ON $L$-PACKETS AND DEPTH FOR $SL_2(K)$ AND ITS INNER FORM

ANNE-MARIE AUBERT, SERGIO MENDES, ROGER PLYMEN, AND MAARTEN SOLLEVELD

Abstract. We consider the group $SL_2(K)$, where $K$ a local non-archimedean field of characteristic two. We prove that the depth of any irreducible representation of $SL_2(K)$ is larger than the depth of the corresponding Langlands parameter, with equality if and only if the L-parameter is essentially tame.

We also work out a classification of all L-packets for $SL_2(K)$ and for its non-split inner form, and we provide explicit formulae for the depths of their L-parameters.

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1. Introduction

Let $K$ be a non-archimedean local field and let $K_s$ be a separable closure of $K$. A central role in the representation theory of reductive $K$-groups is played by the local Langlands correspondence (LLC). It is known to exist in particular for the inner forms of the groups $GL_n(K)$ or $SL_n(K)$, and to preserve interesting arithmetic information, like local L-functions and $\epsilon$-factors.

Another invariant that makes sense on both sides of the LLC is depth. The depth $d(\pi)$ of an irreducible smooth representation $\pi$ of a reductive $p$-adic group $G$ was defined by Moy and Prasad [MoPr] in terms of filtrations $G_{x,r}$ ($r \in \mathbb{R}_{\geq 0}$) of its...
parahoric subgroups $G_x$. The depth of a Langlands parameter $\phi$ is defined to be the smallest number $d(\phi) \geq 0$ such that $\phi$ is trivial on $\text{Gal}(F_s/F)^r$ for all $r > d(\phi)$, where $\text{Gal}(K_s/K)^r$ be the $r$-th ramification subgroup of the absolute Galois group of $K$.

Let $D$ be a division algebra with centre $K$, of dimension $d^2$ over $K$. Then $\text{GL}_m(D)$ is an inner form of $\text{GL}_n(K)$ with $n = dm$. There is a reduced norm map $\text{Nrd}: \text{GL}_m(D) \to K^\times$ and the derived group $\text{SL}_m(D) := \ker(\text{Nrd}: G \to K^\times)$ is an inner form of $\text{SL}_n(K)$. Every inner form of $\text{GL}_n(K)$ or $\text{SL}_n(K)$ is isomorphic to one of this kind. When $n = 2$, the only possibilities for $d$ are 1 or 2, and so the inner forms are, up to isomorphism, $\text{GL}_2(K)$ and $\text{D}_2(K)$, and $\text{SL}_2(K)$ and $\text{SL}_1(D)$.

The LLC for $\text{GL}_m(D)$ preserves the depth, that is, for every smooth irreducible representation $\pi$ of $\text{GL}_m(D)$, we have $d(\pi) = d(\varphi_\pi)$, where $\varphi_\pi$ corresponds to $\pi$ by the LLC [ABPS1, Theorem 2.9].

The situation is different for $\text{SL}_m(D)$. All the irreducible representations in a given $L$-packet $\Pi_\phi$ have the same depth, so the depth is an invariant of the $L$-packet, say $d(\Pi_\phi)$. We have $d(\Pi_\phi) = d(\varphi)$ where $\varphi$ is a lift of $\phi$ which has minimal depth among the lifts of $\phi$, and the following holds:

\[(1) \quad d(\phi) \leq d(\Pi_\phi)\]

for any Langlands parameter $\phi$ for $\text{SL}_m(D)$ [ABPS1, Proposition 3.4 and Corollary 3.4]. Moreover [1] is an equality if $\phi$ is essentially tame, that is, if the image of the wild inertia subgroup $P_K$ of the Weil group $W_K$ of $K$ lies in a maximal torus of $\text{PGL}_m(\mathbb{C})$.

We observe that this notion of essentially tameness is consistent with the usual notion for Langlands parameters for $\text{GL}_n(K)$. Indeed, any lift $\varphi: W_K \to \text{GL}_n(\mathbb{C})$ of $\phi$, is called essentially tame if its restriction to $P_K$ is a direct sum of characters. Clearly $\varphi$ is essentially tame if and only if $\varphi(P_K)$ lies in a maximal torus of $\text{GL}_n(\mathbb{C})$, which in turn is equivalent to $\phi(P_K)$ lying in a maximal torus of $\text{PGL}_n(\mathbb{C})$.

We denote by $t(\varphi)$ the torsion number of $\varphi$, that is, the number of unramified characters $\chi$ of $W_K$ such $\varphi_\chi \approx \varphi$. Then $\phi$ and $\varphi$ are essentially tame if and only if the residual characteristic $p$ of $K$ does not divide $n/t(\varphi)$ [BuiHe2, Appendix].

In this article we take $K$ to be a local non-archimedean field $K$ of characteristic 2. In positive characteristic, $K$ is of the form $K = \mathbb{F}_q((t))$, the field of Laurent series with coefficients in $\mathbb{F}_q$, with $q = 2^f$. This case is particularly interesting because there are countably many quadratic extensions of $\mathbb{F}_q((t))$. These quadratic extensions are parametrised by the cosets in $K/\varphi(K)$ where $\varphi$ is the map, familiar from Artin-Schreier theory, given by $\varphi(X) = X^2 - X$.

We first show that equality holds in [1] only if $\phi$ is essentially tame (i.e., $t(\varphi) = 2$):

**Theorem 1.1.** Let $K$ be a non-archimedean local field of characteristic 2, and let $\pi$ be an irreducible representation of an inner form of $\text{SL}_2(K)$, with Langlands parameter $\phi$. If $\phi$ is not essentially tame then we have $d(\pi) > d(\phi)$.

Let $\varphi$ be a lift of $\phi$ with minimal depth among the lifts of $\phi$. In the proof we distinguish the cases where $\varphi$ is imprimitive, respectively primitive.

An irreducible Langlands parameter $\varphi: W_K \to \text{GL}_2(\mathbb{C})$ is called imprimitive if there exists a separable quadratic extension $L$ of $K$ and a character $\xi$ of $L^\times$ such as...
that $\varphi \simeq \text{ind}^{W_L}_{W_K}(\xi)$. Then the depth of $\varphi$ and $\phi$ may be expressed in terms of that of $\xi$ and $\xi^2$, respectively, as

$$d(\varphi) = \frac{d(\xi) + d(L/K)}{2} \quad \text{and} \quad d(\phi) = \frac{d(\xi^2) + d(L/K)}{2},$$

where $p_{K}^{1+d(L/K)}$ is the relative discriminant of $L/K$. Let $\mathcal{S}(\varphi)$ be the group of characters $\chi$ of $W_K$ such that $\chi \otimes \varphi \simeq \varphi$. As in [BuHe1, 41.4], we call $\varphi$ totally ramified if $\mathcal{S}(\phi)$ does not contain any unramified character. If $\varphi$ is not essentially tame, then it is totally ramified. We check that if this case we have $d(\xi) > d(\xi^2)$, and hence $d(\Pi_\phi) > d(\phi)$.

We obtain in Proposition 3.2 the following characterization of $L$-packets for $\text{SL}_2(K)$ or $\text{SL}_1(D)$: an $L$-packet is a minimal set of irreducible representations from which a stable distribution can be constructed.

Next we give the explicit classification of the $L$-packets for both $\text{SL}_2(K)$ and $\text{SL}_1(D)$.

In particular, to each biquadratic extension $L/K$, there is attached a Langlands parameter $\phi = \phi_{L/K}$, and an $L$-packet $\Pi_\phi$ of cardinality 4. The depth of the parameter $\phi_{L/K}$ depends on the extension $L/K$. More precisely, the numbers $d(\phi)$ depend on the breaks in the upper ramification filtration of the Galois group $\text{Gal}(L/K) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Let $D$ be a central division algebra of dimension 4 over $K$. The parameter $\phi$ is relevant for the inner form $\text{SL}_1(D)$, which admits singleton $L$-packets.

**Theorem 1.2.** Let $L/K$ be a biquadratic extension, let $\phi$ be the Langlands parameter $\phi_{L/K}$. If the highest break in the upper ramification of the Galois group $\text{Gal}(L/K)$ is $t$ then we have $d(\phi) = t$. For every $\pi \in \Pi_\phi(\text{SL}_2(K)) \cup \Pi_\phi(\text{SL}_1(D))$ these integers provide lower bounds:

$$d(\pi) \geq d(\phi).$$

Depending on the extension $L/K$, all the odd numbers $1, 3, 5, 7, \ldots$ are achieved as such breaks.

This contrasts strikingly with the case of $\text{SL}_2(Q_p)$ with $p > 2$. Here there is a unique biquadratic extension $L/K$, and a unique tamely ramified discrete parameter $\phi : \text{Gal}(L/K) \to \text{SO}_3(\mathbb{R})$ of depth zero.

Let $E/K$ be the quadratic extension given by

$$E = K(\varpi^{-1}(\varpi^{-2n-1}))$$

with $\varpi$ a uniformizer and $n = 0, 1, 2, 3, \ldots$ and let $\phi_E$ be the associated $L$-parameter. We prove in Subsection 3.4 that the depth of $\phi_E$ is given by

$$d(\phi_E) = 2n + 1.$$ 

For the $L$-packets considered in this article, the depths $d(\pi)$ can be arbitrarily large.

We have included an Appendix on aspects of the Artin-Schreier theory. This Appendix goes a little further than the exposition in [FeVo, p.146–151] and the article of Dalawat [Da]. We have the occasion to refer to the Appendix at several points in our article.

We thank Chandan Dalawat for a valuable exchange of emails and for bringing the reference [Da] to our attention.
2. Depth of $L$-parameters

The field $K$ possesses a central division algebra $D$ of dimension 4 and, up to isomorphism, only one. The group $D^\times$ is locally profinite and is compact modulo its centre $K^\times$, see [BuHe1] p.325. Let $\text{Nrd}$ denote the reduced norm on $D^\times$. Define $\text{SL}_1(D) = \{x \in D^\times : \text{Nrd}(x) = 1\}$.

Then $\text{SL}_1(D)$ is an inner form of $\text{SL}_2(K)$. The articles [HiSa, ABPS2] finalize the local Langlands correspondence for any inner form of $\text{SL}_n$ over all local fields.

