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THE GAUSSIAN MOMENTS CONJECTURE AND THE JACOBIAN CONJECTURE

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ABSTRACT. We first propose what we call the *Gaussian Moments Conjecture*. We then show that the Jacobian Conjecture follows from the Gaussian Moments Conjecture. Note that the the Gaussian Moments Conjecture is a special case of ([11, Conjecture 3.2]). The latter conjecture was referred as *Moment Vanishing Conjecture* in ([9, Conjecture A]) and *Integral Conjecture* in [6, Conjecture 3.1] (for the one-dimensional case). We also give a counter-example to show that ([11, Conjecture 3.2]) fails in general for polynomials in more than two variables.

1. Introduction

For a random variable X we denote its expected value by $\mathbb{E}(X)$. Suppose that $X = (X_1, \dots, X_n)$ is a random vector with a multi-variate normal distribution. We make the following conjecture:

Conjecture 1.1 (Gaussian Moments Conjecture **GMC**(n)). *Suppose that $P(x_1, \dots, x_n) \in \mathbb{C}[x_1, \dots, x_n]$ is a complex-valued polynomial such that the moments $\mathbb{E}(P(X)^m)$ are equal to 0 for all $m \geq 1$. Then for every polynomial $Q(x_1, \dots, x_n) \in \mathbb{C}[x_1, \dots, x_n]$ we have $\mathbb{E}(P(X)^m Q(X)) = 0$ for $m \gg 0$.*

By using translations and linear maps, we can normalize the random vector X such that X_1, \dots, X_n are independent, with mean 0 and variance 1.

The *Gaussian Moments Conjecture* is a special case of [11, Conjecture 3.2]. Furthermore, because of Proposition 3.3 and relation (3.2) in [11], the *Gaussian Moments Conjecture* is the special case of [11, Conjecture 3.1] for Hermite polynomials. Note that ([11, Conjecture 3.2]) was later referred as *Moment Vanishing Conjecture* in ([9, Conjecture

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A]), and *Integral Conjecture* in [6, Conjecture 3.1] (for one-dimensional case). Unfortunately, this conjecture is false in general, as can be seen from the following

Proposition 1.2. *Let $B = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0, x^2 + y^2 \leq 1\}$, $P(x, y) = (x + iy)^2$ and $Q(x, y) = x + iy$. Then $\int_B P(x, y)^m dx dy = 0$ for all $m \geq 1$, but $\int_B Q(x, y)P(x, y)^m dx dy \neq 0$ for all $m \geq 1$.*

Proof. For each $m \geq 1$, by using the polar coordinates (r, θ) we have

$$\begin{aligned} \int_B P(x, y)^m dx dy &= \int_0^1 \int_0^\pi r^{2m} e^{2mi\theta} r dr d\theta = 0; \\ \int_B Q(x, y)P(x, y)^m dx dy &= \int_0^1 \int_0^\pi r^{2m+1} e^{(2m+1)i\theta} r dr d\theta \\ &= \frac{2i}{(2m+3)(2m+1)} \neq 0. \end{aligned}$$

□

Remark 1.3. Note that Conjecture 3.2 in [11] is still open for univariate polynomials. It is also open for the (whole) disks or squares centered at the origin for polynomials in two variables.

Remark 1.4. The function $X_1^2 + X_2^2$ has an exponential distribution and more generally, $X_1^2 + \dots + X_{2k}^2$ has a χ^2 distribution. So, if the Gaussian Moments Conjecture is true for all $n \geq 1$, then the conjecture is also true when we replace the Gaussian distributions by exponential or χ^2 distributions. The Moments Conjecture for exponential distributions is equivalent to [5, Conjecture 4.1], which is a weaker form of the Factorial Conjecture ([5, Conjecture 4.2]).

One of the main open conjectures in affine algebraic geometry is the notorious Jacobian Conjecture, which was first proposed by O. H. Keller [7] in 1939. See also [1] and [3].

