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Sharp Concentration Inequalities for Deviations from the Mean for Sums of Independent Rademacher Random Variables

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Abstract. For a fixed unit vector $a = (a_1, a_2, \ldots, a_n) \in S^{n-1}$, that is, $\sum_{i=1}^n a_i^2 = 1$, we consider the 2^n signed vectors $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in \{-1, 1\}^n$ and the corresponding scalar products $a \cdot \varepsilon = \sum_{i=1}^n a_i \varepsilon_i$. In [3] the following old conjecture has been reformulated. It states that among the 2^n sums of the form $\sum \pm a_i$ there are not more with $\left|\sum_{i=1}^n \pm a_i\right| > 1$ than there are with $\left|\sum_{i=1}^n \pm a_i\right| \leq 1$. The result is of interest in itself, but has also an appealing reformulation in probability theory and in geometry. In this paper we will solve an extension of this problem in the uniform case where $a_1 = a_2 = \cdots = a_n = n^{-1/2}$. More precisely, for S_n being a sum of n independent Rademacher random variables, we will give, for several values of ξ, precise lower bounds for the probabilities

$$P_n := \mathbb{P}\left\{-\xi\sqrt{n} \leq S_n \leq \xi\sqrt{n}\right\}$$

or equivalently for

$$Q_n := \mathbb{P}\{-\xi \leq T_n \leq \xi\},$$

where T_n is a standardized binomial random variable with parameters n and $p = 1/2$. These lower bounds are sharp and much better than for instance the bound that can be obtained from application of the Chebyshev inequality. In case $\xi = 1$ Van Zuijlen solved this problem in [5]. We remark that our bound will have nice applications in probability theory and especially in random walk theory (cf. [1, 2]).

Keywords: sums of independent Rademacher random variables, tail probabilities, lower bounds, concentration inequalities, random walk, finite samples

1. Introduction and Result

Recall a Rademacher random variable is a random variable that takes the values $+1$ and -1 both with probability $1/2$. Let $\varepsilon_1, \varepsilon_2, \ldots$ be a sequence of independent identically distributed Rademacher random variables and for positive integers n let $a_n = (a_{1n}, a_{2n}, \dots, a_{nn})$ be a unit-vector in $\mathbb{R}^n$, so that $\sum_{i=1}^n a_{in}^2 = 1$. The following problem has been presented in [4] and is attributed to B. Tomaszewski. In [3], Conjecture 1.1, this problem has been reformulated as follows:

$$\mathbb{P}\{|a_{1n}\varepsilon_1 + a_{2n}\varepsilon_2 + \cdots + a_{nn}\varepsilon_n| \leq 1\} \geq \frac{1}{2}, \quad \text{for } n = 1, 2, \dots.$$

This conjecture is at least 25 years old and seems still to be unsolved. In the uniform case where,

$$a_{1n} = a_{2n} = \cdots = a_{nn} = n^{-1/2},$$

the maximum possible value of $\frac{S_n}{\sqrt{n}}$ is $\sqrt{n}$, where

$$S_n := \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n \tag{1.1}$$

and the conjecture, stating that for integers $n \geq 1$,

$$\mathbb{P}\{|S_n| \leq \sqrt{n}\} \geq 1/2,$$

has been solved recently by Van Zuijlen [5]. It means that at least 50% of the probability mass is between minus one and plus one standard deviation from the mean, which is quite remarkable. In [3], Theorem 1.2, the following sharp inequality has been shown for all $a = (a_1, \dots, a_n)$ with $\sum_{i=1}^n a_i^2 = 1$ and $\forall i\ |a_i| < 1$:

$$\mathbb{P}\{|a_1\varepsilon_1 + a_2\varepsilon_2 + \cdots + a_n\varepsilon_n| < 1\} \geq \frac{3}{8}, \quad \text{for } n = 2, 3, \dots.$$