**Depth of an $L$-parameter for $\text{GL}_2(K)$.** Let $W_K$ denote the Weil group of $K$, and let $\Phi(\text{GL}_2(K))$ be the set of $L$-parameters $\varphi : W_K \times \text{SL}_2(\mathbb{C}) \to \text{GL}_2(\mathbb{C})$ for inner forms of $\text{GL}_2(K)$. Let $t$ be a real number, $t \geq 0$, let $\text{Gal}(K_s/K)^t$ be the $t$-th ramification subgroup of the absolute Galois group of $K$. We define

$$\Phi_t(\text{GL}_2(K)) := \{\varphi \in \Phi(\text{GL}_2(K)) : \text{Gal}(K_s/K)^t \subset \ker(\varphi)\}.$$  

Notice that $\Phi_t(x, \text{GL}_2(K)) \subset \Phi_t(\text{GL}_2(K))$, if $t' \leq t$. It is known that the set of $t$'s at which $\text{Gal}(F_s/F)^t$ breaks consists of rational numbers and is discrete [Ser, Chap. IV, §3]. In particular there exists a unique rational number $d(\varphi)$, called the depth of $\varphi$, such that

$$\varphi \notin \Phi_{d(\varphi)}(\text{GL}_2(K)) \quad \text{and} \quad \varphi \in \Phi_t(\text{GL}_2(K)) \quad \text{for any} \quad t > d(\varphi).$$

**Depth of an $L$-parameter for $\text{SL}_2(K)$.** The depth of an $L$-parameter $\phi : W_K \times \text{SL}_2(\mathbb{C}) \to \text{PGL}_2(\mathbb{C})$ for an inner form of $\text{SL}_2(K)$ is defined as:

$$d(\phi) = \inf\{t \in \mathbb{R}_{\geq 0} | \text{Gal}(K_s/K)^t \subset \ker \phi\},$$

where

$$\text{Gal}(K_s/K)^t := \bigcap_{r > t} G^r.$$

Each projective representation $\phi : W_K \to \text{PGL}_2(\mathbb{C})$ lifts to a Galois representation $\varphi : W_K \to \text{GL}_2(\mathbb{C})$.

For any such lift $\varphi$ of $\phi$ we have $\ker(\varphi) \subset \ker \phi$, so

$$d(\varphi) \geq d(\phi).$$

Let $\varphi : W_K \to \text{GL}_2(\mathbb{C})$ be a 2-dimensional irreducible representation of $W_K$, and let $\Sigma(\varphi)$ be the group of characters $\chi$ of $W_K$ such that $\chi \otimes \varphi \simeq \varphi$. Then $\varphi$ is primitive if $\Sigma(\varphi) = \{1\}$, simply imprimitive if $\Sigma(\varphi)$ has order 2, and triply imprimitive if $\Sigma(\varphi)$ has order 4, as in [BuHe1, 41.3]. Comparing determinants, we see that every nontrivial element of $\Sigma(\varphi)$ has order 2.

As in [BuHe1, 41.4], we call $\phi$ and $\varphi$ unramified if $\Sigma(\varphi) \setminus \{1\}$ contains an unramified character, and totally ramified if $\Sigma(\varphi) \setminus \{1\}$ does not contain any unramified character. By definition, a primitive representation is totally ramified. Thus every imprimitive irreducible representation of dimension 2 of $W_K$ which is not totally ramified is essentially tame.

Let $\phi : W_K \times \text{SL}_2(\mathbb{C}) \to \text{PGL}_2(\mathbb{C})$ with trivial restriction to $\text{SL}_2(\mathbb{C})$, and such that $\varphi$ is a lift of $\phi$. If $\varphi$ is essentially tame and has minimal depth among the lifts of $\phi$, then we have $d(\phi) = d(\varphi)$ [ABPS1, Theorem 3.8]. Thus we are reduced to computing the depths of the projective representations of $W_K$ which lift to totally ramified representations.
We recall how the depth of an irreducible representation \((\varphi, V)\) of \(W_K\) can be computed. Put \(E = (K_s)^{\ker \varphi}\), so that \(\phi\) factors through \(\text{Gal}(E/K)\). Let \(g_j\) be the order of the ramification subgroup \(\text{Gal}(E/K)_j\) (in the lower numbering). The Artin conductor \(a(\varphi) = a(V)\) is given by
\[
ad(\varphi) = g_0^{-1} \sum_{j \geq 0} g_j \dim(V^{\text{Gal}(E/K)_j}) \in \mathbb{Z}_{\geq 0}.
\]
(6)

Since \((\varphi, V)\) is irreducible and \(\text{Gal}(E/K)_j\) is normal in \(\text{Gal}(E/K)\), \(V^{\text{Gal}(E/K)_j} = 0\) whenever \(g_j > 1\). Thus (6) simplifies to the formula [GrRe, (1)]:
\[
a(\varphi) = \frac{\dim V}{g_0} \sum_{j \geq 0, g_j > 1} g_j = \dim V + \frac{\dim V}{g_0} \sum_{j \geq 1, g_j > 1} g_j.
\]
(7)

It was shown in [ABPS2, Lemma 4.1] that
\[
d(\varphi) := \begin{cases} a(\varphi) & \text{if } 1_F \subset \ker(\phi), \\ \dim V - 1 & \text{otherwise.} \end{cases}
\]
(8)

Let \(\varphi : W_K \to \text{GL}_2(\mathbb{C})\) be a totally ramified irreducible representation. Let \(\phi : W_K \to \text{PGL}_2(\mathbb{C})\) be its projection. We will show that \(d(\varphi) > d(\phi)\). To this end we may and will assume that \(\varphi\) has minimal depth among the lifts of \(\phi\).

**Theorem 2.1.** Let \(\varphi\) be an irreducible totally ramified representation \(W_K \to \text{GL}_2(\mathbb{C})\), let \(\phi : W_K \to \text{PGL}_2(\mathbb{C})\) be its projection. Then we have
\[
d(\varphi) > d(\phi).
\]

**Proof.** Primitive representations. Let \(\varphi\) be primitive. Put \(E = K_s^{\ker \phi}\) and \(E^+ = K_s^{\ker \varphi}\). By [BuHe1, §42.3] there exists a unique intermediate field \(K \subset L \subset E\) such that \(E/L\) is a wildly ramified biquadratic extension. Then \(\phi(\text{Gal}(E/L))\) is a subgroup of \(\text{PGL}_2(\mathbb{C})\) isomorphic to the Klein four group. Up to conjugacy \(\text{PGL}_2(\mathbb{C})\) has only one such subgroup. After a suitable change of basis, we may assume that it is
\[
D_2 := \{ (\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}), (\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}), (\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}), (\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}) \} \subset \text{PGL}_2(\mathbb{C}).
\]
(9)

The three subextensions of \(E/L\) are conjugate under \(\text{Gal}(E/K)\) because the conjugation action of \(A_4\) on its normal subgroup \(V_4\) of order four is transitive on the nontrivial elements of \(V_4\). Hence there is a unique \(r \in \mathbb{Z}\) such that \(\text{Gal}(E/L)_r = \text{Gal}(E/L)\) and \(\text{Gal}(E/L)_{r+1} = \{1\}\). In section A.2 we will see that \(r\) is odd. We call this \(r\) the **ramification depth** of \(E/L\).

The nontrivial elements of \(\text{Gal}(E/L)\) are the deepest elements of \(\text{Gal}(E/K)\) outside the kernel of \(\phi\), and therefore the depth of \(\phi\) can be expressed in terms of \(r\).

Let us compare this to what happens for the lift \(\varphi\) of \(\phi\). Since \(\text{SL}_2(\mathbb{C}) \to \text{PGL}_2(\mathbb{C})\) is a surjection with kernel of order 2, the preimage of \(\phi(W_K)\) in \(\text{SL}_2(\mathbb{C})\) has order \(2|\phi(W_K)|\). The matrices in (9) do not yet form a group in \(\text{GL}_2(\mathbb{C})\), for that we really need the nontrivial element of \(\ker(\text{SL}_2(\mathbb{C}) \to \text{PGL}_2(\mathbb{C}))\). In other words, \(\text{SL}_2(\mathbb{C})\) contains a unique subgroup of order \(2[E : K]\) which projects onto \(\phi(W_K)\). As \(\varphi\) has minimal depth among the lifts of \(\phi\), \(\varphi(W_K)\) is precisely this subgroup. Thus \(|E^+ : E| = 2\) and \(\text{Gal}(E^+ / K)\) is a nontrivial index two central extension of \(\text{Gal}(E/K)\). In particular \(\text{Gal}(E^+ / L)\) is isomorphic to the quaternion group of order eight.
Choose a subset \( \{w_1 = 1, w_2, w_3, w_4\} \subset \text{Gal}(E^+/L) \) which projects onto \( \text{Gal}(E/L) \). We may assume that the \( \varphi(w_i) \) are ordered as in \([9]\). As \( \ker(\text{GL}_2(\mathbb{C}) \to \text{PGL}_2(\mathbb{C})) \) is central,

\[
[\varphi(w_3), \varphi(w_4)] = \left[ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \text{GL}_2(\mathbb{C}).
\]

Write

\[
(10) \quad z = [w_3, w_4] \in \text{Gal}(E^+/L),
\]

so that \( \varphi(z) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \). It follows from the definition of \( r \) and the condition on \( \varphi \) that

\[
\text{Gal}(E^+/L)_r = \text{Gal}(E^+/L) \text{ and } \text{Gal}(E^+/L)_{r+1} = \text{Gal}(E^+/E).
\]

By \([\text{Ser}], \text{Proposition IV.2.10}\) \( z \in \text{Gal}(E^+/L)_{2r+1} \). Now \( z \notin \ker(\varphi) \) and it lies deeper in \( \text{Gal}(E^+/K) \) than \( w_2, w_3 \) and \( w_4 \). On the other hand, \( z \) does lie in the kernel of \( \phi \), which explains why \( \varphi \) has larger depth than \( \phi \).

In the sequel of this section, we assume that the depth of the element \( z \) defined in \([10]\) is exactly \( 2r + 1 \). This is allowed because, in the above setting, it constitutes the worst possible case for the theorem.