Conjecture 1.5 (Jacobian Conjecture **JC**(n)). *If $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is a polynomial map that is locally invertible, then it is globally invertible.*

The main result of this paper is:

Theorem 1.6. *If **GMC**(n) is true for all $n \geq 1$, then **JC**(n) is true for all $n \geq 1$.*

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2. Background

Suppose that A is a unital commutative \mathbb{C} -algebra.

Definition 2.1. A *Mathieu-Zhao space* (or *MZ space*) is a \mathbb{C} -linear subspace $V \subseteq A$ with the property that $f^m \in V$ for all $m \geq 1$ implies that for every $g \in A$, $f^m g \in V$ for $m \gg 0$.

Observe that in this definition we have changed the name *Mathieu subspace*, which was introduced by the third author in [11, 12], into *Mathieu-Zhao space* or *MZ space*. This follows a suggestion of the second author in [4]. For some more general studies of this new notion, see [12].

With the definition above we can now reformulate our main conjecture as follows.

Conjecture 2.2 (**GMC**(n), reformulation). *The subspace*

$$\{P(x_1, \dots, x_n) \in \mathbb{C}[x_1, \dots, x_n] \mid \mathbb{E}(P(X_1, \dots, X_n)) = 0\}$$

is an MZ space of $\mathbb{C}[x_1, \dots, x_n]$.

Suppose that G is a complex reductive algebraic group acting regularly on an affine variety Z . Then G also acts on the ring $\mathbb{C}[Z]$ of polynomial functions on Z . Let $K \subseteq G$ be a maximal compact subgroup. Then K is Zariski dense in G . The Reynolds operator $\mathcal{R}_Z : \mathbb{C}[Z] \rightarrow \mathbb{C}$ is the averaging operator:

$$\mathcal{R}_Z(f) = \int_{g \in K} g \cdot f \, d\mu.$$

where $d\mu$ is the Haar measure on K , normalized such that $\int_K d\mu = 1$.

Conjecture 2.3 (**Mathieu Conjecture MC**(Z)). *The kernel $\text{Ker}(\mathcal{R}_Z)$ of the Reynolds operator is an MZ space of $\mathbb{C}[Z]$.*

This conjecture is equivalent to the conjecture $C(\mathbb{C}[Z])$ of [8] (see [8, Corollary 1.3]). The group G acts on its own coordinate ring, and **MC**(G) implies **MC**(Z) ([8, Corollary 1.7]). The following theorem was proven in [8, Theorem 5.5]:

Theorem 2.4 (Mathieu). *If **MC**($\text{SL}_n(\mathbb{C})/\text{GL}_{n-1}(\mathbb{C})$) is true for all $n \geq 1$, then **JC**(n) is true for all $n \geq 1$.*

For later purposes, here we also point out that J. Duistermaat and W. van der Kallen [2] in 1998 had proved the Mathieu conjecture for the case of tori, which can be re-stated in terms of MZ spaces as follows.

Theorem 2.5 (Duistermaat and van der Kallen). *Let $x = (x_1, x_2, \dots, x_n)$ be n commutative free variables and M the subspace of the Laurent polynomial algebra $\mathbb{C}[x_1^{-1}, \dots, x_n^{-1}, x_1, \dots, x_n]$ consisting of the Laurent polynomials with no constant term. Then M is an MZ space of $\mathbb{C}[x_1^{-1}, \dots, x_n^{-1}, x_1, \dots, x_n]$.*

Let $\partial_i = \frac{\partial}{\partial z_i}$ be the partial derivative with respect to z_i . Define

$$\mathcal{E}_n : \mathbb{C}[w, z] = \mathbb{C}[w_1, \dots, w_n, z_1, \dots, z_n] \rightarrow \mathbb{C}[z]$$

such that

$$\mathcal{E}_n(P(w)Q(z)) = P(\partial)Q(z) \in \mathbb{C}[z].$$

Zhao made the following conjecture in [10]:

