Abstract

Binary relational algebra provides semantic foundations for major areas of computing, such as database design, state-based modeling and functional programming. Remarkably, static checking support in these areas fails to exploit the full semantic content of relations. In particular, properties such as the simplicity or injectivity of relations are not statically enforced in operations such as database queries, state transitions, or composition of functional components. When data models, their constraints and operations are represented by point-free binary relational expressions, proof obligations can be expressed as inclusions between relational expressions. We developed a type-directed, strategic term rewriting system that can be used to simplify relational proof obligations and ultimately reduce them to tautologies. Such reductions can be used to provide extended static checking for design constraints commonly found in software modeling and development.

Keywords: Models verification; Symbolic execution; Abstract model verification; Extended static checking; Strategic term rewriting

1 Introduction

Software design is error-prone. The negative impact of programming errors on software productivity can be limited by catching them early. Static checkers (e.g., syntax and type checkers) are tools which catch errors at compile-time, i.e. before running the program. Examples of such errors are unmatched parentheses (wrong syntax) and adding integers to booleans (wrong typing). Errors such as null dereferencing, division by 0, and array bound overflow, are not caught by standard static checking; detecting their presence requires extensive testing, and if their presence cannot be excluded with certainty, they must be handled at run-time via exception mechanisms.

Software formalists will argue that error checking in the coding phase is too late: first a formal model should be written, queried, reasoned about, and possibly animated (using e.g. a symbolic interpreter). Formal modeling relies on “rich” datatypes such as finite mappings, finite sequences, and recursive data structures, which abstract from much of the complexity found in common imperative programming languages (e.g. pointers, loop boundaries). However, such rich structures are not able to capture all properties, meaning that additional constraints need to be added to models such as invariants (attached to types) and pre-conditions (attached to operations). Checking such constraints is once again a process which falls outside standard static type-checking, leading to a so-called dynamic type checking process, typical of model animation tools such as the VDMTools system [1]. Static checking of formal models involving such constraints is a complex process, relying on generation and discharge of proof obligations [2]. While proof obligations can be generated mechanically, their discharge is in general above the decidability ceiling in requiring full-fledged formal verification (theorem proving) [3]. Between these two extremes of standard, cheap, decidable static checking and costly theorem proving, extended static checking (ESC) [4] aims to catch more errors at compile-time at the relatively moderate cost of adding annotations to the code which record design decisions which were lost throughout the programming process (if ever explicitly recorded).

Extended static checking tools have been developed for imperative programming languages such as Java (ESC/Java [4]). At the heart of these tools we find a verification condition generator and the Simplify theorem prover [5]. Verification conditions are predicates...
in first-order logic which are computed in weakest precondition style. Theorem proving is performed by a combination of techniques, including SAT solvers, matching algorithms, and heuristics to guide proof search.

In the current paper we follow the spirit of this approach but intend to apply it much earlier in the design process: we wish to perform extended static checking at abstract model level. Formal modeling notations such as Alloy, Z and VDM include “rich data structuring” primitives which already preclude (by construction) the occurrence of errors such as null pointers and array bound overflow. In verifying designs at abstract model level we aim to catch errors higher on the semantic scale. A pertinent example is the finite mapping data type (also called finite partial function), which covers a number of interesting situations in both formal modeling (e.g. in VDM-like notations) and programming (e.g. SQL). For instance, relational tables with a primary key can be modeled as finite mappings. UPDATE and INSERT are examples of operations which in general put primary keys at risk, something that model animation tools and database systems can only check at run-time. Maintenance of primary key relationships by these operations will be one of the targets of our extended static checks.

The main novelty of our approach resides in the chosen method of proof construction, whereby first-order proof obligations are subject to the PF-transform [6] before they are reasoned about. (See reference [7] for the theory behind this blending of ESC with the PF-transform, suggestively referred to as the ESC/PF proof obligation calculus.) Such a transformation eliminates quantifiers and bound variables (so-called points), resulting in a pointfree notation which is more agile to calculate with.

