THE GAUSSIAN MOMENTS CONJECTURE AND THE JACOBIAN CONJECTURE

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Abstract. We first propose what we call the *Gaussian Moments Conjecture*. We then show that the Jacobian Conjecture follows from the Gaussian Moments Conjecture. Note that the the Gaussian Moments Conjecture is a special case of ([11, Conjecture 3.2]). The latter conjecture was referred as *Moment Vanishing Conjecture* in ([9, Conjecture A]) and *Integral Conjecture* in [6, Conjecture 3.1] (for the one-dimensional case). We also give a counter-example to show that ([11, Conjecture 3.2]) fails in general for polynomials in more than two variables.

1. Introduction

For a random variable X we denote its expected value by $\mathbb{E}(X)$. Suppose that $X = (X_1, \ldots, X_n)$ is a random vector with a multi-variate normal distribution. We make the following conjecture:

Conjecture 1.1 (Gaussian Moments Conjecture $\mathbf{GMC}(n)$)**.** *Suppose that $P(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n]$ is a complex-valued polynomial such that the moments $\mathbb{E}(P(X)^m)$ are equal to 0 for all $m \geq 1$. Then for every polynomial $Q(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n]$ we have $\mathbb{E}(P(X)^m Q(X)) = 0$ for $m \gg 0$.*

By using translations and linear maps, we can normalize the random vector X such that $X_1, \ldots, X_n$ are independent, with mean 0 and variance 1.

The *Gaussian Moments Conjecture* is a special case of [11, Conjecture 3.2]. Furthermore, because of Proposition 3.3 and relation (3.2) in [11], the *Gaussian Moments Conjecture* is the special case of [11, Conjecture 3.1] for Hermite polynomials. Note that ([11, Conjecture 3.2]) was later referred as *Moment Vanishing Conjecture* in ([9, Conjecture

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A]), and *Integral Conjecture* in [6, Conjecture 3.1] (for one-dimensional case). Unfortunately, this conjecture is false in general, as can be seen from the following

Proposition 1.2. *Let* $B = \{(x,y) \in \mathbb{R}^2 \mid y \geq 0,\ x^2+y^2 \leq 1\}$, $P(x,y) = (x+iy)^2$ *and* $Q(x,y) = x+iy$. *Then* $\int_B P(x,y)^m\, dxdy = 0$ *for all* $m \geq 1$, *but* $\int_B Q(x,y)P(x,y)^m\, dxdy \neq 0$ *for all* $m \geq 1$.

Proof. For each $m \geq 1$, by using the polar coordinates (r,θ) we have

$$\int_B P(x,y)^m\, dxdy = \int_0^1 \int_0^\pi r^{2m} e^{2mi\theta} r\, drd\theta = 0;$$

$$\begin{aligned} \int_B Q(x,y)P(x,y)^m\, dxdy &= \int_0^1 \int_0^\pi r^{2m+1} e^{(2m+1)i\theta} r\, drd\theta \\ &= \frac{2i}{(2m+3)(2m+1)} \neq 0. \end{aligned}$$

□

Remark 1.3. Note that Conjecture 3.2 in [11] is still open for univariate polynomials. It is also open for the (whole) disks or squares centered at the origin for polynomials in two variables.

Remark 1.4. The function $X_1^2 + X_2^2$ has an exponential distribution and more generally, $X_1^2 + \cdots + X_{2k}^2$ has a χ^2 distribution. So, if the Gaussian Moments Conjecture is true for all $n \geq 1$, then the conjecture is also true when we replace the Gaussian distributions by exponential or χ^2 distributions. The Moments Conjecture for exponential distributions is equivalent to [5, Conjecture 4.1], which is a weaker form of the Factorial Conjecture ([5, Conjecture 4.2]).

One of the main open conjectures in affine algebraic geometry is the notorious Jacobian Conjecture, which was first proposed by O. H. Keller [7] in 1939. See also [1] and [3].

Conjecture 1.5 (Jacobian Conjecture $\mathbf{JC}(n)$)**.** *If* $F : \mathbb{C}^n \to \mathbb{C}^n$ *is a polynomial map that is locally invertible, then it is globally invertible.*

The main result of this paper is:

Theorem 1.6. *If* $\mathbf{GMC}(n)$ *is true for all* $n \geq 1$, *then* $\mathbf{JC}(n)$ *is true for all* $n \geq 1$.

