi'' \neq \varphi'$, so $\varphi' \in \Theta$. This is impossible since the θ_i form a list of all elements of Θ. Hence $\varphi' = \theta_{j-1}$, thus $f(\theta_{j-1} \to \varphi) = 0$, thus $f(\varphi \to \theta_j) = \frac{m(\theta_j)}{n}$. Similarly one proves that $f(\theta_k \to \psi) = 0$, so $f(\varphi \to \psi) = \frac{1}{n}\left(m(\theta_j) + m(\theta_{j+1}) + \cdots + m(\theta_k)\right) = \Phi(\Psi(f))(\varphi \to \psi)$.

Finally we show that $\Psi \circ \Phi = \mathrm{id}$. If $(x_1, \ldots, x_n)$ is a list of points in $(\mathbb{S}^1)^n/S_n$, then we may assume that they are ordered cyclically. Then the collection $\{\theta \in \mathbb{S}^1 \mid \Phi(x_1, \ldots, x_n)(\varphi \to \theta) \text{ for all } \varphi \neq \theta\}$ is equal to $\{x_1, \ldots, x_n\}$, which gives $\Psi(\Phi(x_1, \ldots, x_n)) = (x_1, \ldots, x_n)$, modulo the action of the symmetric group.

Chapter 6

Cohomology

Cohomology groups can be assigned to various mathematical structures, such as topological spaces or groups, and are frequently helpful to classify certain properties of the structure. For example, the cohomology groups of a topological space provide information about the holes in the space, and the second cohomology group of a group classifies its extensions. The main purpose of this chapter is to define and study cohomology of effect algebroids.

The theory will be especially interesting in the case of effect algebras, because it has applications to no-go theorems in quantum foundations. There are two reasons for this. Firstly, as shown in [4] based on earlier work in [1], sheaf cohomology of measurement covers has proven to be fruitful in the investigation of non-locality and contextuality. Measurement covers are loosely related to effect algebras via the framework of test spaces. Therefore one expects that the techniques used in [4] have analogues in the world of effect algebras. Secondly, in [117] it has been shown that the Bell paradox can be formulated in terms of (non)-existence of factorizations in the category of effect algebras. Since cohomology is often used to determine whether factorizations exist, a cohomology theory of effect algebras will allow us to examine Bell's Theorem in a new way.

We will propose two different cohomology theories for effect algebroids. The first definition of cohomology is based on Connes' cyclic cohomology from [28]. This is a natural choice because it is defined for any cyclic sets, and effect algebroids can be embedded in cyclic sets.

Most cohomology theories are obtained by assigning a sequence of abelian groups to a mathematical object. Since effect algebras are ordered structures, it will turn out to be productive to use a sequence of ordered abelian groups instead. This will lead to the second definition of cohomology, which we call order cohomology. It is loosely related to Pulmannová's classification of extensions of certain ordered algebraic structures in [106].

Both approaches for defining cohomology have advantages and disadvantages. Cyclic cohomology is more suited for theoretical investigations, since it opens up the possibility of using the powerful techniques from homological algebra. For example, we will show how cyclic cohomology interacts with products, coproducts, intersections, and unions of effect algebras. For order cohomology, it is less clear what the interactions are, due to a lack of general theory of homological algebra for ordered abelian groups. On the other hand, order cohomology lends itself better to applications to quantum mechanical no-go theorems. We will provide cohomological characterizations for when a state on a certain probabilistic system is classically realizable, for both cyclic and order cohomology. In the cyclic case, we only obtain a necessary condition for realizability, so in certain scenarios false positives may arise. A similar phenomenon occurs in the cohomological analysis of contextuality in [4]. Order cohomology repairs this defect of cyclic cohomology, since the order allows us to obtain a necessary and sufficient condition for realizability of states.

The results in this chapter appeared first in [110].

6.1 Cyclic cohomology of an effect algebroid

Effect algebroids embed in cyclic sets, and cyclic sets admit a natural cohomology theory called *cyclic cohomology*. Therefore it is reasonable to use cyclic cohomology also for effect algebroids. Cyclic cohomology was introduced by Connes in [28, 29], see also [92]. The book [91] contains an overview of the theory.

The cohomology groups arising from a cyclic set are defined from a cochain complex associated to the cyclic set. We will describe this construction for the cyclic set $\mathcal{C}(A)$, where A is an effect algebroid. We will take coefficients in the field $\mathbb{R}$, since some of our results only hold over this field of coefficients. There are two versions of the definition of cyclic cohomology: Connes' version from [28] is simpler, but only valid over fields containing the rational numbers. Tsygan's version from [121] uses a double complex and is more complicated, but also more general. Since we will only be concerned with coefficients in $\mathbb{R}$, we will work with Connes' definition.

Let $C^\bullet(A)$ be the complex

$$\mathbb{R}^{\mathcal{C}(A)_0} \xrightarrow{\delta^0} \mathbb{R}^{\mathcal{C}(A)_1} \xrightarrow{\delta^1} \cdots$$

Elements of $\mathbb{R}^{\mathcal{C}(A)_n}$ are functions from the n-cycles to $\mathbb{R}$ and are called *n-cocycles*. The boundary maps are given by an alternating sum over the face maps δ_i. Define maps $\lambda : \mathbb{R}^{\mathcal{C}(A)_n} \to \mathbb{R}^{\mathcal{C}(A)_n}$ by $\lambda(\varphi)(\alpha) = (-1)^n \varphi(\tau \cdot \alpha)$, where τ is the cyclic permutation in $\mathbf{\Lambda}$. We wish to consider only cocycles that are invariant under the action of λ. In other words, take a subcomplex of $C^\bullet(A)$ consisting of those cocycles φ for which $\varphi = \lambda(\varphi)$. The boundary maps δ^n send invariant cocycles to invariant cocycles, so this indeed gives a well-defined subcomplex, denoted $C^\bullet_\lambda(A)$. The cyclic cohomology of the effect algebroid A is the cohomology of $C^\bullet_\lambda(A)$, that is, $\mathrm{HC}^n(A) = \ker(\delta^n)/\operatorname{im}(\delta^{n-1})$.

Sometimes we will also be interested in the cohomology of the complex $C^\bullet(A)$ itself, i.e. without taking the subcomplex of invariant cocycles. The cohomology of $C^\bullet(A)$ is called the *Hochschild cohomology* of A and denoted $\mathrm{HH}^n(A)$. We will see that the Hochschild cohomology of an effect algebra is not as well-behaved as its cyclic cohomology. However, there are useful relations between Hochschild cohomology and cyclic cohomology, for instance Connes' exact sequence connecting the two. Therefore computing Hochschild cohomology is sometimes a practical intermediate step for computing cyclic cohomology.

Remark. According to [40, 80, 116], the cyclic cohomology of a cyclic set is isomorphic to the $\mathbb{S}^1$-equivariant cohomology of its geometric realization. Hence we can use the theory of realizations of effect algebroids developed in Section 5.4 to find cohomology groups. This equivariant cohomology of an $\mathbb{S}^1$-space X is easiest to describe in the case where the circle group acts freely, since then the equivariant cohomology of X is just the ordinary cohomology of the orbit space $X/\mathbb{S}^1$.

As an example, the geometric realization of $\Lambda_{n_1} \times \cdots \times \Lambda_{n_k}$ is $(\mathbb{S}^1)^k \times \Delta_{n_1} \times \cdots \times \Delta_{n_k}$, following part 1 of Example 5.4.4. The circle group acts diagonally on $(\mathbb{S}^1)^k$ and trivially on $\Delta_{n_1} \times \cdots \times \Delta_{n_k}$, hence the action is free. The orbit space is $(\mathbb{S}^1)^{k-1} \times \Delta_{n_1} \times \cdots \times \Delta_{n_k}$, and the m^{th} cohomology group of this space (over $\mathbb{R}$) is $\mathbb{R}^{\binom{k-1}{m}}$. Therefore the m^{th} cyclic cohomology group of $\Lambda_{n_1} \times \cdots \times \Lambda_{n_k}$ is also $\mathbb{R}^{\binom{k-1}{m}}$.

This approach is rather indirect and relies on knowledge of geometric realizations of effect algebroids. Since these are sometimes hard to compute, we will develop more direct methods for determining cohomology groups in the following sections.

Many results about cohomology of effect algebroids only hold for the special case of effect algebras, so we will describe this case in more detail.

Recall that an n-test on an effect algebra is a sequence of elements $a_1, \ldots, a_n$ whose sum is 1, and that $\mathcal{T}_n(A)$ denotes the $(n+1)$-tests on A. Then $\mathcal{C}(A)_n \cong \mathcal{T}_n(A)$, so the cochain complex assigned to an effect algebra is

$$\mathbb{R}^{\mathcal{T}_0(A)} \xrightarrow{\delta^0} \mathbb{R}^{\mathcal{T}_1(A)} \xrightarrow{\delta^1} \cdots$$

where

$$\begin{aligned}\delta^n(\alpha)(a_0, a_1, \ldots, a_n, a_{n+1}) = &\sum_{i=0}^{n}(-1)^i \alpha(a_0, \ldots, a_i \boxplus a_{i+1}, \ldots, a_{n+1}) \\ &+ (-1)^{n+1}\alpha(a_{n+1} \boxplus a_0, a_1, \ldots, a_n).\end{aligned}$$

Observe that every effect algebra has exactly one 1-test, so $\mathbb{R}^{\mathcal{T}_0(A)}$ can be identified with $\mathbb{R}$. Also, in a 2-test, each entry determines the other one via complementation, so $\mathbb{R}^{\mathcal{T}_1(A)}$ can be identified with $\mathbb{R}^A$.

The subcomplex $C^\bullet_\lambda(A)$ consists of those cocycles α for which

$$\alpha(a_0, a_1, \ldots, a_n) = (-1)^n \alpha(a_n, a_0, \ldots, a_{n-1}).$$

Example 6.1.1. We will determine the cohomology groups of the effect algebra $L_1 = \{0, 1\}$ via a direct computation. The n-tests on L_1 have a 1 at exactly one position, and are zero at all other positions. If $\alpha \in C^n_\lambda(L_1)$, then α is determined by its value on the test $(1, 0, \ldots, 0)$ by invariance. Hence each $C^n_\lambda(L_1)$ is a one-dimensional vector space.

If n is even, then $(\delta^n\alpha)(a_0, \ldots, a_{n+1})$ is an alternating sum with $n+2$ terms. By invariance, all terms in the sum are equal, so because the sum is alternating and has an even number of terms, it is zero. Hence we have $\delta^n = 0$ for even n, and similarly δ^n is non-zero for odd n. Thus

$$\ker(\delta^n) = \begin{cases}\mathbb{R} & \text{if } n \text{ is even}\\ 0 & \text{if } n \text{ is odd}\end{cases}$$

and

$$\operatorname{im}(\delta^n) = \begin{cases}0 & \text{if } n \text{ is even}\\ \mathbb{R} & \text{if } n \text{ is odd.}\end{cases}$$

Therefore

$$\mathrm{HC}^n(L_1) = \begin{cases}\mathbb{R} & \text{if } n = 0\\ 0 & \text{if } n > 0.\end{cases}$$

6.2 First cohomology group

We will look at the cohomology groups of an effect algebra A in low degrees. For $n = 0$, the definition reduces to $\mathrm{HC}^0(A) = \ker(\delta^0 : \mathbb{R} \to \mathbb{R}^A)$, since there is only one 1-test, and 2-tests correspond to elements of A. The boundary map δ^0 satisfies $\delta^0(\alpha)(a) = \alpha(a \boxplus a^\perp) - \alpha(a^\perp \boxplus a) = 0$, hence $\mathrm{HC}^0(A)$ is always the ground field $\mathbb{R}$.