<table>
<thead>
<tr>
<th>$\phi$</th>
<th>$PF\phi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\exists a :: b R a \land a S c)$</td>
<td>$b(R \cdot S)c$</td>
</tr>
<tr>
<td>$\forall a,b :: b R a \Rightarrow b S a)$</td>
<td>$R \subseteq S$</td>
</tr>
<tr>
<td>$\forall a :: a R a)$</td>
<td>$id \subseteq R$</td>
</tr>
<tr>
<td>$b R a \land c S a)$</td>
<td>$(b,c)(R,S)a$</td>
</tr>
<tr>
<td>$b R a \land b S a)$</td>
<td>$b (R \cap S) a$</td>
</tr>
<tr>
<td>$b R a \lor b S a)$</td>
<td>$b (R \cup S) a$</td>
</tr>
<tr>
<td>$b = a)$</td>
<td>$b id a$</td>
</tr>
<tr>
<td>$TRUE)$</td>
<td>$b \top a$</td>
</tr>
<tr>
<td>$FALSE)$</td>
<td>$b \bot a$</td>
</tr>
</tbody>
</table>

Figure 1: The PF-transform. In analogy to the well-known Laplace transform [11], the PF-transform takes expressions from a mathematical problem space, in this case first order logic formula, into a mathematical solution space, in this case relational algebra expressions [12]. The PF-transform eliminates quantifiers and bound variables (so-called points), resulting in a pointfree notation which is more agile to calculate with.

2 Motivating example

The UML class diagram in Fig. 2 depicts a simplified model of a system for trading non-consumable (uniquely identifiable) items. A user can put an item for sale for a given price, and other users can express their interest in these items for a price they are willing to pay. If a match between a seller and a buyer is established, this leads to a deal with an agreed price.

The specification of queries, predicates, and transformations on this model may present some pitfalls. Suppose the following operations are desired:

- $\text{listWantedItems :: Wanted} \rightarrow \text{MapIidPrice}$
- $\text{wantedItemsAreForSale :: Wanted} \rightarrow \text{ForSale} \rightarrow \text{Bool}$
- $\text{putBatchForSale :: (Uid,MapIidPrice) \rightarrow ForSale \rightarrow ForSale}$
- $\text{settleDeal :: (Iid,Uid,Price) \rightarrow Deal \rightarrow Deal}$

The $\text{listWantedItems}$ query produces a map of item identifiers together with the price that has been offered for them. The $\text{wantedItemsAreForSale}$ predicate tests whether each item listed in $\text{Wanted}$ is actually for sale, thus testing referential integrity. The transformation $\text{putBatchForSale}$ adds a batch of items belonging to a given user to the $\text{ForSale}$ relation. The $\text{settleDeal}$ transformation adds an entry to the $\text{Deal}$ collection.

When specifying these operations, the designer could benefit from the feedback of an extended static checker. For example, the checker should tell her/him that query $\text{listWantedItems}$ should only return a map if the $\text{Wanted}$ collection contains no two offers for the same item with different prices. Rather than adding a precondition to that effect, he will likely decide to change the return type to a general relation $\text{Rel Iid Price}$ or, equivalently, to $\text{Set (Iid,Price)}$. In
The identity position is defined in the usual way: \( R \cap S \subseteq R \) means that pair \( a \in S \) such that \( b[R]a \equiv (b = a) \land (p \ a) \) holds, that is, the relation that maps every \( a \) which satisfies \( p \) onto itself. Negation is modeled by \( \neg \Phi = id - \Phi \). A set \( S \subseteq A \) is modeled by \( \{ \lambda a.a \in S \} \), that is \( b[S]a \equiv (b = a) \land a \in S \).

### Taxonomy

To establish a fundamental taxonomy of relations (illustrated in Fig. 3), let us first define the kernel of a relation, \( \ker R = R \cdot R \) and its dual, \( \img R = \ker (R^*) = R \cdot R^* \), called the image of \( R \).