In this paper we shall generalize Van Zuijlen's result and derive sharp lower bounds for probabilities concerning ξ standard deviations:

$$P_n := \mathbb{P}\{|S_n| \leq \xi\sqrt{n}\}, \tag{1.2}$$

where $\xi \in (0, 1]$. We notice that S_n can be easily expressed in terms of sums of independent Bernoulli(1/2) random variables since $(\varepsilon_i + 1)/2$ are independent Bernoulli random variables and hence S_n is distributed as $2B_n - n$, where B_n is a binomial random variable with parameters n and $1/2$. In particular,

$$\frac{1 - P_n}{2} = \sum_{k < \frac{1}{2}n - \frac{1}{2}\xi\sqrt{n}} \binom{n}{k} 2^{-n}.$$

Easy calculations show that the sequence $(P_n)_n$ is not monotone in n. Note that trivially

$$P_1 = \begin{cases} 1, & \text{for } \xi = 1; \\ 0, & \text{for } \xi < 1. \end{cases}$$

Throughout the paper n and k will denote integers, with $n \geq 1$ and $k \geq 0$. Our result is as follows.

Theorem 1.1. *Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ be independent Rademacher random variables. Let $\xi \in (0,1]$ and S_n and P_n be defined as in* (1.1) *and* (1.2)*. For $k \geq 1$, let*

$$n_k := 2\left\lceil \frac{\frac{k^2}{\xi^2} - k}{2} \right\rceil + k - 1, \quad C_k := \{n\colon n_k \leq n < n_{k+1}\}, \text{ and } Q_k^- := P_{n_{k+1}-1}.$$

Then the sequence $\{n_k\}$ is strictly increasing and with Φ indicating the standard normal distribution function, we have

a. $P_n = \mathbb{P}\{|S_n| \leq \xi\sqrt{n}\} = \mathbb{P}\{|S_n| \leq k\}$*, for $n \in C_k$,*
b. $Q_k^- = \min_{n\in C_k} P_n$*,*
c. the sequence (Q_k^-) is strictly monotone increasing in k,
d. $\lim_{n\to\infty} P_n = \lim_{k\to\infty} Q_k^- = 2\Phi(\xi) - 1$*,*
e. $Q_1^- = P_{n_2-1} \leq P_n$*, for all $n \geq n_1$.*

Notice that a Lyapunov type bound, the Berry-Esseen bound, with explicit constants for the remainder term in the Central Limit Theorem gives (see, for instance, [1, p. 38])

$$\sup_{\xi>0} |\mathbb{P}\{|S_n| < \xi\sqrt{n}\} - (2\Phi(\xi) - 1)| \leq \frac{1.12}{\sqrt{n}}, \quad n = 1, 2, \ldots.$$

A consequence of Theorem 1.1 is the following result.

Corollary 1.2. *Let $0 < \xi \leq 1$. Then $n_2 \geq 3$,*

$$\frac{2}{\sqrt{n_2+1}} \leq \xi < \frac{2}{\sqrt{n_2-1}}, \qquad n_1 = 2\left\lceil \frac{n_2 - 3}{8} \right\rceil$$

and for $n \geq n_1$ we have

$$P_n \geq P_{n_2-1} = \binom{n_2 - 1}{(n_2-1)/2} 2^{-(n_2-1)}.$$

For instance, for $n \geq 2$ we have the sharp lower bounds

$$P_n \geq \begin{cases} 1/2, & \text{for } \xi = 1; \\ 3/8, & \text{for } \xi \in \left[\sqrt{2/3}, 1\right); \\ 5/16, & \text{for } \xi \in \left[\sqrt{1/2}, \sqrt{2/3}\right). \end{cases}$$

It is worthwhile to clarifying in a plot the structure of the probabilities $P_n(\ell) = \mathbb{P}\{|S_n| \leq \ell\}$, where n and ℓ are nonnegative integers such that $n + \ell$ is even. See Figure 1. In the next section we will study the behavior of $P_n(\ell)$ depending on n in detail culminating in Theorem 2.3 which states the key property of binomial coefficients that is needed in our result. Corollaries 2.2 and 2.4 are intended to characterize the features of Figure 1.

