THE GAUSSIAN MOMENTS CONJECTURE
AND THE JACOBIAN CONJECTURE

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Abstract. We first propose what we call the Gaussian Moments Conjecture. We then show that the Jacobian Conjecture follows from the Gaussian Moments Conjecture. Note that the Gaussian Moments Conjecture is a special case of ([11, Conjecture 3.2]). The latter conjecture was referred as Moment Vanishing Conjecture in ([9, Conjecture A]) and Integral Conjecture in [6, Conjecture 3.1] (for the one-dimensional case). We also give a counter-example to show that ([11, Conjecture 3.2]) fails in general for polynomials in more than two variables.

1. Introduction

For a random variable $X$ we denote its expected value by $E(X)$. Suppose that $X = (X_1, \ldots, X_n)$ is a random vector with a multi-variate normal distribution. We make the following conjecture:

**Conjecture 1.1** (Gaussian Moments Conjecture GMC(n)). Suppose that $P(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n]$ is a complex-valued polynomial such that the moments $E(P(X)^m)$ are equal to 0 for all $m \geq 1$. Then for every polynomial $Q(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n]$ we have $E(P(X)^m Q(X)) = 0$ for $m \gg 0$.

By using translations and linear maps, we can normalize the random vector $X$ such that $X_1, \ldots, X_n$ are independent, with mean 0 and variance 1.

The Gaussian Moments Conjecture is a special case of ([11 Conjecture 3.2]). Furthermore, because of Proposition 3.3 and relation (3.2) in [11], the Gaussian Moments Conjecture is the special case of [11 Conjecture 3.1] for Hermite polynomials. Note that ([11 Conjecture 3.2]) was later referred as Moment Vanishing Conjecture in ([9 Conjecture 3.2])...
A), and Integral Conjecture in [6, Conjecture 3.1] (for one-dimensional case). Unfortunately, this conjecture is false in general, as can be seen from the following

**Proposition 1.2.** Let \( B = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0, x^2 + y^2 \leq 1\} \), \( P(x, y) = (x + iy)^2 \) and \( Q(x, y) = x + iy \). Then \( \int_B P(x, y)^m \, dx \, dy = 0 \) for all \( m \geq 1 \), but \( \int_B Q(x, y)P(x, y)^m \, dx \, dy \neq 0 \) for all \( m \geq 1 \).

**Proof.** For each \( m \geq 1 \), by using the polar coordinates \((r, \theta)\) we have

\[
\int_B P(x, y)^m \, dx \, dy = \int_0^1 \int_0^\pi r^{2m} e^{2mi\theta} r \, dr \, d\theta = 0;
\]

\[
\int_B Q(x, y)P(x, y)^m \, dx \, dy = \int_0^1 \int_0^\pi r^{2m+1} e^{(2m+1)i\theta} r \, dr \, d\theta
= \frac{2i}{(2m + 3)(2m + 1)} \neq 0.
\]

\[\Box\]

**Remark 1.3.** Note that Conjecture 3.2 in [11] is still open for univariate polynomials. It is also open for the (whole) disks or squares centered at the origin for polynomials in two variables.

**Remark 1.4.** The function \( X_1^2 + X_2^2 \) has an exponential distribution and more generally, \( X_1^2 + \cdots + X_k^2 \) has a \( \chi^2 \) distribution. So, if the Gaussian Moments Conjecture is true for all \( n \geq 1 \), then the conjecture is also true when we replace the Gaussian distributions by exponential or \( \chi^2 \) distributions. The Moments Conjecture for exponential distributions is equivalent to [5, Conjecture 4.1], which is a weaker form of the Factorial Conjecture ([5, Conjecture 4.2]).

One of the main open conjectures in affine algebraic geometry is the notorious Jacobian Conjecture, which was first proposed by O. H. Keller [7] in 1939. See also [11] and [3].

**Conjecture 1.5 (Jacobian Conjecture JD(n)).** If \( F : \mathbb{C}^n \to \mathbb{C}^n \) is a polynomial map that is locally invertible, then it is globally invertible.