A relation \( R \) is said to be \( \text{entire} \) (or total) if its kernel is reflexive; and \( \text{simple} \) (or functional) iff its image is coreflexive. Simple relations are denoted with capital letters \( M, N \), etc. Dually, \( R \) is \( \text{surjective} \) iff \( \img R \) is reflexive, and \( R \) is \( \text{injective} \) iff \( \ker R \) is coreflexive. This terminology is recorded in the summary table in Fig. 3. The coreflexive fragments of kernel and image are named \( \text{domain} \) (\( \delta \)) and \( \text{range} \) (\( \rho \)).

### Functions

As the taxonomy indicates, a relation is a function iff it is both simple and entire. Functions will be denoted by lowercase letters \( f, g, \) etc. and are such that \( bfa \) means \( b = f a \). The constant function which maps every value of its domain to the value \( k \) is denoted by \( k \). It can be easily seen that, given two data values \( a, b \), relation \( b \cdot g^a \) is the PF-version of singleton relation \( \{(b, a)\} \).

### Algebraic properties

A rich set of algebraic properties is available for the various operators of relational algebra [12], of which a small sample is listed in Table 1. Of particular interest for the current paper are the various shunting laws. They allow the ‘shunting’ of relations (functions and simple relations in the listed cases) from one side of the inclusion to the other, similar to the shunting rules we learned in high school, such as \( x - y \leq z \iff x \leq z + y \). The utility of such laws will become evident below.
Table 1: Some laws of the binary relational algebra.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Property</th>
</tr>
</thead>
<tbody>
<tr>
<td>((R \cdot S) \cup T = R \cdot (S \cup T))</td>
<td>\text{comp_assoc}</td>
</tr>
<tr>
<td>(R \cdot id = R)</td>
<td>\text{comp_id}</td>
</tr>
<tr>
<td>((R \cdot S) \circ = S \circ R)</td>
<td>\text{inv_comp}</td>
</tr>
<tr>
<td>((R^o)^o = R)</td>
<td>\text{inv_inv}</td>
</tr>
<tr>
<td>(R \perp = \perp )</td>
<td>\text{comp_empty}</td>
</tr>
<tr>
<td>((R \cup S)^o = R^o \cup S^o)</td>
<td>\text{inv_union}</td>
</tr>
<tr>
<td>((R \cup S) \cdot T = (R \cdot T) \cup (S \cdot T))</td>
<td>\text{union_fusion}</td>
</tr>
<tr>
<td>(T \cdot (R \cup S) = (T \cdot R) \cup (T \cdot S))</td>
<td>\text{union_fusion}</td>
</tr>
<tr>
<td>(\Phi^o = \Phi)</td>
<td>\text{corefl_symm}</td>
</tr>
<tr>
<td>(\Phi \cdot \Phi = \Phi)</td>
<td>\text{corefl_trans}</td>
</tr>
<tr>
<td>(R \cdot \delta R = R)</td>
<td>\text{dom_elim}</td>
</tr>
<tr>
<td>(k \cdot R = k \cdot \delta R)</td>
<td>\text{const_fusion}</td>
</tr>
<tr>
<td>(\neg \Phi \cdot \Phi = \perp \cdot \neg \Phi = \perp)</td>
<td>\text{negCo_comp}</td>
</tr>
<tr>
<td>(R \cdot \neg (\delta R) = \perp)</td>
<td>\text{not_dom_cancel}</td>
</tr>
<tr>
<td>(R \cup S \subseteq T \equiv (R \subseteq T \wedge S \subseteq T))</td>
<td>\text{union_univ}</td>
</tr>
<tr>
<td>(\perp \subseteq R \equiv \text{True})</td>
<td>\text{incl_empty}</td>
</tr>
<tr>
<td>(R \subseteq R \equiv \text{True})</td>
<td>\text{incl_refl}</td>
</tr>
<tr>
<td>(R \cdot S \subseteq T \leftarrow R \cdot S \subseteq T)</td>
<td>\text{monotonicity}</td>
</tr>
<tr>
<td>(R \cdot f^o \subseteq S \equiv R \subseteq S \cdot f)</td>
<td>\text{shunt_funginv}</td>
</tr>
<tr>
<td>(f \cdot R \subseteq S \equiv R \subseteq f^o \cdot S)</td>
<td>\text{shunt_fung}</td>
</tr>
<tr>
<td>(R \cdot M^o \subseteq S \equiv R \cdot \delta M \subseteq S \cdot M)</td>
<td>\text{shunt_mapinv}</td>
</tr>
<tr>
<td>(M \cdot R \subseteq S \equiv \delta M \cdot R \subseteq M^o \cdot S)</td>
<td>\text{shunt_map}</td>
</tr>
<tr>
<td>(\text{img } R = R \cdot R^o)</td>
<td>\text{img_def}</td>
</tr>
<tr>
<td>(\ker R = R^o \cdot R)</td>
<td>\text{ker_def}</td>
</tr>
</tbody>
</table>