Conjecture 2.6 (Special Image Conjecture **SIC**(n)). *Ker(\mathcal{E}_n) is an MZ space of $\mathbb{C}[w, z]$.*

Zhao proved the following result ([10, Theorem 3.6, Theorem 3.7]):

Theorem 2.7 (Zhao). *If **SIC**(n) is true for all $n \geq 1$, then **JC**(n) is true for all $n \geq 1$.*

3. Reduction of the Jacobian Conjecture to the Gaussian Moments Conjecture

We define the linear map

$$\mathcal{F}_n : \mathbb{C}[w, z] = \mathbb{C}[w_1, \dots, w_n, z_1, \dots, z_n] \rightarrow \mathbb{C}$$

by setting

$$\mathcal{F}_n(P) = \mathcal{E}_n(P) |_{z=0}.$$

For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, set $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ and $\alpha! = \alpha_1! \alpha_2! \cdots \alpha_n!$. Then we have

$$\mathcal{F}_n(w^\alpha z^\beta) = \begin{cases} \alpha! & \text{if } \alpha = \beta; \\ 0 & \text{if } \alpha \neq \beta. \end{cases}$$

Proposition 3.1. *If Ker(\mathcal{F}_n) is an MZ space of $\mathbb{C}[w, z]$, then Ker(\mathcal{E}_n) is an MZ space of $\mathbb{C}[w, z]$, i.e. **SIC**(n) is true.*

Proof. Assume that $P^m \in \text{Ker}(\mathcal{E}_n)$ for $m \geq 1$. Then for each $\alpha \in \mathbb{C}^n$ we have

$$\mathcal{E}_n(P^m(w, z)) |_{z=\alpha} = \mathcal{E}_n(P^m(w, z + \alpha)) |_{z=0} = \mathcal{F}_n(P^m(w, z + \alpha)) = 0.$$

Hence $P^m(w, z + \alpha) \in \text{Ker}(\mathcal{F}_n)$ for all $m \geq 1$. Since Ker(\mathcal{F}_n) is an MZ space of $\mathbb{C}[w, z]$, for any $Q \in \mathbb{C}[w, z]$ and $\alpha \in \mathbb{C}^n$ we have $Q(w, z + \alpha)P(w, z + \alpha)^m \in \text{ker}(\mathcal{F}_n)$ for all $m \gg 0$. Therefore, for all $m \gg 0$ we have

$$\mathcal{E}_n(Q(w, z)P(w, z)^m) |_{z=\alpha} = \mathcal{F}_n(Q(w, z + \alpha)P(w, z + \alpha)^m) = 0.$$

Define $Z_N \subseteq \mathbb{C}^n$ to be the zero set of all $\mathcal{E}_n(Q(w, z)P(w, z)^m)$ with $m \geq N$. Clearly, Z_N is Zariski closed for all N , and $\bigcup_{N=1}^{\infty} Z_N = \mathbb{C}^n$. It follows that $Z_N = \mathbb{C}^n$ for some integer N , because a countable union of Zariski closed proper subsets cannot be the whole affine space. So for $m \geq N$, $\mathcal{E}_n(Q(w, z)P(w, z)^m)$ is the zero function. \square

Proposition 3.2. *If GMC(2n) is true, then $\text{Ker}(\mathcal{F}_n)$ is an MZ space of $\mathbb{C}[w, z]$.*

Proof. Let $X_1, \dots, X_n, Y_1, \dots, Y_n$ are $2n$ independent random variables with the normal distribution and with mean 0 and variance 1. Define complex-valued random variables W_j, Z_j and real-valued random variables R_j and T_j by

$$W_j = \frac{X_j - Y_j i}{\sqrt{2}} = R_j e^{-iT_j} \quad \text{and} \quad Z_j = \frac{X_j + Y_j i}{\sqrt{2}} = R_j e^{iT_j}.$$