The main result of this paper is:

**Theorem 1.6.** If GMC(n) is true for all \( n \geq 1 \), then JC(n) is true for all \( n \geq 1 \).

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2. Background

Suppose that $A$ is a unital commutative $\mathbb{C}$-algebra.

**Definition 2.1.** A Mathieu-Zhao space (or MZ space) is a $\mathbb{C}$-linear subspace $V \subseteq A$ with the property that $f^m \in V$ for all $m \geq 1$ implies that for every $g \in A$, $f^m g \in V$ for $m \gg 0$.

Observe that in this definition we have changed the name Mathieu subspace, which was introduced by the third author in [11, 12], into Mathieu-Zhao space or MZ space. This follows a suggestion of the second author in [4]. For some more general studies of this new notion, see [12].

With the definition above we can now reformulate our main conjecture as follows.

**Conjecture 2.2** (GMC($n$), reformulation). The subspace

$$\{ P(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n] \mid \mathbb{E}(P(X_1, \ldots, X_n)) = 0 \}$$

is an MZ space of $\mathbb{C}[x_1, \ldots, x_n]$.

Suppose that $G$ is a complex reductive algebraic group acting regularly on an affine variety $Z$. Then $G$ also acts on the ring $\mathbb{C}[Z]$ of polynomial functions on $Z$. Let $K \subseteq G$ be a maximal compact subgroup. Then $K$ is Zariski dense in $G$. The Reynolds operator $R_Z : \mathbb{C}[Z] \to \mathbb{C}$ is the averaging operator:

$$R_Z(f) = \int_{g \in K} g \cdot f \, d\mu,$$

where $d\mu$ is the Haar measure on $K$, normalized such that $\int_K d\mu = 1$.

**Conjecture 2.3** (Mathieu Conjecture MC($Z$)). The kernel $\text{Ker}(R_Z)$ of the Reynolds operator is an MZ space of $\mathbb{C}[Z]$.

This conjecture is equivalent to the conjecture $C(\mathbb{C}[Z])$ of [8] (see [8, Corollary 1.3]). The group $G$ acts on its own coordinate ring, and MC($G$) implies MC($Z$) ([8, Corollary 1.7]). The following theorem was proven in [8, Theorem 5.5]:

**Theorem 2.4** (Mathieu). If MC(SL$_n(\mathbb{C})$/GL$_{n-1}(\mathbb{C}))$ is true for all $n \geq 1$, then JC($n$) is true for all $n \geq 1$.

For later purposes, here we also point out that J. Duistermaat and W. van der Kallen [2] in 1998 had proved the Mathieu conjecture for the case of tori, which can be re-stated in terms of MZ spaces as follows.
Theorem 2.5 (Duistermaat and van der Kallen). Let \( x = (x_1, x_2, \ldots, x_n) \) be \( n \) commutative free variables and \( M \) the subspace of the Laurent polynomial algebra \( \mathbb{C}[x_1^{-1}, \ldots, x_n^{-1}, x_1, \ldots, x_n] \) consisting of the Laurent polynomials with no constant term. Then \( M \) is an MZ space of \( \mathbb{C}[x_1^{-1}, \ldots, x_n^{-1}, x_1, \ldots, x_n] \).

Let \( \partial_i = \frac{\partial}{\partial z_i} \) be the partial derivative with respect to \( z_i \). Define 
\[
\mathcal{E}_n : \mathbb{C}[w, z] = \mathbb{C}[w_1, \ldots, w_n, z_1, \ldots, z_n] \to \mathbb{C}[z]
\]
such that 
\[
\mathcal{E}_n(P(w)Q(z)) = P(\partial)Q(z) \in \mathbb{C}[z].
\]
Zhao made the following conjecture in \([10]\):

Conjecture 2.6 (Special Image Conjecture SIC\((n)\)). Ker\((\mathcal{E}_n)\) is an MZ space of \( \mathbb{C}[w, z] \).