4 Rewriting relational expressions and propositions

We developed a type-safe, type-directed rewriting system for normalization of relational expressions that harnesses the various algebraic laws of binary relations presented in Table 1 in the functional programming language Haskell. In this section and the next, we will provide a high-level description independent of the programming language.

Terms. The terms to be rewritten by our term rewriting system will be the expression of binary relational calculus with some additional annotations. The following outlines the grammar:

\[
\begin{align*}
P & ::= R \subseteq T | \text{True} | P \wedge P \\
R & ::= id | R \circ R | R^o | V_L | \Delta_T | \ldots \\
T & ::= 1 | \text{Int} | \text{Bool} | \text{String} | [T] | T \times T | \ldots \\
L & ::= \text{entire} | \text{simple} | \text{injective} | \text{surjective} | \text{reflexive} | \text{coreflexive} \\
V & ::= \text{variable names} \\
A & ::= \text{values}
\end{align*}
\]

Thus, some of the relation operators are annotated with type information (shown as subscript). Relation variables are annotated with properties as they appear in the leaves of the taxonomy of Figure 3. For example, a function \(f\) is an entire and simple relation and is therefore annotated as \(f_{\text{entire,simple}}\). Also, endo-relations can be annotated to be (co-)reflexive.

Predicates on relations. The first ingredient into our rewriting system are predicates for testing the various properties that relations may have, such as simplicity, surjectivity, etc. The various properties declared on relational variables propagate through relational operators. For example, the composition of two surjective relations is surjective, and the inverse of an injective relation is simple. This gives rise to predicates on relations that inductively check their properties. For example:

\[
\begin{align*}
isSimple(id) & = \text{True} \\
isSimple(r_l) & = \text{simple} \in l \\
isSimple(r^o) & = \text{isInjective}(r) \\
isSimple(s \cdot b\ r) & = \text{isSimple}(r) \land \text{isSimple}(s) \\
\ldots \\
isSimple(r) & = \text{False}
\end{align*}
\]

Similar predicates are supplied for the remaining properties. These predicates test for properties by induction over the structure of relational expressions, but do not attempt to derive proofs for the properties. In this sense, they are approximations and may fail to discover that a certain relational expression enjoys particular properties.

Type-directed and property-aware rewriting rules.

The predicates above are used in the definition of rewrite rules. Each rewrite rule encodes particular laws of the relational calculus. Since our rewrite system is type-directed, rewrite rules are annotated with types. Here is an encoding of the \text{inv\_comp} law, applied in the left-to-right direction:

\[
\text{inv\_comp} : (r \cdot s)^o \mapsto_{(c \cdot a)} (s^o \cdot b)^o
\]

Pattern matching is performed on a relational expression and, on successful match, a resulting expression is returned.