**Octahedral representations.** Let \( \varphi \) be octahedral, that is, it is primitive and \( \phi(W_K) \cong S_4 \). Let \( \text{Ad} \) denote the adjoint representation of \( \text{PGL}_2(\mathbb{C}) \) on \( \mathfrak{sl}_2(\mathbb{C}) = \text{Lie}(\text{PGL}_2(\mathbb{C})) \). Then \( \text{Ad} \circ \phi \) is an irreducible 3-dimensional representation of \( W_K \). Since \( \text{PGL}_2(\mathbb{C}) \) is the adjoint group of \( \mathfrak{sl}_2(\mathbb{C}) \), \( \text{Ad} \circ \phi \) has the same kernel and hence the same depth as \( \phi \).

By \([\text{BuHe1}], \text{Theorem 42.2}\) \( L/K \) is Galois with automorphism group \( S_3 \) and residue degree 2. Thus \( \text{Ad}(\phi(I_K)) \subset \text{Ad}(\phi(W_K)) \) is a normal subgroup of index two, isomorphic to \( A_4 \). As \( L/K \) has tame ramification index 3, the image of the wild inertia subgroup \( \text{P}_K \) under \( \text{Ad} \circ \phi \) equals the image of \( \text{Gal}(E/L) \). By our convention \([9]\) it is \( \text{Ad}(D_2) \). By the definition of \( r \) as the ramification depth of \( E/L \), we have

\[
g_0 = 12, \quad g_1 = \cdots = g_r = 4 \quad \text{and} \quad g_{r+1} = 1
\]

With the formula \([7]\) we find

\[
a(\text{Ad} \circ \phi) = \frac{3}{12} (12 + r \cdot 4) = 3 + r,
\]

and from \([8]\) we conclude that

\[
d(\phi) = d(\text{Ad} \circ \phi) = r/3.
\]

On the other hand, \( \varphi \) is an irreducible two-dimensional representation of \( W_K \), and we must base our calculations on the Galois group of \( E^+/K \). The numbers

\[
g_j = |\text{Gal}(E^+/K)_j| = |\varphi(\text{Gal}(E^+/K)_j)|
\]

can be computed from those for \( \phi \) by means of the twofold covering \( \varphi(W_K) \to \phi(W_K) \). We find

\[
g_0 = 24, \quad g_1 = \cdots = g_r = 8 \quad \text{and} \quad g_{r+1} = \cdots = g_{2r+1} = 2.
\]

Assuming that the depth of \( z \) is precisely \( 2r + 1 \) (see above), we can also say that \( g_{2r+2} = 1 \). Then \([7]\) gives

\[
a(\varphi) = \frac{2}{24} (24 + r \cdot 8 + (r + 1) \cdot 2) = 2 + \frac{5r + 1}{6}.
\]
Now (8) says that
\[ d(\varphi) = \frac{(5r + 1)}{12}. \]
We note that this is strictly larger than \( d(\phi) = \frac{r}{3} \). (As \( a(\phi) \in \mathbb{Z}_{\geq 0} \), we must have \( r - 1 \in 6\mathbb{Z} \). This means that above not all biquadratic extensions can occur.)

**Tetrahedral representations.** Let \( \varphi \) be tetrahedral, that is, it is primitive and \( \phi(W_K) \cong A_4 \). By [BuHe1, Theorem 42.2] \( L/K \) is a cubic Galois extension. It is of prime order, so either it is unramified or it is totally ramified.

First we consider the case that \( L/K \) ramifies totally. Then \( I_K \) surjects onto \( \text{Gal}(E/K) \), so \( \varphi(I_K) = \varphi(W_K) \). This means that within \( I_K \) everything is similar to octahedral representations. The same calculations as above show that
\[ d(\phi) = \frac{r}{3} < d(\varphi) = \frac{(5r + 1)}{12}. \]
Now we look at the case where \( L/K \) is unramified. Then \( \phi(I_K) = \phi(\text{Gal}(E/K)) = D_2 \).

To compute the depth, we replace \( \phi \) by the 3-dimensional representation \( \text{Ad} \circ \phi \) of \( W_K \) on \( \mathfrak{sl}_2(\mathbb{C}) \). With \( r \) as before, \( g_0 = \cdots = g_r = 4 \) and \( g_{r+1} = 1 \). With (7) and (8) we calculate
\[ a(\text{Ad} \circ \varphi) = \frac{3}{4}(r + 1) \cdot 4 = 3(r + 1), \]
\[ d(\phi) = d(\text{Ad} \circ \phi) = \frac{3(r + 1)}{3} - 1 = r. \]
Like in the octahedral case, the numbers \( \text{Gal}(E^+/K)_j \) for \( \varphi \) are related to those for \( \phi \) via the twofold covering \( \text{SL}_2(\mathbb{C}) \to \text{PGL}_2(\mathbb{C}) \). We find
\[ g_0 = \cdots = g_r = 8 \quad \text{and} \quad g_{r+1} = \cdots = g_{2r+1} = 2. \]
Moreover \( g_{2r+2} = 1 \) if we assume that the depth of \( z \) is \( 2r + 1 \). Now (7) says
\[ a(\varphi) = \frac{2}{3}((r + 1) \cdot 2 + (r + 1) \cdot 2) = \frac{5(r + 1)}{2} \in \mathbb{Z}, \]
and from (8) we obtain
\[ d(\varphi) = \frac{5(r + 1)}{2} - 1 = \frac{5r + 4}{4}. \]
Again, this is larger than \( d(\phi) = r \).

**Imprimitive representations.** Consider an imprimitive totally ramified representation \( \varphi : W_K \to \text{GL}_2(\mathbb{C}) \). By [BuHe1, §41.4] there exists a separable totally ramified quadratic extension \( L/K \) and a character \( \xi \) of \( W_L \) such that \( \varphi = \text{ind}_{W_K}^{W_L}(\xi) \). Let \( \mathfrak{p}_K^{1+d(L/K)} \) be the discriminant of \( L/K \). If \( L \cong K[X]/(X^2 + X + b) \), then one deduces from [BuHe1, §41.1] that \( d(L/K) = -\nu_K(b) > 0 \).

From the proof of [BuHe1, Lemma 41.5] one sees that the level of \( \varphi \) equals \( d(\xi) + d(E/F) \). By construction the level of a \( n \)-dimensional irreducible representation of \( W_K \) equals \( n \) times its depth, so
\[ d(\varphi) = (d(\xi) + d(L/K))/2. \]
As before we assume that \( \varphi \) is minimal among the lifts of \( \phi \). Then [BuHe1] [§41.4] says that \( d(\xi) > d(L/K) \), and in particular \( d(\xi) \geq 2 \). Since \( \Gal(K_s/L)^2 \) is a pro-2-group, the image of \( \xi \) in \( \mathbb{C}^\times \) is a subgroup of even order.

Let \( \sigma \) be the nontrivial element of \( \Gal(L/K) \), so that the restriction of \( \varphi \) to \( W_L \) is \( \xi \oplus \sigma(\xi) \). If \( \xi(w) = -1 \), then also \( \xi(\sigma(w)) = -1 \). As \( \xi(W_L) \) is even, this means that \( (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}) \in \phi(W_L) \). We note that, as every \( W_K \setminus W_L \) interchanges \( \xi \) and \( \sigma(\xi) \), the kernel of \( \phi \) equals the kernel of \( \xi \oplus \sigma(\xi) \) composed with the projection \( \GL_2(\mathbb{C}) \rightarrow \PGL_2(\mathbb{C}) \). Thus the kernel of \( \phi \) contains the kernel of \( \varphi \) with index two. More precisely

\[
\ker(\phi) = (\xi \oplus \sigma(\xi))^{-1} \{ (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) , (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}) \} = \xi^{-1} \{ 1, -1 \} = \ker(\xi^2).
\]

By the same argument as above also \( \ker(\ind_{W_L}^{W_K} \xi^2) = \ker(\xi^2) \). Hence \( \phi \) and \( \ind_{W_L}^{W_K} (\xi^2) \) have the same kernel, and in particular the same depth. With (11) we can express it as

\[
(12) \quad d(\phi) = (d(\xi^2) + d(L/K)) / 2.
\]

The depth (or level) of \( \xi \) is the least \( l \) such that \( \xi \) (or rather its composition with the Artin reciprocity isomorphism) is nontrivial on the higher units group \( U_L^l = 1 + p_L^l \subset L^\times \). For \( l > 0 \) the group \( U_L^l/U_L^{l+1} \) has exponent 2, so \( \xi(U_L^{d(\xi)}) = \{ 1, -1 \} \). Consequently \( U_L^{d(\xi)} \subset \ker \xi^2 \) and \( d(\xi^2) < d(\xi) \). Comparing (11) and (12), we get

\[
(13) \quad d(\varphi) - d(\phi) = (d(\xi) - d(\xi^2)) / 2 > 0. \quad \square
\]

3. L-packets

According to a classical result of Shelstad [She, p.200], for \( F \) of characteristic zero all the L-packets \( \Pi_\varphi(\SL_2(F)) \) have cardinality 1, 2 or 4. We will check below, after [15], that the same holds for the L-packets for \( \SL_2(K) \). It will follow from the classification in this section that L-packets for \( \SL_1(D) \) have cardinality 1 or 2.

**Theorem 3.1.** [ABPS] Let \( \phi : W_K \times \SL_2(\mathbb{C}) \rightarrow \PGL_2(\mathbb{C}) \) be an L-parameter for \( \SL_2(K) \), and let \( \varphi : W_K \times \SL_2(\mathbb{C}) \rightarrow \GL_2(\mathbb{C}) \) be a lift of minimal depth. For any \( \pi \) in one of the L-packets \( \Pi_\varphi(\GL_2(K)), \Pi_\varphi(\GL_1(D)), \Pi_\phi(\SL_2(K)) \) and \( \Pi_\phi(\SL_1(D)) \):

\[
d(\phi) \leq d(\varphi) = d(\pi).
\]

Moreover \( d(\phi) = d(\varphi) = d(\pi) \) if \( \varphi \) is essentially tame, in particular whenever \( \varphi \) is unramified.