We continue with the first cohomology group $\mathrm{HC}^1(A)$. We will first rewrite the definition of HC^1. Since the boundary map δ^0 is zero, $\mathrm{HC}^1(A)$ reduces to the kernel of δ^1. We identify 1-cocycles α with maps from A to $\mathbb{R}$, by letting a 2-test (a, b) correspond to the element $b \in A$. Invariance under the cyclic permutation map λ then means that $\alpha(a^\perp) = -\alpha(a)$, and $\alpha \in \ker(\delta^1)$ means that $\alpha(b) - \alpha(a \boxplus b) + \alpha(a) = 0$ whenever $a \boxplus b$ is defined. Therefore

$$\mathrm{HC}^1(A) \cong \{\alpha : A \to \mathbb{R} \mid \alpha(a \boxplus b) = \alpha(a) + \alpha(b),\ \alpha(a^\perp) = -\alpha(a)\}.$$

Recall that the state space of A is the convex space of morphisms $\sigma : A \to [0, 1]$. These satisfy $\sigma(a \boxplus b) = \sigma(a) + \sigma(b)$ and $\sigma(a^\perp) = 1 - \sigma(a)$. With the definition of the first cohomology group written as above, we see that the state space is similar to the first cohomology group. We will make the connection more precise. The state space of an effect algebra is always a compact convex space, and hence it embeds in a vector space over $\mathbb{R}$. We would like to prove that $\mathrm{HC}^1(A)$ (with coefficients in $\mathbb{R}$) is the *smallest* vector space that contains a copy of $\mathrm{St}(A)$. This means that there exists an affine injection $i : \mathrm{St}(A) \hookrightarrow \mathrm{HC}^1(A)$, such that for all affine injections $j : \mathrm{St}(A) \hookrightarrow V$ into a vector space there exists a unique affine injection $\varphi : \mathrm{HC}^1(A) \hookrightarrow V$ that makes the triangle

$$\begin{array}{ccc} \mathrm{St}(A) & \overset{i}{\hookrightarrow} & \mathrm{HC}^1(A) \\ & {\scriptstyle j}\searrow & \downarrow{\scriptstyle \varphi} \\ & & V \end{array}$$

commute. Note that φ is a map between vector spaces, but it is usually not linear. We can only obtain an affine map between the vector spaces.

Unfortunately this result does not hold for all effect algebras, for instance it fails for the effect algebra of projections on a Hilbert space. However, the result holds for many classes of well-behaved effect algebras. We will first present a general result that provides a sufficient condition on A that makes the statement true. This sufficient condition is hard to prove in practice, so after proving the general result we will mention a large class of effect algebras that satisfy the condition.

Definition 6.2.1. A map φ from an effect algebra A into $\mathbb{R}$ is *additive* if $\varphi(a \boxplus b) = \varphi(a) + \varphi(b)$ whenever $a \boxplus b$ is defined. It is positive whenever $\varphi(a) \geq 0$ for all a.

Theorem 6.2.2. *Let A be an effect algebra whose state space is non-empty. Suppose that every additive map $\alpha : A \to \mathbb{R}$ can be written as a difference of two positive additive maps $\alpha = \alpha_1 - \alpha_2$. Then $\mathrm{HC}^1(A)$ is the smallest vector space that contains a copy of the state space $\mathrm{St}(A)$.*

Proof. Fix a state σ_0 and use this to define an embedding $i : \mathrm{St}(A) \to \mathrm{HC}^1(A)$ by $i(\sigma) = \sigma - \sigma_0$. Then $i(\sigma)$ is linear because σ and σ_0 are, and $i(\sigma)$ satisfies

$$\begin{aligned} i(\sigma)(a^\perp) = \sigma(a^\perp) - \sigma_0(a^\perp) &= 1 - \sigma(a) - (1 - \sigma_0(a)) \\ &= -\sigma(a) + \sigma_0(a) = -i(\sigma)(a). \end{aligned}$$

Thus i maps states to cocycles in $\mathrm{HC}^1(A)$, and i is clearly injective and affine.

Let $j : \mathrm{St}(A) \to V$ be an arbitrary affine injection. To define a map $\varphi : \mathrm{HC}^1(A) \to V$, take any $\alpha \in \mathrm{HC}^1(A)$. Since α is additive and σ_0 is a state, $\alpha + \sigma_0$ is also additive. Using the hypothesis, express $\alpha + \sigma_0$ as a difference $\alpha + \sigma_0 = \alpha_1 - \alpha_2$, where α_1 and α_2 are positive additive maps. To define $\varphi(\alpha)$, we distinguish several cases.

- Suppose that $\alpha_1(1)$ and $\alpha_2(1)$ are both non-zero. Define $\sigma_i(a) = \frac{\alpha_i(a)}{\alpha_i(1)}$ for $i = 1, 2$, which is a state. Then define φ via $\varphi(\alpha) = \alpha_1(1) j(\sigma_1) - \alpha_2(1) j(\sigma_2)$.
- If $\alpha_1(1) = 0$ and $\alpha_2(1)$ is non-zero, define $\sigma_2(a) = \frac{\alpha_2(a)}{\alpha_2(1)}$ and put $\varphi(\alpha) = -\alpha_2(1) j(\sigma_2)$.
- Similarly, if $\alpha_1(1) \neq 0$ and $\alpha_2(1) = 0$, define $\sigma_1(a) = \frac{\alpha_1(a)}{\alpha_1(1)}$ and put $\varphi(\alpha) = \alpha_1(1) j(\sigma_1)$.
- Finally, if $\alpha_1(1) = \alpha_2(1) = 0$, then let $\varphi(\alpha) = 0$.

The decomposition of $\alpha + \sigma_0$ need not be unique, so we have to prove that φ is well-defined by showing that it does not depend on the choice of decomposition. Suppose that $\alpha + \sigma_0 = \alpha_1 - \alpha_2 = \alpha_1' - \alpha_2'$. We will assume that all of $\alpha_1(1)$, $\alpha_2(1)$, $\alpha_1'(1)$ and $\alpha_2'(1)$ are non-zero; the other cases are easier. We have to prove that $\alpha_1(1)j(\sigma_1) - \alpha_2(1)j(\sigma_2) = \alpha_1'(1)j(\sigma_1') - \alpha_2'(1)j(\sigma_2')$. For this we use that j preserves convex combinations, and that linear combinations can be made convex by normalization:

$$\begin{aligned}\frac{\alpha_1(1)}{\alpha_1(1)+\alpha_2'(1)}j(\sigma_1) + \frac{\alpha_2'(1)}{\alpha_1(1)+\alpha_2'(1)}j(\sigma_2') &= j\left(\frac{\alpha_1(1)\sigma_1 + \alpha_2'(1)\sigma_2'}{\alpha_1(1)+\alpha_2'(1)}\right)\\ &= j\left(\frac{\alpha_1+\alpha_2'}{\alpha_1(1)+\alpha_2'(1)}\right)\end{aligned}$$

Now using $\alpha_1 + \alpha_2' = \alpha_1' + \alpha_2$ and rewriting back shows that this equals

$$\frac{\alpha_1'(1)}{\alpha_1'(1)+\alpha_2(1)}j(\sigma_1') + \frac{\alpha_2(1)}{\alpha_1'(1)+\alpha_2(1)}j(\sigma_2),$$

so $\alpha_1(1)j(\sigma_1) - \alpha_2(1)j(\sigma_2) = \alpha_1'(1)j(\sigma_1') - \alpha_2'(1)j(\sigma_2')$.

The next step is to show that φ makes the triangle commute, that is, $\varphi(i(\sigma)) = j(\sigma)$ for all states σ. A decomposition of $i(\sigma)+\sigma_0$ is just $\sigma - 0$, since σ is a state and hence positive. Then $\varphi(i(\sigma)) = \sigma(1)j(\sigma) = j(\sigma)$, as required.

It is easy to see that φ is affine. To show that it is injective, suppose that $\varphi(\alpha) = \varphi(\alpha')$. Then $\alpha_1(1)j(\sigma_1) - \alpha_2(1)j(\sigma_2) = \alpha_1'(1)j(\sigma_1') - \alpha_2'(1)j(\sigma_2')$. By using normalization and affinity of j, we obtain

$$j\left(\frac{\alpha_1+\alpha_2'}{\alpha_1(1)+\alpha_2'(1)}\right) = j\left(\frac{\alpha_1'+\alpha_2}{\alpha_1'(1)+\alpha_2(1)}\right),$$

and since j is injective this gives $\alpha_1 - \alpha_2 = \alpha_1' - \alpha_2'$. This means $\alpha = \alpha'$, proving injectivity of φ.

Finally we have to prove that φ is the unique morphism with this property. Suppose that an affine map $\psi : \mathrm{HC}^1(A) \to V$ satisfies $\psi \circ i = j$. Take $\alpha \in \mathrm{HC}^1(A)$ and decompose $\alpha + \sigma_0$ as $\alpha_1 - \alpha_2$ where both α_i are positive. We assume that $\alpha_1(1)$ and $\alpha_2(1)$ are both non-zero; the other cases are similar. Define $\sigma_i = \frac{\alpha_i}{\alpha_i(1)}$ as before. We have to show that $\psi(\alpha) = \alpha_1(1)j(\sigma_1) - \alpha_2(1)j(\sigma_2)$. Since $\psi \circ i = j$, we have $\psi(\sigma - \sigma_0) = j(\sigma)$ for all states σ. Therefore we are done if we can establish that $\alpha_1(1)\psi(\sigma_1 - \sigma_0) = \psi(\alpha) + \alpha_2(1)\psi(\sigma_2 - \sigma_0)$. We will prove a normalized version of this equality, that is,

$$\frac{\alpha_1(1)}{1+\alpha_2(1)}\psi(\sigma_1 - \sigma_0) = \frac{1}{1+\alpha_2(1)}\psi(\alpha) + \frac{\alpha_2(1)}{1+\alpha_2(1)}\psi(\sigma_2 - \sigma_0).$$

To prove this, first note that

$$1 + \alpha_2(1) = \alpha(1) + \sigma_0(1) + \alpha_2(1) = \alpha_1(1),$$

where we used that σ_0 is a state and that $\alpha(1) = -\alpha(0) = 0$ because α lies in $\mathrm{HC}^1(A)$. Furthermore,

$$\begin{aligned}\alpha + \alpha_2(1)(\sigma_2 - \sigma_0) &= \alpha + \alpha_2 - \alpha_2(1)\sigma_0 \\ &= \alpha_1 - \sigma_0 - \alpha_2(1)\sigma_0 \\ &= \alpha_1 - (1 + \alpha_2(1))\sigma_0 \\ &= \alpha_1 - \alpha_1(1)\sigma_0 \\ &= \alpha_1(1)(\sigma_1 - \sigma_0).\end{aligned}$$

Because ψ preserves convex combinations, it follows that

$$\begin{aligned}\frac{1}{1+\alpha_2(1)}\psi(\alpha) + \frac{\alpha_2(1)}{1+\alpha_2(1)}\psi(\sigma_2 - \sigma_0) &= \psi\left(\frac{\alpha + \alpha_2(1)(\sigma_2 - \sigma_0)}{1+\alpha_2(1)}\right) \\ &= \psi\left(\frac{\alpha_1(1)(\sigma_1 - \sigma_0)}{\alpha_1(1)}\right) \\ &= \psi(\sigma_1 - \sigma_0) \\ &= \frac{\alpha_1}{1+\alpha_2(1)}\psi(\sigma_1 - \sigma_0).\end{aligned}$$

This finishes the proof that φ is unique. □

The next result shows that all finite Archimedean interval effect algebras satisfy the assumption in the previous theorem. By Corollary 3.4.12, the state space of any such algebra is non-empty. Therefore, for all finite Archimedean interval effect algebras A, the first cohomology group $\mathrm{HC}^1(A)$ is the smallest vector space surrounding $\mathrm{St}(A)$.

Proposition 6.2.3. *If A is a finite Archimedean interval effect algebra, then every additive map $\alpha : A \to \mathbb{R}$ can be expressed as the difference of two positive additive maps.*

Proof. The following proof is inspired by an analogous result for complemented lattices in [35], but modified to be suitable for effect algebras.

Since A is finite, it can be presented by a finite number of generators and relations. Let X be a finite set of generators. The state space of A consists of maps $X \to [0,1]$ subject to the relations. Therefore the state space is a compact convex space generated by a finite number of points. Let $\sigma_1, \dots, \sigma_n$ be generators for the state space and define a state β by $\beta = \frac{1}{n}\sigma_1 + \cdots + \frac{1}{n}\sigma_n$. We will show that β is a faithful state, which means that $\beta(a) \neq 0$ for all $a \neq 0$. Assume that $a \neq 0$ but $\beta(a) = 0$. Then $\sigma_i(a) = 0$ for all $i = 1, \dots, n$. But since the state space is generated by the states σ_i, this implies that all states σ map a to zero. By Corollary 3.4.12, this is only possible if $a = 0$, contradicting our assumption that $a \neq 0$.

We will use the faithful state β to prove the proposition. Let $\alpha : A \to \mathbb{R}$ be an additive map. We may assume that $\alpha(a) < 0$ for some $a \in A$, since otherwise the claim is proven immediately. Let

$$K = \frac{-\min\{\alpha(a) \mid a \in A\}}{\min\{\beta(a) \mid a \neq 0\}} \in \mathbb{R}.$$

Both minimums exists since A is finite. The denominator is strictly positive, because β is a faithful state. Also the numerator is strictly positive, since there is an $a \in A$ for which $\alpha(a) < 0$. Hence $K > 0$.