The \text{const\_fusion} rule provides an example of rewriting directed by properties:

\[
\text{const\_fusion} : (s \cdot b\ r) \mapsto_{(c \cdot a)} (s^o \cdot b)^o
\]

The rule works on a composition and, if the first argument \(s\) constant as required by the guarding predicate, then it replaces the second argument \(r\) by its domain. If the second argument \(r\) is entire, \(\delta r = id\) then the rule return just the first (constant) argument. When \(r\) is coreflexive, the rule does not trigger, because the domain of a coreflexive is that relation itself.

An example of a rewrite rule on the level of relational propositions is offered by the shunting rule for functions:
shunt_fun_inv : 
(\langle x \cdot b f \rangle \subseteq_{(c-e-a)} y) \longmapsto_{(c-e-a)} \langle x \subseteq_{(c-e-b)} (y \cdot a f) \rangle 

if isEntire(f) \land isSimple(f)

Note the use of a guarding predicate that tests whether the relation \( f \) is indeed a function (entire and simple).

**Combinators for strategic rewriting.** To compose rewriting systems out of individual rewrite rules, we employ the following set of rule combinators known from strategic term rewriting:\(^1\):

- **nop** :: Rule -- identity rule
- **(▔)** :: Rule → Rule → Rule -- sequential comp.
- **(⊕)** :: Rule → Rule → Rule -- choice (based on mplas)
- **(⊘)** :: Rule → Rule → Rule -- choice (bas. on mcatch)
- **all** :: Rule → Rule -- map on all children
- **one** :: Rule → Rule -- map on one child
- **run** :: Rule → R r → (R r, Derivation) -- top-lev. app.

The implementation of each of these combinators is straightforward, and omitted here for brevity. The top-level application function \( \text{run} \) takes the result of rewriting and the derivation (proof trace) out of the \( \text{Rewrite} \) monad; in case of failure it returns the original term and an empty derivation.

Using the basic rule combinators, more sophisticated ones can be defined:

- **many** \( r \) = (\( r ▔ (\text{many} r) ⌫ \) \( \text{nop} \)) -- repeat until failure
- **once** \( r \) = \( r ◦ \text{one} (\text{oncer}) \) -- apply once, at any depth
- **innermost** \( r \) = all (innermost \( r ▔ (r ▔ \text{innermost} r ⌫ \) \( \text{nop} \))

The derived combinator \( \text{innermost} \) performs exhaustive rewrite rule application according to the leftmost innermost rewriting strategy.

**5 Rewriting strategies**

Having defined individual rules and rule combinators, we can proceed to the composition of rewrite systems for various purposes.

**Normalization of relational expressions.** The following definitions express that a relational expression can be normalized by exhaustive application of individual association, desugaring, and normalization rules:

- **simplify** = innermost simplify1
- **simplify1** = comp_assoc1 ◦ desugar1 ◦ applylaw1
- **desugar1** = ker_def ◦ img_def ◦ ...
- **applylaw1** =
  - inv_comp ◦ inv_inv ◦ comp_id ◦
  - comp_empty ◦ dom_elim ◦ corefl_symm ◦
  - const_fusion ◦ not_dom_cancel ◦ ...

We use the convention of postfixing the names of single-step rule combinations with 1 in order to distinguish them from rule combinations that rewrite repetitively until a fixpoint is reached. Note that the \( \text{comp_assoc} \) rule is employed to bring relational compositions into left-associative form. Since the normalization rules together form a confluent and terminating rewrite system, the left-catching combinator \( ◦ \) is sufficient to combine them — no need for backtracking.

For example, the following derivation is constructed when applying the \( \text{simplify} \) strategy to \((N \cdot (\neg (\delta N))^o \cdot M^o)^o\), where \( N \) and \( M \) are simple relations:

\[
\begin{align*}
(N \cdot (\neg (\delta N))^o \cdot M^o)^o &= \{ \text{corefl_symm} \} \\
(N \cdot (\neg (\delta N))^o \cdot M^o)^o &= \{ \text{not_dom_cancel} \} \\
(N \cdot (\neg (\delta N))^o \cdot M^o)^o &= \{ \text{comp_empty} \} \\
(N \cdot (\neg (\delta N))^o \cdot M^o)^o &= \{ \text{corefl_symm} \}
\end{align*}
\]

This normalization proof trace demonstrates that the original expression is equal to \( \perp \). (Recall that proof traces are generated by our \( \text{Rewrite} \) monad.)