We define the groups

\[
(14) \quad C(\phi) := Z_{\SL_2(\mathbb{C})}(\im \phi), \quad S_\phi := C(\phi)/C(\phi)^0 = \pi_0(Z_{\SL_2(\mathbb{C})}(\phi)),
\]

\[
Z_\phi := Z(\SL_2(\mathbb{C}))/Z(\SL_2(\mathbb{C})) \cap C(\phi)^0, \quad S_\phi := \pi_0(Z_{\PGL_2(\mathbb{C})}(\phi)).
\]

The group \( S_\phi \) is abelian, \( S_\phi \) can be nonabelian, and there is a short exact sequence

\[
1 \rightarrow Z_\phi \rightarrow \pi_0(Z_{\SL_2(\mathbb{C})}(\phi)) \rightarrow \pi_0(Z_{\PGL_2(\mathbb{C})}(\phi)) \rightarrow 1.
\]
It is easily seen that $|Z_\phi| = 2$ if and only if $\phi$ is relevant for $\text{SL}_1(D)$. By Theorem 3.3 there are bijections

\begin{align}
\text{Irr}(\pi_0(Z_{\text{PGL}_2(C)}(\phi))) &\leftrightarrow \Pi_\phi(\text{SL}_2(K)), \\
\text{Irr}(\pi_0(Z_{\text{SL}_2(C)}(\phi))) &\leftrightarrow \Pi_\phi(\text{SL}_2(K)) \cup \Pi_\phi(\text{SL}_1(D)).
\end{align}

(15)

We remark for $\text{SL}_2(F)$ with $\text{char}(F) = 0$, (15) was shown in [GeKn, Theorem 4.2] and [HiSa, Theorem 12.7]. Recall that $\mathfrak{T}(\varphi)$ is the abelian group of characters $\chi$ of $W_K$ with $\varphi \otimes \chi \cong \varphi$. By [GeKn, Theorem 4.3] and by [ABPS2, (21)]

\begin{align}
\mathfrak{T}(\varphi) \cong \pi_0(Z_{\text{PGL}_2(C)}(\phi)).
\end{align}

(16)

By [Balle1, Proposition 41.3], and by the classification of $L$-parameters for the principal series in Subsection 3.2, $\mathfrak{T}(\varphi)$ has order dividing four. This shows that all $L$-packets for $\text{SL}_2(K)$ have order 1, 2 or 4.

3.1. Stability.

Before we proceed with the classification of $L$-packets, some remarks about the stability of the associated distributions are in order. In this subsection $K$ can be any local non-archimedean field. Recall that a class function on an algebraic $K$-group $G(K)$ is called stable if it is constant on the intersection of any $G(K_s)$-conjugacy class with $G(K)$. For an invariant distribution on $G(K)$ one would like to use a similar definition of stability, but that does not work well in general. Instead, stable distributions are usually defined in terms of stable orbital integrals. But, whenever an invariant distribution $\delta$ on $G(K)$ is represented by a class function on an open dense subset of $G(K)$, we can use the easier criterion for stability of functions to determine whether or not $\delta$ is stable.

Harish-Chandra proved that the trace of an admissible representation is a distribution which is represented by a locally constant function on the set of regular semisimple elements of $G(K)$, see [DBHCS]. So the study the stability of traces of $G(K)$-representations, it suffices to look at (regular) semisimple elements of $G(K)$.

For semisimple elements in $\text{GL}_2(K)$ conjugacy is the same as stable conjugacy, it is determined by characteristic polynomials. Hence every irreducible (admissible) representation of $\text{GL}_2(K)$ defines a stable distribution.

The semisimple conjugacy classes in $\text{GL}_1(D)$ are naturally in bijection with the elliptic conjugacy classes in $\text{GL}_2(K)$, i.e. those semisimple classes for which the characteristic polynomials are irreducible over $K$. Moreover any irreducible essentially square-integrable representation of $\text{GL}_2(K)$ is already determined by the values of its trace on elliptic elements. These observations constitute some of the foundations of the Jacquet–Langlands correspondence [JaLa]. In fact the Jacquet–Langlands correspondence can be defined as the unique bijection between $\text{Irr}(G_1(D))$ and the essentially square-integrable representations in $\text{Irr}(G_2(K))$ which preserves the traces on elliptic conjugacy classes, up to a sign. Consequently the trace of any irreducible representation $\pi$ of $G_1(D)$ is the restriction of a stable distribution on $G_2(K)$ to the set of elliptic elements. In particular the trace of $\pi$ is itself a stable distribution.

Proposition 3.2. Let $\phi$ be a $L$-parameter for $\text{SL}_2(K)$.

(a) Write $\Pi_\phi(\text{SL}_2(K)) = \{\pi_1, \ldots, \pi_m\}$. The trace of $\pi := \pi_1 \oplus \cdots \oplus \pi_m$ is a stable distribution on $\text{SL}_2(K)$. Any other stable distribution that can be obtained from $\Pi_\phi(\text{SL}_2(K))$ is a scalar multiple of the trace of $\pi$. 


(b) Suppose that $\phi$ is relevant for $\text{SL}_1(D)$ and write $\Pi_\phi(\text{SL}_1(D)) = \{\pi'_1, \ldots, \pi'_m\}$. The trace of $\pi' := \pi'_1 \oplus \cdots \oplus \pi'_m$ is a stable distribution on $\text{SL}_1(D)$. Any other stable distribution that can be obtained from $\Pi_\phi(\text{SL}_1(D))$ is a scalar multiple of the trace of $\pi'$.

Proof. (a) Since the restriction of irreducible representations from $\text{GL}_2(K)$ to $\text{SL}_2(K)$ is multiplicity-free [BuKn §1], $\pi = \pi_1 \oplus \cdots \oplus \pi_m$ is the restriction of some irreducible representation of $\text{GL}_2(K)$. If $\varphi : W_K \times \text{SL}_2(\mathbb{C}) \to \text{GL}_2(\mathbb{C})$ is any lift of $\phi$, the image of $\phi$ under the local Langlands correspondence is such a representation. We denote this representation of $\text{GL}_2(K)$ again by $\pi$. By the above remarks, its trace is a stable distribution on $\text{GL}_2(K)$, and hence also on $\text{SL}_2(K)$.

The different $\pi_i$ are inequivalent, but they are $\text{GL}_2(K)$ conjugate, because $\pi$ is irreducible. If a linear combination $\sum_{i=1}^m \lambda_i \text{tr}(\pi_i)$ is a stable distribution, then it must be invariant under conjugation by $\text{GL}_2(K)$. Hence all the $\lambda_i \in \mathbb{C}$ must be equal.

(b) The restriction of representations from $\text{GL}_1(D)$ to $\text{SL}_1(D)$ can have multiplicities, but still every constituent will appear with the same multiplicity [GeKn Lemma 2.1.d]. So there exists an integer $\mu$ such that $\mu \pi' = \mu \pi'_1 \oplus \cdots \oplus \mu \pi'_m$ lifts to an irreducible representation of $\text{GL}_1(D)$. The $L$-parameter of such a representation is a lift of $\phi$, so we can take $\text{JL}(\pi)$, the image of $\pi$ under the Jacquet–Langlands correspondence.

As remarked above, $\text{tr}(\text{JL}(\pi))$ is stable distribution on $\text{GL}_1(D)$ and by restriction also on $\text{SL}_1(D)$. Thus $\text{tr}(\pi') = \mu^{-1} \text{tr}(\text{JL}(\pi))$ is also a stable distribution on $\text{SL}_1(D)$. By the same argument as for part (a), any linear combination of the tr $(\pi'_i)$ which is stable, must be a scalar multiple of tr $(\pi')$. \qed

We remark that Proposition 3.2 also holds for inner forms of $\text{SL}_n(F)$ with $n > 2$. The proof is the same, one only has to replace the elliptic conjugacy classes by the conjugacy classes that correspond to elements of that particular inner form.

3.2. L-packets of cardinality one.

First we consider the case that $\varphi : W_K \to \text{GL}_2(\mathbb{C})$ is irreducible, so the $L$-packet consists of supercuspidal representations. By [16] and [15], $\Pi_{\phi}(\text{SL}_2(K))$ is a singleton if and only if $\varphi$ is primitive. The $L$-parameter $\phi$ is relevant for $\text{SL}_1(D)$, so $\Pi_{\phi}(\text{SL}_1(D))$ is nonempty. It follows from (14) and (15) that $Z_{\phi} \cong \pi_0(Z_{\text{SL}_2(\mathbb{C})(\phi)}) \cong \mathbb{Z}/2\mathbb{Z}$, and then from (15) that $\Pi_{\phi}(\text{SL}_1(D))$ is also a singleton. Any primitive representation of $W_K$ is either octahedral or tetrahedral, as in Section 2. See [BuHeC] §42 for more background.

Suppose now that $\varphi : W_K \to \text{GL}_2(\mathbb{C})$ is reducible, so $\phi$ is a $L$-parameter for the principal series of $\text{SL}_2(K)$. If $\phi(W_K) = 1$ and $\phi|_{\text{SL}_2(\mathbb{C})} : \text{SL}_2(\mathbb{C}) \to \text{PGL}_2(\mathbb{C})$ is the canonical projection, then $\phi$ is relevant for $\text{SL}_1(D)$. In this case $\Pi_{\phi}(\text{SL}_1(D))$ is just the trivial representation of $\text{SL}_1(D)$, and $\Pi_{\phi}(\text{SL}_2(K))$ consists of the Steinberg representation of $\text{SL}_2(K)$ – the unique irreducible square-integrable, non-supercuspidal representation.

All other principal series $L$-parameters are trivial on $\text{SL}_2(\mathbb{C})$ and are irrelevant for $\text{SL}_1(D)$. By conjugating $\phi$, we may assume that its image is contained in the diagonal torus of $\text{PGL}_2(\mathbb{C})$. One checks that $Z_{\text{PGL}_2(\mathbb{C})}(\phi)$ is connected unless the image of $\phi$ is $\{1, (-1, 0)\}$. Whenever $Z_{\text{PGL}_2(\mathbb{C})}(\phi)$ is disconnected, its $L$-packet has two elements, see Subsection 3.5.
If \( Z_{\text{PGL}_2}(\mathbb{C})(\phi) \) is connected, then \( \Pi_\phi(\text{SL}_2(K)) \) consists of precisely one principal series representation. Let \( T \) be the diagonal torus of \( \text{SL}_2(K) \), and let \( \chi_\phi \) be the character of \( T \) determined by local class field theory. Then \( \Pi_\phi(\text{SL}_2(K)) \) is the Langlands quotient of the parabolic induction of \( \chi_\phi \), and the depth of that representation equals the depth of \( \chi_\phi \).

