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The use of Pauli-Villars' regularization in string theory

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Abstract: The proper-time regularization of bosonic string reproduces the results of canonical quantization in a special scaling limit where the length in target space has to be renormalized. We repeat the analysis for the Pauli-Villars regularization and demonstrate the universality of the results. In the mean-field approximation we compute the susceptibility anomalous dimension and show it equals 1/2. We discuss the relation with the previously known results on lattice strings.

Contents

1 Introduction

Regularization plays an important role in quantum string theory. In the early days people mainly used the regularization by a cut-over-modes in the mode expansion, which was intimately linked to canonical quantization. It was already then recognized that the regularization is to be done in a covariant way to comply with diffeomorphism invariance. A beautiful example of why this is important is the Brink-Nielsen computation [1] of the energy due to zero-point string fluctuations, which contributes both to the string tension and to the lowest mass. The results for the lowest (tachyonic) mass are reproduced also by the zeta-function regularization which has no dimensionful cut-off.

The modern path-integral string quantization [2] is naturally associated with the proper-time regularization, as the Seeley expansion of the matrix element of the heat kernel operator [3–6] is used for the computation of the determinants. It was believed for the long time that the proper-time regularization gives the same string spectrum as canonical quantization and the zeta-function regularization.

We have recently showed [7–9] that this is indeed the case, but only if one renormalizes length scales in target space. Using the path integral formulation of string theory, this is to be expected from a field theoretical point of view since the target space variables X^μ are treated as ordinary quantum fields living on the two-dimensional world sheet of the string. Generically, one would expect such fields to receive a wave function renormalization

$$Z^{1/2} X_R^\mu = X^\mu, \tag{1.1}$$

and this is precisely what was needed when we performed a mean-field calculation in bosonic string theory. From the point of view of perturbation theory such a mean-field calculation involves a certain summation to all orders of α'_0, and a corresponding renormalization of α'_0:

$$Z\alpha'_R = \alpha'_0, \quad Z = 1 - c_1\alpha'_0\Lambda^2 + \dots \tag{1.2}$$

Equation (1.1) – (1.2) is just a standard perturbative expansion around $\alpha'_0 = 0$, telling us that since the only coupling constant in the theory, α'_0, is dimensionful, the perturbative expansion is in $\alpha'_0\Lambda^2$ while target space length X is naturally measured in units of $1/\sqrt{\alpha'_0}$. However, as shown in [7–9], since the proper-time regularization provides with a diffeomorphism invariant cut-off Λ and since we can calculate Z exactly, the condition $0 \leq Z \leq 1$ forces us to have $\alpha'_0 < c'_1/\Lambda^2$. If we insist that α'_R is finite when $\Lambda \to \infty$, then $Z \to 0$ in that limit, and from the explicit expression for Z we obtained $Z \propto 1/\alpha_R\Lambda^2$. This implies that if X_R is finite in the limit $\Lambda \to \infty$, X will be of the order of $1/\Lambda$.

The fact that the bare quantities α'_0 and X becomes singular in a limit where the renormalized quantities are kept finite and where the cut-off is taken to infinity should not come as a surprise from a field theoretical point of view. However, there are few intriguing points associated with this in relation to string theory.

Firstly we have used the proper-time regularization in our calculation. It had build in an explicit diffeomorphism invariant cut-off Λ. In principle one could use a hypercubic lattice in target space as a regularization, the random plaquette surfaces being the string world sheets which appear in the path integral. One is allowed to use such a regularization, as well as any other regularization. It has the virtue that the Nambu-Goto action is simply the number of plaquettes in random surface multiplied by a^2/α'_0. The lattice spacing a plays the role of $1/\Lambda$, but contrary to the proper-time regularization the cut-off refers to distances in target space. Conceptually this is nice since the world sheet is not a physical quantity which can be measured, but it makes it difficult to understand a relation like (1.1).

To be explicit let us consider the two-point function, i.e. the sum over all random surfaces in the path integral formalism, where two points are fixed in target space and the random surfaces pass through these points. The standard way to define a continuum theory from a lattice theory is to fix a distance $|X|$ in target space, $|X| = n \cdot a$, then let $a \to 0$, $n \to \infty$ such that $|X|$ is fixed, and then investigate how one should scale the dimensionless lattice coupling constants such that the two-point function $G(|X|)$ has a sensible limit for $a \to 0$. For the lattice string theory the dimensionless coupling constant is $\mu = a^2/\alpha'_0$ and it is indeed possible to show [10] that one can renormalize μ much like in (1.2) such that $G(|X|)$ exists when $a \to 0$. As shown in [7–9] such a scaling can also be done in the context of our mean field theory. What is absent is the rescaling (1.1) of X. Now X *is* the physical continuum distance and the consequence is that the two-point function $G(|X|)$

is different from the string two-point function. It is rather an ordinary particle two-point function [10]. We called this continuum world, coming from the regularized lattice string theory, "Gulliver's world". This world is in contrast to the string world where a scaling (1.1) of X takes place. In Gulliver's world $X = n \cdot a$ has the extension of (infinitely) many lattice spacings when $a \to 0$, but the scaling (1.1) where X_R is finite brings X back to be of order a: the string world becomes a "Lilliputian world" with the extension of a few lattice spacings, seen from Gulliver's perspective.

Secondly, even from a purely stringy point of view there is something surprising about the renormalization (1.1). Let us consider the N-point function for a closed string

$$G(x_1, \ldots, x_N) = \int \mathcal{D}X(\omega)\, \mathrm{e}^{-S[X(\omega)]} \prod_{i=1}^{N} \int d^2\omega_i\, \delta(X(\omega_i) - x_i). \tag{1.3}$$

For $N = 2$ it is the one discussed in the previous paragraph. While x_1, x_2 in ordinary quantum field theory just refer to spacetime coordinates and $G(x_1, x_2)$ becomes a function of $x_1 - x_2$, in the string case they are promoted to background fields, which like the quantum fields need a wavefunction renormalization. Thus we have the situation where the distances $|x_1 - x_2|$, on which $G(x_1, , x_2)$ depends, becomes a function of the cut-off of the theory, and in fact a singular function as our calculations in [7–9] explicitly showed. Usually one performs a Fourier transformation of $G(x_1, \ldots, x_N)$ in order to replace the δ-functions in (1.3) with vertex operators, but even then the renormalization (1.1) should be implemented on the corresponding momenta P_μ as

$$P_R = Z^{1/2} P \tag{1.4}$$

in order to obtain the standard string scattering amplitude, as described in detail in [9]. The string limit is thus much more delicate than the field theoretical limit. Again, this is of course not a surprise since the definition (1.3) is in principle an off-shell definition of the N-point function of string theory, and off-shell definitions are known to be problematic.

The intriguing points mentioned above arise because we have used a regularization with an explicit, dimensionful cut-off. In order to ensure that the results are not an artifact of the proper-time regularization, we have repeated the calculations using another regularization with an explicit, diffeomorphism invariant cut-off, namely the Pauli-Villars regularization. In addition we calculate the string susceptibility exponent for our partition function. There are several advantages of using the Pauli-Villars regularization. First of all and probably the most important one is that the dependence on the cut-off then becomes explicit which allows us to use standard techniques of quantum field theory, in particular the Schwinger-Dyson equations. Another one is that the determinants involved can be exactly computed for certain metrics, including those for which the standard results based on the Seeley expansion are not applicable.

We consider the Nambu-Goto string formulation, where the intrinsic world-sheet metric and the induced metric are treated independently by introducing a Lagrange multiplier. They are generically different, so only their quantum averages coincide. We perform the path integral over target-space coordinates to obtain the effective action for the intrinsic metric and the Lagrange multiplier, whose minimum determines their values in the mean-field approximation which becomes exact at large number of target-space dimensions d. Fluctuations around the mean-field values are also governed by the effective action that can easily be computed to quadratic order in the fluctuations. Not surprisingly, the critical dimension $d = 26$ plays then a crucial role. For $d < 26$ the mean field is stable under quantum fluctuations to quadratic order. For $d > 26$ it is also stable before the scaling limit is taken and in Gulliver's scaling limit, but for $d > 26$ the effective action is no longer positive definite in the Lilliputian scaling limit. This may be associated with the presence of negative-norm states for $d > 26$.

In Sect. 2 we formulate our setup. In Sect. 3 we introduce the Pauli-Villars regularization of the bosonic string and compute the effective action to quadratic order in the fluctuations around the mean-field values. In Sect. 4 we demonstrate that this action is positive for $2 \le d < 26$ and thus the mean-field vacuum is stable under fluctuations. In Sect. 5 we show how an analog of the Seeley expansion of the heat kernel looks for the Pauli-Villars regularization. In Sect. 6 we use the standard technique of quantum field theory to calculate an "effective potential" and then demonstrate an instability of the classical vacuum. We also compute the string susceptibility exponent and obtain $\gamma_{\rm str} = 1/2$ in the mean-field approximation. Sect. 7 is devoted to a discussion of the obtained results and some speculations. In Appendix A we consider a more general Pauli-Villars regularization of determinants and demonstrate the universality of the results. In Appendix B we use the Gel'fand-Yaglom technique to compute the determinants exactly for certain metrics and compare with the results based on the Seeley expansion.