Then $R_1, \dots, R_n, T_1, \dots, T_n$ are independent, and for every $1 \leq j \leq n$, R_j^2 has an exponential distribution with mean 1 and $\mathbb{E}(R_j^{2k}) = k!$. Now consider

$$\mathbb{E}(W^\alpha Z^\beta) = \mathbb{E}(R^{\alpha+\beta} e^{i \sum_j (\beta_j - \alpha_j) T_j}) = \prod_{j=1}^n \left(\mathbb{E}(R^{\alpha_j + \beta_j}) \mathbb{E}(e^{i(\beta_j - \alpha_j) T_j}) \right).$$

If $\beta \neq \alpha$, then $\beta_j \neq \alpha_j$ for some j , whence $\mathbb{E}(e^{i(\beta_j - \alpha_j) T_j}) = 0$ and $\mathbb{E}(W^\alpha Z^\beta) = 0$. If $\alpha = \beta$, then we have

$$\mathbb{E}(W^\alpha Z^\alpha) = \mathbb{E}(R^{2\alpha}) = \prod_j \mathbb{E}(R_j^{2\alpha_j}) = \prod_j \alpha_j! = \alpha!$$

It follows that $\mathbb{E}(W^\alpha Z^\beta) = \mathcal{F}_n(w^\alpha z^\beta)$ for all $\alpha, \beta \in \mathbb{N}^n$. By linearity, we get $\mathbb{E}(Q(W, Z)) = \mathcal{F}_n(Q(w, z))$ for every polynomial $Q(w, z) \in \mathbb{C}[w, z]$. It follows readily from GMC(2n) that $\text{Ker } \mathcal{F}_n$ is an MZ space of $\mathbb{C}[w, z]$. \square

Now we can prove our main result Theorem 1.6:

Proof of Theorem 1.6. It follows directly from Proposition 3.1, Proposition 3.2 and Theorem 2.7. \square

4. Some Special Cases of the Gaussian Moments Conjecture

We view $\mathbb{C}[x_1, \dots, x_n]$ as the coordinate ring of $V \cong \mathbb{C}^n$, where V is viewed as the standard representation of $O(n)$.

Proposition 4.1. *For homogeneous polynomials $P(x)$, GMC(n) follows from MC(V).*

Proof. Let $\Phi : \mathbb{C}[x_1, \dots, x_n] \rightarrow \mathbb{C}$ be given by $\Phi(P(x)) = \mathbb{E}(P(X))$. Any linear map $\mathbb{C}[x_1, \dots, x_n]_d \rightarrow \mathbb{C}$ is determined by an element of $S^d(V)$. Since Φ is invariant under the action of $O(n)$ it is given by an element of $S^d(V)^{O(n)}$. But $S^d(V)^{O(n)}$ is at most one dimensional and is spanned by the restriction of the Reynolds operator \mathcal{R}_V . So up to a constant, $\Phi(P(x)^m)$ is equal to $\mathcal{R}_V(P(x)^m)$. If $\mathbb{E}(P(X)^m) = 0$ for $m \geq 1$, then $\mathcal{R}_V(P(X)^m) = 0$ for $m \geq 1$. If $Q(x)$ is homogeneous, then $\mathcal{R}_V(P(x)^m Q(x)) = 0$ for $m \gg 0$. So $\mathbb{E}(P(X)^m Q(X)) = 0$ for $m \gg 0$. If $Q(X)$ is non-homogeneous then $\mathbb{E}(P(X)^m Q(X)) = 0$ for $m \gg 0$, because $\mathbb{E}(P(X)Q_d(X)) = 0$ for $m \gg 0$ for every homogeneous summand $Q_d(x)$ of $Q(x)$. \square

Proposition 4.2. *Suppose that X is a Gaussian Random Variable, and $P(x) \in \mathbb{C}[x]$ is a univariate polynomial such that $\mathbb{E}(P(X)^m) = 0$ for $m \geq 1$, then $P(x) = 0$. In particular, **GMC**(n) is true for $n = 1$.*

Proof. As observed in the beginning of this paper, **GMC**(n) is a special case of the Image Conjecture for Hermite polynomials. For $n = 1$ the case of Hermite polynomials is proved in Corollary 4.3 of [6]. \square

For a different proof of **GMC**(1), see Proposition 4.7 and Remark 4.8 of this section.