### 3.3. Supercuspidal L-packets of cardinality two.

For such L-parameters \([16]\) shows that
\[
\mathcal{X}(\varphi) \cong \pi_0(Z_{\text{PGL}_2}(\mathbb{C})(\phi)) \cong \mathbb{Z}/2\mathbb{Z}.
\]
The L-parameter \( \phi \) is relevant for \( \text{SL}_1(D) \), so by \([14]\) \( |\pi_0(Z_{\text{SL}_2}(\mathbb{C})(\phi))| = 4 \). Then \( \pi_0(Z_{\text{SL}_2}(\mathbb{C})(\phi)) \) is either \( \mathbb{Z}/4\mathbb{Z} \) or \( (\mathbb{Z}/2\mathbb{Z})^2 \). In any case, it is abelian and has precisely four inequivalent characters. Now \([15]\) says that
\[
|\Pi_\phi(\text{SL}_1(D))| = |\Pi_\phi(\text{SL}_2(K))| = 2.
\]

Now we classify the discrete L-parameters \( \phi \) for which the packet \( \Pi_\phi(\text{SL}_2(K)) \) is not a singleton. We note that every L-parameter for a supercuspidal representation of \( \text{SL}_2(K) \) has to be trivial on \( \text{SL}_2(\mathbb{C}) \). For if it were nontrivial on \( \text{SL}_2(\mathbb{C}) \), then the image of \( W_K \) would be in the centre of \( \text{PGL}_2(\mathbb{C}) \), and we would get the L-parameter for the Steinberg representation, as discussed in the previous subsection. Since we want \( \phi \) to be discrete, it has to be an irreducible projective two-dimensional representation of \( W_K \).

Let \( \varphi \) be an irreducible two-dimensional representation of \( W_K \) which lifts \( \phi \). By \([16]\) and \([15]\) the associated L-packet \( \Pi_\phi(\text{SL}_2(K)) \) has more than one element if and only if \( \varphi \) is imprimitive. By \([\text{BuHe}1] \) \( \varphi \) is imprimitive if and only if there exists a separable quadratic extension \( E/K \) and a character \( \xi \) of \( E^\times \) such that \( \varphi \cong \text{Ind}_{E/K}\xi \). By the irreducibility \( \xi^\sigma \neq \xi \), where \( \sigma \) is the nontrivial automorphism of \( E \) over \( K \).

**Lemma 3.3.** Let \( \phi \) and \( \varphi \cong \text{Ind}_{E/K}\xi \) be as above.

(a) Suppose that the character \( \xi^\sigma \xi^{-1} \) of \( E^\times \) has order two. Then \( \varphi \) is triply imprimitive and there exists a biquadratic extension \( L/K \) such that \( \ker(\varphi) = W_L \) and \( L \supset E \).

(b) Suppose that \( \xi^\sigma \xi^{-1} \) has order \( > 2 \). Then \( \varphi \) is simply imprimitive.

**Proof.** Let \( \chi_E \) be the unique character of \( W_K \) with kernel \( W_E \). Then \( \chi_E \in \mathcal{X}(\varphi) \), this holds in general for induction of irreducible representations from subgroups of index two. In particular \( |\mathcal{X}(\varphi)| \in \{2, 4\} \). From \([\text{BuHe}1\text{ Corollary 41.3}] \) we see that \( \mathcal{X}(\varphi) = \{1, \chi_E\} \) if and only if the character \( \xi^\sigma \xi^{-1} \) of \( W_E \) cannot be lifted to a character of \( W_F \). Since the target group \( \mathbb{C}^\times \) is divisible, this happens if and only if \( \xi^\sigma \xi^{-1} \) does not equal
\[
(\xi^\sigma \xi^{-1})^\sigma = \xi \xi^{-\sigma} = (\xi^\sigma \xi^{-1})^{-1}.
\]

We conclude that the representation \( \varphi = \text{Ind}_{E/K}\xi \) is triply imprimitive if \( \xi^\sigma \xi^{-1} \) has order two and is simply imprimitive otherwise.

Now we focus on the triply imprimitive case. By local class field theory there exists a unique separable quadratic extension \( L/E \) such that \( \xi^\sigma \xi^{-1} \) is the associated character \( \chi_L \) of \( E^\times \). We consider it also as a character of \( W_E \). Then
\[
W_L = \ker(\chi_L) = \{w \in W_K : \varphi(w) \in Z(\text{GL}_2(\mathbb{C}))\}.\]
Hence \( W_L = \ker(\phi) \) is a normal subgroup of \( W_K \), which means that \( L/K \) is a Galois extension. The explicit form of \( \varphi \) entails that the image of \( \phi \) is the Klein four group. Consequently
\[
\text{Gal}(L/K) \cong W_K/W_L \cong \phi(W_K) \cong (\mathbb{Z}/2\mathbb{Z})^2,
\]
which says that \( L/K \) is biquadratic. \( \square \)

We remark that the depth of \( \varphi = \text{Ind}_{E/K}\xi \) can be computed in the same way as for the imprimitive representations in Section 2, see in particular \([11]\).

3.4. **Supercuspidal \( L \)-packets of cardinality four.** We continue with the case when \( \varphi \) is triply imprimitive, as in \([17]\). This means that we have a biquadratic extension \( L/K \) and the Langlands parameter
\[
\phi : W_K \to \text{Gal}(L/K) \cong (\mathbb{Z}/2\mathbb{Z})^2 \subset \text{PGL}_2(\mathbb{C}).
\]
We also have
\[
Z_{\text{PGL}_2(\mathbb{C})}(\text{im} \phi) = \pi_0(Z_{\text{PGL}_2(\mathbb{C})}(\text{im} \phi)) = S_{\phi} \cong (\mathbb{Z}/2\mathbb{Z})^2.
\]
This implies, by \([15]\), that \( \Pi_{\phi}(\text{SL}_2(K)) \) is a supercuspidal packet of cardinality four.

We note the isomorphism \( \text{PGL}_2(\mathbb{C}) = \text{PSL}_2(\mathbb{C}) \), and the morphism
\[
\text{SL}_2(\mathbb{C}) \to \text{PSL}_2(\mathbb{C}).
\]

As in \([\text{We}, \S 14]\), the pull-back \( S_{\phi} \) of \( S_{\phi} \) is isomorphic to the quaternion group of order eight. This group admits four characters and one irreducible representation of degree two. Only the two-dimensional representation has nontrivial central character.

The parameter \( \phi \) creates a packet with five elements, which are allocated to \( \text{SL}_2(K) \) or \( \text{SL}_1(D) \) according to central characters. So \( \phi \) gives rise to an \( L \)-packet \( \Pi_{\phi}(\text{SL}_2(K)) \) with four elements, and a singleton packet to the inner form \( \text{SL}_1(D) \).

**Theorem 3.4.** Let \( L/K \) be a biquadratic extension, let \( \phi \) be the Langlands parameter \([18]\). If \( t \) is the highest break in the upper ramification of \( \text{Gal}(L/K) \) then \( d(\phi) = t \). The allowed values of \( d(\phi) \) are \( 1, 3, 5, 7, \ldots \) except in Case 2.2 (see Appendix A.3), when the allowed values are \( 3, 5, 7, \ldots \).

**Proof.** From the inclusion \( L \subset K_s \) we obtain a natural surjection
\[
\pi_{L/K} : \text{Gal}(K_s/K) \to \text{Gal}(L/K).
\]
Let \( K_{ur} \) be the maximal unramified extension of \( K \) in \( K_s \) and let \( K_{ab} \) be the maximal abelian extension of \( K \) in \( K_s \). We have a commutative diagram, where the horizontal maps are the canonical maps and the vertical maps are the natural projections
\[
\begin{array}{cccccccccc}
1 & \longrightarrow & I_{K_s/K} & \longrightarrow & \text{Gal}(K_s/K) & \longrightarrow & \text{Gal}(K_{ur}/K) & \longrightarrow & 1 \\
& \downarrow & \alpha_1 & & \downarrow & \pi_1 & & \downarrow & \text{id} \\
1 & \longrightarrow & I_{K_{ab}/K} & \longrightarrow & \text{Gal}(K_{ab}/K) & \longrightarrow & \text{Gal}(K_{ur}/K) & \longrightarrow & 1 \\
& \downarrow & \alpha_2 & & \downarrow & \pi_2 & & \downarrow & \beta \\
1 & \longrightarrow & I_{L/K} & \longrightarrow & \text{Gal}(L/K) & \longrightarrow & \text{Gal}(L \cap K_{ur}/K) & \longrightarrow & 1
\end{array}
\]
In the above notation, we have \( \pi_{L/K} = \pi_2 \circ \pi_1 \). Let
\[
\cdots \subset I^{(2)} \subset I^{(1)} \subset I^{(0)} \subset G = \text{Gal}(L/K)
\]
be the filtration of the relative inertia subgroup $I^{(0)} = I_{L/K}$ of $\text{Gal}(L/K)$, $I^{(1)}$ is the
wild inertia subgroup, and so on. Note that $I^{(r)}$ is the restriction of the filtration
$G^r$ of $G = \text{Gal}(L/K)$ to the subgroup $I_{L/K}$, i.e., $I^{(r)} = t_3(G^r)$. Let
\[(20) \quad \cdots \subset I^{(2)} \subset I^{(1)} \subset I^{(0)} \subset G = \text{Gal}(K/K)\]
be the filtration of the absolute inertia subgroup $I^{(0)} = I_{K_s/K}$ of $\text{Gal}(K_s/K)$, $I^{(1)}$
is the wild inertia subgroup, and so on.