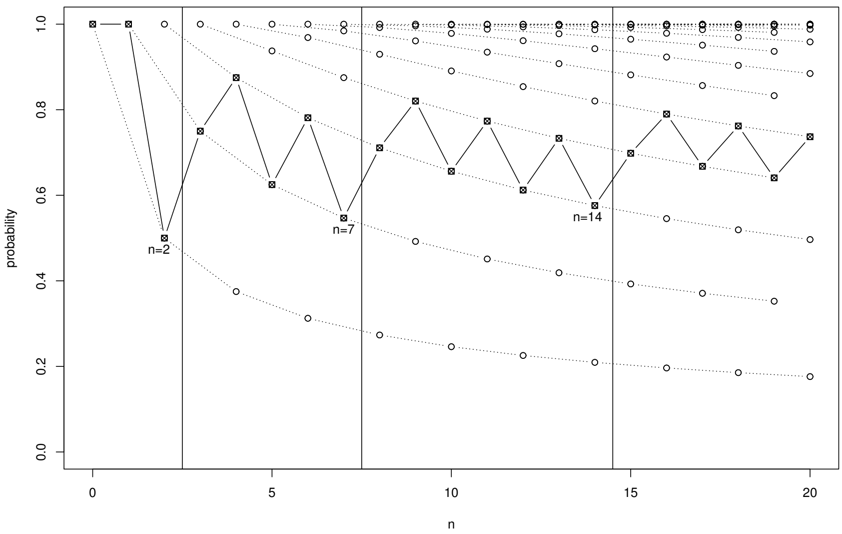

Figure 1: Graph of probabilities $P_n(\ell)$, $n+\ell$ even. (Dotted lines connect points with constant $\ell = 0, 1, 2, \ldots$, upwards in graph. The square symbols indicate the points (n, P_n) for $\xi = 1$. The vertical lines separate the regions C_k, $k = 1, 2, \ldots$.)

2. Preliminaries

Given independent Rademacher random variables ε_i, $i = 1, 2, \ldots$, as defined in Theorem 1.1 and let $S_n = \sum_{i=1}^n \varepsilon_i$ with $S_0 = 0$. Clearly, $S_n = S_{n-1} + \varepsilon_n$. Define for integers n, k with $k \geq 0$, $n \geq 0$ and $k \leq n$,

$$P_n(k) = \mathbb{P}\{|S_n| \leq k\},$$

so that $P_0(k) = 1$. In the sequel we adopt the convention $P_n(-1) = 0$. A basic property is the symmetry of the distribution of S_n:

$$\mathbb{P}\{S_n = k\} = \mathbb{P}\{S_n = -k\}.$$

Using this symmetry and the independence of S_{n-1} and ε_n we obtain for $n \geq 1$ and $k \geq 0$,

$$P_n(k) = \mathbb{P}\{|S_n| \leq k\} = \mathbb{P}\{S_{n-1} \in [1-k, 1+k]\} = \mathbb{P}\{S_{n-1} \in [-1-k, -1+k]\},$$

so that in case $n+k$ is even (hence $\mathbb{P}\{S_{n-1} = k\} = 0$) we have

$$\mathbb{P}\{|S_n| \leq k\} = \mathbb{P}\{|S_{n-1}| \leq k-1\} + \mathbb{P}\{S_{n-1} = k+1\}$$

and

$$\mathbb{P}\{|S_n| \leq k\} = \mathbb{P}\{|S_{n-1}| \leq k+1\} - \mathbb{P}\{S_{n-1} = k+1\}.$$

This leads to the following properties for $P_n(k)$.

Remark 2.1. Let k, n be integers with $k \geq 0$, $n \geq 1$, $k \leq n$ and $n+k$ even. Then

$$\begin{aligned} P_n(k) &= P_{n-1}(k-1) + \mathbb{P}\{S_{n-1} = k+1\} \\ &= P_{n-1}(k+1) - \mathbb{P}\{S_{n-1} = k+1\}, \end{aligned} \tag{2.1}$$

so that

$$P_{n-1}(k-1) \leq P_n(k) = \frac{P_{n-1}(k-1) + P_{n-1}(k+1)}{2} \leq P_{n-1}(k+1).$$