We wish to write α as the difference of two positive additive maps $\alpha = \alpha_1 - \alpha_2$. Take $\alpha_2(a) = K\beta(a)$, which is positive since K and β are positive, and additive since β is additive. Then let $\alpha_1 = \alpha + \alpha_2$. Clearly α_1 is additive and $\alpha = \alpha_1 - \alpha_2$, so it is left to check that α_1 is positive. Take any $b \neq 0$ in A. Then $\min\{\beta(a) \mid a \neq 0\} \leq \beta(b)$, and since $\min\{\alpha(a) \mid a \in A\}$ is negative, it follows that

$$K \geq \frac{-\min\{\alpha(a) \mid a \in A\}}{\beta(b)}.$$

Therefore $\alpha_1(b) = \alpha(b) + K\beta(b) \geq \alpha(b) - \min\{\alpha(a) \mid a \in A\} \geq 0$, where the last inequality uses that $\alpha(a)$ is negative for some a. This proves that α_1 is a positive map, hence α is the difference of two positive maps. □

6.3 Relative cohomology

We shall define relative cohomology of effect algebras and discuss some applications.

Let B be an effect algebra and $A \subseteq B$ a subalgebra. Each test on A is in particular a test on B, so the collection of $(n+1)$-tests $\mathcal{T}_n(A)$ on A can be considered as a subset of $\mathcal{T}_n(B)$. This gives a surjection $p^n : \mathrm{Hom}(\mathcal{T}_n(B), \mathbb{R}) \to \mathrm{Hom}(\mathcal{T}_n(A), \mathbb{R})$ by restriction:

$$p^n(\alpha) = \alpha|_{\mathcal{T}_n(A)}$$

Since the map p^n is compatible with cyclic permutations, it restricts to a surjection $C^n_\lambda(B) \to C^n_\lambda(A)$, also denoted p^n or p.

The kernel of p^n consists of all invariant cochains $\mathcal{T}_n(B) \to \mathbb{R}$ that are zero on A-tests, but not necessarily on B-tests. It fits in a short exact sequence

$$0 \longrightarrow \ker(p^n) \longrightarrow C^n_\lambda(B) \xrightarrow{p^n} C^n_\lambda(A) \longrightarrow 0.$$

The coboundary maps of the cochain complex $C^\bullet_\lambda(B)$ restrict to $\ker(p^n)$, so the above is in fact a short exact sequence of cochain complexes. The *relative cohomology* of the pair (B, A) is defined to be the cohomology of $\ker(p^\bullet)$. By general results from homological algebra (see e.g. [122]), the short exact sequence above gives rise to a long exact sequence in cohomology:

$$\begin{aligned}\cdots \longrightarrow \mathrm{HC}^{n-1}(A) \longrightarrow \mathrm{HC}^n(B, A) \longrightarrow \mathrm{HC}^n(B) \\ \longrightarrow \mathrm{HC}^n(A) \longrightarrow \mathrm{HC}^{n+1}(B, A) \longrightarrow \cdots\end{aligned}$$

As a first application of relative cohomology, we will show that trivial tests can be ignored when calculating the cohomology of an effect algebra. A trivial test is a test $(a_0, \ldots, a_n)$ in which exactly one a_i is one and all others are zero. To make the statement precise, consider the effect algebra $L_1 = \{0, 1\}$. This can be embedded in any effect algebra A, since all effect algebras have a zero and a one. The relative cohomology of the pair (A, L_1) is the cohomology of $\ker(p^n)$, where $p^n : C^n_\lambda(A) \to C^n_\lambda(L_1)$ is the restriction map. Since the tests on L_1 are exactly the trivial tests, the kernel of p^n consists of those cocycles that are zero on trivial tests. Hence the claim that trivial tests can be ignored in the calculation of cohomology groups amounts to the following.

Proposition 6.3.1. *For any effect algebra A and any $n > 0$, $\mathrm{HC}^n(A, L_1) \cong \mathrm{HC}^n(A)$.*

Proof. Look at the long exact sequence for the pair (A, L_1). We have seen that the cohomology of L_1 is $\mathbb{R}$ in degree 0 and zero elsewhere. Hence around degree 1 the long exact sequence looks like:

$$\mathrm{HC}^0(A) \cong \mathbb{R} \xrightarrow{\alpha} \mathrm{HC}^0(L_1) \cong \mathbb{R} \xrightarrow{\beta} \mathrm{HC}^1(A, L_1) \xrightarrow{\gamma} \mathrm{HC}^1(A) \longrightarrow 0$$

The group $\mathrm{HC}^0(A)$ consists of all cocycles that map the trivial 1-test (1) to a real number, and the same holds for the group $\mathrm{HC}^0(L_1)$. Since α is a restriction map, it is the identity on $\mathbb{R}$ here. From exactness at $\mathrm{HC}^0(L_1)$ it follows that $\beta = 0$. This in turn implies that $\ker(\gamma) = \mathrm{im}(\beta) = 0$, so γ is injective. But γ is also surjective since the sequence is exact at $\mathrm{HC}^1(A)$, so $\mathrm{HC}^1(A, L_1) \cong \mathrm{HC}^1(A)$. This proves the result for $n = 1$.

For an arbitrary $n > 1$, consider the fragment of the long exact sequence around degree n:

$$\mathrm{HC}^{n-1}(L_1) \to \mathrm{HC}^n(A, L_1) \to \mathrm{HC}^n(A) \to \mathrm{HC}^n(L_1)$$

Since $\mathrm{HC}^{n-1}(L_1) = \mathrm{HC}^n(L_1) = 0$, we conclude that $\mathrm{HC}^n(A, L_1) \cong \mathrm{HC}^n(A)$. □

The above proposition is useful to show that cyclic cohomology preserves coproducts of effect algebras. For this property, it is essential that we use cyclic cohomology. For Hochschild cohomology the analogous result is false.

Corollary 6.3.2. *For any $n > 0$, $\mathrm{HC}^n(A + B) = \mathrm{HC}^n(A) \oplus \mathrm{HC}^n(B)$.*

Proof. We will show that $\mathrm{HC}^n(A + B, L_1) \cong \mathrm{HC}^n(A, L_1) \oplus \mathrm{HC}^n(B, L_1)$; the result will then follow from the previous proposition. Call the cochain complex that computes $\mathrm{HC}^n(A, L_1)$ $D^\bullet(A)$. Similarly there are cochain complexes $D^\bullet(B)$ and $D^\bullet(A + B)$. These complexes consist of all invariant cocycles that map trivial tests to zero.

A test on a coproduct $A + B$ is either a trivial test, or a non-trivial test on A, or a non-trivial test on B. (Beware that we do not have $\mathcal{T}_n(A+B) \cong \mathcal{T}_n(A)+\mathcal{T}_n(B)$, since $\mathcal{T}_n(A+B)$ contains n trivial tests, while the coproduct on the right-hand side contains $2n$ trivial tests.) Therefore $D^n(A + B) \cong D^n(A) \oplus D^n(B)$, from which the desired statement follows. □

6.4 Künneth sequence

To compute the cohomology groups of a product of two effect algebras, the Künneth sequence is helpful. As before, we only consider cohomology with coefficients in $\mathbb{R}$.

Theorem 6.4.1. *Let A and B be effect algebras. There is a long exact sequence*

$$\cdots \longrightarrow \mathrm{HC}^{n-1}(A \times B) \longrightarrow \bigoplus_{p+q=n-2} \mathrm{HC}^p(A) \otimes \mathrm{HC}^q(B)$$
$$\longrightarrow \bigoplus_{p+q=n} \mathrm{HC}^p(A) \otimes \mathrm{HC}^q(B) \longrightarrow \mathrm{HC}^n(A \times B) \longrightarrow \cdots$$

Proof. Tests on a product algebra satisfy $\mathcal{T}_n(A \times B) \cong \mathcal{T}_n(A) \times \mathcal{T}_n(B)$. Therefore $\mathrm{Hom}(\mathcal{T}_n(A \times B), \mathbb{R}) \cong \mathrm{Hom}(\mathcal{T}_n(A), \mathbb{R}) \otimes \mathrm{Hom}(\mathcal{T}_n(B), \mathbb{R})$, so to compute the cohomology of the product, we have to look at the cohomology of a tensor product of cyclic modules. According to the dual of [91, Thm. 4.3.11], this can be computed using the sequence in the theorem. □

Example 6.4.2. We will compute the cohomology of the power set effect algebra $\mathcal{P}(2) = L_1 \times L_1$. We have already seen in Example 6.1.1 that the cohomology of L_1 is $\mathbb{R}$ in degree zero, and vanishes elsewhere. The fragment of the Künneth sequence around degree 1 looks like:

$$\begin{aligned}
&\mathrm{HC}^0(L_1) \otimes \mathrm{HC}^1(L_1) \oplus \mathrm{HC}^1(L_1) \otimes \mathrm{HC}^0(L_1) \\
&\quad \longrightarrow \mathrm{HC}^1(\mathcal{P}(2)) \longrightarrow \mathrm{HC}^0(L_1) \otimes \mathrm{HC}^0(L_1) \\
&\quad \longrightarrow \mathrm{HC}^0(L_1) \otimes \mathrm{HC}^2(L_1) \oplus \mathrm{HC}^1(L_1) \otimes \mathrm{HC}^1(L_1) \oplus \mathrm{HC}^2(L_1) \otimes \mathrm{HC}^0(L_1)
\end{aligned}$$

The outer groups in this sequence are zero, by the computation of the cohomology of L_1. It follows that $\mathrm{HC}^1(\mathcal{P}(2)) \cong \mathrm{HC}^0(L_1) \otimes \mathrm{HC}^0(L_1) \cong \mathbb{R}$. Furthermore, the cohomology in degree zero is $\mathbb{R}$ since this holds for all effect algebras, and from the Künneth sequence it can be deduced that it is zero in degrees at least two.

If the connecting morphisms in the above sequence are unknown, then applying the theorem can be problematic. In this case, it may be easier to compute cyclic cohomology using Hochschild cohomology as an intermediate step. In the remainder of this section, we will use this technique to compute the cyclic cohomology of a power set effect algebra $\mathcal{P}(m)$, which is a product of m copies of L_1. First we observe that the Künneth formula for Hochschild cohomology assumes a particularly simple form.

Proposition 6.4.3. *Let A and B be effect algebras. Then*

$$\mathrm{HH}^n(A \times B) \cong \bigoplus_{p+q=n} \mathrm{HH}^p(A) \otimes \mathrm{HH}^q(B).$$

Proof. This follows from e.g. [122, Thm. 3.6.3], using that we take coefficients in a field. □

We will also need a connection between cyclic and Hochschild cohomology, in the case where we work with a product of copies of L_1.

Lemma 6.4.4. *For any effect algebra A, $\mathrm{HH}^n(A) \cong \mathrm{HC}^n(A \times L_1)$.*

Proof. We will show that the complex computing $\mathrm{HH}(A)$ is isomorphic to the complex computing $\mathrm{HC}(A \times L_1)$. Define a map $f : \mathbb{R}^{\mathcal{T}_n(A)} \to C^n_\lambda(A \times L_1)$ in the following way. Take an arbitrary $\alpha : \mathcal{T}_n(A) \to \mathbb{R}$ and an arbitrary test $((a_0, k_0), \ldots, (a_n, k_n))$ on $A \times L_1$. A test on L_1 has a 1 at exactly one position, and zeroes elsewhere. Let i be the unique index for which $k_i = 1$. Then put

$$(f\alpha)((a_0, k_0), \ldots, (a_n, k_n)) = (-1)^{in} \alpha(a_i, a_{i+1}, \ldots, a_n, a_0, \ldots, a_{i-1}).$$

To show that $f\alpha$ actually lies in $C^n_\lambda(A \times L_1)$, we have to prove that it is invariant under cyclic permutations, i.e.

$$\begin{aligned}
&(f\alpha)((a_n, k_n), (a_0, k_0), \ldots, (a_{n-1}, k_{n-1})) \\
&\quad = (-1)^n (f\alpha)((a_0, k_0), \ldots, (a_n, k_n)).
\end{aligned}$$

Suppose that the i^{th} entry of the test $(k_0, \dots, k_n)$ satisfies $k_i = 1$, and $i < n$. Then the $(i+1)^{\text{th}}$ entry of $(k_n, k_0, \dots, k_{n-1})$ has value 1. Hence

$$\begin{aligned}&(f\alpha)((a_n, k_n), (a_0, k_0), \dots, (a_{n-1}, k_{n-1}))\\ &\quad = (-1)^{(i+1)n}\alpha(a_i, \dots, a_n, a_0, \dots, a_{i-1})\\ &\quad = (-1)^n(-1)^{in}\alpha(a_i, \dots, a_n, a_0, \dots, a_{i-1})\\ &\quad = (-1)^n(f\alpha)((a_0, k_0), \dots, (a_n, k_n))\end{aligned}$$

A similar computation shows that the result still holds if $i = n$.