**Deriving proofs and proof obligations.** We define a more sophisticated strategy to simplify or dispatch proof obligations:

\[
\begin{align*}
\text{derive} &= \text{simplify} ▔ \text{all} \land \text{process_conjunct ▔ innermost} \land \text{true}\\
\text{where} \\
\text{process_conjunct} &= (\text{shunt_conjunct} \oplus \text{strengthen_conjunct}) \oplus \text{nop} \\
\text{shunt_conjunct} &= \text{shunt} ▔ \text{derive} \\
\text{strengthen_conjunct} &= \text{strengthen} ▔ \text{derive} ▔ \text{qed} \\
\text{shunt} &= (\text{shunt_fun_inv} ▔ \text{shunt_map_inv}) \oplus (\text{shunt_fun} ▔ \text{shunt_map}) \\
\text{strengthen} &= \text{corefl_cancel} \\
\text{all} \land ▔ \text{Rule} ▔ \text{Rule} &= \text{apply arg. rule on all cons} \\
\text{qed} &= \text{Rule} ▔ \text{test whether the current exp. is True}
\end{align*}
\]

The initial application of \( \text{simplify} \) brings a given proposition into conjunctive normal form, where each conjunct is a normalized relational inclusion. The \( \text{all} \land ▔ \text{Rule} \) combinator applies \( \text{process_conjunct} \) to all conjuncts. After processing each conjunct separately, \( \text{and_true} \) \((p \land True \Rightarrow True \land p \Rightarrow p)\) is applied to absorb the propositions that have been rewritten to \( True \). The processing of each conjunct makes a non-deterministic choice (using the backtracking operator \( \oplus \)) between starting with a shunting step (\( \text{shunt_conjunct} \)) or starting with a strengthening step (\( \text{strengthen_conjunct} \)); the conjunct is left unchanged if neither is possible.

---

\(^1\)These rules and our representation technique are inspired on the 2LT system [13]. For related work using a similar approach see e.g. [14].
When starting π with shunting, the choice between shunting a left-composed relation or shunting a right-composed converse of a relation is again made non-deterministically (shunt). After the shunting step, a recursive call is made to the overal derive strategy. When starting with strengthening, the subsequent recursive call to derive is required to lead to a full proof (qed), since we are interested in strengthened propositions only for the purpose of discharging proof obligations.

The use of backtracking entails that several results may be obtained or the same result through different derivations. In the implementation, lazy evaluation is employed to ensure that only a single derivation is actually constructed.

6 Application scenarios

We now explain how our rewriting system can be used in concrete scenarios, such as the ones in our motivation example (Section 2). The overall operation of the developed tool is based on transforming and rewriting PF-relational expressions using the ESC/PF calculus described in [7].

List wanted items. The operation listWantedItems can be specified in binary relational terms as listWantedItems = Wanted · π₁, where π₁ is the first projection on pairs, i.e. π₁ (a, b) = a. Note that we leave the argument Wanted implicit in the definition of the operation. Regarding Wanted as a set of pairs, the definition converts to the pointwise \( \{ (p, i) \mid (p, (i, u)) \in \text{Wanted} \} \), where p, i, u range over Price, Id and Uid, respectively. Clearly, this won’t be a simple relation in general, even if Wanted is so, because dropping u from the input may lead to the same i related to different p. Since this operation is specified to produce a finite map (thus simple), it gives rise to the proof obligation \( \text{img}(\text{Wanted} \cdot \pi_1^i) \subseteq \text{id} \), which in turn leads to the following derivation when applying our derive strategy:

\[
\text{img}(\text{Wanted} \cdot \pi_1^i) \subseteq \text{id} \\
\Leftrightarrow \{ \text{img}_{\text{def}} \} \\
\text{Wanted} \cdot \pi_1^i (\text{Wanted} \cdot \pi_1^i) \circ \subseteq \text{id} \\
\Leftrightarrow \{ \text{inv}_{\text{comp}} \} \\
\text{Wanted} \cdot \pi_1^i (\pi_1^i) \circ \cdot \text{Wanted} \circ \subseteq \text{id} \\
\Leftrightarrow \{ \text{inv}_{\text{inv}} \} \\
\text{Wanted} \cdot \pi_1^i \cdot \pi_1 \cdot \text{Wanted} \circ \subseteq \text{id} \\
\Leftrightarrow \{ \text{shunt}_{\text{map}_{\text{inv}}} \} \\
\text{Wanted} \cdot \pi_1^i \cdot \pi_1 \cdot \text{δ Wanted} \subseteq \text{id} \cdot \text{Wanted} \\
\Leftrightarrow \{ \text{comp}_{\text{id}} \} \\
\text{Wanted} \cdot \pi_1^i \cdot \pi_1 \cdot \text{δ Wanted} \subseteq \text{Wanted} \\
\Leftrightarrow \{ \text{shunt}_{\text{map}} \} \\
\text{δ Wanted} \cdot \pi_1^i \cdot \pi_1 \cdot \text{δ Wanted} \subseteq \text{Wanted} \circ \text{Wanted}
\]

What does the last line above mean? We simply have to apply the rules of the PF-transform the other way round and find the corresponding, more descriptive logic expression:

\[
\forall x, y . x \in \text{dom} (\text{Wanted}) \land \langle E \rangle \\
y \in \text{dom} (\text{Wanted}) \land \pi_1 (x) = \pi_1 (y) \\
\Rightarrow \text{Wanted} (x) = \text{Wanted} (y)
\]

This formula expresses that query listWantedItems only returns a finite map if the Wanted collection contains no two offers for the same item with different prices. This feedback should lead the designer to broaden the output type of the operation to general binary relations.

Settle deal. Using singleton relation notation as described in Section 3, we can define settleDeal \((i, u, p) = \text{Deal} \cup (u, p) \circ i^\circ \). (Again we leave the old value of Deal implicit in the definition.) Checking the simplicity of its output gives rise to the following derivation (condensed):

\[
\text{img}(\text{Deal} \cup (u, p) \circ i^\circ) \subseteq \text{id} \\
\Leftrightarrow \{ \text{img}_{\text{def}}, \text{various union laws} \} \\
\text{Deal} \cdot \text{Deal}^\circ \subseteq \text{id} \land \\
\text{Deal} \cdot i \cdot (u, p) \circ \subseteq \text{id} \land \\
(u, p) \circ i^\circ \cdot \text{Deal}^\circ \subseteq \text{id} \land \\
(u, p) \circ i^\circ \cdot \text{Deal} \subseteq \text{id} \\
\Leftrightarrow \{ \text{various shunting laws, dom}_{\text{elim}} \} \\
\text{δ Deal} \cdot i \subseteq \text{Deal}^\circ \cdot (u, p) \land \\
i^\circ \cdot \text{δ Deal} \subseteq (u, p) \circ \text{Deal}
\]

Thus, the simplification of this proof obligation leads to an intermediate conjunction of four proof obligations, of which two are subsequently discharged. The remaining two obligations actually express the same property (they can be converted into each other by taking their inverse). Conversion back to pointwise notation gives the following precondition:

\[
i \in \text{dom} (\text{Deal}) \Rightarrow (u, p) = \text{Deal} (i)
\]

Note that the proof obligation we derived is weaker than the over-defensive precondition that is typically added to an operation such as settleDeal, namely that \(i \notin \text{dom} (\text{Deal})\).