Moreover,

$$\mathbb{P}\{S_{n-1} = k-1\} = \binom{n-1}{\frac{n+k}{2}-1} 2^{-(n-1)} = \frac{n+k}{n}\binom{n}{\frac{n+k}{2}} 2^{-n} = \frac{n+k}{n}\mathbb{P}\{S_n = k\},$$

$$\mathbb{P}\{S_{n-1} = k+1\} = \mathbb{P}\{S_{n-1} = -k-1\} = \frac{n-k}{n}\mathbb{P}\{S_n = -k\} = \frac{n-k}{n}\mathbb{P}\{S_n = k\},$$

so that for $n \geq k \geq 1$,

$$\begin{aligned} &P_{n-1}(k-1) - P_{n+1}(k-1) \\ &= P_n(k-2) + \mathbb{P}\{S_{n-1} = k-1\} - P_n(k-2) - \mathbb{P}\{S_n = k\} \\ &= \mathbb{P}\{S_{n-1} = k-1\} - \mathbb{P}\{S_n = k\} \\ &= \frac{k}{n}\mathbb{P}\{S_n = k\} \\ &> 0, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} \frac{\mathbb{P}\{S_n = k\}}{\mathbb{P}\{S_{n+2} = k\}} &= \frac{\mathbb{P}\{S_n = k\}}{\mathbb{P}\{S_{n+1} = k+1\}} \times \frac{\mathbb{P}\{S_{n+1} = k+1\}}{\mathbb{P}\{S_{n+2} = k\}} \\ &= \frac{n+1+k+1}{n+1} \times \frac{n+2-k}{n+2} \\ &= \frac{(n+2)^2 - k^2}{(n+2)^2 - (n+2)}. \end{aligned}$$

Therefore,

$$[\mathbb{P}\{S_n = k\} > \mathbb{P}\{S_{n+2} = k\}] \Leftrightarrow \left[n+2 > k^2\right]$$

and

$$[\mathbb{P}\{S_n = k\} < \mathbb{P}\{S_{n+2} = k\}] \Leftrightarrow \left[k \leq n+2 < k^2\right].$$

More explicitly, for $k \geq 3$,

$$\mathbb{P}\{S_k = k\} < \mathbb{P}\{S_{k+2} = k\} < \cdots < \mathbb{P}\left\{S_{k^2-2} = k\right\}$$

and, for $k \geq 2$,

$$\mathbb{P}\left\{S_{k^2-2} = k\right\} = \mathbb{P}\{S_{k^2} = k\} > \mathbb{P}\left\{S_{k^2+2} = k\right\} > \mathbb{P}\left\{S_{k^2+4} = k\right\} > \cdots. \tag{2.3}$$

(The inequalities (2.3) also appear in [1, pp. 23–24].)

Corollary 2.2. *Let k, n, and m be integers with $n+k$ even and $n > m \geq k \geq 0$. Then,*

$$P_n(k) = P_n(k+1) < P_m(k+1). \tag{2.4}$$

Proof. In case $n-m$ is even we have $P_m(k+1) = P_m(k)$ and according to (2.2)

$$P_n(k+1) = P_n(k) < P_{n-2}(k) < \cdots < P_{k+2}(k) < P_k(k) = 1.$$

In case $n-m$ is odd, using (2.1) and (2.2) we have

$$P_n(k) < P_{n-1}(k+1) < P_{n-3}(k+1) < \cdots < P_{k+1}(k+1) = 1. \qquad \blacksquare$$

Theorem 2.3. *Let k, n be positive integers with $n+k$ even and $n+2 \geq k^2$. If $\ell \geq 0$ is an integer such that $n+1+2\ell < \frac{(k+1)^2}{k^2}(n+2)$, then*

$$P_n(k-2) < P_{n+1+2\ell}(k-1).$$

Proof. For $k = 1$ the statement is obvious, so we suppose that $k > 1$. Notice that for $k \geq 2$, $n+2 \geq k^2$ implies $n \geq k$. Let $s_i := \mathbb{P}\{S_{n+2i} = k\}$. For fixed $k \geq 2$ and n with $n+2 \geq k^2$ we have from (2.3)