Now we will verify that f is a chain map from the Hochschild complex to the cyclic complex. To achieve this, we have to check that

$$(f\delta\alpha)((a_0, k_0), \dots, (a_n, k_n)) = (\delta f\alpha)((a_0, k_0), \dots, (a_n, k_n)).$$

First assume that $k_0 = 1$. Then the left-hand side of this equation becomes

$$\begin{aligned}(\delta\alpha)(a_0, \dots, a_n) = &\sum_{j=0}^{n-1}(-1)^j\alpha(a_0, \dots, a_j \boxplus a_{j+1}, \dots, a_n)\\ &+ (-1)^n\alpha(a_n \boxplus a_0, a_1, \dots, a_{n-1}).\end{aligned}$$

The right-hand side equals

$$\begin{aligned}&\sum_{j=0}^{n-1}(-1)^j(f\alpha)((a_0, k_0), \dots, (a_j \boxplus a_{j+1}, k_j \boxplus k_{j+1}), \dots, (a_n, k_n))\\ &\quad + (-1)^n(f\alpha)((a_n \boxplus a_0, k_n \boxplus k_0), (a_1, k_1), \dots, (a_{n-1}, k_{n-1})).\end{aligned}$$

In each term of this sum, the first entry of the test has 1 as second component. Therefore it is equal to the left-hand side.

Now assume that $k_i = 1$ for some $i \neq 0$. We can reduce this to the previous case by permuting the tests cyclically. Since $f(\delta\alpha)$ is invariant under cyclic permutations, we have

$$\begin{aligned}&(f\delta\alpha)((a_0, k_0), \dots, (a_n, k_n))\\ &\quad = (-1)^{in}(f\delta\alpha)((a_i, k_i), \dots, (a_n, k_n), (a_0, k_0), \dots, (a_{n-1}, k_{n-1})).\end{aligned}$$

Furthermore, since $f\alpha$ is invariant and δ maps invariant cochains to invariant cochains, also $\delta f\alpha$ is invariant under cyclic permutations. Hence

$$\begin{aligned}&(\delta f\alpha)((a_0, k_0), \dots, (a_n, k_n))\\ &\quad = (-1)^{in}(\delta f\alpha)((a_i, k_i), \dots, (a_n, k_n), (a_0, k_0), \dots, (a_{n-1}, k_{n-1})).\end{aligned}$$

But in the test $((a_i, k_i), \ldots, (a_n, k_n), (a_0, k_0), \ldots, (a_{n-1}, k_{n-1}))$, the first entry has a 1 as second component, so we are back in the previous case. This shows that f is a chain map.

The final step is proving that f is a bijection. For injectivity, suppose that $f\alpha = f\beta$. Then for each test $((a_0, k_0), \ldots, (a_n, k_n))$ we have

$$\alpha(a_i, \ldots, a_n, a_0, \ldots, a_{i-1}) = \beta(a_i, \ldots, a_n, a_0, \ldots, a_{i-1}).$$

Let $(a_0, \ldots, a_n)$ be an arbitrary test on A. Take the test $(k_0, \ldots, k_n)$ on L_1 defined by $k_0 = 1$ and $k_i = 0$ for $i \neq 0$. This yields $\alpha(a_0, \ldots, a_n) = \beta(a_0, \ldots, a_n)$. Since $(a_0, \ldots, a_n)$ was arbitrary, f is injective.

To establish surjectivity, let $\beta : \mathcal{T}_n(A \times L_1) \to \mathbb{R}$ be a map invariant under cyclic permutations. Define $\alpha : \mathcal{T}_n(A) \to \mathbb{R}$ by

$$\alpha(a_0, \ldots, a_n) = \beta((a_0, 1), (a_1, 0), \ldots, (a_n, 0)).$$

In order to show that $f\alpha = \beta$, take a test $((a_0, k_0), \ldots, (a_n, k_n))$ with $k_i = 1$. Then

$$\begin{aligned}
&(f\alpha)((a_0, k_0) \ldots, (a_n, k_n)) \\
&\quad = (-1)^{in}\alpha(a_i, \ldots, a_n, a_0, \ldots, a_{i-1}) \\
&\quad = (-1)^{in}\beta((a_i, 1), (a_{i+1}, 0), \ldots, (a_n, 0), (a_0, 0), \ldots, (a_{i-1}, 0)) \\
&\quad = \beta((a_0, k_0), \ldots, (a_n, k_n))
\end{aligned}$$

where we used invariance of β in the final step. □

Example 6.4.5. We will compute the cyclic cohomology groups of all power set effect algebras $\mathcal{P}(m)$. First we will determine their Hochschild cohomology. For $m = 1$, apply Lemma 6.4.4 and Example 6.4.2 to find $\mathrm{HH}^n(\mathcal{P}(1)) \cong \mathrm{HC}^n(\mathcal{P}(2))$, which is $\mathbb{R}$ in degrees 0 and 1, and zero in all higher degrees. By applying the Künneth formula from Proposition 6.4.3 with induction to m, we obtain $\mathrm{HH}^n(\mathcal{P}(m)) \cong \mathbb{R}^{\binom{m}{n}}$. From Lemma 6.4.4 it now follows that $\mathrm{HC}^n(\mathcal{P}(m)) \cong \mathrm{HH}^n(\mathcal{P}(m-1)) \cong \mathbb{R}^{\binom{m-1}{n}}$.

6.5 Mayer–Vietoris sequence

A finite orthoalgebra is the union of its maximal Boolean subalgebras, as discussed in Section 2.4. Since these are generated by the maximal tests, the orthoalgebra is completely determined by its atoms and maximal tests. In this section we will establish a Mayer–Vietoris sequence for the cyclic cohomology of an effect algebra, which relates the cohomology of a union to the cohomology of the constituents and their intersection. Since we already know the cohomology of finite Boolean algebras, this yields a technique for computing the cohomology of any finite orthoalgebra. Using the Mayer–Vietoris sequence is usually a very efficient way to determine the cohomology groups, since it only involves the atoms and the maximal tests, instead of the collection of all tests on the effect algebra.

Theorem 6.5.1. *Let A and B be subalgebras of an effect algebra E, such that $E = A \cup B$. Then there is a long exact sequence*

$$\begin{aligned} \cdots \longrightarrow \mathrm{HC}^{n-1}(A \cap B) \longrightarrow \mathrm{HC}^n(E) \longrightarrow \mathrm{HC}^n(A) \oplus \mathrm{HC}^n(B) \\ \longrightarrow \mathrm{HC}^n(A \cap B) \longrightarrow \mathrm{HC}^{n+1}(E) \longrightarrow \cdots \end{aligned}$$

Proof. We shall construct a short exact sequence

$$0 \longrightarrow C^n_\lambda(E) \xrightarrow{\varphi} C^n_\lambda(A) \oplus C^n_\lambda(B) \xrightarrow{\psi} C^n_\lambda(A \cap B) \longrightarrow 0,$$

which will induce the desired long exact sequence in cohomology. Define $\varphi : C^n_\lambda(E) \to C^n_\lambda(A) \oplus C^n_\lambda(B)$ by restricting to tests on the subalgebras, i.e. $\varphi(\alpha) = \big(\alpha|_{\mathcal{T}_n(A)}, \alpha|_{\mathcal{T}_n(B)}\big)$. The map ψ is defined by $\psi(\alpha, \beta) = \alpha|_{\mathcal{T}_n(A\cap B)} - \beta|_{\mathcal{T}_n(A\cap B)}$.

Now we will show that the maps φ and ψ yield a short exact sequence. To show that φ is injective, suppose that $\varphi(\alpha) = \varphi(\beta)$. Then $\alpha(t) = \beta(t)$ for all tests t on A, and all tests t on B. Hence, by Proposition 2.4.7, $\alpha = \beta$, establishing injectivity.

We continue by proving surjectivity of ψ. Take any $\gamma \in C^n_\lambda(A \cap B)$. Define $\alpha \in C^n_\lambda(A)$ and $\beta \in C^n_\lambda(B)$ as follows: for any test t on $A \cap B$, define $\alpha(t) = \frac{1}{2}\gamma(t)$ and $\beta(t) = -\frac{1}{2}\gamma(t)$. On all tests that do not lie completely inside $A \cap B$, α and β are zero. Then, for each test t on $A \cap B$, $\psi(\alpha, \beta)(t) = \alpha(t) - \beta(t) = \frac{1}{2}\gamma(t) + \frac{1}{2}\gamma(t) = \gamma(t)$, so ψ is surjective.

Finally we will show that the sequence is exact in the middle. If $\alpha \in C^n_\lambda(E)$, then $\alpha|_{\mathcal{T}_n(A)}$ and $\alpha|_{\mathcal{T}_n(B)}$ agree on the intersection $\mathcal{T}_n(A \cap B)$. It follows that $(\psi \circ \varphi)(\alpha) = 0$, hence $\mathrm{im}(\varphi) \subseteq \ker(\psi)$. Conversely, suppose that $\alpha \in C^n_\lambda(A)$ and $\beta \in C^n_\lambda(B)$ agree on $\mathcal{T}_n(A \cap B)$. We have to show that both are restrictions of some $\gamma \in C^n_\lambda(E)$. Let t be a test on E. By Proposition 2.4.7, t is either a test on A or a test on B. If it is a test on A, define $\gamma(t) = \alpha(t)$; if it is a test on B, define $\gamma(t) = \beta(t)$. Then γ is well-defined because α and β agree on the intersection, and it restricts to α and β on $\mathcal{T}_n(A)$ and $\mathcal{T}_n(B)$, respectively. This concludes the proof that $\mathrm{im}(\varphi) = \ker(\psi)$. □

Example 6.5.2. We will compute the cohomology groups of the effect algebra from Example 2.4.6. Call the effect algebra E, let A be the subalgebra generated by the atoms a, b, e, and let B be the subalgebra generated by c, d, e. Then $E = A \cup B$, and $A \cong B \cong \mathcal{P}(3)$. Furthermore, $A \cap B$ consists of the four elements 0, e, $a \boxplus b = c \boxplus d$, and $a \boxplus b \boxplus e = c \boxplus d \boxplus e = 1$, so it is isomorphic to $\mathcal{P}(2)$. Plugging this information into the Mayer–Vietoris sequence gives

$$\begin{aligned} &\mathrm{HC}^0(\mathcal{P}(2)) \xrightarrow{\partial_0} \mathrm{HC}^1(E) \xrightarrow{\alpha} \mathrm{HC}^1(\mathcal{P}(3)) \oplus \mathrm{HC}^1(\mathcal{P}(3)) \xrightarrow{\beta} \mathrm{HC}^1(\mathcal{P}(2)) \\ &\quad \xrightarrow{\partial_1} \mathrm{HC}^2(E) \xrightarrow{\gamma} \mathrm{HC}^2(\mathcal{P}(3)) \oplus \mathrm{HC}^2(\mathcal{P}(3)) \xrightarrow{\delta} \mathrm{HC}^2(\mathcal{P}(2)) \end{aligned}$$

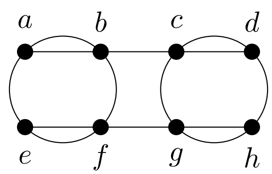
Recall from Example 6.4.5 that $\mathrm{HC}^n(\mathcal{P}(m)) \cong \mathbb{R}^{\binom{m-1}{n}}$.

Since the coboundary map δ^0 is always zero, the connecting homomorphism ∂_0 is zero as well. From exactness of the Mayer–Vietoris sequence it follows that $\mathrm{HC}^1(E) \cong \mathrm{im}(\alpha) = \ker(\beta)$. The first cohomology group of an effect algebra consists of additive maps into $\mathbb{R}$ that map 1 to 0. Since every additive map $\mathcal{P}(2) \to \mathbb{R}$ can be extended to an additive map $\mathcal{P}(3) \to \mathbb{R}$, β is surjective, hence $\mathrm{HC}^1(E) \cong \mathbb{R}^3$.