Batch addition of items to sell. Once PF-transformed, our last function is defined by

\[
\text{putBatchForSale} (u, m) = \text{ForSale} \uplus x
\]

where \(x = \text{withUser} u m\) and \(\text{withUser} u m = (u, m)\). This model illustrates the use of two other useful binary operators on relations, \text{override} (\uplus) and \text{split} ((\cdot, \cdot)) [15]. The latter pairs the outputs of two relations (recall Fig. 1) and the former overrides one relation by another. Checking the simplicity of the output of putBatchForSale leads to a 32-step derivation of which we show only the starting and closing steps, the latter condensed for space economy:
from our approach: Grok is a calculator for finite relational algebra expressions and RelView uses BDDs to implement relations in an efficient way.

**Typed strategic rewriting.** Strategic programming [10] was first supported in the non-typed setting of the Stratego language [8]. A strongly-typed combinator suite was introduced as a Haskell library by the Strafnuki system [9] and later generalized into the so-called ‘scrap-your-boilerplate’ generic programming library [20]. We developed GADT-based strategic combinator suites, similar to the one presented here, for two-level data transformation [21] and transformation of pointfree and structure-shy functions [22]. A similar use of GADTs can be found in [14].

**Description logics.** Description logic (DL) [23] is a family of variable-free notations intended for knowledge representation. General knowledge as well as transient facts are represented by unary predicates (concepts, akin to co-reflexive binary relations) and binary predicates among them (relationships, akin to general binary relations). Operations include union, conjunction (intersection), negation (complement), role restriction (denoted by $\forall R. \Phi$ and having binary-relational interpretation $\sim (\delta (R \cdot \Phi))$), and existential restriction (denoted by $\exists R. \top$ and interpretable as $\delta (R \cdot \Phi)$).

Description logics are weaker than classical first-order logic, making them decidable and amenable to automatic reasoning. The DL community has explored many variations of the formalism and has produced a range of corresponding reasoning tools [24, 25]. These tools are generally aimed at solving subsumption of one concept by another (corresponding to the inclusion relation between (co-reflexive) binary relations) for knowledge bases with large numbers of facts. Reasoning and optimization methods include rule-based rewriting of DL-expressions.

**8 Concluding remarks**

We have implemented a type-directed strategic rewrite system for normalization of pointfree relational expressions and simplification or discharge of relational propositions. We have demonstrated the utility of the system in the context of extended static checking of common model and program properties. We intend to elaborate our approach in various directions.

So far, we have limited ourselves to rewriting of pointfree expressions, relying on manual transformation of logic formulæ into relational algebra expressions and back. We intend to also automate this point-free transform.

The suite of operators and laws implemented in the system is currently under study with respect to minimality, confluence and termination. We are especially interested in a comparison on expressiveness of the
supported binary relational algebra with the numerous variants of description logic.

The strategy for proof search is likely to further evolve as well, for instance to include short cut derivations for special common cases or to eliminate duplication of proof obligations due to converse inclusions. A thorough analysis of the formal properties of the rewriting system we are building is one of our current concerns.

When achieving a good degree of maturity, an assessment will be needed as to whether this approach can indeed be an alternative or supplement to existing ESC approaches based on theorem proving. A good test will be to try and discharge complex ESC/PF proof obligations such as those arising from the Verified File System project [26, 7]. Besides ESC, we envision to apply our relational algebra rewriting system to areas such as program optimization, program verification, relational programming, and more.

As suggested in [26], ESC/PF could be integrated in a Alloy-centric tool-chain complementing the bounded verification performed by the Alloy Analyzer with unbounded verification, thus enabling its application to the certification of highly-critical software systems.

Acknowledgements

The work of Claudia M. Necco was carried out within the Universidad Nacional de San Luis (UNSL), Project PROICO 3-40214 Problemas de Decisión y aplicaciones. The work by José N. Oliveira was funded by ERDF - European Regional Development Fund through the COMPETE Programme (operational programme for competitiveness) and by National Funds through the Portuguese Foundation for Science and Technology (FCT) within project POCI-01-0145-FEDER-016826.

Competing interests

The authors have declared that no competing interests exist.

References