$$s_0 \geq s_1 \geq \cdots \geq s_\ell > 0.$$

We have to show that $P_n(k-2) < P_{n+1+2\ell}(k-1)$ or equivalently that $P_{n+1}(k-1) - P_{n+1+2\ell}(k-1) < P_{n+1}(k-1) - P_n(k-2)$. The left-hand side of this inequality (use (2.2)) equals $\sum_{i=1}^{\ell} \frac{k}{n+2i} s_i$ and the right-hand side equals s_0, so that we have to show that

$$\sum_{i=1}^{\ell} \frac{k}{n+2i} s_i < s_0.$$

Since

$$\sum_{i=1}^{\ell} \frac{k}{n+2i} s_i \leq \sum_{i=1}^{\ell} \frac{k}{n+2i} s_1,$$

it is sufficient to show that

$$\sum_{i=1}^{\ell} \frac{k}{n+2i} < \frac{s_0}{s_1} = \frac{\mathbb{P}\{S_n = k\}}{\mathbb{P}\{S_{n+2} = k\}} = \frac{(n+2)^2 - k^2}{(n+2)^2 - (n+2)}.$$

Since $s_0 \geq s_1 > 0$, Theorem 2.3 now follows from Lemma 5.1. $\blacksquare$

Corollary 2.4. *Let n_k, $k = 1, 2, \ldots$ be an increasing sequence of integers with $n_1 \geq 0$, $n_k + k$ odd, $n_k + 1 \geq k^2$ and $n_{k+1} - 1 < \frac{(k+1)^2}{k^2}(n_k + 1)$. Then,*

$$\mathbb{P}\{S_{n_2-1} = 0\} = P_{n_2-1}(0) < \cdots < P_{n_k-1}(k-2) < P_{n_{k+1}-1}(k-1) \cdots .$$

Proof. Apply Theorem 2.3 with $n = n_k - 1$ and $\ell = \frac{n_{k+1} - n_k - 1}{2}$. $\blacksquare$

3. The Original Context and Proofs

In this section we prove that the sequence n_k defined in Theorem 1.1 satisfies the conditions of Theorem 2.3 and we prove Theorem 1.1.a). For positive integers k, in Section 1 we defined

$$n_k := 2\left\lceil \frac{\frac{k^2}{\xi^2}-k}{2} \right\rceil + k - 1. \tag{3.1}$$

It satisfies

$$\xi^2(n_k-1) < k^2 \le \xi^2(n_k+1).$$

Notice that the n_k's satisfy one of the conditions in Corollary 2.4, namely,

$$\begin{aligned} n_{k+1} &< \frac{(k+1)^2}{\xi^2} - (k+1) + (k+1) - 1 + 2 = \frac{(k+1)^2}{k^2}\frac{k^2}{\xi^2} + 1 \\ &\le \frac{(k+1)^2}{k^2}\left\{ 2\left\lceil \frac{\frac{k^2}{\xi^2}-k}{2} \right\rceil + k - 1 + 1 \right\} + 1 = \frac{(k+1)^2}{k^2}(n_k+1) + 1. \end{aligned}$$

A sufficient condition for n_k to be *increasing* in k is that $k^2/\xi^2 - k$ is non-decreasing in k for integer $k \ge 1$. This is the case if for all $k \ge 1$ we have $\xi^2 \le 2k+1$, that is, if $\xi \le \sqrt{3}$.

Proof of Theorem 1.1. Consider the block $C_k = [n_k, n_{k+1})$ given in the theorem. Let n be such that $n_k < n < n_{k+1}$, then $\lfloor \xi\sqrt{n} \rfloor = k$ since

$$k^2 \le \xi^2(n_k+1) \le \xi^2 n \le \xi^2(n_{k+1}-1) < (k+1)^2.$$

We need still consider $\xi\sqrt{n_k}$. In the situation that $n_{k-1} < n_k < n_{k+1}$ we see that

$$(k-1)^2 \le \xi^2(n_{k-1}+1) \le \xi^2 n_k \le \xi^2(n_{k+1}-1) < (k+1)^2.$$

Then $k-1 \le \xi\sqrt{n_k} < k+1$ so that $\left\{|S_{n_k}| \le \xi\sqrt{n_k}\right\} = \left\{|S_{n_k}| \le k-1\right\} = \left\{|S_{n_k}| \le k\right\}$. Since already for $\xi \le \sqrt{3}$ the sequence n_k is increasing we have proven Theorem 1.1.a).