2 The setup

Let us consider a closed bosonic string in a target space with one compactified dimension of length β, whose world sheet wraps once around this compactified dimension. There is no tachyon with this setup if β is larger than a certain value of the order of the cut-off. The Nambu-Goto action is given by the area of the embedded surface. We rewrite it, using a Lagrange multiplier λ^{ab} and an independent intrinsic metric ρ_{ab}, as[1]

$$\begin{aligned} S &= K_0 \int d^2\omega \sqrt{\det \partial_a X \cdot \partial_b X} = K_0 \int d^2\omega \sqrt{\det \rho} + \frac{K_0}{2} \int d^2\omega\, \lambda^{ab} \left(\partial_a X \cdot \partial_b X - \rho_{ab}\right), \\ K_0 &= \frac{1}{2\pi\alpha_0'}. \end{aligned} \tag{2.1}$$

It is convenient to choose the world-sheet coordinates ω_1 and ω_2 inside an $\omega_L \times \omega_\beta$ rectangle in the parameter space. Then the classical solution $X^\mu_{\rm cl}$ minimizing the action (2.1) depends on ω linearly while the classical induced metric is ω-independent.

Using the path-integral quantization, we integrate over the quantum fluctuations of the X-fields by splitting $X^\mu = X^\mu_{\rm cl} + X^\mu_{\rm q}$ and then performing the Gaussian path integral

[1] We denote $\det \rho = \det \rho_{ab}$ and $\det \lambda = \det \lambda^{ab}$.

over $X_{\rm q}^\mu$. We thus obtain the effective action, governing the fields λ^{ab} and ρ_{ab},

$$\begin{aligned} S_{\rm eff} &= K_0 \int {\rm d}^2\omega \sqrt{\det \rho} + \frac{K_0}{2} \int {\rm d}^2\omega\, \lambda^{ab} \left(\partial_a X_{\rm cl} \cdot \partial_b X_{\rm cl} - \rho_{ab}\right) + \frac{d}{2} {\rm tr} \log(-\mathcal{O}), \\ \mathcal{O} &:= \frac{1}{\sqrt{\det \rho}} \partial_a \lambda^{ab} \partial_b. \end{aligned} \tag{2.2}$$

The operator $\mathcal{O}$ reproduces the usual two-dimensional Laplacian for $\lambda^{ab} = \rho^{ab}\sqrt{\det \rho}$. Quantum observables are given by the path integral over the fields λ^{ab} and ρ_{ab}. It runs over the functions $\lambda^{ab}(\omega)$ and $\rho_{ab}(\omega)$ taking on imaginary and real values, respectively.

A very important property of the quantum system with the action (2.2) first pointed out in [11] is that the field λ^{ab} does *not* propagate and is localized at the value

$$\bar\lambda^{ab} = C \rho^{ab} \sqrt{\det \rho}, \tag{2.3}$$

where C is constant for the world-sheet parametrization we use. We can thus rewrite the right-hand side of Eq. (2.1) as

$$S = K_0 (1-C) \int {\rm d}^2\omega \sqrt{\det \rho} + \frac{K_0 C}{2} \int {\rm d}^2\omega \sqrt{\det \rho}\, \rho^{ab} \partial_a X \cdot \partial_b X, \tag{2.4}$$

which reproduces the Polyakov string formulation [2] for $C = 1$. As shown in [7] the action (2.4) is consistent only for a certain value of C which is regularization-dependent. One has $C = 1$ for the zeta-function regularization but $C < 1$ for the proper-time regularization where

$$C = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{d\Lambda^2}{2K_0}} \tag{2.5}$$

as $d \to \infty$.

Instead of the proper-time regularization used in [7, 8] we can consider a regularization of the Pauli-Villars type, introducing the ratio of massless to massive determinants

$$\mathcal{R} \equiv \frac{\det(-\mathcal{O}) \det(-\mathcal{O} + 2M^2)}{\det(-\mathcal{O} + M^2)^2}, \tag{2.6}$$

when

$${\rm tr} \log \mathcal{R} = -\int_0^\infty \frac{{\rm d}\tau}{\tau}\, {\rm tr}\, {\rm e}^{\tau \mathcal{O}} \left(1 - {\rm e}^{-\tau M^2}\right)^2 \tag{2.7}$$

is convergent. Here $M \to \infty$ is the regulator mass

We have added in (2.6) the additional ratio of the determinants for the masses $\sqrt{2}M$ and M to cancel the logarithmic divergence at small τ, because the Seeley expansion

$$\left\langle \omega \middle| {\rm e}^{\tau \mathcal{O}} \middle| \omega \right\rangle = \frac{1}{4\pi\tau} \frac{\sqrt{\det \rho}}{\sqrt{\det \lambda}} + \frac{R}{24\pi} + \ldots \tag{2.8}$$

starts with the term proportional to $1/\tau$. This is specific to the two-dimensional case. In Appendix A we consider a more general ratio of the determinants applicable in multi-dimensional cases as well and demonstrate universality of the results.

A nice feature of the ratio (2.6) is that for some metrics, depending on only one variable, it can be exactly computed using the Gel'fand–Yaglom technique as is described in Appendix (B). The results are compared with the ones based on the Seeley expansion (2.8) to understand when this expansion works.

It is convenient (but not necessary) to fix the conformal gauge when $\rho_{ab} = \rho\delta_{ab}$, so that $\sqrt{\det \rho} = \rho$. Then the log of the determinant of the ghost operator

$$\mathcal{O}^a_b = \left[\Delta - \frac{1}{2}(\Delta \log \rho)\right] \delta^a_b \tag{2.9}$$

is to be added to the effective action (2.2) [or (2.4)]. Equation (2.3) in the conformal gauge reduces to

$$\bar{\lambda}^{ab} = C\delta^{ab} \tag{2.10}$$

because $\rho^{ab}\sqrt{\det \rho} = \delta^{ab}$ in the conformal gauge.

A subtlety with the computation of the determinants involved in the conformal gauge is now immediately seen: the fields X^μ and ρ do not interact in the action (2.4) since

$$S = K_0(1 - C) \int \mathrm{d}^2\omega\, \rho + \frac{K_0 C}{2} \int \mathrm{d}^2\omega\, \delta^{ab}\partial_a X \cdot \partial_b X \tag{2.11}$$

in the conformal gauge. But the dependence of the determinants on ρ appears because the world-sheet regularization

$$\varepsilon = \frac{1}{\Lambda^2\sqrt{\det \rho}} = \frac{1}{\Lambda^2 \rho} \tag{2.12}$$

depends on ρ owing to diffeomorphism invariance. For smooth ρ the determinants are given by the conformal anomaly [2]. An advantage of using the Pauli-Villars regularization in the conformal gauge is that the implicit dependence on the metric becomes explicit as we shall immediately see.

3 Computation with the Pauli-Villars regularization

Let us repeat the computation of the effective action of the Nambu-Goto string to quadratic order in fluctuations for the Pauli-Villars regularization, where M in Eq. (2.6) plays the role of a regulator mass. The ratio in Eq. (2.6) can be rewritten in the conformal gauge as

$$\mathcal{R} = \frac{\det\left(-\partial_a \lambda^{ab} \partial_b\right) \det\left(-\partial_a \lambda^{ab} \partial_b + 2M^2\rho\right)}{\det\left(-\partial_a \lambda^{ab} \partial_b + M^2\rho\right)^2}, \tag{3.1}$$

which is analogous to that for a quantum-mechanical problem in flat space with the potential $V = M^2\rho$. It is important that this ratio is finite at finite M and we do not have to take care of a cut-off.