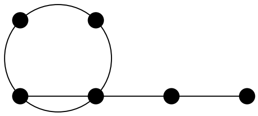
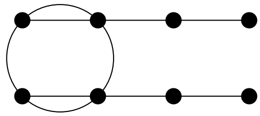
Similarly we can compute the second cohomology group. Surjectivity of β gives $\partial_1 = 0$. Furthermore $\mathrm{HC}^2(\mathcal{P}(2)) = 0$, hence $\mathrm{HC}^2(E) \cong \mathrm{HC}^2(\mathcal{P}(3)) \oplus \mathrm{HC}^2(\mathcal{P}(3)) \cong \mathbb{R}^2$. Since all higher cohomology groups of $\mathcal{P}(3)$ are zero, all groups $\mathrm{HC}^n(E)$ for $n \geq 3$ are zero as well.

The Mayer–Vietoris sequence can be applied repeatedly to find the cohomology of orthoalgebras with more than two blocks. However, one has to be careful that all unions of blocks encountered at intermediate stages are actual subalgebras, since otherwise Theorem 6.5.1 does not apply. We give an example where this phenomenon plays a role.

Example 6.5.3. Consider the orthoalgebra E with Greechie diagram

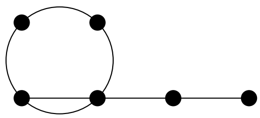
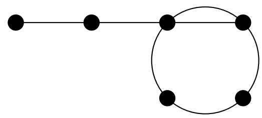


Naively, one could try to compute the cohomology of E by adding one block at the time, for instance by first using Mayer–Vietoris to obtain the cohomology of the left diagram, and then using the result to obtain the cohomology of the right diagram:

Finally, use the cohomology of the right diagram to obtain the cohomology of E. However, this fails because the diagram on the right is not a subalgebra of E. Consider the atoms labeled c and g in E. Their sum is defined in E, since both lie on the right circle. But $c \boxplus g$ is not defined in the diagram on the right, since there is no hyperedge containing both c and g. Therefore this diagram does not represent a subalgebra of E, and the Mayer–Vietoris sequence cannot be applied.

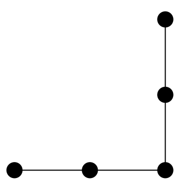
To solve this problem, we have to build up E in a different way. Consider the following subalgebras of E:

Call the one on the left A and the one on the right B. Note that both A and B are actual subalgebras of E. The diagrams represent isomorphic algebras, and their cohomology can be computed in the same way as in Example 2.4.6, yielding:

n	0	1	2	3	≥ 4
$\mathrm{HC}^n(A),\ \mathrm{HC}^n(B)$	$\mathbb{R}$	$\mathbb{R}^4$	$\mathbb{R}^5$	$\mathbb{R}^2$	0

Since A and B are subalgebras and $E = A \cup B$, the Mayer–Vietoris sequence applies. The intersection $A \cap B$ is generated under addition by the elements $a, b, (a \boxplus b)^\perp, g, h, (g \boxplus h)^\perp$. Since $(a \boxplus b)^\perp = c \boxplus d = (g \boxplus h)^\perp$, the intersection has 5 atoms, and its Greechie diagram is



We determined the cohomology of this algebra in the previous example. From a Mayer–Vietoris argument it follows that E has the following cohomology:

n	0	1	2	3	≥ 4
$\mathrm{HC}^n(E)$	$\mathbb{R}$	$\mathbb{R}^5$	$\mathbb{R}^8$	$\mathbb{R}^4$	0

6.6 Generalized Mayer–Vietoris principle

Theorem 6.5.1 only gives information about unions of two subalgebras. Applying the theorem repeatedly to get information about unions of more than two subalgebras can be problematic, as witnessed by Example 6.5.3. The problem is that the union of two subalgebras need not be a subalgebra again. Therefore it is sometimes desirable to have a generalization of the above statement applicable to unions of an arbitrary number of subalgebras. We will use an effect algebraic version of the generalized Mayer–Vietoris principle from [22]. It applies to finite orthoalgebras, and gives a method to determine their cohomology from the cohomology of their blocks.

Let E be a finite orthoalgebra. Then E can be written as a union $E = B_1 \cup \cdots \cup B_m$ of its blocks. We consider cocycles on the intersections $B_{i_1} \cap \cdots \cap B_{i_k}$, for $1 \leq i_1 < \cdots < i_k \leq m$. Our goal will be to prove that there is a long exact sequence

$$\begin{aligned} 0 \to C^n_\lambda(E) \to \bigoplus_i C^n_\lambda(B_i) \to \bigoplus_{i_1<i_2} C^n_\lambda(B_{i_1} \cap B_{i_2}) \\ \to \bigoplus_{i_1<i_2<i_3} C^n_\lambda(B_{i_1} \cap B_{i_2} \cap B_{i_3}) \to \cdots \end{aligned}$$

This sequence generalizes the short exact sequence constructed in the proof of the binary Mayer–Vietoris sequence by also including terms for intersections of more than two subalgebras.

First we describe the maps involved in the sequence. There is a restriction map $r : C^n_\lambda(E) \to \bigoplus_i C^n_\lambda(B_i)$, whose i^{th} component maps $\alpha \in C^n_\lambda(E)$ to $\alpha|_{\mathcal{T}_n(B_i)}$. Furthermore, we define maps

$$\delta_k : \bigoplus_{i_1<\cdots<i_k} C^n_\lambda(B_{i_1} \cap \cdots \cap B_{i_k}) \to \bigoplus_{i_1<\cdots<i_{k+1}} C^n_\lambda(B_{i_1} \cap \cdots \cap B_{i_{k+1}})$$

for $k = 1, 2, \ldots$. To define δ_k on a sequence $\overline{\alpha} = (\alpha_{i_1 \ldots i_k})_{i_1<\cdots<i_k}$, let the component of $\delta_k(\overline{\alpha})$ with index $i_1 < \cdots < i_{k+1}$ be

$$\sum_{j=1}^{k+1} (-1)^{j+1} \, \alpha_{i_1 \ldots \widehat{i_j} \ldots i_{k+1}} \Big|_{\mathcal{T}_n(B_{i_1} \cap \cdots \cap B_{i_{k+1}})}$$

Here the hat $\widehat{i_j}$ means that the index i_j has been omitted.

It is helpful to work out what this map does in low degrees. Firstly, the map

$$\delta_1 : \bigoplus_i C^n_\lambda(B_i) \to \bigoplus_{i<j} C^n_\lambda(B_i \cap B_j)$$

takes as input a sequence (α_i) of maps $\mathcal{T}_n(B_i) \to \mathbb{R}$, for $i = 1, \ldots, m$. The output is a sequence (β_{ij}) for $i < j$, where $\beta_{ij} : \mathcal{T}_n(B_i \cap B_j) \to \mathbb{R}$ is the map $\alpha_j - \alpha_i$ restricted to tests on the intersection $B_i \cap B_j$. Secondly, the map

$$\delta_2 : \bigoplus_{i<j} C^n_\lambda(B_i \cap B_j) \to \bigoplus_{i<j<k} C^n_\lambda(B_i \cap B_j \cap B_k)$$

maps a sequence (α_{ij}), indexed by $i < j$, to the sequence (β_{ijk}), indexed by $i < j < k$, where β_{ijk} is the restriction of $\alpha_{jk} - \alpha_{ik} + \alpha_{ij}$.

Proposition 6.6.1 (Generalized Mayer–Vietoris Principle)**.** *Let E be a finite orthoalgebra with blocks $B_1, \ldots, B_m$. Then the sequence*

$$0 \longrightarrow C^n_\lambda(E) \xrightarrow{r} \bigoplus_i C^n_\lambda(B_i) \xrightarrow{\delta_1} \bigoplus_{i_1<i_2} C^n_\lambda(B_{i_1} \cap B_{i_2})$$
$$\xrightarrow{\delta_2} \bigoplus_{i_1<i_2<i_3} C^n_\lambda(B_{i_1} \cap B_{i_2} \cap B_{i_3}) \xrightarrow{\delta_3} \cdots$$

is exact.

Proof. To prove that r is injective, suppose that $r(\alpha) = r(\beta)$ for certain $\alpha, \beta \in C^n_\lambda(E)$. Then, for each $i = 1, \ldots, m$ and each test s on B_i, we have $\alpha(s) = \beta(s)$. We have to show that α and β are the same on all tests on E. But if t is a test on E, then its entries generate a Boolean subalgebra of E. By a standard application of Zorn's Lemma, this subalgebra can be enlarged to a block, which has to be one of the blocks B_i. Thus t is a test on B_i, and hence $\alpha(t) = \beta(t)$.

The next step is proving exactness at $\bigoplus_i C^n_\lambda(B_i)$. Using the explicit description of δ_1 preceding the proposition, we see that

$$(\delta_1(r(\alpha)))_{i<j} = r(\alpha)_j - r(\alpha)_i|_{\mathcal{T}_n(B_i \cap B_j)}.$$

The maps $r(\alpha)_i$ and $r(\alpha)_j$ agree on the intersection $B_i \cap B_j$, since they are both restrictions of the same map α. Therefore $\delta_1 \circ r = 0$, or equivalently, $\operatorname{im}(r) \subseteq \ker(\delta_1)$.

For the reverse inclusion, suppose that $\bar{\alpha} \in \ker(\delta_1)$. Then $\alpha_i(t) = \alpha_j(t)$ for all tests t on $B_i \cap B_j$. We seek an $\alpha \in C^n_\lambda(E)$ such that $\alpha|_{\mathcal{T}_n(B_i)} = \alpha_i$ for all i. For a test t on E, define $\alpha(t)$ as follows: since t is a test on E, it is a test on some block B_i. Define $\alpha(t)$ to be $\alpha_i(t)$. The condition $\alpha_i(t) = \alpha_j(t)$ shows that this is independent of the choice of block, making α well-defined. It is clear that α restricts to α_i on B_i, finishing the proof that $\operatorname{im}(r) = \ker(\delta_1)$.

Now we will show that $\operatorname{im}(\delta_{k-1}) = \ker(\delta_k)$ for $k \geq 2$. From a standard computation it follows that $\delta_k \circ \delta_{k-1} = 0$. Suppose that a sequence $(\alpha_{i_1 \dots i_k})_{i_1 < \dots < i_k}$ lies in $\ker(\delta_k)$. That means that

$$\sum_{j=1}^{k+1} (-1)^{j+1} \alpha_{i_1 \dots \widehat{i_j} \dots i_{k+1}} = 0 \tag{6.1}$$

on $\mathcal{T}_n(B_{i_1} \cap \dots \cap B_{i_k})$, for all $i_1 < \dots < i_{k+1}$.

First we extend the definition of α to not necessarily increasing sequences of indices by stipulating that interchanging two indices gives a minus sign:

$$\alpha_{i_1 \dots i_j \dots i_{j'} \dots i_k} = -\alpha_{i_1 \dots i_{j'} \dots i_j \dots i_k}$$

In particular that means that a repeated index always gives zero.

Define $\beta_{i_1 \dots i_{k-1}}$ on $B_{i_1} \cap \dots \cap B_{i_{k-1}}$ in the following way: given a test $t \in \mathcal{T}_n(B_{i_1} \cap \dots \cap B_{i_{k-1}})$, let $N(t) = \{j \mid t \in \mathcal{T}_n(B_j)\}$. Then define

$$\beta_{i_1 \dots i_{k-1}}(t) = \frac{1}{\# N(t)} \sum_{j \in N(t)} \alpha_{j i_1 \dots i_{k-1}}(t).$$

Here we implicitly used the convention about not necessarily increasing sequences of indices.

To check that $\delta_{k-1}(\beta) = \alpha$, observe that

$$\begin{aligned}
(\delta_{k-1}(\beta)(t))_{i_1 < \dots < i_k} &= \sum_{j=1}^{k} (-1)^{j+1} \beta_{i_1 \dots \widehat{i_j} \dots i_k}(t) \\
&= \sum_{j=1}^{k} \sum_{\ell \in N(t)} \frac{(-1)^{j+1}}{\# N(t)} \alpha_{\ell i_1 \dots \widehat{i_j} \dots i_k}.
\end{aligned}$$

Condition (6.1) with indices $\ell, i_1, \dots, i_k$ becomes

$$\alpha_{i_1 \dots i_k} - \sum_{j=1}^{k} (-1)^{j+1} \alpha_{\ell i_1 \dots \widehat{i_j} \dots i_k} = 0.$$

Consequently,

$$(\delta_{k-1}(\beta)(t))_{i_1 < \dots < i_k} = \frac{1}{\# N(t)} \sum_{\ell \in N(t)} \alpha_{i_1 \dots i_k} = \alpha_{i_1 \dots i_k} \qquad \square$$

In Example 6.5.2, the cohomology groups become zero above a certain degree. This is reminiscent of topological cohomology theories, where cohomology groups in degree higher than the dimension of a space are zero. There is a similar result for cohomology of effect algebras, where the dimension is replaced by the height.