From Corollary 2.2 it follows for all $k \le m < n_{k+1} - 1$ that

$$Q_k^- = P_{n_{k+1}-1} = P_{n_{k+1}-1}(k) = P_{n_{k+1}-1}(k-1) < P_m(k),$$

so that in particular $Q_k^- = \min\{P_n \mid n \in C_k\}$. Notice that $P_m(k) = 1$ for $m < k$. This proves Theorem 1.1.b).

If moreover $\xi \le 1$ then $n_k + 1 \ge k^2$ and Corollary 2.4 applies, proving Theorem 1.1.c).

The first interesting block is C_1 with minimal value $P_{n_2-1} = P_{n_2-1}(1) = P_{n_2-1}(0) = \mathbb{P}\{S_{n_2-1} = 0\}$. This proves Theorem 1.1.e).

Finally, Theorem 1.1.d) follows from the Central Limit Theorem: $\lim_{n\to\infty} P_n = \lim_{n\to\infty} \mathbb{P}\{|S_n| \le \xi\sqrt{n}\} = 2\Phi(\xi) - 1$. ∎

Proof of Corollary 1.2. Let $0 < \xi \leq 1$. We have from definition (3.1) that $n_2 \geq 3$ and moreover

$$\frac{4}{\xi^2} - 1 \leq n_2 = 2\left\lceil \frac{\frac{4}{\xi^2} - 2}{2} \right\rceil + 2 - 1 = 2\left\lceil \frac{2}{\xi^2} \right\rceil - 1 < 2\left(\frac{2}{\xi^2} + 1\right) - 1 = \frac{4}{\xi^2} + 1.$$

Hence,

$$\frac{2}{\sqrt{n_2+1}} \leq \xi < \frac{2}{\sqrt{n_2-1}}$$

and

$$\frac{n_2-5}{8} < \frac{\frac{1}{\xi^2} - 1}{2} \leq \frac{n_2-3}{8}.$$

Since n_2 is odd, the open interval $\left(\frac{n_2-5}{8}, \frac{n_2-3}{8}\right)$ does not contain an integer. Hence,

$$n_1 = 2\left\lceil \frac{\frac{1}{\xi^2} - 1}{2} \right\rceil + 1 - 1 = 2\left\lceil \frac{n_2-3}{8} \right\rceil$$

and for all $n \geq n_1$, we have from Theorem 1.1

$$P_n \geq P_{n_2-1} = \mathbb{P}\{S_{n_2-1} = 0\} = \binom{n_2-1}{(n_2-1)/2} 2^{-(n_2-1)}.$$

It is straightforward to see that

$$n_2 = \begin{cases} 3, & \text{for } \xi = 1, \\ 5, & \text{for } \xi \in \left[\sqrt{2/3}, 1\right), \\ 7, & \text{for } \xi \in \left[\sqrt{1/2}, \sqrt{2/3}\right), \end{cases}$$

so that

$$\mathbb{P}\{-1 \leq S_{n_2-1} \leq 1\} = \mathbb{P}\{S_{n_2-1} = 0\} = \begin{cases} 1/2, & \text{for } \xi = 1, \\ 3/8, & \text{for } \xi \in \left[\sqrt{2/3}, 1\right), \\ 5/16, & \text{for } \xi \in \left[\sqrt{1/2}, \sqrt{2/3}\right). \end{cases} \quad \blacksquare$$

4. Examples

It is the condition $\xi \leq 1$ that implies $n_k + 1 \geq k^2$, needed in Corollary 2.4. For $\xi > 1$ it is no longer true that $P_{n_{k+1}-1}(k-1) > P_{n_k-1}(k-2)$, for all k, as can be seen from the following Example 4, giving the case $\xi = \sqrt{2}$. In general, the situation for $\xi > 1$ is very irregular and quite complicated. However, in case $\xi = \sqrt{2}$ we are able to prove that for $k \geq 6$ the sequence $P_{n_k-1}(k-2)$ is monotonically increasing in k. We believe, also based on numerical computations, that this is the only value for $\xi > 1$ where the sequence $P_{n_k-1}(k-2)$ is asympotically monotonically increasing in k.