For $L \gg \beta$ we can replace one summation over modes by an integration and use Plana's summation formula for the other sum over the modes. The finite part (the Lüscher term) then comes as the difference between the latter sum and the integral as is demonstrated in Appendix B, while the divergent as $M \to \infty$ part for constant $\rho = \bar{\rho}$ and $\lambda = \bar{\lambda}$ reads

$$[\log \mathcal{R}]_{\mathrm{div}} = \omega_\beta \omega_L \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \log\left[\frac{\bar{\lambda}k^2\left(\bar{\lambda}k^2 + 2M^2\bar{\rho}\right)}{\left(\bar{\lambda}k^2 + M^2\bar{\rho}\right)^2}\right] = -M^2 \frac{\omega_\beta \omega_L \bar{\rho}}{2\pi\bar{\lambda}} \log 2. \tag{3.2}$$

It is the same as that for the proper-time regularization with

$$\Lambda^2 = \frac{M^2}{2\pi}\log 2. \tag{3.3}$$

To compute the effective action to quadratic order, we expand

$$\rho = \bar\rho + \delta\rho, \quad \lambda^{ab} = \bar\lambda\delta^{ab} + \delta\lambda^{ab}. \tag{3.4}$$

Every determinant can be written as the path integral

$$\det\left(-\partial_a\lambda^{ab}\partial_b + M^2\rho\right)^{-d/2} = \int \mathcal{D}X_M^\mu \mathrm{e}^{-\frac{1}{2}\int \mathrm{d}^2\omega\,\left(\lambda^{ab}\partial_a X_M\cdot\partial_b X_M + M^2\rho X_M\cdot X_M\right)} \tag{3.5}$$

over the fields $X_M^\mu(\omega)$ with normal statistics or $Y_M^\mu(\omega)$ with ghost statistics. This generates the propagator of the X_M field

$$\left\langle X_M^\mu(k)X_M^\nu(-k)\right\rangle = \frac{\delta^{\mu\nu}}{\bar\lambda k^2 + M^2\bar\rho} \tag{3.6}$$

while the two triple vertices of the $\delta\lambda^{ab}X^\mu X^\nu$ and $\delta\rho X^\mu X^\nu$ interactions are

$$\begin{aligned} \left\langle \delta\lambda^{ab}(-p)X_M^\mu(k+p)X_M^\nu(-k)\right\rangle_{\text{truncated}} &= -(k+p)^a k^b\delta^{\mu\nu}, \\ \left\langle \delta\rho(-p)X_M^\mu(k+p)X_M^\nu(-k)\right\rangle_{\text{truncated}} &= -M^2\delta^{\mu\nu}. \end{aligned} \tag{3.7}$$

The latter vanishes for $M=0$ as it should owing to conformal invariance.

For the $\delta\rho\delta\rho$, $\delta\rho\delta\lambda$ and $\delta\lambda\delta\lambda$ terms in the effective action we find, respectively,

$$-\frac{d}{2}\times\frac{\delta\rho(p)\delta\rho(-p)}{2}\int\frac{d^2k}{(2\pi)^2}\frac{M^4}{(\bar\lambda k^2+M^2\bar\rho)[\bar\lambda(k+p)^2+M^2\bar\rho]}, \tag{3.8}$$

$$-\frac{d}{2}\times\delta\lambda^{ab}(p)\delta\rho(-p)\int\frac{d^2k}{(2\pi)^2}\frac{M^2(k+p)_a k_b}{(\bar\lambda k^2+M^2\bar\rho)[\bar\lambda(k+p)^2+M^2\bar\rho]}, \tag{3.9}$$

$$-\frac{d}{2}\times\frac{\delta\lambda^{ab}(p)\delta\lambda^{cd}(-p)}{2}\int\frac{d^2k}{(2\pi)^2}\frac{(k+p)_a k_b(k+p)_c k_d}{(\bar\lambda k^2+M^2\bar\rho)[\bar\lambda(k+p)^2+M^2\bar\rho]}. \tag{3.10}$$

Let us first consider the M^2-term in the ratio (3.1), which is divergent as $M\to\infty$. For $\delta\lambda^{ab} = \delta\lambda\delta^{ab}$ we find, respectively,

$$\frac{1}{2}\int\frac{d^2k}{(2\pi)^2}\left[\frac{4M^4}{(\bar\lambda k^2+2M^2\bar\rho)^2} - \frac{2M^4}{(\bar\lambda k^2+M^2\bar\rho)^2}\right] = 0, \tag{3.11}$$

$$\int\frac{d^2k}{(2\pi)^2}\,k^2\left[\frac{2M^2}{(\bar\lambda k^2+2M^2\bar\rho)^2} - \frac{2M^2}{(\bar\lambda k^2+M^2\bar\rho)^2}\right] = -\frac{\Lambda^2}{\bar\lambda^2}, \tag{3.12}$$

$$\frac{1}{2}\int\frac{d^2k}{(2\pi)^2}\,(k^2)^2\left[\frac{1}{(\bar\lambda k^2)^2} + \frac{1}{(\bar\lambda k^2+2M^2\bar\rho)^2} - \frac{2}{(\bar\lambda k^2+M^2\bar\rho)^2}\right] = \frac{\Lambda^2\bar\rho}{\bar\lambda^3}, \tag{3.13}$$

reproducing the expansion of

$$-\frac{d}{2}\Lambda^2\int \mathrm{d}^2\omega\frac{\rho(\omega)}{\lambda(\omega)}. \tag{3.14}$$

Let us now consider the terms $\mathcal{O}(p^2)$ – the only which survive as $M \to \infty$. It is convenient first to integrate (3.8), (3.9) and (3.10) over the angle between the vectors p_a and k_b, then to expand in $1/M$ and finally to integrate over k^2. For (3.8) we find

$$-\frac{d}{96\pi\bar{\rho}^2}\int \mathrm{d}^2\omega\,(\partial_a\delta\rho)^2 \tag{3.15}$$

which coincides with the standard conformal anomaly

$$-\frac{d}{96\pi}\int \mathrm{d}^2\omega\,(\partial_a\log\rho)^2 \tag{3.16}$$

to quadratic order in $\delta\rho$. Analogously, we obtain from (3.9)

$$-\frac{d}{24\pi\bar{\lambda}\bar{\rho}}\int \mathrm{d}^2\omega\,(\partial_a\delta\rho)(\partial_a\delta\lambda), \tag{3.17}$$

which looks like quadratic order of the anomaly

$$-\frac{d}{24\pi}\int \mathrm{d}^2\omega\,(\partial_a\log\rho)(\partial_a\log\lambda), \tag{3.18}$$

Finally, for the final part of (3.10) we find

$$\int\frac{\mathrm{d}^2p}{(2\pi)^2}\,\delta\lambda(p)\delta\lambda(-p)\frac{p^2d\left[(5-\log 8)+3\log\frac{M^2\bar{\rho}}{p^2\bar{\lambda}}\right]}{96\pi\bar{\lambda}^2}, \tag{3.19}$$

where we have assumed that $p \gg 2\pi/\omega_\beta$. Notice this term is normal rather than anomalous (i.e. regularization dependent).

The computation of the determinant of the ghost operator (2.9) is similar. It gives only the term $(\delta\rho)^2$

$$\frac{13}{48\pi\bar{\rho}^2}\int \mathrm{d}^2\omega\,(\partial_a\delta\rho)^2. \tag{3.20}$$

Combining Eqs. (3.16), (3.18), (3.19) and (3.20), we find for the effective action to quadratic order in fluctuations

$$\begin{aligned}\delta S =& -\left(K_0-\frac{d\Lambda^2}{2\bar{\lambda}^2}\right)\int \mathrm{d}^2\omega\,\delta\rho\delta\lambda-\frac{d\Lambda^2\bar{\rho}}{2\bar{\lambda}^3}\int \mathrm{d}^2\omega\,(\delta\lambda)^2+\frac{(26-d)}{96\pi\bar{\rho}^2}\int \mathrm{d}^2\omega\,(\partial_a\delta\rho)^2\\ &-\frac{d}{24\pi\bar{\lambda}\bar{\rho}}\int \mathrm{d}^2\omega\,(\partial_a\delta\rho)(\partial_a\delta\lambda)+\int\frac{\mathrm{d}^2p}{(2\pi)^2}\,\delta\lambda(p)\delta\lambda(-p)\frac{p^2d\log\left(\frac{M^2\bar{\rho}}{cp^2\bar{\lambda}}\right)}{32\pi\bar{\lambda}^2},\end{aligned} \tag{3.21}$$

where c is fixed by Eq. (3.19). This reproduces the result [8] for the proper-time regularization, except for the constant c in the last term which is regularization dependent.

Applying to (3.21) the variational derivative $-\rho(\omega)\delta/\delta\rho(\omega)$, we reproduce the Seeley expansion (2.8) in the conformal gauge:

$$\left\langle\omega\middle|\mathrm{e}^{\tau\rho^{-1}\partial_a\lambda^{ab}\partial_b}\middle|\omega\right\rangle=\frac{1}{4\pi\tau}\frac{\rho}{\lambda}+\frac{1}{4\pi}\left[-\frac{1}{6}\partial_a^2\ln\rho-\frac{1}{3}\partial_a^2\ln\lambda-\frac{1}{4}(\partial_a\ln\lambda)^2\right]+\mathcal{O}(\tau), \tag{3.22}$$

including the term $\mathcal{O}(\tau^0)$. The last term on the right-hand side of Eq. (3.21) does not depend on ρ and thus does not contribute to the Seeley expansion.