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2. **Background**

Suppose that A is a unital commutative $\mathbb{C}$-algebra.

Definition 2.1. A *Mathieu-Zhao space* (or *MZ space*) is a $\mathbb{C}$-linear subspace $V \subseteq A$ with the property that $f^m \in V$ for all $m \geq 1$ implies that for every $g \in A$, $f^m g \in V$ for $m \gg 0$.

Observe that in this definition we have changed the name *Mathieu subspace*, which was introduced by the third author in [11, 12], into *Mathieu-Zhao space* or *MZ space*. This follows a suggestion of the second author in [4]. For some more general studies of this new notion, see [12].

With the definition above we can now reformulate our main conjecture as follows.

Conjecture 2.2 (**GMC**(n), reformulation)**.** *The subspace*

$$\{P(x_1,\ldots,x_n) \in \mathbb{C}[x_1,\ldots,x_n] \mid \mathbb{E}(P(X_1,\ldots,X_n)) = 0\}$$

is an MZ space of $\mathbb{C}[x_1,\ldots,x_n]$.

Suppose that G is a complex reductive algebraic group acting regularly on an affine variety Z. Then G also acts on the ring $\mathbb{C}[Z]$ of polynomial functions on Z. Let $K \subseteq G$ be a maximal compact subgroup. Then K is Zariski dense in G. The Reynolds operator $\mathcal{R}_Z : \mathbb{C}[Z] \to \mathbb{C}$ is the averaging operator:

$$\mathcal{R}_Z(f) = \int_{g\in K} g \cdot f \, d\mu.$$

where $d\mu$ is the Haar measure on K, normalized such that $\int_K d\mu = 1$.

Conjecture 2.3 (Mathieu Conjecture **MC**(Z))**.** *The kernel* $\mathrm{Ker}(\mathcal{R}_Z)$ *of the Reynolds operator is an MZ space of* $\mathbb{C}[Z]$.

This conjecture is equivalent to the conjecture $C(\mathbb{C}[Z])$ of [8] (see [8, Corollary 1.3]). The group G acts on its own coordinate ring, and $\mathbf{MC}(G)$ implies $\mathbf{MC}(Z)$ ([8, Corollary 1.7]). The following theorem was proven in [8, Theorem 5.5]:

Theorem 2.4 (Mathieu)**.** *If* $\mathbf{MC}(\mathrm{SL}_n(\mathbb{C})/\mathrm{GL}_{n-1}(\mathbb{C}))$ *is true for all* $n \geq 1$, *then* $\mathbf{JC}(n)$ *is true for all* $n \geq 1$.

For later purposes, here we also point out that J. Duistermaat and W. van der Kallen [2] in 1998 had proved the Mathieu conjecture for the case of tori, which can be re-stated in terms of MZ spaces as follows.

Theorem 2.5 (Duistermaat and van der Kallen)**.** *Let $x = (x_1, x_2, ..., x_n)$ be n commutative free variables and M the subspace of the Laurent polynomial algebra $\mathbb{C}[x_1^{-1}, \dots, x_n^{-1}, x_1, \dots, x_n]$ consisting of the Laurent polynomials with no constant term. Then M is an MZ space of $\mathbb{C}[x_1^{-1}, \dots, x_n^{-1}, x_1, \dots, x_n]$.*

Let $\partial_i = \frac{\partial}{\partial z_i}$ be the partial derivative with respect to z_i. Define

$$\mathcal{E}_n : \mathbb{C}[w, z] = \mathbb{C}[w_1, \dots, w_n, z_1, \dots, z_n] \to \mathbb{C}[z]$$

such that

$$\mathcal{E}_n\big(P(w)Q(z)\big) = P(\partial)Q(z) \in \mathbb{C}[z].$$

Zhao made the following conjecture in [10]:

Conjecture 2.6 (Special Image Conjecture **SIC**(n))**.** Ker$(\mathcal{E}_n)$ *is an MZ space of* $\mathbb{C}[w, z]$.