4.1. EXAMPLE 1, the Case $\xi = \sqrt{1/2}$

In case $\xi = \sqrt{1/2}$, we obtain for $k \in \{1, 2, \ldots\}$

$$n_k = \begin{cases} 2k^2 - 1, & \text{for } k = \text{even}, \\ 2k^2, & \text{for } k = \text{odd}. \end{cases}$$

In this case $n_1 = 2$, $n_2 = 7$, $n_3 = 18$, $n_4 = 31$, so that $C_1 = [2, 6]$, $C_2 = [7, 17]$, $C_3 = [18, 30]$ and the minimal value in C_1 is

$$Q_1^- = P_{n_2-1} = P_6 = P_6(1) = P_6(0) = \frac{5}{16}.$$

Also,

$$Q_2^- = P_{n_3-1} = P_{17}(2) = P_{17}(1) = \frac{12155}{32768} \geq \frac{10240}{32768} = \frac{5}{16}.$$

We notice that these probabilities (as in the other Examples) can easily be computed by using the binomial representations of probabilities which involve Radermacher sums.

4.2. EXAMPLE 2, the Case $\xi = \sqrt{2/3}$

In case $\xi = \sqrt{2/3}$, we obtain for $k \in \{1, 2, \ldots\}$

$$n_k = \begin{cases} \frac{3}{2}k^2 - 1, & \text{for } k = \text{even}, \\ \frac{3}{2}k^2 + \frac{1}{2}, & \text{for } k = \text{odd}. \end{cases}$$

Hence, $n_1 = 2$, $n_2 = 5$, $n_3 = 14$, $n_4 = 23$, $n_5 = 38$, $n_6 = 53$ with blocks $C_1 = [2, 4]$, $C_2 = [5, 13]$, $C_3 = [14, 22]$, $C_4 = [23, 37]$, $C_5 = [38, 52]$. The minimal value in C_1 is obtained for

$$P_{n_2-1} = P_4 = \frac{3}{8}.$$

Also,

$$P_{n_3-1} = P_{13} = \frac{429}{1024} \geq \frac{384}{1024} = \frac{3}{8}.$$

4.3. EXAMPLE 3, the Case $\xi = 1$

In case $\xi = 1$ we obtain for $k \in \{1, 2, \ldots\}$

$$n_k = k^2 - 1.$$

We obtain for integers $k \geq 2$, $C_k = \left\{k^2 - 1, k^2, \ldots, (k+1)^2 - 2\right\}$, with length $m_k = 2k + 1$. Now $n_1 = 0$, $n_2 = 3$, $n_3 = 8$, $n_4 = 15$, so that $C_1 = [0, 2]$, $C_2 = [3, 7]$, $C_3 = [8, 14]$. The minimal value in C_1 is obtained for

$$P_{n_2-1} = P_2 = \frac{1}{2}.$$

The minimal value in C_2 is obtained for $n = n_3 - 1 = 7$ and equals

$$P_{n_3-1} = P_7 = \frac{35}{64} \geq \frac{32}{64} = \frac{1}{2}.$$

4.4. EXAMPLE 4, the Case $\xi = \sqrt{2}$

We give this example to show that the condition $\xi \leq 1$ in Theorem 1.1 is necessary. The case $\xi = \sqrt{2}$ is the only example we know, where the monotonicity of P_{n_k-1} is violated only finitely often. Recall that in case $\xi = \sqrt{2}$, for $k \in \{1, 2, \ldots\}$ we obtain

$$n_k = \begin{cases} k^2/2 - 1, & \text{for } k = \text{even}, \\ k^2/2 - 1/2, & \text{for } k = \text{odd}. \end{cases}$$