We have
\[(21) \quad (\forall r) \pi_{L/K} I^{(r)} = I^{(r)}\]
This follows immediately from the above diagram. Here, we identify $I^{(r)}$ with $t_1(I^{(r)})$
and $I^{(r)}$ with $t_3(I^{(r)})$. (Note that $\alpha$ is injective. Therefore, by (21), we have
\[\phi(I^{(r)}) = 1 \iff (\alpha \circ \pi_{L/K})(I^{(r)}) = 1 \iff \alpha(I^{(r)}) = 1 \iff I^{(r)} = 1.\]
The highest break $t$ has the property that $I^{(t+1)} = 1$ and $I^{(t)} \neq I^{(t+1)}$. It follows
that $d(\phi) = t$.

**Case 1:** There are two ramification breaks occurring at $-1$ and some odd integer $t > 0$:
\[\{1\} = \cdots = I^{(t+1)} \subset I^{(t)} \subset \cdots = I^{(0)} = I_{L/K} \subset \text{Gal}(L/K), \quad d(\phi) = t.\]
The allowed depths are $1, 3, 5, 7, \ldots$.

**Case 2.1:** One single ramification break occurs at some odd integer $t > 0$:
\[\{1\} = \cdots = I^{(t+1)} \subset I^{(t)} \subset \cdots = I^{(0)} = I_{L/K} = \text{Gal}(L/K); \quad d(\phi) = t.\]
The allowed depths are $1, 3, 5, 7, \ldots$.

**Case 2.2:** There are two ramification breaks occurring at some odd integers $t_1 < t_2$ (with $I^{(0)} = I_{L/K}$):
\[\{1\} = \cdots = I^{(t_2+1)} \subset I^{(t_2)} \subset \cdots = I^{(t_1+1)} \subset I^{(t_1)} = \cdots = I^{(0)} = I_{L/K} = \text{Gal}(L/K); \quad d(\phi) = t_2.\]
The allowed depths are $3, 5, 7, 9, \ldots$ \[\square\]

Theorem 3.4 contrasts with the case of $\text{SL}_2(\mathbb{Q}_p)$ with $p > 2$. Here there is a
unique biquadratic extension $L/K$, and the associated $L$-parameter $\phi : \text{Gal}(L/K) \to
\text{SO}_3(\mathbb{R})$ has depth zero.

3.5. **Principal series $L$-packets of cardinality two.**

Recall from Subsection 3.2 that a principal series $L$-parameter whose $L$-packet is
not a singleton has image $\{1, (\frac{-1}{0})\}$ in the diagonal torus $T^r$ of $\text{PGL}_2(\mathbb{C})$. Thus
it comes from a character $W_K \to \mathbb{C}^\times$ of order two. Define
\[W_K \times \text{SL}_2(\mathbb{C}) \to K^\times\]
to be the projection $(g, M) \mapsto g$ followed by the Artin reciprocity map
\[a_K : W_K \to K^\times.\]
Let $E/K$ be a quadratic extension and let $\chi_E$ be the associated quadratic character of $K^\times$. Consider the map

$$K^\times \to \text{PGL}_2(\mathbb{C}), \quad \alpha \mapsto \begin{pmatrix} \chi_E(\alpha) & 0 \\ 0 & 1 \end{pmatrix}$$

The composite map

$$\phi_E: \mathbf{W}_K \times \text{SL}_2(\mathbb{C}) \to K^\times \to \text{PGL}_2(\mathbb{C})$$

is then an $L$-parameter attached to $\chi_E$. For the centralizer of the image, we have

$$Z_{\text{PGL}_2(\mathbb{C})}(\text{im} \, \phi_E) = N_{\text{PGL}_2(\mathbb{C})}(T^\vee), \quad S_\phi \cong S_{\phi} = \{1, w\},$$

where $w$ generates the Weyl group of the dual group $\text{PGL}_2(\mathbb{C})$. As there are two characters $1, \epsilon$ of $W = \{1, w\}$, \cite{15} says that the $L$-packet has cardinality two. There are two enhanced parameters $(\phi_E, 1)$ and $(\phi_E, \epsilon)$, which parametrize the two elements in the $L$-packet $\Pi_{\phi_E} = \Pi_{\phi_E}(\text{SL}_2(K))$. We will write

$$\Pi_{\phi_E} = \{\pi_{E, 1}, \pi_{E, 2}\}.$$  

If $\gamma \in K_s$ is a root of $X^2 - X - \beta \in K[X]$, the quadratic extension $K(\gamma)$ is denoted also by $K(\varphi^{-1}(\beta))$, with $\beta \in K$, where $\varphi(X) = X^2 - X$. So the quadratic character

$$\chi_{n,j} = (-, u_j \varpi^{-2n-1} + \varphi(K))$$

is associated with the quadratic extension $E = K(\varphi^{-1}(u_j \varpi^{-2n-1}))$, see \cite{27} in the Appendix.

Let $E/K$ be a quadratic extension. There are two kinds: the unramified one $E_0 = K(\gamma_0)$ and countably many totally (and wildly) ramified $E = K(\gamma)$. The unramified quadratic extension has a single ramification break for $t = -1$.

Let $E/K$ be a quadratic totally ramified extension. According to \cite{Da} Proposition 11, p.411 and Proposition 14, p.413], there is a single ramification break for $t = 2n + 1$. Each value $2n + 1$ occurs as a break, with $n \geq 0, 1, 2, 3, \ldots$. By Theorem \cite{34} adapted to the present case, we have

$$d(\phi_E) = 2n + 1.$$  

Fix a basis $\mathcal{B} = \{u_1, \ldots, u_f\}$ of $\mathbb{F}_q / \mathbb{F}_2$ and let $u_j \in \mathcal{B}$. The next result shows how to realise the extension $E/K$.

**Theorem 3.5.** If $E = K(\varphi^{-1}(u_j \varpi^{-2n-1}))$ then

$$d(\phi_E) = 2n + 1$$

with $n = 0, 1, 2, 3, 4, \ldots$

**Proof.** Let $a_K : \mathbf{W}_K \to K^\times$ be the Artin reciprocity map. Then we have \cite{ABPS1} Theorem 3.6:

$$a_K(\text{Gal}(K_s/K)^l) = U_K^{[l]}$$

for all $l \geq 0$, where $[l]$ denotes the least integer greater than or equal to $l$, and $U_K^i$ is the $i$th higher unit group.

We are concerned here with the quadratic character $\chi = \chi_E$ and the associated $L$-parameter $\phi = \phi_E$. The level $\ell(\chi)$ of $\chi$ is the least integer $n \geq 0$ for which $\chi(U_K^{n+1}) = 1$. Call this integer $N$. For this integer $N$, we have

$$N < l \leq N + 1 \implies a_K(\text{Gal}(K_s/K)^l) = U_K^{[l]} = U_K^{N+1}$$

on which $\chi$ is trivial.
$N - 1 < l \leq N \implies a_K(\text{Gal}(K_s/K)^l) = U_K^{[l]} = U_K^{N}$ on which $\chi$ is nontrivial

The $L$-parameter $\phi$ will factor through $K^\times$ and we have to consider its depth $d(\phi)$. Recall: the depth of $\phi$ is the smallest number $d(\phi) \geq 0$ such that $\phi$ is trivial on $\text{Gal}(K_s/K)^l$ for all $l > d(\phi)$. Then $d(\phi) = N$ in view of the above two implications. We infer that

$$\ell(\chi_E) = d(\phi_E).$$

If $\chi$ is the unramified quadratic character given by $\chi(x) = (-1)^{\text{val}_K(x)}$ then we will have to allow $N = -1$ in which case $\phi$ has negative depth.

If $E = K(\wp^{-1}(u_j \varpi^{-2n-1}))$ then $\chi_E = \chi_{n,j}$ and so we have

$$\ell(\chi_E) = \ell(\chi_{n,j}).$$

We now compute the level of the quadratic character $\chi_{n,j}$ defined in (27). Every $\alpha \in U_K$ has the form $\alpha = 1 + \varepsilon \varpi^i$, with $\varepsilon \in \mathfrak{o}$, and can be expanded in the convergent product

$$\alpha = \prod_{i \geq 1} (1 + \theta_i \varpi^i)$$

for unique $\theta_i \in \mathbb{F}_q$. As we can see in the proof of Theorem [A.2]

$$d_\varpi(1 + \theta_{2n+1} \varpi^{2n+1}, u_j \varpi^{-2n-1}) = \text{Tr}_{\mathbb{F}_q/F_2}(u_j \theta_{2n+1})$$

and

$$d_\varpi(1 + \theta_i \varpi^i, u_j \varpi^{-2n-1}) = 0$$

if $i \nmid 2n + 1$. There exists, therefore, an element $\alpha \in U_K^{2n+1}$ such that $\chi_{n,j}(\alpha) \neq 0$ and $\chi_{n,j}(U_K^{2n+2}) = 1$. We infer that

$$\ell(\chi_{n,j}) = 2n + 1.$$

The theorem now follows from (23), (24) and (25). \hfill \square

We conclude that, if $E = K(\wp^{-1}(u_j \varpi^{-2n-1}))$, then

$$d(\pi_E) \geq 2n + 1$$

with $i = 1, 2$.

It follows that the depths of the irreducible representations $\pi_E^1, \pi_E^2$ in the $L$-packet $\Pi_{\wp_E}$ can be arbitrarily large. For representations of enormous depth, such as the ones encountered in this article, the term *hadopelagic* commends itself, in contrast to the currently accepted term *epipelagic* for representations of modest depth, see [en.wikipedia.org/wiki/Epipelagic](http://en.wikipedia.org/wiki/Epipelagic).

**Appendix A. Artin-Schreier Symbol**

Let $K$ be a local field of characteristic $p$ with finite residue field $k$. The field of constants $k = \mathbb{F}_q$ is a finite extension of $\mathbb{F}_p$, with degree $[k : \mathbb{F}_p] = f$ and $q = p^f$. Let $\mathfrak{o}$ be the ring of integers in $K$ and $\mathfrak{p} \subseteq \mathfrak{o}$ the maximal ideal. A choice of uniformizer $\varpi \in \mathfrak{o}$ determines isomorphisms $K \cong \mathbb{F}_q((\varpi)), \mathfrak{o} \cong \mathbb{F}_q[[\varpi]]$ and $\mathfrak{p} = \varpi \mathfrak{o} \cong \varpi \mathbb{F}_q[[\varpi]]$.