Definition 6.6.2. The *height* of an effect algebra A is the highest n for which there is a chain $0 = a_0 < a_1 < \cdots < a_n = 1$ in A. If such n does not exist, we say that A has infinite height. The height of A is denoted $h(A)$.

If A is a finite orthoalgebra, then it can be represented using its atoms and maximal tests. The height of A is then the length of the longest test, since a maximal test $(a_0, \dots, a_n)$ gives a chain

$$0 < a_0 < a_0 \boxplus a_1 < \ldots < a_0 \boxplus \cdots \boxplus a_n = 1.$$

Theorem 6.6.3 (Height Theorem)**.** *Let E be a finite orthoalgebra. For any $n \geq h(E)$, the cohomology group $\mathrm{HC}^n(E)$ is zero.*

Proof. First note that the Height Theorem holds for finite Boolean algebras: any finite Boolean algebra is a power set $\mathcal{P}(m)$, and according to Example 6.4.5, the Height Theorem holds for $\mathcal{P}(m)$.

If E is a finite orthoalgebra, then it can be written as a union of blocks $E = B_1 \cup \cdots \cup B_m$. Proposition 6.6.1 gives a long exact sequence

$$0 \longrightarrow C^n_\lambda(E) \xrightarrow{\delta_0} A_1 \xrightarrow{\delta_1} A_2 \xrightarrow{\delta_2} \cdots ,$$

where $A_k = \bigoplus_{i_1 < \cdots < i_k} C^n_\lambda(B_{i_1} \cap \cdots \cap B_{i_k})$, and $\delta_0 = r$. For each $k \geq 1$, this gives a short exact sequence

$$0 \longrightarrow \mathrm{im}(\delta_{k-1}) \longrightarrow A_k \xrightarrow{\delta_k} \mathrm{im}(\delta_k) \longrightarrow 0.$$

This in turn gives for each k a long exact sequence in cohomology:

$$\begin{aligned} \cdots \to \mathrm{HC}^{n-1}(\mathrm{im}\,\delta_k) \to \mathrm{HC}^n(\mathrm{im}\,\delta_{k-1}) \to \mathrm{HC}^n(A_k) \\ \to \mathrm{HC}^n(\mathrm{im}\,\delta_k) \to \mathrm{HC}^{n+1}(\mathrm{im}\,\delta_{k-1}) \to \cdots \end{aligned}$$

Since E is finite, there exists k such that $A_{k'} = 0$ for all $k' > k$. We will show that $\mathrm{HC}^{n-k+j}(\mathrm{im}\,\delta_{k-j}) = 0$ for each $j = 1, \dots, k-1$, by induction to j. To prove the claim for $j = 1$, first we will show that $\mathrm{HC}^{n-k+1}(A_k) = 0$. Finite Boolean algebras are fixed by their height, so if B and B' are different Boolean subalgebras of E, then $h(B \cap B') \leq h(B) - 1, h(B') - 1$. Using this fact repeatedly yields

$$h(B_{i_1} \cap \cdots \cap B_{i_k}) \leq h(B_{i_1}) - k + 1 \leq h(E) - k + 1 \leq n - k + 1.$$

Therefore, by the Height Theorem for finite Boolean algebras, $\mathrm{HC}^{n-k+1}(A_k)$ is zero. Now look at the following fragment of the long exact sequence obtained earlier:

$$\mathrm{HC}^{n-k}(\mathrm{im}\,\delta_k) \to \mathrm{HC}^{n-k+1}(\mathrm{im}\,\delta_{k-1}) \to \mathrm{HC}^{n-k+1}(A_k)$$

Since $A_{k+1} = 0$, the map δ_k must be the zero map, hence $\mathrm{HC}^{n-k}(\mathrm{im}\,\delta_k) = 0$. We just showed that $\mathrm{HC}^{n-k+1}(A_k)$ is zero as well. By exactness, the term in the middle must also be zero, proving the first step in the induction.

Now suppose that $\mathrm{HC}^{n-k+j}(\mathrm{im}\,\delta_{k-j}) = 0$ for a certain j. Then, using a similar argument as in the base case, it can be shown that $\mathrm{HC}^{n-k+j+1}(A_{k-j})$ is zero. Look at the following fragment of the long exact sequence:

$$\mathrm{HC}^{n-k+j}(\mathrm{im}\,\delta_{k-j}) \to \mathrm{HC}^{n-k+j+1}(\mathrm{im}\,\delta_{k-(j+1)}) \to \mathrm{HC}^{n-k+j+1}(A_{k-j})$$

The outer terms are zero, so the inner term is zero too, finishing the induction argument.

We know that $\mathrm{HC}^{n-k+j}(\mathrm{im}\,\delta_{k-j}) = 0$ for each $j = 1, \ldots, k-1$. In particular, taking $j = k-1$, we obtain $\mathrm{HC}^{n-1}(\mathrm{im}\,\delta_1) = 0$. There is a short exact sequence

$$0 \to C^n_\lambda(E) \to A_1 \to \mathrm{im}(\delta_1) \to 0,$$

hence a fragment of a long exact sequence

$$\mathrm{HC}^{n-1}(\mathrm{im}\,\delta_1) \to \mathrm{HC}^n(E) \to \mathrm{HC}^n(A_1)$$

We already noted that the term on the left is zero. By the Height Theorem for Boolean algebras, the term on the right is zero, hence $\mathrm{HC}^n(E) = 0$, which is what we wanted to prove. □

6.7 Applications

Many no-go theorems in physics can be phrased in terms of morphisms between effect algebras. We will show how cohomology helps to study these no-go theorems.

To keep the setting concrete, we will focus on the Bell scenario. The following description of the Bell experiment is based on [117]. In the setup there are two observers, Alice and Bob. Alice can perform either of two measurements a and a', with possible outcomes 0 and 1. The event "Alice performs measurement a and obtains outcome i" will be denoted by a_i, and similarly we define a'_i. Bob can also perform either of two measurements b and b', again with possible outcomes 0 and 1. The notations b_i and b'_i

have the expected meanings. After both Alice and Bob have chosen a measurement, there are four possible joint outcomes: $(0,0)$, $(0,1)$, $(1,0)$, and $(1,1)$. Each of these is obtained with a certain probability, indicated in the following table:

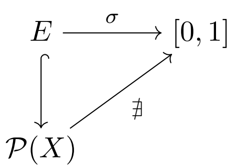
	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a,b)	$1/2$	0	0	$1/2$
(a,b')	$3/8$	$1/8$	$1/8$	$3/8$
(a',b)	$3/8$	$1/8$	$1/8$	$3/8$
(a',b')	$1/8$	$3/8$	$3/8$	$1/8$

This table of probabilities cannot be reproduced by classical physics, but there is a quantum mechanical setup realizing exactly these probabilities. This is the content of Bell's famous theorem showing that quantum mechanics is fundamentally different from classical mechanics, see [16, 1].

The effect algebraic description of the Bell experiment is as follows. All events for Alice can be collected in an effect algebra E_A with elements $0, a_0, a_1, a'_0, a'_1, 1$. Since Alice always obtains outcome 0 or 1, the sums $a_0 \boxplus a_1$ and $a'_0 \boxplus a'_1$ are defined and equal to 1. All other non-trivial sums are undefined, since Alice cannot perform the measurements a and a' at the same time. Thus E_A is isomorphic to the coproduct effect algebra $\mathcal{P}(2) + \mathcal{P}(2)$, which is the free effect algebra on two elements. Similarly we construct an effect algebra E_B for Bob's measurements, with elements $0, b_0, b_1, b'_0, b'_1, 1$. Since Bob can perform essentially the same measurements as Alice, E_B is isomorphic to E_A. The effect algebra representing the full experiment is $E := E_A \otimes E_B$, since composite systems are modeled by tensor products.

Bell's Theorem states that there is a probability distribution on this system that cannot be reproduced by classical physics. The probability distribution amounts to a state on E. More precisely, the above table of probabilities gives rise to a state that maps e.g. $a_i \otimes b'_j$ to the probability that Alice obtains outcome i when she picks measurement a, and Bob obtains outcome j when he picks measurement b'.

The measurements on a classical physical system are given by an effect algebra of the form $\mathcal{P}(X)$ for some set X. Thus Bell's Theorem says that there exists a state $\sigma : E \to [0,1]$ that does not factor through any $\mathcal{P}(X)$:



In general, no-go theorems are about extending a state $\sigma : A \to [0,1]$ to a state on a larger effect algebra B, via an inclusion $i : A \hookrightarrow B$. This inclusion map may be weak, i.e. it may not be an actual inclusion of a subalgebra. We will now apply the cohomology theory of effect algebras to study when extensions of states exist. Our approach is similar to the one in [4], but we use cyclic cohomology of effect algebras instead of sheaf cohomology.

Let A and B be finite Archimedean interval effect algebras, and let $i : A \hookrightarrow B$ be a weak injective morphism. Note that this assumption is satisfied in the case of the Bell effect algebra: the power set $\mathcal{P}(2)$ is clearly an interval effect algebra. Since the Bell effect algebra E is obtained from $\mathcal{P}(2)$ using coproducts and tensor products, it is an interval effect algebra by Proposition 2.3.8, and it is straightforward to check that E is Archimedean.

Look at the following fragment of the long exact sequence of the pair (B, A):

$$\cdots \longrightarrow \mathrm{HC}^1(B) \longrightarrow \mathrm{HC}^1(A) \xrightarrow{\partial} \mathrm{HC}^2(B, A) \longrightarrow \mathrm{HC}^2(B) \longrightarrow \cdots$$

By Theorem 6.2.2 and Proposition 6.2.3, there exists an embedding $j : \mathrm{St}(A) \to \mathrm{HC}^1(A)$, given by $j(\sigma) = \sigma - \sigma_0$ for some fixed state σ_0. The map j and the connecting homomorphism ∂ from the long exact sequence determine whether a state on A extends to a state on B.

Theorem 6.7.1. *Let $i : A \hookrightarrow B$ be a weak injective morphism between finite Archimedean interval effect algebras, and let $\sigma : A \to [0,1]$ be a state. If σ extends to a state $\tau : B \to [0,1]$ for which $\tau \circ i = \sigma$, then the cohomology class $\partial(j(\sigma)) \in \mathrm{HC}^2(A, B)$ is zero.*

Proof. It is useful to have an explicit description of the connecting homomorphism ∂. Take a cohomology class $x \in \mathrm{HC}^1(A)$ and represent it by a map $\varphi : A \to \mathbb{R}$ satisfying $\varphi(a^\perp) = -\varphi(a)$. Since i is injective, φ extends to a map $\psi : B \to \mathbb{R}$ with $\psi \circ i = \varphi$ and $\psi(b^\perp) = -\psi(b)$. Applying the coboundary map δ to ψ gives the 2-cocycle

$$(\delta\psi)(b, b') = \psi(b') - \psi(b \boxplus b') - \psi(b),$$

which is defined on all pairs (b, b') for which $b \boxplus b'$ exists. Then $\partial(x)$ is the relative cohomology class represented by $\delta\psi$.

Suppose that the state $\sigma \in \mathrm{St}(A)$ extends to a state τ on B. Let τ_0 be any state on B, and let $\sigma_0 = \tau_0 \circ i$. This gives the embedding $j(\sigma) = \sigma - \sigma_0$. Since τ extends σ, we have $(\tau - \tau_0) \circ i = \sigma - \sigma_0$, and $\tau - \tau_0$ is an additive map preserving complements. Therefore, by our description of the connecting homomorphism, $\partial(j(\sigma)) = \delta(\tau - \tau_0)$. But since $\tau - \tau_0$ is additive, its coboundary is zero, as required. $\square$

Unfortunately, the converse does not hold, so false positives may arise.

6.8 Order cohomology

Cyclic cohomology provides a necessary criterion for extending a state on an effect algebra to a larger one, but not a sufficient criterion. The problem is that positivity of the state is not encoded in the first cohomology group. One can show that the coboundary of a state is zero if and only if it extends to a signed state, i.e. one with possibly negative values. We will now define a new cohomology theory of effect algebras that takes order, and hence positivity, into account. This will lead to a necessary and sufficient criterion for extending states.

The ideas behind cohomology of effect algebras that takes order into account have been studied before in [106] and [41], although both of these only define a structure that behaves like a second cohomology group. Our definition is a variation of Pulmannová's cohomology from [106], but generalized to give cohomology in arbitrary degrees.