Proposition 4.3. *Let $P \in \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]$ such that for each $1 \leq k \leq n$ $P(x, y)$ as a polynomial in x_k and y_k is homogeneous. Then **GMC**($2n$) holds for P .*

Proof. For each $1 \leq k \leq n$, let d_k be the degree of f as a polynomial in x_k and y_k .

Making the change of variables for x_i and y_i ($1 \leq i \leq n$):

$$x_i = r_i \cos \theta_i \quad \text{and} \quad y_i = r_i \sin \theta_i,$$

we see that $P = (r_1^{d_1} r_2^{d_2} \dots r_n^{d_n})F$ for some polynomial F in $\cos \theta_i$ and $\sin \theta_i$ ($1 \leq i \leq n$), which is independent on r_i ($1 \leq i \leq n$).

Let $S^n := (S^1)^{\times n}$, where S^1 is the unit circle in \mathbb{C} . Denote by $d\mu_n$ the measure of $d\theta_1 d\theta_2 \dots d\theta_n$, which is a haar measure of the torus S_n . Then F can be viewed as S^n -finite function over the torus S^n . Furthermore, for any $m \geq 1$ we have

$$\begin{aligned} (4.1) \quad \mathbb{E}(P^m(X, Y)) &= \int_{r_1=0}^1 \dots \int_{r_n=0}^1 (r_1^{md_1+1} \dots r_n^{md_n+1}) \left(\int_{S^n} F^m d\mu_n \right) dr_1 \dots dr_n \\ &= A_m \int_{S^n} F^m d\mu_n, \end{aligned}$$

for some nonzero constant A_m .

Hence, if $\mathbb{E}(P^m) = 0$ when $m \gg 0$, then so is $\int_{S_n} F^m$. Since $d\mu_n$ is a Haar measure of the torus S_n , applying the Duistermaat-van der Kallen Theorem 2.5 to F we see that for each polynomial G in $\cos \theta_i$ and $\sin \theta_i$ ($1 \leq i \leq n$), we have $\int_{S_n} F^m G d\mu_n = 0$ when $m \gg 0$.

Now for each monomial $M(x, y)$ in x_i and y_i ($1 \leq i \leq n$), by Eq. (4.1) with P^m replaced by $P^m M$, we see that $\mathbb{E}(P^m M) = 0$ when $m \gg 0$. Hence for each polynomial $Q(x, y)$, we also have $\mathbb{E}(P^m Q) = 0$ when $m \gg 0$. Therefore **GMC**(2n) holds for P . \square

Since every homogeneous polynomial in two variables satisfies the condition of Proposition 4.3, we immediately have the following

Corollary 4.4. **GMC**(2) holds for all homogeneous polynomials P .

By a similar argument as in the proof of Proposition 4.3, we have also the following case of Conjecture 3.2 in [11]:

Corollary 4.5. Let B be the unit disk in \mathbb{R}^2 centered at the origin with the Lebesgue measure $dxdy$. Let $P \in \mathbb{C}[x, y]$ such that P is homogeneous and $\int_B P^m dxdy = 0$ for all $m \gg 0$. Then for every $Q \in \mathbb{C}[x, y]$ we have $\int_B P^m Q dxdy = 0$ for all $m \gg 0$.

In the rest of this section we point out that some results proved in [5] for the Factorial Conjecture ([5, Conjecture 4.2]) can also be proved similarly for **GMC**(n).

First, we give a proof for the following case of **GMC**(n), which is parallel to [5, Proposition 4.8].