4 Positivity of the effective action to quadratic order

In the previous Section we have computed the effective action assuming that $\lambda^{ab} = \lambda\delta^{ab}$. To justify this assumption, let us consider the divergent part of the effective action for nondiagonal λ^{ab}

$$
\begin{aligned}
S_{\rm div} &= \int \mathrm{d}^2\omega \left[\frac{K_0}{2} \lambda^{ab} \partial_a X_{\rm cl} \cdot \partial_b X_{\rm cl} + K_0 \rho \left(1 - \frac{1}{2}\lambda^{aa}\right) - \frac{d\Lambda^2}{2} \frac{\rho}{\sqrt{\det \lambda}} + \Lambda^2 \rho \right], \\
&\lambda^{aa} = \lambda^{11} + \lambda^{22}.
\end{aligned} \tag{4.1}
$$

It is easy to verify this formula for constant $\lambda^{ab} = \bar{\lambda}\delta^{ab}$ and $\rho = \bar{\rho}$, when [8]

$$
\bar{\lambda} = C \equiv \frac{1}{2} + \frac{\Lambda^2}{2K_0} + \sqrt{\frac{1}{4}\left(1 + \frac{\Lambda^2}{K_0}\right)^2 - \frac{d\Lambda^2}{2K_0}}, \tag{4.2a}
$$

$$
\bar{\rho} = \frac{L\beta}{\omega_L \omega_\beta} \frac{C}{\left(2C - 1 - \frac{\Lambda^2}{K_0}\right)}, \tag{4.2b}
$$

$$
\omega_\beta = \frac{\omega_L}{L}\beta \tag{4.2c}
$$

at the minimum for $\beta \gg 1/\sqrt{K_0}$.

Expanding to quadratic order

$$
\begin{aligned}
\sqrt{\det(\bar{\lambda}\delta^{ab} + \delta\lambda^{ab})} &= \bar{\lambda} + \frac{1}{2}\delta\lambda^{aa} - \delta\lambda_2 + \mathcal{O}\left((\delta\lambda)^3\right), \\
\delta\lambda_2 &= \frac{1}{8\bar{\lambda}}(\delta\lambda_{11} - \delta\lambda_{22})^2 + \frac{1}{2\bar{\lambda}}(\delta\lambda_{12})^2,
\end{aligned} \tag{4.3}
$$

we find from (4.1) for $\bar{\lambda} = C$

$$
S^{(2)}_{\rm div} = -\frac{d\Lambda^2\bar{\rho}}{2C} \int \mathrm{d}^2\omega\, \delta\lambda_2 - \left(K_0 - \frac{d\Lambda^2}{2C^2}\right) \int \mathrm{d}^2\omega\, \delta\rho \frac{\delta\lambda^{aa}}{2} - \frac{d\Lambda^2\bar{\rho}}{2C^3} \int \mathrm{d}^2\omega \left(\frac{\delta\lambda^{aa}}{2}\right)^2. \tag{4.4}
$$

Because the path integral over λ^{ab} goes parallel to imaginary axis, i.e. $\delta\lambda^{ab}$ is pure imaginary, the exponential of the first term on the right-hand side of Eq. (4.4) (which is always *positive*) plays the role of a functional delta-function as $\Lambda \to \infty$, forcing $\delta\lambda^{ab} = \delta\lambda\, \delta^{ab}$. The last two terms on the right-hand side of Eq. (4.4) then reproduce the first two terms in (3.21).

From Eq. (3.21) for the effective action to the second order in fluctuations we find the following quadratic form:

$$
\delta S_2 = \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \left[A_{\rho\rho} \frac{\delta\rho(p)\delta\rho(-p)}{\bar{\rho}^2} + 2A_{\rho\lambda} \frac{\delta\rho(p)\delta\lambda(-p)}{\bar{\rho}\bar{\lambda}} + A_{\lambda\lambda} \frac{\delta\lambda(p)\delta\lambda(-p)}{\bar{\lambda}^2} \right] \tag{4.5}
$$

with

$$
A_{ij} = \begin{bmatrix} \frac{(26-d)p^2}{96\pi} & -\frac{1}{2}\left(K_0 - \frac{d\Lambda^2}{2C^2}\right)\bar{\rho}C - \frac{dp^2}{48\pi} \\ -\frac{1}{2}\left(K_0 - \frac{d\Lambda^2}{2C^2}\right)\bar{\rho}C - \frac{dp^2}{48\pi} & -A \end{bmatrix}, \tag{4.6}
$$

where

$$A = \frac{d\Lambda^2\bar{\rho}}{2C} + \frac{dp^2}{32\pi}\log(cp^2/\Lambda^2\bar{\rho}). \tag{4.7}$$

For $p^2 \ll \Lambda^2\bar{\rho}$, we can drop the second term on the right-hand side of Eq. (4.7), so A becomes constant. For $p^2 \gtrsim \Lambda^2\bar{\rho}$, A depends on p^2 but remains positive.

Since $\delta\lambda(\omega)$ is pure imaginary, i.e. $\delta\lambda(-p) = -\delta\lambda^*(p)$, we find for the determinant associated with the matrix in Eq. (4.6)

$$D = \left[\frac{1}{2}\left(K_0 - \frac{d\Lambda^2}{2C^2}\right)\bar{\rho}C + \frac{dp^2}{48\pi}\right]^2 + \frac{(26-d)p^2}{96\pi}A. \tag{4.8}$$

For generic

$$K_0 > K_* = \left(d - 1 + \sqrt{d^2 - 2d}\right)\Lambda^2 \tag{4.9}$$

the first term in (4.8) dominates and D is positive. We thus we a stability of the minimum (4.2) with respect to quantum fluctuation

As described in detail in [7, 8] the limit where the cut-off goes to infinity (the so-called scaling limit) is obtained by letting the bare coupling constant K_0 approach K_* in the following way:

$$K_0 \to K_* + \frac{K_R^2}{2\Lambda^2\sqrt{d^2-2d}} \tag{4.10}$$

for $\Lambda \to \infty$, while K_R, the renormalized coupling, is fixed. In this limit we have in addition

$$K_0 - \frac{d\Lambda^2}{2C^2} \to K_R\left(1 + \sqrt{1 - \frac{2}{d}}\right). \tag{4.11}$$

This scaling is valid both in the “Gulliver” scaling limit and in the “Lilliputian” scaling limit. The difference between the two scaling limits is the following: in the Lilliputian scaling limit we scale in addition the external lengths L and β as $1/\Lambda$, such that $\bar{\rho}$ in (4.2b) is finite. This implies that the second term on the right-hand side of Eq. (4.8) dominates. It is positive for $d < 26$ and negative for $d > 26$. The propagator

$$\frac{1}{\bar{\rho}^2}\langle\delta\rho(p)\delta\rho(-p)\rangle = \frac{48\pi}{(26-d)p^2}. \tag{4.12}$$

then becomes negative which may indicate a negative-norm state.

Gulliver's scaling limit is only intended to work for the two-point function. In this limit L and β are not scaled like $1/\Lambda$. However, β is taken to zero as

$$\beta^2 - \frac{\pi(d-2)}{3K_0C} \propto \frac{m^2}{K_0^2} \tag{4.13}$$

with $m \propto \sqrt{K_R}$ being the particle mass. The interpretation is that the β-boundaries are contracted to points, separated by a distance L in target space. In the Gulliver scaling limit we have to take into account the final part of the effective action, given in the mean-field approximation by the Lüscher term

$$S_{\text{fin}} = -\frac{\pi(d-2)}{6}\frac{\omega_L}{\omega_\beta}. \tag{4.14}$$

This changes Eqs. (4.2b) and (4.2c) to

$$\bar{\rho} = \frac{L}{\omega_L \omega_\beta} \frac{\left(\beta^2 - \frac{\pi(d-2)}{6K_0 C}\right)}{\sqrt{\beta^2 - \frac{\pi(d-2)}{3K_0 C}}} \frac{C}{\left(2C - 1 - \frac{\Lambda^2}{K_0}\right)}, \tag{4.15}$$

$$\omega_\beta = \frac{\omega_L}{L} \sqrt{\beta^2 - \frac{\pi(d-2)}{3K_0 C}}. \tag{4.16}$$

Equations (4.2b) and (4.2c) are recovered for $\beta \gg 1/\sqrt{K_0}$. In the Gulliver limit we have from Eq. (4.15) that $\bar{\rho} \sim \Lambda^4$. The determinant (4.8) is then positive for finite p^2, while it changes the sign for $p^2 \sim \bar{\rho} K_R^2/\Lambda^2$ which diverges as Λ^2.

Both in the Gulliver and Lilliputian scaling limits λ stays localized, i.e. $\lambda(\omega) = \bar{\lambda}$. Thus only ρ fluctuates. This is similar to what is described in the book [11].