The group of units is denoted by $\mathfrak{o}^\times$ and $\nu$ represents a normalized valuation on $K$, so that $\nu(\varpi) = 1$ and $\nu(K) = \mathbb{Z}$.

Following [FeVo. IV.4 - IV.5], we have the reciprocity map

$$\Psi_K : K^\times \to \text{Gal}(K_{ab}/K)$$
We define the map (Artin-Schreier symbol)

$$(-,-) : K^\times \times K \to \mathbb{F}_p$$

by the formula

$$(\alpha, \beta] = \Psi_K(\alpha)(\gamma) - \gamma$$

where \(\gamma\) is a root of the polynomial \(X^p - X - \beta\). The polynomial \(X^p - X\) is denoted \(\wp(X)\). According to [FeVo, p.148] the pairing \((-,-]\) determines the nondegenerate pairing

$$(26) \quad K^\times / K^{\times p} \times K/\wp(K) \to \mathbb{F}_p.$$ 

Let us fix a coset \(\beta + \wp(K) \in K/\wp(K)\). According to (26), this coset determines an element of \(\text{Hom}(K^\times / K^{\times p}, \mathbb{F}_p)\).

Now specialise to \(p = 2\). We will identify the additive group \(\mathbb{F}_2\) with the multiplicative group \(\mu_2(\mathbb{C}) = \{1, -1\} \subset \mathbb{C}\). In that case, the elements of \(\text{Hom}(K^\times / K^\times 2, \mathbb{F}_2)\) are precisely the quadratic characters of \(K^\times\). Since the pairing (26) is nondegenerate, the quadratic characters are parametrised by the cosets \(\beta + \wp(K) \in K/\wp(K)\).

Now the index of \(\wp(K)\) in \(K\) is infinite; in fact, the powers \(\{\wp^{-2n-1} : n \geq 0\}\) are distinct coset representatives, see [FeVo, p.146].

**Lemma A.1.** For \(K = \mathbb{F}_2(\wp)\) the set of powers \(\{\wp^{-2n-1} : n \geq 0\}\) is a complete set of coset representatives.

That is not the case when \(K = \mathbb{F}_q(\wp)\) has residue degree \(f > 1\). Let \(\mathcal{B} = \{u_1, \ldots, u_f\}\) denote a basis of the \(\mathbb{F}_2\)-linear space \(\mathbb{F}_q\). Then,

$$\{u_j \wp^{-2n-1} : n \geq 0, j = 1, \ldots, f\}$$

is a complete set of coset representatives of \(K/\wp(K)\), see 
§5 and §6 of [Da].

The pairing (26) creates a sequence of quadratic characters

$$(27) \quad \chi_{n,j}(\alpha) := (\alpha, u_j \wp^{-2n-1} + \wp(K)]$$

with \(n \geq 0\) and \(j = 1, \ldots, f\).

**A.1. Explicit formula for the Artin-Schreier symbol.**

In [FeVo, Corollary 5.5, p.148], the authors introduce the map \(d_\wp\) which we now describe. Let \(\wp\) be a fixed uniformizer. Using the isomorphism \(K = \mathbb{F}_q(\wp)\), where \(q = 2^f\), every element \(\alpha \in K\) can be uniquely expanded as

$$(28) \quad \alpha = \sum_{i \geq i_0} \vartheta_i \wp^i, \quad \vartheta_i \in \mathbb{F}_q.$$ 

Put

$$\frac{d\alpha}{d\wp} = \sum_{i \geq i_0} i\vartheta_i \wp^i, \quad \text{res}_\wp(\alpha) = \vartheta_{-1}.$$ 

Define the pairing

$$(29) \quad d_\wp : K^\times \times K \to \mathbb{F}_2, \quad d_\wp(\alpha, \beta) = \text{Tr}_{\mathbb{F}_{q/\mathbb{F}_2}} \text{res}_\wp(\beta \alpha^{-1} \frac{d\alpha}{d\wp})$$

By [FeVo, Theorem 5.6. p.149], the pairing \((-,-]\) coincides with the pairing defined in (29). In particular, \(d_\wp\) does not depend on the choice of uniformizer.
We conclude that every quadratic character $\chi_{n,j}$ from (27) is completely described by
\begin{equation}
(30) \quad d_{\varpi}(\alpha, u_j \varpi^{-2n-1}) = \text{Tr}_{F_q/F_2} \text{res}_{\varpi}(u_j \varpi^{-2n-1} \alpha^{-1} \frac{d\alpha}{d\varpi}), \ n \geq 0.
\end{equation}

We seek a formula more explicit than (30).

By [FeVo, Proposition 5.10, p. 17], for every $\alpha \in K^\times$ there exist uniquely determined $k \in \mathbb{Z}$ and $\theta_i \in F_q$ for $i \geq 0$ such that $\alpha$ can be expanded in the convergent product
\begin{equation}
(31) \quad \alpha = \varpi^k \theta_0 \prod_{i \geq 1} \left(1 + \theta_i \varpi^i\right)
\end{equation}

We have
\[
d_{\varpi}(\varpi^k \theta_0 \prod_{i \geq 1} \left(1 + \theta_i \varpi^i\right), u_j \varpi^{-2n-1}) =
d_{\varpi}(\theta_0 \varpi^k, u_j \varpi^{-2n-1}) + d_{\varpi}(\prod_{i \geq 1} \left(1 + \theta_i \varpi^i\right), u_j \varpi^{-2n-1})
\]
Now, $d_{\varpi}(\theta_0 \varpi^k, u_j \varpi^{-2n-1})$ is easy to compute:
\[
d_{\varpi}(\theta_0 \varpi^k, u_j \varpi^{-2n-1}) = \text{Tr}_{F_q/F_2} \text{res}_{\varpi}(u_j \varpi^{-2n-1} \theta_0^{-1} \varpi^{-k} \frac{d(\theta_0 \varpi^k)}{d\varpi})
= \text{Tr}_{F_q/F_2} \text{res}_{\varpi}(k u_j \varpi^{-2n-2})
= 0.
\]

On the other hand,
\[
d_{\varpi}(\prod_{i \geq 1} \left(1 + \theta_i \varpi^i\right), u_j \varpi^{-2n-1}) = \sum_{i \geq 1} d_{\varpi}(1 + \theta_i \varpi^i, u_j \varpi^{-2n-1})
= \sum_{i=1}^{2n+1} d_{\varpi}(1 + \theta_i \varpi^i, u_j \varpi^{-2n-1})
\]
since $d_{\varpi}(1 + \theta_i \varpi^i, u_j \varpi^{-2n-1}) = 0$ if $i > 2n+1$ (see [FeVo, p. 150], proof of Corollary). Moreover, by the same proof of Corollary in [FeVo, p. 150], we have
\begin{equation}
(32) \quad d_{\varpi}(1 + \theta_{2n+1} \varpi^{2n+1}, u_j \varpi^{-2n-1}) = \text{Tr}_{F_q/F_2}(\theta_{2n+1})
= \text{Tr}_{F_q/F_2}(u_j \theta_{2n+1}).
\end{equation}

This last formula is a particular case of a more general formula we are about to prove.

In order to compute $d_{\varpi}(1 + \theta_i \varpi^i, u_j \varpi^{-2n-1})$ for $i = 1, \ldots, 2n+1$, we need to find the Laurent series expansion of $(1 + \theta_i \varpi^i)^{-1}$. This can be done by expanding the geometric series
\[
(1 + \theta_i \varpi^i)^{-1} = \sum_{j \geq 0} (-\theta_i \varpi^i)^j = 1 - \theta_i \varpi i + \theta_i^2 \varpi 2i - \theta_i^3 \varpi 3i + \cdots
\]
We have
\[ u_j \omega^{-2n-1}(1 + \theta_i \omega^i)^{-1} \frac{d}{d\omega} (1 + \theta_i \omega^i) = \]
\[ iu_j \theta_i \omega^{-2n-1-i-1}(1 - \theta_i \omega^i) + \theta_i^2 \omega^{2i} - \theta_i^3 \omega^{3i} + \cdots + (-1)^i \theta_i^r \omega^r + \cdots \]
The residue will be nonzero if
\[ -2n - 1 + i - 1 + ri = -1 \iff r = \frac{2n + 1}{i} - 1 \]
Hence, \( d_\omega (1 + \theta_i \omega^i, u_j \omega^{-2n-1}) = 0 \) if \( i \nmid 2n + 1 \). In particular, \( i \) must be odd.

We have:
\[ d_\omega (1 + \theta_i \omega^i, u_j \omega^{-2n-1}) = \begin{cases} 
0 & \text{if } i \nmid 2n + 1 \\
\text{Tr}_{F_q/F_2}(u_j \theta_i^{(2n+1)/i}) & \text{if } i | 2n + 1 
\end{cases} \]
In particular, we recover formula (32) by taking \( i = 2n + 1 \).

From the above, we have established the following explicit formula.

**Theorem A.2.** Let \( K \) be a local function field of characteristic 2 with residue degree \( f \), and let \( \chi_{n,j} \) denote the quadratic character from (27). Then we have the explicit formula
\[ \chi_{n,j}(\alpha) = \sum_{i | 2n+1} \text{Tr}_{F_q/F_2}(u_j \theta_i^{(2n+1)/i}) \]
where \( \alpha = \omega^k \theta_0 \prod_{i \geq 1} (1 + \theta_i \omega^i) \in K^\times \), \( n \geq 0 \) and \( j = 1, \ldots, f \).

For example, we have
\[ \chi_{0,1}(\alpha) = \text{Tr}_{F_q/F_2} \theta_1, \quad \chi_{1,1}(\alpha) = \text{Tr}_{F_q/F_2} (\theta_1^2 + \theta_3), \quad \chi_{2,1}(\alpha) = \text{Tr}_{F_q/F_2} (\theta_1^2 + \theta_5), \]
where \( \{1, u_2, \ldots, u_f\} \) is a basis of \( F_q/F_2 \).