Zhao proved the following result ([10, Theorem 3.6, Theorem 3.7]):

Theorem 2.7 (Zhao)**.** *If* **SIC**(n) *is true for all $n \geq 1$, then* **JC**(n) *is true for all $n \geq 1$.*

3. Reduction of the Jacobian Conjecture to the Gaussian Moments Conjecture

We define the linear map

$$\mathcal{F}_n : \mathbb{C}[w, z] = \mathbb{C}[w_1, \dots, w_n, z_1, \dots, z_n] \to \mathbb{C}$$

by setting

$$\mathcal{F}_n(P) = \mathcal{E}_n(P) \mid_{z=0} .$$

For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, set $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ and $\alpha! = \alpha_1!\alpha_2! \cdots \alpha_n!$. Then we have

$$\mathcal{F}_n(w^\alpha z^\beta) = \begin{cases} \alpha! & \text{if } \alpha = \beta; \\ 0 & \text{if } \alpha \neq \beta. \end{cases}$$

Proposition 3.1. *If* Ker$(\mathcal{F}_n)$ *is an MZ space of* $\mathbb{C}[w, z]$*, then* Ker$(\mathcal{E}_n)$ *is an MZ space of* $\mathbb{C}[w, z]$*, i.e.* **SIC**(n) *is true.*

Proof. Assume that $P^m \in \text{Ker}(\mathcal{E}_n)$ for $m \geq 1$. Then for each $\alpha \in \mathbb{C}^n$ we have

$$\mathcal{E}_n(P^m(w, z)) \mid_{z=\alpha} = \mathcal{E}_n(P^m(w, z + \alpha)) \mid_{z=0} = \mathcal{F}_n(P^m(w, z + \alpha)) = 0.$$

Hence $P^m(w, z + \alpha) \in \text{Ker}(\mathcal{F}_n)$ for all $m \geq 1$. Since Ker$(\mathcal{F}_n)$ is an MZ space of $\mathbb{C}[w, z]$, for any $Q \in \mathbb{C}[w, z]$ and $\alpha \in \mathbb{C}^n$ we have $Q(w, z + \alpha)P(w, z + \alpha)^m \in \ker(\mathcal{F}_n)$ for all $m \gg 0$. Therefore, for all $m \gg 0$ we have

$$\mathcal{E}_n(Q(w, z)P(w, z)^m) \mid_{z=\alpha} = \mathcal{F}_n(Q(w, z + \alpha)P(w, z + \alpha)^m) = 0.$$

Define $Z_N \subseteq \mathbb{C}^n$ to be the zero set of all $\mathcal{E}_n(Q(w,z)P(w,z)^m)$ with $m \geq N$. Clearly, Z_N is Zariski closed for all N, and $\bigcup_{N=1}^{\infty} Z_N = \mathbb{C}^n$. It follows that $Z_N = \mathbb{C}^n$ for some integer N, because a countable union of Zariski closed proper subsets cannot be the whole affine space. So for $m \geq N$, $\mathcal{E}_n(Q(w,z)P(w,z)^m)$ is the zero function. □

Proposition 3.2. *If* $\mathbf{GMC}(2n)$ *is true, then* $\mathrm{Ker}(\mathcal{F}_n)$ *is an MZ space of* $\mathbb{C}[w,z]$.

Proof. Let $X_1,\dots,X_n,Y_1,\dots,Y_n$ are $2n$ independent random variables with the normal distribution and with mean 0 and variance 1. Define complex-valued random variables W_j, Z_j and real-valued random variables R_j and T_j by

$$W_j = \frac{X_j - Y_j i}{\sqrt{2}} = R_j e^{-iT_j} \text{ and } Z_j = \frac{X_j + Y_j i}{\sqrt{2}} = R_j e^{iT_j}.$$

Then $R_1,\dots,R_n,T_1,\dots,T_n$ are independent, and for every $1 \leq j \leq n$, R_j^2 has an exponential distribution with mean 1 and $\mathbb{E}(R_j^{2k}) = k!$. Now consider

$$\mathbb{E}(W^\alpha Z^\beta) = \mathbb{E}(R^{\alpha+\beta} e^{i\sum_j(\beta_j-\alpha_j)T_j}) = \prod_{j=1}^{n}\Big(\mathbb{E}(R^{\alpha_j+\beta_j})\mathbb{E}(e^{i(\beta_j-\alpha_j)T_j})\Big).$$

If $\beta \neq \alpha$, then $\beta_j \neq \alpha_j$ for some j, whence $\mathbb{E}(e^{i(\beta_j-\alpha_j)T_j}) = 0$ and $\mathbb{E}(W^\alpha Z^\beta) = 0$. If $\alpha = \beta$, then we have

$$\mathbb{E}(W^\alpha Z^\alpha) = \mathbb{E}(R^{2\alpha}) = \prod_j \mathbb{E}(R_j^{2\alpha_j}) = \prod_j \alpha_j! = \alpha!$$