Defining cohomology of effect algebras with coefficients in an ordered abelian group involves morphisms between these two structures. Therefore we need a common generalization of effect algebras and ordered abelian groups, to ensure that both live in the same category. Similar structures have been considered in [106, 123].

A *partial commutative monoid* is a set together with a partial binary operation $\boxplus$ that is commutative and associative, and has a neutral element 0. An *ordered partial commutative monoid* is a partial commutative monoid A equipped with a positive cone $P \subseteq A$, for which:

- $0 \in P$.
- If $a, b \in P$ and $a \boxplus b$ is defined, then $a \boxplus b \in P$.
- For $a, b \in P$, if $a \boxplus b = 0$, then $a = b = 0$.

We will write A^+ for the positive cone P of A. Any ordered partial commutative monoid carries an order defined by $a \leq b$ if and only if there exists $c \in A^+$ such that $a \boxplus c = b$. It is straightforward to show that this forms a partial order.

Examples 6.8.1.

1. Any ordered abelian group is an ordered partial commutative monoid, in which the addition operation is total, and in which every element has an inverse.

2. Any effect algebra A is an ordered partial commutative monoid. The positive cone is simply all of A.

3. Any partial commutative monoid A can be made into an ordered partial commutative monoid by endowing it with the trivial cone $\{0\}$. The resulting order is an antichain. The resulting structure is called a *discrete* partial commutative monoid and denoted $\mathrm{Disc}(A)$.

A morphism of ordered partial commutative monoids is just a morphism of their underlying partial monoids. Such a morphism $f : A \to B$ is called *positive* if $f(A^+) \subseteq B^+$. A morphism is positive if and only if it preserves the order. Furthermore, we say that f is *strong* if the condition that $f(a) \boxplus f(b)$ is defined implies that also $a \boxplus b$ is defined.

Definition 6.8.2. Let $f : A \to B$ be a morphism between ordered partial commutative monoids. The *precone* of f is $\mathrm{prec}(f) = f^{-1}(B^+) \subseteq A$.

The precone of a morphism $f : A \to B$ is again an ordered partial commutative monoid, with addition and order inherited from A. The restricted morphism $f|_{\mathrm{prec}(f)}$ is always a positive morphism, so the precone construction is a way to transform non-positive morphisms into positive morphisms, albeit in a somewhat trivial way.

If B is discrete, then the precone of f is simply its kernel. Hence precones generalize kernels to the ordered setting. The kernel is a fundamental operation for many constructions in homological algebra. We will see that many results from homological algebra generalize to the setting of ordered abelian groups, or ordered partial commutative monoids, by replacing all kernels with precones.

The fundamental notion from homological algebra is a chain complex. Since we will mainly use cohomology, we will work with cochain complexes. In the ordered setting we define a *cochain complex* to be a sequence

$$0 \longrightarrow A_0 \xrightarrow{\delta} A_1 \xrightarrow{\delta} \cdots,$$

where each A_i is an ordered abelian group, each δ is a (not necessarily positive) homomorphism, and $\delta \circ \delta = 0$. Define the collection of n-cocycles by $\mathcal{Z}^n_{\leq}(A) = \{a \in A_n \mid a \in \mathrm{prec}(\delta)\}$. The index $\leq$ indicates that we take the order into account by using a precone instead of a kernel. Since $\delta \circ \delta = 0$, we have $\mathrm{im}(\delta) \subseteq \ker(\delta) \subseteq \mathrm{prec}(\delta)$, so we can define order cohomology as

$$\mathrm{H}^n_{\leq}(A) = \mathrm{prec}(\delta)/\,\mathrm{im}(\delta).$$

The precone of a morphism between ordered abelian groups is an ordered commutative monoid. The equivalence relation defined above is compatible with addition, but not with the order, so $\mathrm{H}^n_{\leq}(A)$ is a commutative monoid.

In ordinary homological algebra, the cohomology of a quotient complex is related to the cohomology of the larger complex via relative cohomology. We will define relative cohomology of ordered abelian groups here, and show that there is a sequence that captures some of its properties.

Let $p : B^\bullet \to A^\bullet$ be a surjective positive morphism of cochain complexes. Then p restricts to a map $\mathcal{Z}^n_\leq(B) \to \mathcal{Z}^n_\leq(A)$ because it is positive. Define the collection of relative cocycles by $\mathcal{Z}^n_\leq(A, B) = \text{prec}(\delta) \cap \text{prec}(p)$. Put an equivalence relation $\sim$ on $\mathcal{Z}^n_\leq(A, B)$ by $a \sim b$ if and only if there exists c such that $a - b = \delta(c)$ and $p(c) = 0$. Then the relative cohomology of the pair $(B^\bullet, A^\bullet)$ is the quotient $\text{H}^n_\leq = \mathcal{Z}^n_\leq / \sim$.

Just like for ordinary cohomology, it is possible to construct a sequence

$$\cdots \to \text{H}^{n-1}_\leq(A) \to \text{H}^n_\leq(B, A) \to \text{H}^n_\leq(B) \to \text{H}^n_\leq(A) \to \text{H}^{n+1}_\leq(B, A) \to \cdots$$

This sequence will not turn out to be exact, but it does satisfy a related property. The maps $\text{H}^n_\leq(B, A) \to \text{H}^n_\leq(B)$ are induced by the inclusions $\mathcal{Z}^n_\leq(B, A) \to \mathcal{Z}^n_\leq(B)$, and the maps $\text{H}^n_\leq(B) \to \text{H}^n_\leq(A)$ by p. The connecting homomorphism $\partial : \text{H}^n_\leq(A) \to \text{H}^{n+1}_\leq(B, A)$ is manufactured as follows. Take any $x \in \text{H}^n_\leq(A)$ and represent it by $a \in \mathcal{Z}^n_\leq(A)$. By surjectivity of p, there exists a $b \in B^n$ for which $p(b) = a$. Then $\delta(b)$ is an element of $\mathcal{Z}^{n+1}_\leq(B, A)$, because $\delta(\delta(b)) = 0$ and $p(\delta(b)) = \delta(p(b)) = \delta(a) \geq 0$, where we used that $a \in \mathcal{Z}^n_\leq(A) = \text{prec}(\delta)$. Let $\partial(x)$ be the cohomology class of $\delta(b)$ in $\text{H}^{n+1}_\leq(B, A)$. This does not depend on the choice of b, since if both $p(b)$ and $p(b')$ are equal to a, then $c := b' - b$ satisfies $\delta(b') - \delta(b) = \delta(c)$ and $p(c) = 0$, so $\delta(b) \sim \delta(b')$.

An exact sequence is a sequence in which the image of each morphism is the kernel of the next one. In accordance with our general theme of replacing kernels with precones, we wish to show that in order cohomology the image of each morphism is the precone of the next one. Observe that the cohomology monoids are not ordered in general, so it is not immediately clear what the precone of a map between them should be. However, there is always a pre-order on $\text{H}^n_\leq(A)$, defined in the following way: let $a, b \in A^n$, and let $[a], [b]$ be the corresponding cohomology classes. We say that $[a] \leq [b]$ if and only if there exists $c \in A^{n-1}$ such that $a + \delta(c) \leq b$ in A^n.

Lemma 6.8.3. *The relation $\leq$ is a well-defined pre-order on $\text{H}^n_\leq(A)$.*

Proof. Suppose that $a \sim a'$ and $b \sim b'$, and that $a + \delta(c) \leq b$. Then there are a'' and b'' such that $a - a' = \delta(a'')$ and $b - b' = \delta(b'')$. Let $c' = a'' - b'' + c$, then

$$a' + \delta(c') = a' + a - a' - b + b' + \delta(c) \leq b - b + b' = b'.$$

Hence the order does not depend on the choice of representatives. It is clear that $\leq$ is reflexive and transitive. □

Likewise, on the relative cohomology monoid $\mathrm{H}^n_{\leq}(B, A)$ we define $[a] \leq [b]$ if and only if there exists $c \in B^{n-1}$ such that $a + \delta(c) \leq b$ and $p(c) = 0$.

Proposition 6.8.4. *In the sequence* $\mathrm{H}^n_{\leq}(B) \xrightarrow{p} \mathrm{H}^n_{\leq}(A) \xrightarrow{\partial} \mathrm{H}^{n+1}_{\leq}(B, A)$, *we have* $\mathrm{prec}(\partial) = \mathrm{im}(p)$.

Proof. Suppose that $x \in \mathrm{prec}(\partial)$. Represent it by $a \in \mathcal{Z}^n_{\leq}(A)$, then there exists $b \in B^n$ such that $\delta(b)$ is positive in cohomology, and $p(b) = a$. Positivity in cohomology means that there exists c such that $\delta(b) \geq \delta(c)$ and $p(c) = 0$. Define $d = b - c$, then d lies in $\mathcal{Z}^n_{\leq}(B)$ because $\delta(b) \geq \delta(c)$. Furthermore $p(d) = p(b) - p(c) = a$, hence $x = [a] \in \mathrm{im}(p)$.

Conversely, take $x \in \mathrm{im}(p)$ and represent x by $a \in \mathcal{Z}^n_{\leq}(A)$. Then $a = p(b)$ for some $b \in \mathcal{Z}^n_{\leq}(B)$. It suffices to show that $[\delta(b)] \geq 0$. Since $b \in \mathcal{Z}^n_{\leq}(B)$, we have $\delta(b) \geq 0$, therefore $[\delta(b)] \geq 0$. □

Similarly one can prove that $\mathrm{prec}(p) = \mathrm{im}(i)$. Unfortunately it is not the case in general that $\mathrm{prec}(i) = \mathrm{im}(\partial)$, but we will only need the property from Proposition 6.8.4.

We will now specialize the homological algebra theory above to obtain order cohomology of an effect algebra. Let E be an effect algebra, and let A be an ordered abelian group. We wish to define order cohomology of E with coefficients in A. Often our coefficient group will be $\mathbb{R}$.

Define the abelian group $C^n(E; A) = A^{\mathcal{T}_n(E)}$ of maps from $(n+1)$-tests on E to A. To avoid cluttered notation, we will often suppress the coefficient group A. The group $C^n(E)$ forms an ordered abelian group with pointwise positive cone $C^n(E; A)^+ = (A^+)^{\mathcal{T}_n(E)}$. We will construct a cochain complex out of the groups

$$\mathcal{C}^n(E; A) = \mathrm{Disc}(C^n(E; A)) \times C^{n-1}(E; A).$$

Each $\mathcal{C}^n(E)$ is an ordered abelian group whose positive cone is $\{0\} \times C^{n-1}(E; A)^+$.

The groups $C^n(E)$ already form a cochain complex with the usual coboundary maps $\delta : C^n(E) \to C^{n+1}(E)$, given by an alternating sum over boundary maps. We make the groups $\mathcal{C}^n(E)$ into a cochain complex by defining coboundaries

$$\delta^{\mathcal{C}}(\varphi, \psi) = (\delta\varphi, \varphi - \delta\psi).$$

When no confusion is possible, we will write $\delta^{\mathcal{C}}$ simply as δ. From the fact that $\delta^2 = 0$ it easily follows that also $(\delta^{\mathcal{C}})^2 = 0$, so this is indeed a cochain complex. The resulting order cohomology monoids $\mathrm{H}^n_{\leq}(E;A) = \mathrm{prec}(\delta)/\mathrm{im}(\delta)$ are the cohomology of E with coefficients in A. From now on we will assume that our coefficient group A is $\mathbb{R}$ and write $\mathrm{H}^n_{\leq}(E;\mathbb{R})$ as $\mathrm{H}^n_{\leq}(E)$.

We will determine the order cohomology monoids of an effect algebra E in low degrees. We have $\mathcal{C}^0(E) = \mathrm{Disc}(C^0(E)) \cong \mathrm{Disc}(\mathbb{R})$. For the cochain complex in degree 1, we will use that $\mathcal{T}_1(E)$ can be identified with E, by letting $(a_0, a_1) \in \mathcal{T}_1(E)$ correspond to $a_1 \in E$. Hence $\mathcal{C}^1(E) \cong \mathrm{Disc}(\mathbb{R}^E) \oplus \mathbb{R}$. The coboundary map $\delta : \mathcal{C}^0(E) \to \mathcal{C}^1(E)$ is given by

$$\delta^0 : \mathrm{Disc}(\mathbb{R}) \to \mathrm{Disc}(\mathbb{R}^E) \oplus \mathbb{R}, \quad r \mapsto (\delta(r), r) = (0, r)$$

The zeroth cohomology monoid is $\mathrm{H}^0_{\leq}(E) = \mathrm{prec}(\delta^0) = \mathbb{R}_{\geq 0}$.