5 Not-Seeley expansion

The standard computation of the determinant in the proper-time regularization is based on the Seeley expansion of the heat kernel, which emerges after applying the variational derivative $-\delta/\delta \log \rho(\omega)$ to the regularized determinant. An analogous formula for the Pauli-Villars regularization reads

$$-\rho(\omega) \frac{\delta}{\delta \rho(\omega)} \log \mathcal{R} = \left\langle \omega \left| \frac{2M^2 \rho}{-\partial_a \lambda^{ab} \partial_b + M^2 \rho} \right| \omega \right\rangle - \left\langle \omega \left| \frac{2M^2 \rho}{-\partial_a \lambda^{ab} \partial_b + 2M^2 \rho} \right| \omega \right\rangle. \tag{5.1}$$

The operator on the right-hand side of Eq. (5.1) is nothing but the limit of coinciding arguments of the matrix element

$$G(\omega, \omega') = \langle \omega \left| \boldsymbol{G} \right| \omega' \rangle \tag{5.2}$$

of the operator

$$\boldsymbol{G} = \frac{2M^2 \rho}{-\partial_a \lambda^{ab} \partial_b + M^2 \rho} - \frac{2M^2 \rho}{-\partial_a \lambda^{ab} \partial_b + 2M^2 \rho}. \tag{5.3}$$

The role of this operator is to provide a regularization of the products of operators:

$$\boldsymbol{AB} \longrightarrow \boldsymbol{AGB}. \tag{5.4}$$

Using Pauli-Villars regularization $\boldsymbol{G}$ is given by (5.3), while using the proper-time regularization it is given by the heat kernel.

For $\rho = 1$ and $\lambda^{ab} = \delta^{ab}$ we have from (5.2), (5.3) a smearing of the delta function by the difference of modified Bessel's functions

$$G_0(\omega, 0) = \frac{M^2}{\pi} \left[K_0 \left(M|\omega| \right) - K_0 \left(\sqrt{2} M |\omega| \right) \right] \tag{5.5}$$

which substitutes

$$G_0(\omega, 0) = \Lambda^2 \mathrm{e}^{-\pi \Lambda^2 |\omega|^2} \tag{5.6}$$

which appears when one uses the proper-time regularization.

The result of the Seeley expansion can be repeated for the Pauli-Villars regularization and is given by $G(\omega, \omega)$ as shown in Eq. (5.1). The (quadratically) divergent terms are the same provided Λ^2 and M^2 are related by Eq. (3.3). The computation of the finite term (the conformal anomaly) is pretty much similar to that in Sect. 3. They also coincide.

6 The string susceptibility exponent

To understand the properties of the vacuum, it is instructive to compute an "effective potential", like in the studies of symmetry breaking in quantum field theory. For this purpose we add to the action (2.1) the source term

$$S_{\rm src} = \frac{K_0}{2} \int {\rm d}^2\omega \, j^{ab} \rho_{ab} \tag{6.1}$$

and define the partition function $Z[j]$ in the presence of the source by path integration over the fields. It is clear that in the conformal gauge where $\rho_{ab} = \rho\delta_{ab}$ and for constant $j^{ab} = j\delta^{ab}$ this j is a source for the area

$$A = \int {\rm d}^2\omega \, \rho. \tag{6.2}$$

Introducing the Gibbs free energy

$$W[j] = -\frac{1}{K_0 L \beta} \log Z[j] \tag{6.3}$$

and minimizing $W[j]$ with respect to $\bar{\rho}$ for constant $j^{ab} = j\delta^{ab}$, we obtain

$$1 + j + \frac{\Lambda^2}{K_0} - C - \frac{d\Lambda^2}{2K_0 C} = 0. \tag{6.4}$$

Then the solution is the same as before with C from Eq. (4.2a) changing by

$$C(j) = \frac{1}{2}\left(1 + j + \frac{\Lambda^2}{K_0}\right) + \sqrt{\frac{1}{4}\left(1 + j + \frac{\Lambda^2}{K_0}\right)^2 - \frac{d\Lambda^2}{2K_0}} \tag{6.5}$$

while

$$W[j] = C(j) \tag{6.6}$$

in the mean-field approximation.

From Eqs. (6.5), (6.6) we deduce

$$\bar{\rho}(j) \equiv \frac{\partial W[j]}{\partial j} = \frac{\partial C(j)}{\partial j} = \frac{1}{2} + \frac{1 + j + \frac{\Lambda^2}{K_0}}{\sqrt{\left(1 + j + \frac{\Lambda^2}{K_0}\right)^2 - \frac{2d\Lambda^2}{K_0}}} \tag{6.7}$$

for $\omega_L = L$ and $\omega_\beta = \beta \gg 1\sqrt{K_0}$, reproducing (4.2b) for $j = 0$. This determines

$$C(\bar{\rho}) = \sqrt{\frac{d\Lambda^2}{2K_0}} \sqrt{\frac{\bar{\rho}}{\bar{\rho} - 1}} \tag{6.8}$$

and

$$j(\bar{\rho}) = -1 - \frac{\Lambda^2}{K_0} + \sqrt{\frac{d\Lambda^2}{2K_0}} \frac{(2\bar{\rho} - 1)}{\sqrt{\bar{\rho}(\bar{\rho} - 1)}}. \tag{6.9}$$

The effective potential $\Gamma(\bar{\rho})$ is defined in the standard way by the Legendre transformation

$$\Gamma[\bar{\rho}] \equiv W[j] - \frac{1}{2L\beta} \int \mathrm{d}^2\omega \, j^{ab} \bar{\rho}_{ab}. \tag{6.10}$$

In the mean-field approximation we then obtain

$$\bar{\Gamma}(\bar{\rho}) = C(\bar{\rho}) - j(\bar{\rho})\bar{\rho} = \left(1 + \frac{\Lambda^2}{K_0}\right) \bar{\rho} - \sqrt{\frac{2d\Lambda^2}{K_0} \bar{\rho}(\bar{\rho} - 1)}. \tag{6.11}$$

Note that

$$-\frac{\partial \bar{\Gamma}(\bar{\rho})}{\partial \bar{\rho}} = j(\bar{\rho}) \tag{6.12}$$

with $j(\bar{\rho})$ given by Eq. (6.9) as it should.

Near the classical vacuum when $0 < \bar{\rho} - 1 \ll 1$ the potential (6.11) decreases with increasing $\bar{\rho}$ because the second term on the right-hand side has the negative sign. This demonstrates an instability of the classical vacuum. If $K_0 > K_*$ given by Eq. (4.9), the potential (6.11) increases linearly with $\bar{\rho}$ for large $\bar{\rho}$ and thus has a (stable) minimum at

$$\bar{\rho}(0) = \frac{1}{2} + \frac{1 + \frac{\Lambda^2}{K_0}}{2\sqrt{\left(1 + \frac{\Lambda^2}{K_0}\right)^2 - \frac{2d\Lambda^2}{K_0}}} \tag{6.13}$$

which is the same as (4.2b) for $\beta \gg 1/\sqrt{K_0}$. Near the minimum we have

$$\bar{\Gamma}(\bar{\rho}) = C(0) + \frac{K_0}{2d\Lambda^2} \left[\left(1 + \frac{\Lambda^2}{K_0}\right)^2 - \frac{2d\Lambda^2}{K_0} \right]^{3/2} [\bar{\rho} - \bar{\rho}(0)]^2 + \mathcal{O}\left([\bar{\rho} - \bar{\rho}(0)]^3\right). \tag{6.14}$$

The coefficient in front of the quadratic term is positive for $K_0 > K_*$ which explicitly demonstrates the stability of the minimum.

We can now compute a very interesting physical quantity – the string susceptibility. For this purpose we define the Helmholtz free energy $F(\bar{\rho})$ by the inverse Laplace transformation

$$\mathrm{e}^{-K_0 L\beta F(\bar{\rho})} = \int \mathrm{d}j \, \mathrm{e}^{K_0 L\beta (j\bar{\rho} - W[j])}, \tag{6.15}$$

where the integral runs parallel to the imaginary axis. The meaning of this procedure is a passage from grand canonical to canonical ensemble at fixed area A [12].

In the mean-field approximation we use Eq. (6.6). Then the integrand in (6.15) has an extremum at $j(\bar{\rho})$ given by Eq. (6.9). Expanding about the extremum, we find

$$j\bar{\rho} - C(j) = -\bar{\Gamma}(\bar{\rho}) + \sqrt{\frac{2K_0}{d\Lambda^2}} \left[\bar{\rho}(\bar{\rho} - 1)\right]^{3/2} (\Delta j)^2 . \tag{6.16}$$

The integral over $\Delta j = j - j(\bar{\rho})$ goes along the imaginary axis and thus converges. For $F(\bar{\rho})$ we obtain

$$F(\bar{\rho}) = \bar{\Gamma}(\bar{\rho}) + \frac{3}{4K_0 L\beta} \log\left[\bar{\rho}(\bar{\rho}-1)\right] + \text{const.} \tag{6.17}$$

According to the definition of the string susceptibility index [12], we expect

$$K_0 L\beta F(\bar{\rho}) = \text{regular} + (2-\gamma_{\text{str}}) \log \frac{A}{A_{\min}} \tag{6.18}$$

for $A \gg A_{\min}$. Comparing (6.17), this determines $\gamma_{\text{str}} = 1/2$.