Proposition 4.6. Let $F(x) \in \mathbb{C}[x_1, x_2, \dots, x_n]$ such that $F(0) \neq 0$. Then $\mathbb{E}(F^m(X)) \neq 0$ for infinitely many $m \geq 1$.

Proof. Let $\Phi : \mathbb{C}[x_1, \dots, x_n] \rightarrow \mathbb{C}$ be given by $\Phi(P(x)) = \mathbb{E}(P(X))$. Set $(-1)!! := 1$ and $(2k-1)!! := (2k-1)(2k-3)\cdots 3 \cdot 1$ for all $k \geq 1$. Furthermore, for each $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in 2\mathbb{N}$, we set $(\alpha-1)!! := \prod_{i=1}^n (\alpha_i - 1)!!$. Then for each $\alpha \in \mathbb{N}^n$, we have

$$(4.2) \quad \Phi(x^\alpha) = \begin{cases} (\alpha-1)!! & \text{if } \alpha \in 2\mathbb{N}^n; \\ 0 & \text{otherwise.} \end{cases}$$

Now assume that the proposition fails, i.e., there exists $N \geq 1$ such that $\Phi(F^m) = 0$ for all $m \geq N$. Since $F(0) \neq 0$, replacing F by $F/F(0)$ we may assume $F(0) = 1$. Write $F(x) = 1 - \sum_{i=1}^k c_i x^{\beta_i}$ with $c_i \in \mathbb{C}$ and $0 \neq \beta_i \in \mathbb{N}^n$ for all $1 \leq i \leq k$.

Note that if $c_i = 0$ for all $1 \leq i \leq k$, i.e., $F(x) = 1$, the proposition obviously holds. So we assume $c_i \neq 0$ for all $1 \leq i \leq k$. Replacing F by F^2 we may also assume that $0 \neq \beta_i \in 2\mathbb{N}$ for at least one $1 \leq i \leq k$.

Furthermore, by a reduction due to Mitya Boyarchenko (see the proof of [9, Theorem 4.1] or [5, Remarks 4.5 and 4.6]), we may also assume that $c_i \in \mathbb{Q}$ for all $1 \leq i \leq k$.

Let $B = \mathbb{Z}[c_1, c_2, \dots, c_k]$ and p be an odd prime such that $p \geq N$ and $\nu_p(c_i) = 0$ for all $1 \leq i \leq k$, where ν_p denotes an extension of the p -valuation of \mathbb{Z} to B .

Since $p \geq N$ and $F^p \equiv 1 - \sum_{i=1}^k c_i^p x^{p\beta_i} \pmod{pB}$, we have $\Phi(F^p) = 0$ and

$$(4.3) \quad 1 \equiv \sum_{\substack{1 \leq i \leq k \\ 0 \neq \beta_i \in 2\mathbb{N}^n}} c_i^p (p\beta_i - 1)!! \pmod{pB}.$$

Since each $0 \neq \beta_i \in 2\mathbb{N}^n$ in the sum above has at least one nonzero (and even) component, so $(p\beta_i - 1)!!$ is divisible by p . Then applying ν_p to Eq. (4.3) we get $\nu_p(1) = 0$, which is a contradiction. \square

The next proposition is parallel to [5, Proposition 4.10].

Proposition 4.7. *Let $F(x) = c_0 M_0 + \sum_{i=1}^d c_i M_i$ with $M_0 = x_1^{k_1} \cdots x_n^{k_n}$ such that $k_1 \geq 1$ and $k_1 \geq k_j$ for all $2 \leq j \leq n$; $c_i \in \mathbb{C}$ ($0 \leq i \leq d$) with $c_0 \neq 0$; and M_i ($1 \leq i \leq d$) are monomials in x that are divisible by $x_1^{k_1+1}$. Then $\mathbb{E}(F^m(X)) \neq 0$ for infinitely many $m \geq 1$.*