**A.2. Ramification.**

Quadratic extensions \( L/K \) are obtained by adjoining an \( F_2 \)-line \( D \subset K/\varphi(K) \). Therefore, \( L = K(\varphi^{-1}(D)) = K(\gamma) \) where \( D = \text{span}\{\beta + \varphi(K)\} \), with \( \gamma^2 - \gamma = \beta \). In particular, if \( \beta_0 \in \mathfrak{o}/\mathfrak{p} \) such that the image of \( \beta_0 \) in \( \mathfrak{o}/\mathfrak{p} \) has nonzero trace in \( F_2 \), the \( F_2 \)-line \( V_0 = \text{span}\{\beta_0 + \varphi(K)\} \) contains all the cosets \( \beta_1 + \varphi(K) \) where \( \beta_i \) is an integer and so \( K(\varphi^{-1}(\mathfrak{o})) = K(\varphi^{-1}(V_0)) = K(\gamma_0) \) where \( \gamma_0^2 - \gamma_0 = \beta_0 \) gives the unramified quadratic extension, see [Da] Proposition 12, p. 412].

Biquadratic extensions are computed the same way, by considering \( F_2 \)-planes \( W = \text{span}\{\beta_1 + \varphi(K), \beta_2 + \varphi(K)\} \subset K/\varphi(K) \). Therefore, if \( \beta_1 + \varphi(K) \) and \( \beta_2 + \varphi(K) \) are \( F_2 \)-linearly independent then \( K(\varphi^{-1}(W)) := K(\gamma_1, \gamma_2) \) is biquadratic, where \( \gamma_1^2 - \gamma_1 = \beta_1 \) and \( \gamma_2^2 - \gamma_2 = \beta_2 \), \( \gamma_1, \gamma_2 \in K^\times \). Therefore, \( K(\gamma_1, \gamma_2)/K \) is biquadratic if \( \beta_2 - \beta_1 \not\in \varphi(K) \).

A biquadratic extension containing the line \( V_0 \) is of the form \( K(\gamma_0, \gamma)/K \). There are countably many quadratic extensions \( L_0/K \) containing the unramified quadratic extension. They have ramification index \( e(L_0/K) = 2 \). And there are countably many biquadratic extensions \( L/K \) which do not contain the unramified quadratic extension. They have ramification index \( e(L/K) = 4 \).

So, there is a plentiful supply of biquadratic extensions \( K(\gamma_1, \gamma_2)/K \).

The space \( K/\varphi(K) \) comes with a filtration
\[ 0 \subset V_0 \subset V_1 = V_2 \subset V_3 = V_4 \subset \cdots \subset K/\varphi(K) \]

(33)
where $V_t$ is the image of $\sigma_K$ and $V_i (i > 0)$ is the image of $p^{-i}$ under the canonical surjection $K \to K/\wp(K)$. For $K = \mathbb{F}_q((\varpi))$ and $i > 0$, each inclusion $V_{2i} \subset_f V_{2i+1}$ is a sub-$\mathbb{F}_2$-space of codimension $f$. The $\mathbb{F}_2$-dimension of $V_n$ is
\begin{equation}
\dim_{\mathbb{F}_2} V_n = 1 + \lceil n/2 \rceil f,
\end{equation}
for every $n \in \mathbb{N}$, where $\lceil x \rceil$ is the smallest integer bigger than $x$.

Let $L/K$ denote a Galois extension with Galois group $G$. For each $i \geq -1$ we define the $i^{th}$-ramification subgroup of $G$ (in the lower numbering) to be:
\begin{equation}
G_i = \{ \sigma \in G : \sigma(x) - x \in p_L^{i+1}, \forall x \in \mathfrak{a}_L \}.
\end{equation}
An integer $t$ is a break for the filtration $\{G_i\}_{i \geq -1}$ if $G_t \neq G_{t+1}$. The study of ramification groups $\{G_i\}_{i \geq -1}$ is equivalent to the study of breaks of the filtration.

There is another decreasing filtration with upper numbering $\{G^i\}_{i \geq -1}$ and defined by the Hasse-Herbrand function $\psi = \psi_L/K$:
\begin{equation}
G^u = G^{\psi(u)}.
\end{equation}
In particular, $G^{-1} = G_{-1} = G$ and $G^0 = G_0$, since $\psi(-1) = -1$ and $\psi(0) = 0$.

Now, in analogy with the lower notation, a real number $t \geq -1$ is a break for the filtration $\{G^i\}_{i \geq -1}$ if
\begin{equation}
\forall \varepsilon > 0, G^t \neq G^{t+\varepsilon}.
\end{equation}
We define
\begin{equation}
G^{t+} := \bigcap_{r > t} G^r.
\end{equation}
Then $t$ is a break of the filtration if and only if $G^{t+} \neq G^t$. The set of breaks of the filtration is countably infinite and need not consist of integers.

If $G$ is abelian, it follows from Hasse-Arf theorem [FeVo, p.91] that the breaks are integers and (35) is equivalent to $G^t \neq G^{t+1}$.

Let $G_2 = \text{Gal}(K_2/K)$ be the Galois group of the maximal abelian extension of exponent 2, $K_2 = K(\wp^{-1}(K))$. Since $G_2 \cong K^\times/K^\times 2$, the nondegenerate pairing [26] coincides with the pairing $G_2 \times K/\wp(K) \to \mathbb{Z}/2\mathbb{Z}$.

The profinite group $G_2$ comes equipped with a ramification filtration $(G_2^u)_{u \geq -1}$ in the upper numbering, see [Da, p.409]. For $u \geq 0$, we have an orthogonal relation [Da, Proposition 17, p.415]
\begin{equation}
(G_2^u)^ot = p^{-|u|+1} = V_{|u|+1}
\end{equation}
under the pairing $G_2 \times K/\wp(K) \to \mathbb{Z}/2\mathbb{Z}$.

Since the upper filtration is more suitable for quotients, we will compute the upper breaks. By using the Hasse-Herbrand function it is then possible to compute the lower breaks in order to obtain the lower ramification filtration.

According to [Da, Proposition 17], the positive breaks in the filtration $(G^u)_v$ occur precisely at integers prime to $p$. So, for $c(K) = 2$, the positive breaks will occur at odd integers. The lower numbering breaks are also integers. If $G$ is cyclic of prime order, then there is a unique break for any decreasing filtration $(G^u)_v$ (see [Da], Proposition 14). In general, the number of breaks depends on the possible filtration of the Galois group.
Given a plane $W \subset K/\wp(K)$, the filtration \((V_i)\) on $K/\wp(K)$ induces a filtration \((W_i)\) on $W$, where $W_i = W \cap V_i$. There are three possibilities for the filtration breaks on a plane and we will consider each case individually.

**Case 1:** $W$ contains the line $V_0$, i.e. $L_0 = K(\wp^{-1}(W))$ contains the unramified quadratic extension $K(\wp^{-1}(V_0)) = K(\alpha_0)$ of $K$. The extension has residue degree $f(L_0/K) = 2$ and ramification index $e(L_0/K) = 2$. In this case, there is an integer $t > 0$, necessarily odd, such that the filtration \((W_i)\) looks like

$$0 \subset_1 W_0 \subset_2 W_t = W.$$  

By the orthogonality relation (37), the upper ramification filtration on $G = \text{Gal}(L_0/K)$ looks like

$$\{1\} = \cdots \subset G^{t+1} \subset G^t = \cdots \subset G^0 \subset G^{-1} = G$$

Therefore, the upper ramification breaks occur at $-1$ and $t$.

The number of such $W$ is equal to the number of planes in $V_t$ containing the line $V_0$ but not contained in the subspace $V_{t-1}$. This number can be computed and equals the number of biquadratic extensions of $K$ containing the unramified quadratic extensions and with a pair of upper ramification breaks $(-1,t)$, $t > 0$ and odd. Here is an example.

**Example A.3.** The number of biquadratic extensions containing the unramified quadratic extension and with a pair of upper ramification breaks $(-1,1)$ is equal to the number of planes in an $1+f$-dimensional $\mathbb{F}_2$-space, containing the line $V_0$. There are precisely

$$1 + 2 + 2^2 + \cdots + 2^{f-1} = \frac{1 - 2^f}{1 - 2} = q - 1$$

of such biquadratic extensions.

**Case 2.1:** $W$ does not contain the line $V_0$ and the induced filtration on the plane $W$ looks like

$$0 = W_{t-1} \subset_2 W_t = W$$

for some integer $t$, necessarily odd.

The number of such $W$ is equal to the number of planes in $V_t$ whose intersection with $V_{t-1}$ is $\{0\}$. Note that, there are no such planes when $f = 1$. So, for $K = \mathbb{F}_2((\wp))$, **case 2.1** does not occur.

Suppose $f > 1$. By the orthogonality relation, the upper ramification filtration on $G = \text{Gal}(L/K)$ looks like

$$\{1\} = \cdots \subset G^{t+1} \subset G^t = \cdots \subset G^0 \subset G^{-1} = G$$

Therefore, there is a single upper ramification break occurring at $t > 0$ and is necessarily odd.

For $f = 1$ there is no such biquadratic extension. For $f > 1$, the number of these biquadratic extensions equals the number of planes $W$ contained in an $\mathbb{F}_2$-space of dimension $1+fi$, $t = 2i-1$, which are transverse to a given codimension-$f$ $\mathbb{F}_2$-space.

**Case 2.2:** $W$ does not contain the line $V_0$ and the induced filtration on the plane $W$ looks like

$$0 = W_{t_1-1} \subset_1 W_{t_1} = W_{t_2-1} \subset_1 W_{t_2} = W$$

for some integers $t_1$ and $t_2$, necessarily odd, with $0 < t_1 < t_2$. 
The orthogonality relation for this case implies that the upper ramification filtration on $G = \text{Gal}(L/K)$ looks like

$$\{1\} = \cdots = G^{t_2+1} \subset_1 G^{t_2} = \cdots = G^{t_1+1} \subset_1 G^{t_1} = \cdots = G$$

The upper ramification breaks occur at odd integers $t_1$ and $t_2$.

There is only a finite number of such biquadratic extensions, for a given pair of upper breaks $(t_1, t_2)$.

References