Zhao proved the following result (\([10\), Theorem 3.6, Theorem 3.7\]):

Theorem 2.7 (Zhao). If SIC\((n)\) is true for all \( n \geq 1 \), then JC\((n)\) is true for all \( n \geq 1 \).

3. Reduction of the Jacobian Conjecture to the Gaussian Moments Conjecture

We define the linear map 
\[
\mathcal{F}_n : \mathbb{C}[w, z] = \mathbb{C}[w_1, \ldots, w_n, z_1, \ldots, z_n] \to \mathbb{C}
\]
by setting 
\[
\mathcal{F}_n(P) = \mathcal{E}_n(P) \big|_{z=0}.
\]
For \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \), set \( z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n} \) and \( \alpha! = \alpha_1!\alpha_2! \cdots \alpha_n! \).

Then we have 
\[
\mathcal{F}_n(w^\alpha z^\beta) = \begin{cases} 
\alpha! & \text{if } \alpha = \beta; \\
0 & \text{if } \alpha \neq \beta.
\end{cases}
\]

Proposition 3.1. If Ker\((\mathcal{F}_n)\) is an MZ space of \( \mathbb{C}[w, z] \), then Ker\((\mathcal{E}_n)\) is an MZ space of \( \mathbb{C}[w, z] \), i.e. SIC\((n)\) is true.

Proof. Assume that \( P^m \in \text{Ker}(\mathcal{E}_n) \) for \( m \geq 1 \). Then for each \( \alpha \in \mathbb{C}^n \) we have 
\[
\mathcal{E}_n(P^m(w, z)) \big|_{z=\alpha} = \mathcal{E}_n(P^m(w, z + \alpha)) \big|_{z=0} = \mathcal{F}_n(P^m(w, z + \alpha)) = 0.
\]
Hence \( P^m(w, z + \alpha) \in \text{Ker}(\mathcal{F}_n) \) for all \( m \geq 1 \). Since Ker\((\mathcal{F}_n)\) is an MZ space of \( \mathbb{C}[w, z] \), for any \( Q \in \mathbb{C}[w, z] \) and \( \alpha \in \mathbb{C}^n \) we have 
\[
Q(w, z + \alpha)P(w, z + \alpha)^m \in \text{Ker}(\mathcal{F}_n) \text{ for all } m \gg 0.
\]
Therefore, for all \( m \gg 0 \) we have 
\[
\mathcal{E}_n(Q(w, z)P(w, z)^m) \big|_{z=\alpha} = \mathcal{F}_n(Q(w, z + \alpha)P(w, z + \alpha)^m) = 0.
\]
Define $Z_N \subseteq \mathbb{C}^n$ to be the zero set of all $\mathcal{E}_n(Q(w,z)P(w,z)^m)$ with $m \geq N$. Clearly, $Z_N$ is Zariski closed for all $N$, and $\bigcup_{N=1}^{\infty} Z_N = \mathbb{C}^n$. It follows that $Z_N = \mathbb{C}^n$ for some integer $N$, because a countable union of Zariski closed proper subsets cannot be the whole affine space. So for $m \geq N$, $\mathcal{E}_n(Q(w,z)P(w,z)^m)$ is the zero function. □

Proposition 3.2. If GMC(2n) is true, then $\ker(\mathcal{F}_n)$ is an MZ space of $\mathbb{C}[w,z]$.

Proof. Let $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ are 2n independent random variables with the normal distribution and with mean 0 and variance 1. Define complex-valued random variables $W_j, Z_j$ and real-valued random variables $R_j, T_j$ by

$$W_j = \frac{X_j - Y_j}{\sqrt{2}} = R_j e^{-iT_j} \text{ and } Z_j = \frac{X_j + Y_j}{\sqrt{2}} = R_j e^{iT_j}.$$