We continue with the first cohomology monoid. For this we will identify $\mathcal{T}_2(E)$ with $\{(a,b) \mid a, b \in E, a \boxplus b \text{ is defined}\}$, again by letting a 3-test (a,b,c) correspond to (b,c). We have $\mathcal{C}^2(E) \cong \mathrm{Disc}(\mathbb{R}^{\mathcal{T}_2(E)}) \oplus \mathbb{R}^E$, and the coboundary $\delta^1 : \mathcal{C}^1(E) \to \mathcal{C}^2(E)$ satisfies

$$\delta^1(\varphi, r) = (((a,b) \mapsto \varphi(b) - \varphi(a \boxplus b) + \varphi(a)), \varphi)\,.$$

By definition of the positive cone on $\mathcal{C}^2(E)$, the precone of δ^1 consists of those pairs $(\varphi : E \to \mathbb{R}, r \in \mathbb{R})$ for which $\varphi(b) - \varphi(a \boxplus b) + \varphi(a) = 0$ whenever $a \boxplus b$ is defined, and $\varphi \geq 0$. In other words, an element of $\mathrm{prec}(\delta^1)$ is a map $E \to \mathbb{R}_{\geq 0}$ that preserves addition, together with a real number. In cohomology, two of these elements are identified whenever their difference is a coboundary, which happens if and only if it is of the form $(0, r)$. Hence a pair (φ, r) is equivalent to (ψ, s) precisely when $\varphi = \psi$. Thus the second component of the pair collapses in cohomology, i.e.

$$\mathrm{H}^1_{\leq}(E) \cong \{\varphi : E \to \mathbb{R}_{\geq 0} \mid \varphi(a \boxplus b) = \varphi(a) + \varphi(b)\}.$$

In particular, any state on E is a member of the first cohomology monoid, so it is possible to perform a construction similar to the one in Section 6.7. Assume that E lies in a larger effect algebra F, via an inclusion $E \hookrightarrow F$. We wish to know when a state on E can be extended to a state on F. The sequence for relative cohomology obtained earlier gives a connecting homomorphism $\partial : \mathrm{H}^1_{\leq}(E) \to \mathrm{H}^2_{\leq}(F, E)$. Since $\mathrm{St}(E) \subseteq \mathrm{H}^1_{\leq}(E)$, the connecting homomorphism can be applied to any state on E.

Theorem 6.8.5. *Let $i : E \hookrightarrow F$ be an injective morphism of effect algebras, and let $\sigma : E \to [0,1]$ be a state. The following are equivalent:*

1. *The state σ extends to a state τ on F, for which $\tau \circ i = \sigma$.*

2. *The state σ lies in the precone of the connecting homomorphism $\partial : \mathrm{H}^1_{\leq}(E) \to \mathrm{H}^2_{\leq}(F, E)$.*

Proof. If σ extends to a state on F, then σ lies in the image of the restriction map $p : \tau \mapsto \tau \circ i$. By Proposition 6.8.4, σ is an element of $\mathrm{prec}(\partial)$.

Conversely, if $\sigma \in \mathrm{prec}(\partial)$, then by the same proposition, it is of the form $\tau \circ i$ for some $\tau \in \mathrm{H}^1_{\leq}(F)$. It remains to be checked that τ is a state. Since τ lies in the first cohomology monoid, it is an additive map $F \to \mathbb{R}_{\geq 0}$. Furthermore $\tau(1) = \tau(i(1)) = \sigma(1) = 1$, since σ is a state. For any $a \in F$, we have

$$\tau(a) + \tau(a^\perp) = \tau(a \boxplus a^\perp) = 1,$$

hence $\tau(a) \in [0, 1]$ since τ maps into the positive reals. This proves that τ is an additive map $F \to [0, 1]$ preserving 1, in other words, a state. □

We conclude that order cohomology of effect algebras provides a method to check whether states on an effect algebra extend to states on a larger effect algebra, without any false positives.

Example 6.8.6. The Bell state $\sigma : E_A \otimes E_B \to [0, 1]$ is not classically realizable, in the sense that it does not factor through any power set. Therefore, for any set X, the state σ does not lie in $\mathrm{prec}(\partial : \mathrm{H}^1_{\leq}(E_A \otimes E_B) \to \mathrm{H}^2_{\leq}(\mathcal{P}(X), E_A \otimes E_B))$.

On the other hand, the Bell state is quantum realizable. This means that there exists a Hilbert space H such that σ factors through the projection lattice $\mathcal{P}roj(H)$. Observe that $\mathcal{P}roj(H)$ is an effect algebra because it is an orthomodular lattice. The above theorem tells us that $\sigma \in \mathrm{prec}(\partial : \mathrm{H}^1_{\leq}(E_A \otimes E_B) \to \mathrm{H}^2_{\leq}(\mathcal{P}roj(H), E_A \otimes E_B))$.

We have shown that the main advantage of order cohomology is that it provides an equivalent criterion for extendability of states. On the other hand, the advantage of cyclic cohomology is that we can use results from homological algebra to determine the cohomology groups. For example, using the Künneth and Mayer–Vietoris sequences we can find the cohomology of a finite orthoalgebra from its Greechie diagram, as demonstrated in earlier sections. We leave it to future research to find adaptations of homological algebra techniques for the ordered setting.

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Samenvatting

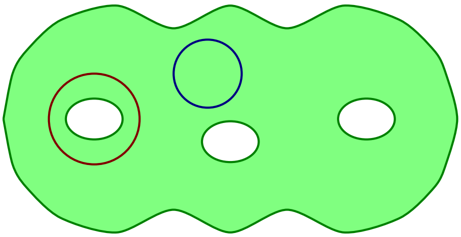
Er is een groot verschil tussen de natuurwetten die gelden voor zeer kleine dingen, zoals atomen en moleculen, en de wetten die gelden voor alledaagse dingen, zoals fietsen en broodroosters. In de alledaagse, macroscopische wereld gedragen de meeste voorwerpen zich zoals verwacht, omdat mensen hiermee opgegroeid zijn en er ervaring mee hebben. Maar op microscopische schaal kunnen er zeer onverwachte fenomenen optreden, omdat we atomen en moleculen niet kunnen zien en er dus geen directe ervaring mee hebben. Zo kan het bijvoorbeeld zijn dat een atoom zich op meerdere plekken tegelijk bevindt, wat met een fiets of een broodrooster niet zo gauw zal gebeuren. Ook kan het gebeuren dat twee atomen met elkaar verstrengeld raken, wat betekent dat de eigenschappen van het ene atoom de eigenschappen van het andere beïnvloeden. Dit gebeurt zelfs als de twee atomen enorm ver van elkaar verwijderd zijn. Ook dit is iets wat niet optreedt in de alledaagse wereld: een fiets op aarde heeft geen invloed op een broodrooster op Mars.

De natuurkunde van kleinschalige fenomenen wordt *kwantummechanica* genoemd, en de natuurkunde van het alledaagse de *klassieke* natuurkunde of klassieke mechanica. Hoewel de vreemde verschijnselen de kwantummechanica erg tegenintuïtief maken, hebben ze ook bepaalde voordelen. Zo zijn er momenteel kwantumcomputers in ontwikkeling, die gebruik maken van deze vreemde fenomenen om berekeningen uit te voeren op een manier die niet mogelijk is met gewone computers.

Stel dat we een experiment bekijken dat gebruik maakt van kwantummechanische processen. Dan kunnen we ons afvragen of dit experiment ook met puur klassieke processen uitgevoerd had kunnen worden, of dat de kwantumprocessen echt essentieel zijn. John Bell heeft bijvoorbeeld een experiment bedacht waarin dit van belang is. In dit experiment worden twee deeltjes met elkaar verstrengeld, waarna beide deeltjes naar verschillende waarnemers op grote afstand van elkaar gestuurd worden. De waarnemers kunnen op een slimme manier gebruik maken van de verstrengeling om bepaalde informatie met elkaar te delen. Dit is in de klassieke natuurkunde niet mogelijk als de twee waarnemers ver van elkaar verwijderd zijn.

Het is soms moeilijk te zien of een kwantumproces ook klassiek uitgevoerd kan worden, of niet. In dit proefschrift ontwikkelen we daarom een techniek waarmee we deze vraag kunnen beantwoorden. De kern van deze techniek is dat we aan een kwantummechanisch systeem een meetkundig figuur toekennen. Dit wordt ook wel de *geometrische realisatie* van het systeem genoemd. Op deze manier krijgen we een link tussen natuurkundige systemen en meetkunde. Vervolgens kunnen we technieken uit de meetkunde gebruiken om onze oorspronkelijke opstelling te analyseren. Deze meetkundige technieken zijn vaak eenvoudiger toe te passen dan technieken uit de kwantummechanica, waardoor dit ons helpt bij het begrijpen van het systeem.

Om meetkundig te bepalen of een kwantumexperiment uitgebreid kan worden naar een klassiek experiment, kijken we of lussen in het bijbehorende meetkundige figuur opgevuld kunnen worden. Om te begrijpen wat dit inhoudt, kunnen we naar het volgende figuur kijken:



Er zijn twee lussen in deze figuur getekend, de blauwe en de rode. De blauwe lus kan geheel opgevuld of ingekleurd worden, omdat de binnenkant hiervan binnen het groene figuur ligt. De rode lus kan echter niet opgevuld worden, omdat er een gat in de weg zit. Bestuderen of lussen opgevuld kunnen worden is van belang in een wiskundig vakgebied dat *homologie* of *cohomologie* wordt genoemd.

Het blijkt dat als we een kwantummechanisch systeem “vertalen” naar een meetkundig figuur, dat dan een experiment correspondeert met een lus in het figuur. We bewijzen in dit proefschrift dat een experiment klassiek uitgevoerd kan worden precies als de bijbehorende lus opgevuld kan worden. Bijvoorbeeld, als de lussen in de bovenstaande afbeelding afkomstig zijn van kwantumexperimenten, dan kan het blauwe experiment wel klassiek uitgevoerd worden, maar het rode niet. Op deze manier is het voordeel van deze vertaling goed te zien: kwantummechanica kan erg ongrijpbaar zijn, maar bepalen of een lus opgevuld kan worden is meestal niet moeilijk als je naar een plaatje kijkt.

Je zou kunnen zeggen dat deze methode werkt omdat we zowel bij de kwantumexperimenten als bij de meetkunde kijken naar uitbreidingen. Onze hoofdvraag was of een kwantumexperiment ook klassiek uitgevoerd kan worden, met andere woorden, of het naar een groter klassiek experiment kan worden uitgebreid. Bij de meetkundige figuren kijken we of lussen kunnen worden opgevuld, en dit is hetzelfde als het uitbreiden van een lus naar een schijf, oftewel een opgevulde lus. Het resultaat hierboven zegt dus eigenlijk dat uitbreidingen van kwantum naar klassiek hetzelfde zijn als uitbreidingen van een lus naar een schijf in de meetkunde.

Tenslotte nog een opmerking over de rol van zogeheten *effectalgebroïden* in dit verhaal. Deze spelen een belangrijke rol in het verbinden van kwantummechanica met meetkunde, en daarom is dit proefschrift ernaar vernoemd. Grof gezegd is een effectalgebroïde een abstracte structuur die zowel kwantumsystemen als bepaalde meetkundige figuren omvat, met name cirkelvormige figuren. Het klinkt misschien vreemd dat zoiets bestaat omdat de twee aspecten ervan zo verschillend zijn, maar dit is mogelijk omdat er op abstract niveau toch bepaalde overeenkomsten zijn tussen kwantummechanica en meetkunde. Op deze manier geven effectalgebroïden een brug tussen de twee vakgebieden, en helpen ze bij het bewijzen van resultaten over verbanden ertussen.

Curriculum Vitae

Frank Roumen was born on April 10, 1989 in Nijmegen, The Netherlands. He started studying mathematics with a strong physics component at the University of Groningen in 2007, and obtained his B.Sc. degree in mathematics with the distinction *cum laude* in 2010. He continued with the M.Sc. program in mathematics at Radboud University in Nijmegen, specializing in the mathematical foundations of computer science. After graduating *cum laude* in this field in 2012, he started as a Ph.D. student at Radboud University, supervised by Prof. Ieke Moerdijk. The Ph.D. project involved an interplay of mathematics, computer science, and mathematical physics. In 2015, Frank spent three months at the University of Ottawa for a research visit to Prof. Philip Scott. He won an award for the best student paper at the conference Quantum Physics and Logic 2016 for his paper "Cohomology of effect algebras". In September 2016, he started as a Research Assistant at the University of Cambridge.