We obtain $n_1 = 0, n_2 = 1, n_3 = 4, n_4 = 7, n_5 = 12, n_6 = 17,$

$$Q_2^- = P_{n_3-1}(1) = P_3(1) = \frac{3}{4} < 1 = P_0(0) = P_{n_2-1}(0) = Q_1^-$$

and we have the second interruption of monotony in block C_4, since

$$Q_4^- = P_{n_5-1}(3) = P_{11}(3) = \frac{99}{128} < \frac{100}{128} = P_6(2) = P_{n_4-1}(2) = Q_3^-$$

and

$$Q_3^- = P_{n_4-1}(2) > Q_2^- = P_{n_3-1}(1) = P_3(1) = \frac{3}{4}.$$

One can prove that these are the only interruptions.

5. Appendix

In this section we state and prove the lemma needed in the proof of Theorem 2.3.

Lemma 5.1. *Let k, n, ℓ be positive integers with $k \geq 2$, $n + k$ even and $n + 2 \geq k^2$. If ℓ is such that $n + 1 + 2\ell < \frac{(k+1)^2}{k^2}(n+2)$, or equivalently, $n + 2 > \frac{2l-1}{2k+1}k^2$, then we have $\sum_{i=1}^{\ell} \frac{k}{n+2i} < 1$.*

Proof. For $\ell \in \{1, 2, \ldots, k\}$ with $k \geq 2$ we trivially have

$$\sum_{i=1}^{\ell} \frac{k}{n+2i} \leq \sum_{i=1}^{k} \frac{k}{n+2i} < k \cdot \frac{k}{k^2 - 2 + 2} = 1.$$

It is easy to see that $\frac{k}{n+2i} + \frac{k}{n+2\ell+2-2i}$ is strictly decreasing in i for $i \leq (\ell+1)/2$. For $\ell \geq k + 1 \geq 3$, it is therefore sufficient to prove $\frac{k}{n+2} + \frac{k}{n+2\ell} \leq \frac{2}{\ell}$, or equivalently,

$$\frac{\ell}{n+2} + \frac{\ell}{n+2\ell} \leq \frac{2}{k}. \tag{5.1}$$

For $\ell = k + 1$ to be allowed we have $n + 2 > k^2$, hence $n + 1 \geq k^2$ and since $n + k$ is even, we even have $n \geq k^2$. Substituting $\ell = k + 1$ and $n = k^2$ we get Inequality (5.1) for $\ell = k + 1$:

$$\frac{2}{k} - \frac{\ell}{n+2} - \frac{\ell}{n+2\ell} \geq \frac{2(k^2 + 2k + 4)}{k(k^2 + 2)(k^2 + 2\ell)} \geq 0.$$

Next, assuming that $\ell \geq k+1$, we remark that

$$\left[n+1+2\ell < \frac{(k+1)^2}{k^2}(n+2)\right]$$

$$\Leftrightarrow \left[n+2 > \frac{2\ell-1}{2k+1}k^2 = k^2 + (\ell-k-1)\left(k-\frac{1}{2}\right) + \frac{\ell-k-1}{2(2k+1)}\right.$$

$$\left.\geq k^2 + (\ell-k-1)\left(k-\frac{1}{2}\right)\right].$$

Hence, the condition $n+1+2\ell < \frac{(k+1)^2}{k^2}(n+2)$ leads to

$$n+2 > k^2 + (\ell-k-1)\left(k-\frac{1}{2}\right). \tag{5.2}$$

We conclude from Inequality (5.2) that $n+2 \geq k^2 + (\ell-k-1)\left(k-\frac{1}{2}\right) + \frac{1}{2}$. Substituting $\ell = k+2+j$ with $j \geq 0$ and $n = k^2 + (\ell-k-1)\left(k-\frac{1}{2}\right) - \frac{3}{2}$ we obtain

$$\frac{2}{k} - \frac{\ell}{n+2} - \frac{\ell}{n+2\ell} = \frac{1}{2}\frac{2j(k^2-2)+j^2(2k-3)}{k(n+2)(n+2\ell)}.$$

Since the right-hand side is nonnegative for $j \geq 0$ and $k \geq 2$ we established Inequality (5.1) for $k \geq 2$ and $\ell \geq k+2$. ∎

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