Then $R_1, \ldots, R_n, T_1, \ldots, T_n$ are independent, and for every $1 \leq j \leq n$, $R_j^2$ has an exponential distribution with mean 1 and $\mathbb{E}(R_j^{2k}) = k!$. Now consider

$$\mathbb{E}(W^\alpha Z^\beta) = \mathbb{E}(R^{\alpha+\beta} e^{i\sum_j (\beta_j - \alpha_j)T_j}) = \prod_{j=1}^{n} \left( \mathbb{E}(R^{\alpha_j + \beta_j}) \mathbb{E}(e^{i(\beta_j - \alpha_j)T_j}) \right).$$

If $\beta \neq \alpha$, then $\beta_j \neq \alpha_j$ for some $j$, whence $\mathbb{E}(e^{i(\beta_j - \alpha_j)T_j}) = 0$ and $\mathbb{E}(W^\alpha Z^\beta) = 0$. If $\alpha = \beta$, then we have

$$\mathbb{E}(W^\alpha Z^\alpha) = \mathbb{E}(R^{2\alpha}) = \prod_j \mathbb{E}(R_j^{2\alpha_j}) = \prod_j \alpha_j! = \alpha!$$

It follows that $\mathbb{E}(W^\alpha Z^\beta) = \mathcal{F}_n(w^\alpha z^\beta)$ for all $\alpha, \beta \in \mathbb{N}^n$. By linearity, we get $\mathbb{E}(Q(W,Z)) = \mathcal{F}_n(Q(w,z))$ for every polynomial $Q(w,z) \in \mathbb{C}[w,z]$. It follows readily from GMC(2n) that $\ker \mathcal{F}_n$ is an MZ space of $\mathbb{C}[w,z]$. □

Now we can prove our main result Theorem 1.6.

Proof of Theorem 1.6. It follows directly from Proposition 3.1, Proposition 3.2 and Theorem 2.7. □

4. Some Special Cases of the Gaussian Moments Conjecture

We view $\mathbb{C}[x_1, \ldots, x_n]$ as the coordinate ring of $V \cong \mathbb{C}^n$, where $V$ is viewed as the standard representation of $O(n)$.

Proposition 4.1. For homogeneous polynomials $P(x)$, GMC(n) follows from MC(V).
Proof. Let \( \Phi : \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C} \) be given by \( \Phi(P(x)) = \mathbb{E}(P(X)) \). Any linear map \( \mathbb{C}[x_1, \ldots, x_n]_d \to \mathbb{C} \) is determined by an element of \( S^d(V) \). Since \( \Phi \) is invariant under the action of \( O(n) \) it is given by an element of \( S^d(V)^{O(n)} \). But \( S^d(V)^{O(n)} \) is at most one dimensional and is spanned by the restriction of the Reynolds operator \( \mathcal{R}_V \). So up to a constant, \( \Phi(P(x)^m) \) is equal to \( \mathcal{R}_V(P(x)^m) \). If \( \mathbb{E}(P(X)^m) = 0 \) for \( m \geq 1 \), then \( \mathcal{R}_V(P(X)^m) = 0 \) for \( m \geq 1 \). If \( Q(x) \) is homogeneous, then \( \mathcal{R}_V(P(x)^mQ(x)) = 0 \) for \( m \gg 0 \). So \( \mathbb{E}(P(X)^mQ(X)) = 0 \) for \( m \gg 0 \). If \( Q(X) \) is non-homogeneous then \( \mathbb{E}(P(X)^mQ(X)) = 0 \) for \( m \gg 0 \), because \( \mathbb{E}(P(X)Q_d(X)) = 0 \) for \( m \gg 0 \) for every homogeneous summand \( Q_d(x) \) of \( Q(x) \). \( \square \)

**Proposition 4.2.** Suppose that \( X \) is a Gaussian Random Variable, and \( P(x) \in \mathbb{C}[x] \) is a univariate polynomial such that \( \mathbb{E}(P(X)^m) = 0 \) for \( m \geq 1 \), then \( P(x) = 0 \). In particular, \( \text{GMC}(n) \) is true for \( n = 1 \).

Proof. As observed in the beginning of this paper, \( \text{GMC}(n) \) is a special case of the Image Conjecture for Hermite polynomials. For \( n = 1 \) the case of Hermite polynomials is proved in Corollary 4.3 of [6]. \( \square \)

For a different proof of \( \text{GMC}(1) \), see Proposition 4.7 and Remark 4.8 of this section.