Because the second term on the right-hand side of Eq. (6.17) is subdominant at large d (and therefore in the mean-field approximation), a question arises whether possible $1/d$ (or one-loop) corrections to the effective potential $\Gamma(\bar{\rho})$ may contribute to γ_{str}. As we shall see momentarily, the answer is "no".

It is easy to compute the one-loop correction to the mean-field result (6.11). As is shown in detail in Sect. 4, the only propagating field is $\delta\rho$ which results for $d < 26$ after performing the path integral over $\delta\rho$ in the standard one-loop correction to the effective action

$$\delta S_{\text{eff}} = -\frac{\Lambda^2}{2} \int \mathrm{d}^2\omega\, \bar{\rho}. \tag{6.19}$$

With the given accuracy we can identify $\bar{\rho}$ in this formula with the variational parameter to be minimized, rather than using its saddle-point value. Then the only effect of this additional term is to change $C(j)$, given at the saddle point by Eq. (4.2a), as

$$C_{1\text{loop}}(j) = \frac{1}{2}\left(1 + j + \frac{\Lambda^2}{2K_0}\right) + \sqrt{\frac{1}{4}\left(1 + j + \frac{\Lambda^2}{2K_0}\right)^2 - \frac{d\Lambda^2}{2K_0}} \tag{6.20}$$

and correspondingly Λ^2/K_0 (coming from the ghosts) is substituted by $\Lambda^2/2K_0$ in the above formulas. Notice, this is not just a simple shift $d \to (d-1)$ as one might expected. The critical value K_*, given at the saddle point by Eq. (4.9), is rather changed as

$$K_{*\,1\text{loop}} = d - \frac{1}{2} + \sqrt{d^2 - d}. \tag{6.21}$$

It now makes sense to consider $d > 1$ like for the Polyakov string.

It is clear from this consideration that the one-loop correction contributes only to the regular part of $F(\bar{\rho})$ as is displayed in Eq. (6.18) and does not change the singular part that gives $\gamma_{\text{str}} = 1/2$.

7 Discussion

We have applied the Pauli-Villars regularization to a relativistic string and showed its convenience and efficiency. The results previously obtained with the proper-time regularization are reproduced this way and this demonstrates their universality. In particular, we have shown an instability of the classical vacuum and the stability of the mean-field vacuum for $2 \le d < 26$.

We have computed the string susceptibility exponent in the mean field approximation and obtained the value $\gamma_{\text{str}} = 1/2$. It remarkably coincides with the one for branched polymers which can be obtained within our consideration in Gulliver's scaling limit. The same value of $\gamma_{\text{str}} = 1/2$ applies also to the Lilliputian scaling limit which corresponds to a string.

An interesting question arises as to whether the value of $\gamma_{\text{str}} = 1/2$ remains valid beyond the mean-field approximation. We may speculate this is the case for $2 \leq d < 26$ if fluctuations are described solely by the Liouville action (see Eq. (B.4)) which is quadratic in the fields. But the problem resides, as usual, in a nonlinearity of the measure for path integration over the Liouville field. This issue deserves future investigation.

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A Universality of Pauli-Villars' regularization

Keeping in mind possible applications to higher dimensions (the membranes), let us generalize Eq. (2.7) as

$$\operatorname{tr} \log \mathcal{R} = -\int_0^\infty \frac{\mathrm{d}\tau}{\tau} \operatorname{tr} \mathrm{e}^{\tau \rho^{-1} \partial_a \lambda^{ab} \partial_b} \left(1 - \mathrm{e}^{-\tau M^2}\right)^N \tag{A.1}$$

which corresponds to

$$\operatorname{tr} \log \mathcal{R} = \sum_{n=0}^{N} (-1)^n C_N^n \log \det \left(-\rho^{-1} \partial_a \lambda^{ab} \partial_b + n M^2\right) \tag{A.2}$$

with

$$C_N^n = \frac{N!}{n!(N-n)!} \tag{A.3}$$

being the binomial coefficients. Above we worked out the case of $N = 2$ but physical results should not depend on N.

Repeating (3.11) – (3.13), we get

$$\frac{1}{2} \int \frac{d^2 k}{(2\pi)^2} \sum_{n=1}^{N} (-1)^n C_N^n \frac{n^2 M^4}{(\bar{\lambda} k^2 + n M^2 \bar{\rho})^2} = 0, \tag{A.4}$$

$$\int \frac{d^2 k}{(2\pi)^2} k^2 \sum_{n=1}^{N} (-1)^n C_N^n \frac{n M^2}{(\bar{\lambda} k^2 + n M^2 \bar{\rho})^2} = -\frac{\Lambda^2}{\bar{\lambda}^2}, \tag{A.5}$$

$$\frac{1}{2} \int \frac{d^2 k}{(2\pi)^2} (k^2)^2 \sum_{n=0}^{N} (-1)^n C_N^n \frac{1}{(\bar{\lambda} k^2 + n M^2 \bar{\rho})^2} = \frac{\Lambda^2 \bar{\rho}}{\bar{\lambda}^3}, \tag{A.6}$$

with non-universal (i.e. N-dependent)

$$\Lambda^2 = M^2 \frac{N!}{4\pi} \int_0^\infty \mathrm{d}x \, \frac{x^2 \left[\psi(1+N+x) - \psi(x)\right]}{\Gamma(1+N+x)} \tag{A.7}$$

and

$$\psi(x) = \frac{\mathrm{d}}{\mathrm{d}x} \log \Gamma(x). \tag{A.8}$$

To prove (A.4) we interchange the integral and the sum and rescale $k^2 \to k^2 n$ in each term of the sum. We then have

$$\sum_{n=1}^{N} (-1)^n C_N^n n = 0. \tag{A.9}$$

It is possible only in (A.4) but not in (A.5) and (A.6) where the integral of each term is divergent and only the integral of the sum is convergent.

To compute the p^2-term, we expand

$$\frac{1}{2} \int \frac{d^2k}{(2\pi)^2} \sum_{n=1}^{N} \frac{(-1)^n C_N^n n^2 M^4}{(\bar\lambda k^2 + nM^2\bar\rho)(\bar\lambda(k+p)^2 + nM^2\bar\rho)} = \frac{p^2}{48\pi\bar\rho^2}, \tag{A.10}$$

$$\int \frac{d^2k}{(2\pi)^2} \sum_{n=1}^{N} \frac{(-1)^n C_N^n n M^2 k(k+p)}{(\bar\lambda k^2 + nM^2\bar\rho)(\bar\lambda(k+p)^2 + nM^2\bar\rho)} = -\frac{\Lambda^2}{\bar\lambda^2} + \frac{p^2}{12\pi\bar\rho\bar\lambda}, \tag{A.11}$$

$$\frac{1}{2} \int \frac{d^2k}{(2\pi)^2} \sum_{n=0}^{N} \frac{(-1)^n C_N^n [k(k+p)]^2}{(\bar\lambda k^2 + nM^2\bar\rho)(\bar\lambda(k+p)^2 + nM^2\bar\rho)} = \frac{\Lambda^2\bar\rho}{\bar\lambda^3} - \frac{p^2}{16\pi\bar\lambda^2} \log \frac{cM^2}{p^2} \tag{A.12}$$

to order p^2. It becomes p^2/n after $k^2 \to k^2 n$ and we find

$$\sum_{n=1}^{N} (-1)^n C_N^n = -1 \tag{A.13}$$

both in (A.10) and (A.11). Thus the p^2-terms there are universal (the conformal anomaly). It is not the case for (A.12), where the result is the log plus a non-universal constant

B Application of the Gel'fand–Yaglom technique

The standard results for the (proper-time regularized) determinants of the two-dimensional Laplacian with the Dirichlet boundary conditions are obtained by Seeley's expansion [5, 6]:

$$\operatorname{tr}\,\log\left(-\Delta\right)\Big|_{\text{div}} = -\frac{1}{4\pi}\left\{\Lambda^2 \int_D - \sqrt{\pi}\Lambda \int_{\partial D} + \frac{1}{3}\log\Lambda^2 \left[\int_D \frac{R}{2} + \int_{\partial D} k\right]\right\} \tag{B.1}$$

for the divergent part and

$$\operatorname{tr}\,\log\left(-\Delta\right)\Big|_{\text{fin}} = -\frac{1}{24\pi}\left[\int_D \frac{1}{2}R\phi + \int_{\partial D} k\phi\right] - \frac{1}{4\pi}\int_{\partial D} k \tag{B.2}$$

for the finite part in the conformal gauge $\rho_{ab} = \mathrm{e}^{\phi}\delta_{ab}$. Here

$$k = -\frac{1}{2} n^a \partial_a \phi \tag{B.3}$$

is the geodesic curvature and n^a is the inward normal unit vector.