It follows that $\mathbb{E}(W^\alpha Z^\beta) = \mathcal{F}_n(w^\alpha z^\beta)$ for all $\alpha,\beta \in \mathbb{N}^n$. By linearity, we get $\mathbb{E}(Q(W,Z)) = \mathcal{F}_n(Q(w,z))$ for every polynomial $Q(w,z) \in \mathbb{C}[w,z]$. It follows readily from $\mathbf{GMC}(2n)$ that $\mathrm{Ker}\,\mathcal{F}_n$ is an MZ space of $\mathbb{C}[w,z]$. □

Now we can prove our main result Theorem 1.6:

Proof of Theorem 1.6. It follows directly from Proposition 3.1, Proposition 3.2 and Theorem 2.7. □

4. Some Special Cases of the Gaussian Moments Conjecture

We view $\mathbb{C}[x_1,\dots,x_n]$ as the coordinate ring of $V \cong \mathbb{C}^n$, where V is viewed as the standard representation of $\mathrm{O}(n)$.

Proposition 4.1. *For homogeneous polynomials* $P(x)$, $\mathbf{GMC}(n)$ *follows from* $\mathbf{MC}(V)$.

Proof. Let $\Phi : \mathbb{C}[x_1, \dots, x_n] \to \mathbb{C}$ be given by $\Phi(P(x)) = \mathbb{E}(P(X))$. Any linear map $\mathbb{C}[x_1, \dots, x_n]_d \to \mathbb{C}$ is determined by an element of $S^d(V)$. Since Φ is invariant under the action of $O(n)$ it is given by an element of $S^d(V)^{\mathrm{O}(n)}$. But $S^d(V)^{\mathrm{O}(n)}$ is at most one dimensional and is spanned by the restriction of the Reynolds operator $\mathcal{R}_V$. So up to a constant, $\Phi(P(x)^m)$ is equal to $\mathcal{R}_V(P(x)^m)$. If $\mathbb{E}(P(X)^m) = 0$ for $m \geq 1$, then $\mathcal{R}_V(P(X)^m) = 0$ for $m \geq 1$. If $Q(x)$ is homogeneous, then $\mathcal{R}_V(P(x)^m Q(x)) = 0$ for $m \gg 0$. So $\mathbb{E}(P(X)^m Q(X)) = 0$ for $m \gg 0$. If $Q(X)$ is non-homogeneous then $\mathbb{E}(P(X)^m Q(X)) = 0$ for $m \gg 0$, because $\mathbb{E}(P(X) Q_d(X)) = 0$ for $m \gg 0$ for every homogeneous summand $Q_d(x)$ of $Q(x)$. □

Proposition 4.2. *Suppose that X is a Gaussian Random Variable, and $P(x) \in \mathbb{C}[x]$ is a univariate polynomial such that $\mathbb{E}(P(X)^m) = 0$ for $m \geq 1$, then $P(x) = 0$. In particular,* $\mathbf{GMC}(n)$ *is true for $n = 1$.*

Proof. As observed in the beginning of this paper, $\mathbf{GMC}(n)$ is a special case of the Image Conjecture for Hermite polynomials. For $n = 1$ the case of Hermite polynomials is proved in Corollary 4.3 of [6]. □

For a different proof of $\mathbf{GMC}(1)$, see Proposition 4.7 and Remark 4.8 of this section.

Proposition 4.3. *Let $P \in \mathbb{C}[x_1, ..., x_n, y_1, ..., y_n]$ such that for each $1 \leq k \leq n$ $P(x, y)$ as a polynomial in x_k and y_k is homogeneous. Then* $\mathbf{GMC}(2n)$ *holds for P.*

Proof. For each $1 \leq k \leq n$, let d_k be the degree of f as a polynomial in x_k and y_k.

Making the change of variables for x_i and y_i $(1 \leq i \leq n)$:

$$x_i = r_i \cos\theta_i \quad \text{and} \quad y_i = r_i \sin\theta_i,$$

we see that $P = (r_1^{d_1} r_2^{d_2} \cdots r_n^{d_n}) F$ for some polynomial F in $\cos\theta_i$ and $\sin\theta_i$ $(1 \leq i \leq n)$, which is independent on r_i $(1 \leq i \leq n)$.