Proof. Replacing F by $c_0^{-1}F$ we may assume $c_0 = 1$ and replacing F by F^2 we may assume that k_1 is an even positive integer. Then under these assumptions the proof of [5, Proposition 4.10] works through similarly for the linear functional Φ of $\mathbb{C}[x_1, \dots, x_n]$ given in Eq. (4.2). \square

Remark 4.8. *Note that when $n = 1$ the conditions of Proposition 4.7 hold automatically for all nonzero univariate polynomials $F(x)$. Hence **GMC**(1) also follows directly from Proposition 4.7.*

Proposition 4.9. *Let $d \geq 1$ and $P(x) = \sum_{i=1}^n c_i x_i^d \in \mathbb{C}[x_1, \dots, x_n]$ for some $c_i \in \mathbb{C}$ ($1 \leq i \leq n$). Assume that $\mathbb{E}(P^m(X)) = 0$ for all $m \gg 0$. Then $P = 0$. In particular, **GMC**(n) holds for $P(x)$.*

This proposition can be proved similarly as Proposition 4.16 in [5] if we choose the integer m there to be even, and the prime p to be $(m+2)d-1$ or $(m+1)d-1$, depending d is odd or even, respectively. Note that the components k_i 's in the proof of Proposition 4.16 in [5] for our case must be even when m is chosen to be even.

5. Moment Vanishing Polynomials

Let again $X = (X_1, \dots, X_n)$ be a random vector with joint Gaussian distribution. For $n \geq 2$, there exist many polynomials $P(x) \in \mathbb{C}[x]$

for which $\mathbb{E}(P(X)^m) = 0$ for all $m \geq 1$: if 0 lies in the closure of the $O(n)$ orbit of $P(x)$, then $\mathbb{E}(P(x)^m) = 0$ for all $m \geq 1$. Indeed, if there exists a sequence of orthogonal matrices A_1, A_2, \dots such that $\lim_{k \rightarrow \infty} P(A_k(x)) = 0$, then we have $\mathbb{E}(P(X)) = \lim_{k \rightarrow \infty} \mathbb{E}(P(A_k(X))) = \mathbb{E}(\lim_{k \rightarrow \infty} P(A_k(X))) = \mathbb{E}(0) = 0$. A 1-parameter subgroup is a homomorphism $\lambda : \mathbb{C}^* \rightarrow O_n(\mathbb{C})$ of algebraic groups. We can view λ as an orthogonal matrix with entries in $\mathbb{C}[t, t^{-1}]$. If $P(\lambda(t)(x))$ lies in $t\mathbb{C}[t][x]$, then $\lim_{t \rightarrow 0} P(\lambda(t)x) = 0$ and 0 lies in the closure of the $O_n(\mathbb{C})$ orbit of $P(x)$. Conversely, the Hilbert-Mumford criterion states that if 0 lies in the $O_n(\mathbb{C})$ -orbit closure of $P(x)$, then there exists such a 1-parameter subgroup $\lambda : \mathbb{C}^* \rightarrow O_n(\mathbb{C})$ such that $P(\lambda(t)(x)) \in t\mathbb{C}[t][x]$. If $Q(x) \in \mathbb{C}[x]$, then for large m , $Q(\lambda(t)(x))P(\lambda(t)x)^m \in t\mathbb{C}[t][x]$ and

$$\mathbb{E}(Q(X)P(X)^m) = \mathbb{E}(\lim_{t \rightarrow 0} Q(\lambda(t)(X))P(\lambda(t)X) = \mathbb{E}(0) = 0.$$

We make the following conjecture:

Conjecture 5.1. *If $\mathbb{E}(P(X)^m) = 0$ for all $m \geq 1$, then there exists a 1-parameter subgroup $\lambda : \mathbb{C}^* \rightarrow O_n(\mathbb{C})$ such that $P(\lambda(t)(x)) \in t\mathbb{C}[t][x]$.*

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