The action describing dynamics of the Liouville field ϕ in the Polyakov string formulation emerges from path integration over X^μ (and the ghosts) due to ultraviolet divergences regularized by a cut-off. For smooth ϕ its finite bulk part is given by the conformal anomaly

$$S_{\mathrm{L}} = \frac{d-26}{96\pi} \int R\Delta^{-1}R = \frac{26-d}{96\pi} \int d^2\omega \, (\partial_a \phi)^2. \tag{B.4}$$

This formula is applicable for smooth metrics with the curvature

$$R \ll \Lambda^2, \tag{B.5}$$

when the determinants result in the conformal anomaly. However, in the path integral over ϕ we integrate, in particular, over ϕ's for which the inequality (B.5) is not satisfied.

A simplest example of these are discontinuous metrics when R is infinite at the discontinuities, so that (B.5) is not satisfied. For $d > 26$ they will dominate the path integral with the action (B.4) because of the negative sign. This is a disastrous feature of the Liouville action for $d > 26$. It is to be compared with the role plays by discontinuous trajectories in the Brownian motion, where they are suppressed because of the positive sign of the action. Thus a question arises as to whether we can indeed approximate the exact effective action by the conformal anomaly for this kind of metrics.

Below in this Appendix we shall exactly compute the determinants for particular metrics: both for the case of a smooth ϕ where Eq. (B.4) works (Subsect. B.2) and a discontinuous ϕ where Eq. (B.4) does not work (Subsect. B.3), using the Gel'fand-Yaglom technique reviewed in in Subsect. B.1. We shall compare the results with Eqs. (B.1), (B.2) and find an agreement when ϕ is smooth. If ϕ is discontinuous, we shall see an essential difference between an exact result for the determinant and Eq. (B.2).

B.1 The Gel'fand-Yaglom technique

The ratio in Eq. (3.1)

$$\mathcal{R} = \frac{\det\left(-\partial^2\right) \det\left(-\partial^2 + 2M^2\mathrm{e}^{\phi}\right)}{\det\left(-\partial^2 + M^2\mathrm{e}^{\phi}\right)^2}, \tag{B.6}$$

is analogous to that for a quantum-mechanical problem in flat space with the potential $V = M^2\mathrm{e}^{\phi}$. It is important that this ratio is finite and we do not have to take care of a cut-off.

The ratio of the determinants in Eq. (B.6) can be computed for the Dirichlet boundary conditions in some cases by the Gel'fand–Yaglom technique. Let us consider the coordinates on a strip: $x \in [x_0, x_1]$, $\theta \in [0, 2\pi]$, and choose

$$\phi(x,\theta) = \varphi(x) \tag{B.7}$$

that depends only on x. Expanding in modes $\mathrm{e}^{in\theta}$, we can rewrite the ratio of 2d determinants in Eq. (B.6) as a product of the ratios of 1d determinants

$$\mathcal{R}^{(1)} \equiv \frac{\det\left(-\partial^2\right)}{\det\left(-\partial^2 + M^2\mathrm{e}^{\varphi}\right)} = \prod_n \frac{\det\left(-\partial_x^2 + n^2\right)}{\det\left(-\partial_x^2 + n^2 + M^2\mathrm{e}^{\varphi}\right)}. \tag{B.8}$$

The ratio of the 1d determinants on the right-hand side of Eq. (B.8) is then given by the ratio

$$\frac{\det\left(-\partial_x^2 + n^2 + M^2\mathrm{e}^{\varphi}\right)}{\det\left(-\partial_x^2 + n^2\right)} = \frac{\Psi_n(x_1)}{\Psi_n^{\mathrm{free}}(x_1)} \tag{B.9}$$

of the properly normalized solutions ($\Psi_n(x_0) = 0, \Psi_n'(x_0) = 1$) to the Schrödinger equations with zero eigenvalues, while the solution in the free case reads

$$\Psi_n^{\mathrm{free}}(x) = \frac{\sinh[n(x - x_0)]}{n}. \tag{B.10}$$

B.2 Continuous metric

We shall elaborate on the case, when

$$\mathrm{e}^{\varphi} = x \qquad (0 < x_0 \le x \le x_1). \tag{B.11}$$

The metric (B.11) is increasing with x, so we might expect no deviations from the standard results unless x_0 is very small, i.e. when $M^2 x_0$ is no longer large. For the metric (B.11) the curvature $R = x^{-3}$ becomes large as $x_0 \to 0$ violating (B.5). We shall fix the scaling factor of the metric by requiring the area to be equal to 2π. This implies $x_1 = \sqrt{2}$ for (B.11).

The associated solution reads

$$\Psi_n(x) = \frac{2}{3M^{2/3}}\sqrt{\xi}\sqrt{\xi_0}\left[I_{1/3}\left(\frac{2}{3}\xi^{3/2}\right)K_{1/3}\left(\frac{2}{3}\xi_0^{3/2}\right) - K_{1/3}\left(\frac{2}{3}\xi^{3/2}\right)I_{1/3}\left(\frac{2}{3}\xi_0^{3/2}\right)\right]. \tag{B.12}$$

We have used here a representation of the Airy functions through the modified Bessel functions and denoted

$$\xi = \frac{n^2 + M^2 x}{M^{4/3}}, \qquad \xi_0 = \frac{n^2 + M^2 x_0}{M^{4/3}}. \tag{B.13}$$

The usual quadratic divergence of 2d determinants cancels in the ratio on the right-hand side of Eq. (B.8), while the logarithmic divergence is of the form

$$\mathcal{R}^{(1)}\Big|_{\mathrm{div}} = \exp\left[\frac{1}{2\pi}\left(\sum_{n=1}^{\infty}\frac{1}{n} - \gamma_{\mathrm{E}}\right)\int \mathrm{d}^2 z\, M^2 \mathrm{e}^{\phi}\right], \tag{B.14}$$

where the Euler constant γ_{E} emerges because of the difference between the sum and the integral. This logarithmic divergence will cancel out in the ratio (B.6), but is present for the ratio (B.8) as is already mentioned.

The coefficient of the logarithmic divergence in Eq. (B.1) involves the Euler character

$$\chi = \frac{1}{2\pi}\left(\frac{1}{2}\int_D R + \int_{\partial D} k\right). \tag{B.15}$$

For $\phi = \varphi(x)$ we have

$$\frac{1}{4\pi}\int_D R = -\frac{1}{2}\int_{x_0}^{x_1} \mathrm{d}x\, \partial_x^2 \varphi(x) = -\frac{1}{2}\partial_x \varphi(x)\Big|_{x_0}^{x_1} = -\frac{1}{2\pi}\int_{\partial D} k \tag{B.16}$$

so that $\chi = 0$. This means that we deal with an upper half plane for a periodical real axis, which is then conformally mapped onto a strip. Analogously, integrating by parts, we have

$$\frac{1}{2\pi}\left(\frac{1}{2}\int_D R\varphi + \int_{\partial D} k\varphi\right) = \frac{1}{2}\int_{x_0}^{x_1} \mathrm{d}x\, (\partial_x \varphi)^2 . \tag{B.17}$$

Subtracting the logarithmic divergence (B.14), we arrive for the ratio of the determinants at the products

$$\mathcal{R}^{(1)}\Big|_{\text{fin}} = \mathrm{e}^{\gamma_{\mathrm{E}} \int_{x_0}^{x_1} dx\, M^2 e^{\varphi(x)}} \prod_n \frac{n}{\sinh[n(x_1 - x_0)]}\, \mathrm{e}^{-\int_{x_0}^{x_1} dx\, M^2 e^{\varphi(x)}/2n} \Psi_n(x_1) \tag{B.18}$$

which are convergent. For the solution (B.12) this can be explicitly verified by substituting its proper asymptote as $n \to \infty$.

For the solution (B.12) with $M \gg 1$ we obtain

$$\begin{aligned} \mathcal{R} &= \mathrm{e}^{\gamma_{\mathrm{E}} M^2 (x_1^2 - x_0^2)/2} \\ &\times \prod_n \frac{n}{\sinh[n(x_1 - x_0)]} \mathrm{e}^{-M^2(x_1^2 - x_0^2)/4n} \frac{2}{3M^{2/3}} \sqrt{\xi_1}\sqrt{\xi_0}\, I_{1/3}\left(\frac{2}{3}\xi_1^{3/2}\right) K_{1/3}\left(\frac{2}{3}\xi_0^{3/2}\right) . \end{aligned} \tag{B.19}$$

This product can hopefully be evaluated using Plana's summation formula

$$\frac{1}{2} f(0) + \sum_{n=1}^{\infty} f(n) = \int_0^{\infty} \mathrm{d}\omega\, f(\omega) + \imath \int_0^{\infty} \mathrm{d}t\, \frac{f(\imath t) - f(-\imath t)}{\mathrm{e}^{2\pi t} - 1}, \tag{B.20}$$

which holds when $f(z)$ is analytic for $\mathrm{Re}\, z \geq 0$, in particular, at the imaginary axis.