Let $S^n := (S^1)^{\times n}$, where S^1 is the unit circle in $\mathbb{C}$. Denote by $d\mu_n$ the measure of $d\theta_1 d\theta_2 \cdots d\theta_n$, which is a haar measure of the torus S_n. Then F can be viewed as S^n-finite function over the torus S^n. Furthermore, for any $m \geq 1$ we have

(4.1)
$$\begin{aligned}\mathbb{E}(P^m(X,Y)) &= \int_{r_1=0}^{1} \cdots \int_{r_n=0}^{1} (r_1^{md_1+1} \cdots r_n^{md_n+1}) \Big(\int_{S^n} F^m d\mu_n \Big)\, dr_1 \cdots dr_n \\ &= A_m \int_{S^n} F^m d\mu_n,\end{aligned}$$

for some nonzero constant A_m.

Hence, if $\mathbb{E}(P^m) = 0$ when $m \gg 0$, then so is $\int_{S_n} F^m$. Since $d\mu_n$ is a Haar measure of the torus S_n, applying the Duistermaat-van der Kallen Theorem 2.5 to F we see that for each polynomial G in $\cos\theta_i$ and $\sin\theta_i$ $(1 \le i \le n)$, we have $\int_{S^n} F^m G \, d\mu_n = 0$ when $m \gg 0$.

Now for each monomial $M(x, y)$ in x_i and y_i $(1 \le i \le n)$, by Eq. (4.1) with P^m replaced by $P^m M$, we see that $\mathbb{E}(P^m M) = 0$ when $m \gg 0$. Hence for each polynomial $Q(x, y)$, we also have $\mathbb{E}(P^m Q) = 0$ when $m \gg 0$. Therefore $\mathbf{GMC}(2n)$ holds for P. $\square$

Since every homogeneous polynomial in two variables satisfies the condition of Proposition 4.3, we immediately have the following

Corollary 4.4. $\mathbf{GMC}(2)$ *holds for all homogeneous polynomials* P.

By a similar argument as in the proof of Proposition 4.3, we have also the following case of Conjecture 3.2 in [11]:

Corollary 4.5. *Let* B *be the unit disk in* $\mathbb{R}^2$ *centered at the origin with the Lebesgue measure* $dxdy$. *Let* $P \in \mathbb{C}[x, y]$ *such that* P *is homogeneous and* $\int_B P^m \, dxdy = 0$ *for all* $m \gg 0$. *Then for every* $Q \in \mathbb{C}[x, y]$ *we have* $\int_B P^m Q \, dxdy = 0$ *for all* $m \gg 0$.

In the rest of this section we point out that some results proved in [5] for the Factorial Conjecture ([5, Conjecture 4.2]) can also be proved similarly for $\mathbf{GMC}(n)$.

First, we give a proof for the following case of $\mathbf{GMC}(n)$, which is parallel to [5, Proposition 4.8].

Proposition 4.6. *Let* $F(x) \in \mathbb{C}[x_1, x_2, ..., x_n]$ *such that* $F(0) \neq 0$. *Then* $\mathbb{E}(F^m(X)) \neq 0$ *for infinitely many* $m \ge 1$.

Proof. Let $\Phi : \mathbb{C}[x_1, \dots, x_n] \to \mathbb{C}$ be given by $\Phi(P(x)) = \mathbb{E}(P(X))$. Set $(-1)!! := 1$ and $(2k-1)!! := (2k-1)(2k-3)\cdots 3 \cdot 1$ for all $k \ge 1$. Furthermore, for each $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in 2\mathbb{N}$, we set $(\alpha - 1)!! := \prod_{i=1}^n (\alpha_i - 1)!!$. Then for each $\alpha \in \mathbb{N}^n$, we have

$$\Phi(x^\alpha) = \begin{cases} (\alpha - 1)!! & \text{if } \alpha \in 2\mathbb{N}^n; \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

Now assume that the proposition fails, i.e., there exists $N \ge 1$ such that $\Phi(F^m) = 0$ for all $m \ge N$. Since $F(0) \neq 0$, replacing F by $F/F(0)$ we may assume $F(0) = 1$. Write $F(x) = 1 - \sum_{i=1}^k c_i x^{\beta_i}$ with $c_i \in \mathbb{C}$ and $0 \neq \beta_i \in \mathbb{N}^n$ for all $1 \le i \le k$.

Note that if $c_i = 0$ for all $1 \le i \le k$, i.e., $F(x) = 1$, the proposition obviously holds. So we assume $c_i \neq 0$ for all $1 \le i \le k$. Replacing F by F^2 we may also assume that $0 \neq \beta_i \in 2\mathbb{N}$ for at least one $1 \le i \le k$.

Furthermore, by a reduction due to Mitya Boyarchenko (see the proof of [9, Theorem 4.1] or [5, Remarks 4.5 and 4.6]), we may also assume that $c_i \in \bar{\mathbb{Q}}$ for all $1 \le i \le k$.

Let $B = \mathbb{Z}[c_1, c_2, ..., c_k]$ and p be an odd prime such that $p \ge N$ and $\nu_p(c_i) = 0$ for all $1 \le i \le k$, where ν_p denotes an extension of the p-valuation of $\mathbb{Z}$ to B.

Since $p \ge N$ and $F^p \equiv 1 - \sum_{i=1}^k c_i^p x^{p\beta_i} \pmod{pB}$, we have $\Phi(F^p) = 0$ and

$$1 \equiv \sum_{\substack{1 \le i \le k \\ 0 \ne \beta_i \in 2\mathbb{N}^n}} c_i^p \, (p\beta_i - 1)!! \pmod{pB}. \tag{4.3}$$

Since each $0 \ne \beta_i \in 2\mathbb{N}^n$ in the sum above has at least one nonzero (and even) component, so $(p\beta_i - 1)!!$ is divisible by p. Then applying ν_p to Eq. (4.3) we get $\nu_p(1) = 0$, which is a contradiction. □

The next proposition is parallel to [5, Proposition 4.10].

Proposition 4.7. *Let $F(x) = c_0 M_0 + \sum_{i=1}^d c_i M_i$ with $M_0 = x_1^{k_1} \cdots x_n^{k_n}$ such that $k_1 \ge 1$ and $k_1 \ge k_j$ for all $2 \le j \le n$; $c_i \in \mathbb{C}$ $(0 \le i \le d)$ with $c_0 \ne 0$; and M_i $(1 \le i \le d)$ are monomials in x that are divisible by $x_1^{k_1+1}$. Then $\mathbb{E}(F^m(X)) \ne 0$ for infinitely many $m \ge 1$.*

Proof. Replacing F by $c_0^{-1}F$ we may assume $c_0 = 1$ and replacing F by F^2 we may assume that k_1 is an even positive integer. Then under these assumptions the proof of [5, Proposition 4.10] works through similarly for the linear functional Φ of $\mathbb{C}[x_1, \ldots, x_n]$ given in Eq. (4.2). □

Remark 4.8. *Note that when $n = 1$ the conditions of Proposition 4.7 hold automatically for all nonzero univariate polynomials $F(x)$. Hence* **GMC**(1) *also follows directly from Proposition 4.7.*

Proposition 4.9. *Let $d \ge 1$ and $P(x) = \sum_{i=1}^n c_i x_i^d \in \mathbb{C}[x_1, ..., x_n]$ for some $c_i \in \mathbb{C}$ $(1 \le i \le n)$. Assume that $\mathbb{E}(P^m(X)) = 0$ for all $m \gg 0$. Then $P = 0$. In particular,* **GMC**(n) *holds for $P(x)$.*

This proposition can be proved similarly as Proposition 4.16 in [5] if we choose the integer m there to be even, and the prime p to be $(m+2)d - 1$ or $(m+1)d - 1$, depending d is odd or even, respectively. Note that the components k_i's in the proof of Proposition 4.16 in [5] for our case must be even when m is chosen to be even.