For the solution (B.12) the limits $n \to \infty$ and $M \to \infty$ commute if $x_0 \gg M^{-2/3}$, which is *precisely* when (B.5) is satisfied, and we can substitute the (modified) Bessel functions by their asymptotic expansions

$$\begin{aligned} I_{1/3}\left(\frac{2}{3}\xi^{3/2}\right) &= \sqrt{\frac{3}{4\pi\xi^{3/2}}}\mathrm{e}^{2\xi^{3/2}/3}\left(1 + \frac{5}{48\xi^{3/2}} + \mathcal{O}\left(\xi^{-3}\right)\right), \\ K_{1/3}\left(\frac{2}{3}\xi^{3/2}\right) &= \sqrt{\frac{3\pi}{4\xi^{3/2}}}\mathrm{e}^{-2\xi^{3/2}/3}\left(1 - \frac{5}{48\xi^{3/2}} + \mathcal{O}\left(\xi^{-3}\right)\right). \end{aligned} \tag{B.21}$$

The next terms of the expansions will not effect the $M \to \infty$ limit to be taken after the computation of the product over n by using Eq. (B.20). Analogously, the second integral on the right-hand side of Eq. (B.20) is exponentially suppressed as $M \to \infty$. Inserting the expansion (B.21) into the first integral on the right-hand side of Eq. (B.20), we obtain as $M \to \infty$ for the final part of the product

$$\log \mathcal{R}^{(1)}\Big|_{\text{fin}} = \frac{5}{24}\left(\frac{1}{x_0} - \frac{1}{x_1}\right). \tag{B.22}$$

The same formula obviously holds for the ratio (B.6).

This is to be compared with the value of the Liouville action

$$-\frac{1}{48\pi}\int \mathrm{d}^2 z\,\partial_a\phi\partial_a\phi = -\frac{1}{24}\int_{x_0}^{x_1}\mathrm{d}x\,\partial_x\varphi\partial_x\varphi = -\frac{1}{24}\left(\frac{1}{x_0}-\frac{1}{x_1}\right) \tag{B.23}$$

for $\phi = \log x$. The obtained structure is similar, while the difference of the coefficients is due to the boundary term that reads [see Eq. (4.42) of [6] with $\sigma = \phi/2$]

$$\frac{1}{8\pi}\int \mathrm{d}s\,\mathrm{e}^{\phi/2}n^a\partial_a\phi = \frac{1}{8\pi}\int_0^{2\pi}\mathrm{d}\theta\left[\partial_x\varphi(x)\Big|_{x=x_0} - \partial_x\varphi(x)\Big|_{x=x_1}\right] = \frac{1}{4}\left(\frac{1}{x_0}-\frac{1}{x_1}\right). \tag{B.24}$$

The sum of (B.23) and (B.24) indeed coincides with (B.22), so is that it agrees with the standard result (B.1) and (B.2) when (B.5) is satisfied.

B.3 Discontinuous metric

Let us consider the case when ϕ is constant along ω_2 and has a discontinuity from $\varphi_1 = 0$ to $\varphi_2 > 0$ at a certain value of ω_1. Since

$$\int d^2\omega(\partial_a\phi)^2 \propto \frac{\beta}{\delta}(\varphi_2-\varphi_1)^2 \tag{B.25}$$

is divergent when we vanish smearing δ of the discontinuity, one might think this leads to an instability for $d > 26$. But $\det(-\Delta)$ for such a discontinuous metric is larger than one for constant $\varphi = \varphi_2$, because all eigenvalues are larger. It cannot thus be zero as the Liouville action says.

Using the Gel'fand-Yaglom technique, we can explicitly compute the ratio (B.6) for such a metric, which is constant along the periodic coordinate and discontinuous along another one. Let $\omega_1 \equiv x$ ranges from 0 to L and the metric has a step from e^{φ_1} to e^{φ_2} at a certain intermediate value $x = x_i$. The proper solution reads

$$\Psi_n(x) = \begin{cases} \frac{1}{m_1}\sinh m_1 x & 0 \le x \le x_i \\ \frac{1}{m_2}\cosh(m_1x_i)\sinh[m_2(x-x_i)] & \\ \quad + \frac{1}{m_1}\sinh(m_1x_i)\cosh[m_2(x-x_i)] & x_i \le x \le L \end{cases} \tag{B.26}$$

where we set $\beta = 2\pi$ and

$$m_1 = \sqrt{n^2+\mathrm{e}^{\varphi_1}M^2}, \qquad m_2 = \sqrt{n^2+\mathrm{e}^{\varphi_2}M^2}. \tag{B.27}$$

For $x_i = L/2$ this gives

$$\Psi_n(L) = \frac{1}{2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\sinh\frac{L(m_1+m_2)}{2} + \frac{1}{2}\left(\frac{1}{m_1}-\frac{1}{m_2}\right)\sinh\frac{L(m_1-m_2)}{2}. \tag{B.28}$$

For asymptotically large L and $x_i \sim L$ we get from Eq. (B.26) for the bulk term

$$\ln\frac{\Psi_n(L)}{\Psi_n^{\mathrm{free}}(L)} = (m_2-n)L + (m_1-m_2)x_i \tag{B.29}$$

and

$$\ln \mathcal{R} = \sum_{n=-\infty}^{+\infty} [(n-m_2)L + (m_2 - m_1)x_i]\,. \tag{B.30}$$

To compute the sum, we use Plana's summation formula (B.20), where the second term on the right-hand side describes the difference between the sum and the integral. In our case this term gives the standard result

$$-2L\int_0^\infty \mathrm{d}t \frac{t}{\mathrm{e}^{2\pi t}-1} = -\frac{L}{12} \tag{B.31}$$

modulo exponentially small terms $\sim L\mathrm{e}^{-2\pi M}$, which arise from the domain $t > M$.

The computation of the M-dependent part is based on the integral

$$\int_0^\infty \mathrm{d}x \left(x - 2\sqrt{x^2+A} + \sqrt{x^2+2A}\right) = -\frac{A}{2}\ln 2 \tag{B.32}$$

For the bulk part of the ratio (B.6) it gives

$$\mathrm{tr}\,\ln(-\Delta)|_{\mathrm{reg}} = -\frac{\beta M^2}{4\pi}\ln 2\,[\mathrm{e}^{\varphi_1}x_i + \mathrm{e}^{\varphi_2}(L-x_i)] - \frac{\pi L}{6\beta} \tag{B.33}$$

The singular at $M\to\infty$ part is of the type $M^2\int \mathrm{d}^2\omega\, \mathrm{e}^{\phi}$ as it should.

The boundary term, which may potentially diverge like in Eq. (B.25), comes from the pre-exponential in Eq. (B.26):

$$\begin{aligned} &\sum_n \log\left[\frac{n}{2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\right] \\ &\quad = \sum_n \log\left[\frac{n}{2}\left(\frac{1}{\sqrt{n^2+\beta^2M^2\mathrm{e}^{2\varphi_1}/4\pi^2}} + \frac{1}{\sqrt{n^2+\beta^2M^2\mathrm{e}^{2\varphi_2}/4\pi^2}}\right)\right]. \end{aligned} \tag{B.34}$$

For the part which becomes divergent as $M\to\infty$, we can replace the sum by an integral to obtain the complete elliptic integral of the second kind:

$$\begin{aligned} &\frac{\beta M}{\pi}\int_0^\infty \mathrm{d}x\,\log\left[\frac{1}{2}\left(\frac{1}{\sqrt{1+\mathrm{e}^{2\varphi_1}/x^2}}+\frac{1}{\sqrt{1+\mathrm{e}^{2\varphi_2}/x^2}}\right)\right] \\ &\quad = \frac{\beta M}{\pi}\mathrm{e}^{\varphi_2}\left[2E\left(\sqrt{1-\mathrm{e}^{2(\varphi_1-\varphi_2)}}\right) - \frac{\pi}{2}\left(1+\mathrm{e}^{\varphi_1-\varphi_2}\right)\right]. \end{aligned} \tag{B.35}$$

When the discontinuity vanishes (i.e. $\varphi_2 = \varphi_1$), we find

$$(B.35) \to \frac{\beta M}{2}\mathrm{e}^{\varphi_1} \tag{B.36}$$

which determines the boundary term in $\mathrm{tr}\,\log\mathcal{R}^{(2)}$ to be

$$\left(2-\sqrt{2}\right)\frac{\beta M}{2}\mathrm{e}^{\varphi_1}. \tag{B.37}$$

The sign is positive as it should be for the Derichlet boundary condition.

Equation (B.35) shows how this term is modified for $\varphi_2 > \varphi_1$, but it definitely remains finite as was anticipated by the inequality below Eq. (B.25). Therefore, the reason why the Liouville action was divergent for the discontinuous metric is that the limit of $\Lambda \to \infty$ is *not* interchangeable with the limit of the smearing parameter $\delta \to 0$. In Eq. (B.25) the limit $\Lambda \to \infty$ was taken first, while in Eq. (B.35) the limit $\delta \to 0$ was taken first. Thus there is no divergence for the above discontinuous metric in a regularized theory.

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