5. Moment Vanishing Polynomials

Let again $X = (X_1, \ldots, X_n)$ be a random vector with joint Gaussian distribution. For $n \ge 2$, there exist many polynomials $P(x) \in \mathbb{C}[x]$

for which $\mathbb{E}(P(X)^m) = 0$ for all $m \geq 1$: if 0 lies in the closure of the $O(n)$ orbit of $P(x)$, then $\mathbb{E}(P(x)^m) = 0$ for all $m \geq 1$. Indeed, if there exists a sequence of orthogonal matrices $A_1, A_2, \ldots$ such that $\lim_{k\to\infty} P(A_k(x)) = 0$, then we have $\mathbb{E}(P(X)) = \lim_{k\to\infty} \mathbb{E}(P(A_k(X))) = \mathbb{E}(\lim_{k\to\infty} P(A_k(X))) = \mathbb{E}(0) = 0$. A 1-parameter subgroup is a homomorphism $\lambda : \mathbb{C}^\star \to \mathrm{O}_n(\mathbb{C})$ of algebraic groups. We can view λ as an orthogonal matrix with entries in $\mathbb{C}[t, t^{-1}]$. If $P(\lambda(t)(x))$ lies in $t\mathbb{C}[t][x]$, then $\lim_{t\to 0} P(\lambda(t)x) = 0$ and 0 lies in the closure of the $O_n(\mathbb{C})$ orbit of $P(x)$. Conversely, the Hilbert-Mumford criterion states that if 0 lies in the $\mathrm{O}_n(\mathbb{C})$-orbit closure of $P(x)$, then there exists such a 1-parameter subgroup $\lambda : \mathbb{C}^\star \to \mathrm{O}_n(\mathbb{C})$ such that $P(\lambda(t)(x)) \in t\mathbb{C}[t][x]$. If $Q(x) \in \mathbb{C}[x]$, then for large m, $Q(\lambda(t)(x))P(\lambda(t)x)^m \in t\mathbb{C}[t][x]$ and

$$\mathbb{E}(Q(X)P(X)^m) = \mathbb{E}(\lim_{t\to 0} Q(\lambda(t)(X))P(\lambda(t)X) = \mathbb{E}(0) = 0.$$

We make the following conjecture:

Conjecture 5.1. *If $\mathbb{E}(P(X)^m) = 0$ for all $m \geq 1$, then there exists a 1-parameter subgroup $\lambda : \mathbb{C}^\star \to \mathrm{O}_n(\mathbb{C})$ such that $P(\lambda(t)(x)) \in t\mathbb{C}[t][x]$.*

References

[1] Hyman Bass, Edwin H. Connell and David Wright, *The Jacobian conjecture, reduction of degree and formal expansion of the inverse*. Bull. Amer. Math. Soc. **7**, (1982), 287–330.

[2] J. J. Duistermaat and Wilberd van der Kallen, *Constant terms in powers of a Laurent polynomial*. Indag. Math. (N.S.) 9 (1998), no. 2, 221231. [MR1691479].

[3] Arno van den Essen, *Polynomial Automorphisms and the Jacobian Conjecture*. Progress in Mathematics, 190. Birkhöuser Verlag, Basel, 2000.

[4] Arno van den Essen, *An Introduction to Mathieu Subspaces*, Lectures given at the International Conference/Short-School on Affine Algebraic Geometry, held at the Chern Institute of Mathematics, Tianjin, China, 14-25 July, 2014.

[5] Arno van den Essen, David Wright and Wenhua Zhao, *On the Image Conjecture*, J. of Algebra **340** (2011), 211–224.

[6] Arno van den Essen and Wenhua Zhao, *Mathieu subspaces of univariate polynomial algebras*, J. of Pure and Applied Algebra **217** (2013), no. 7, 1316–1324.

[7] O. H. Keller, *Ganze Gremona-Transformationen*. Monats. Math. Physik **47** (1939), no. 1, 299-306. [MR1550818].

[8] Olivier Mathieu, *Some conjectures about Invariant Theory and their applications*, Algèbre non commutative, groupes qunatiques et invariants (Reims, 1995), Sémin. Congr. **2**, Soc. Math. France, Paris, 1997.

[9] Jean-Pierre Françoise, Fedor Pakovich, Yosef Yomdin and Wenhua Zhao, *Moment Vanishing Problem and positivity; some examples*, Bull. des Sciences Mathématiques **135**, issue 1, 10–32.

[10] Wenhua Zhao, *Images of commuting differential operators of order one with constant leading coefficients*, J. of Algebra **324** (2010), no. 2, 231–247.

[11] Wenhua Zhao, *Generalizations of the Image Conjecture and the Mathieu Conjecture*, J. of Pure and Applied Algebra **214** (2010), no. 7, 1200–1216.
[12] Wenhua Zhao, *Mathieu Subspaces of Associative Algebras*. J. Algebra **350** (2012), no. 2, 245-272. [MR2859886].

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