**Proposition 4.3.** Let \( P \in \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] \) such that for each \( 1 \leq k \leq n \) \( P(x, y) \) as a polynomial in \( x_k \) and \( y_k \) is homogeneous. Then \( \text{GMC}(2n) \) holds for \( P \).

Proof. For each \( 1 \leq k \leq n \), let \( d_k \) be the degree of \( f \) as a polynomial in \( x_k \) and \( y_k \).

Making the change of variables for \( x_i \) and \( y_i \) (\( 1 \leq i \leq n \)):

\[
x_i = r_i \cos \theta_i \quad \text{and} \quad y_i = r_i \sin \theta_i,
\]

we see that \( P = (r_1^{d_1} r_2^{d_2} \cdots r_n^{d_n}) F \) for some polynomial \( F \) in \( \cos \theta_i \) and \( \sin \theta_i \) (\( 1 \leq i \leq n \)), which is independent of \( r_i \) (\( 1 \leq i \leq n \)).

Let \( S^n := (S^1)^{\times n} \), where \( S^1 \) is the unit circle in \( \mathbb{C} \). Denote by \( d\mu_n \) the measure of \( d\theta_1 d\theta_2 \cdots d\theta_n \), which is a haar measure of the torus \( S^n \). Then \( F \) can be viewed as \( S^n \)-finite function over the torus \( S^n \). Furthermore, for any \( m \geq 1 \) we have

\[
\mathbb{E}(P^m(X, Y)) = \int_{r_1=0}^{1} \cdots \int_{r_n=0}^{1} (r_1^{md_1} \cdots r_n^{md_n} + 1) \left( \int_{S^n} F^m d\mu_n \right) dr_1 \cdots dr_n
\]

\[
= A_m \int_{S^n} F^m d\mu_n,
\]

for some nonzero constant \( A_m \).
Hence, if $\mathbb{E}(P^m) = 0$ when $m \gg 0$, then so is $\int_{S^n} F^m$. Since $d\mu_n$ is a Haar measure of the torus $S_n$, applying the Duistermaat-van der Kallen Theorem [2.5] to $F$ we see that for each polynomial $G$ in $\cos \theta_i$ and $\sin \theta_i$ ($1 \leq i \leq n$), we have $\int_{S^n} F^m G d\mu_n = 0$ when $m \gg 0$.

Now for each monomial $M(x, y)$ in $x_i$ and $y_i$ ($1 \leq i \leq n$), by Eq. (4.1) with $P^m$ replaced by $P^m M$, we see that $\mathbb{E}(P^m M) = 0$ when $m \gg 0$. Hence for each polynomial $Q(x, y)$, we also have $\mathbb{E}(P^m Q) = 0$ when $m \gg 0$. Therefore $\text{GMC}(2n)$ holds for $P$.

Since every homogeneous polynomial in two variables satisfies the condition of Proposition 4.3, we immediately have the following

**Corollary 4.4.** $\text{GMC}(2)$ holds for all homogeneous polynomials $P$.

By a similar argument as in the proof of Proposition 4.3 we have also the following case of Conjecture 3.2 in [11]:

**Corollary 4.5.** Let $B$ be the unit disk in $\mathbb{R}^2$ centered at the origin with the Lebesgue measure $dxdy$. Let $P \in \mathbb{C}[x, y]$ such that $P$ is homogeneous and $\int_B P^m dxdy = 0$ for all $m \gg 0$. Then for every $Q \in \mathbb{C}[x, y]$ we have $\int_B P^m Q dxdy = 0$ for all $m \gg 0$.

In the rest of this section we point out that some results proved in [5] for the Factorial Conjecture ([5, Conjecture 4.2]) can also be proved similarly for $\text{GMC}(n)$.

First, we give a proof for the following case of $\text{GMC}(n)$, which is parallel to [5, Proposition 4.8].

**Proposition 4.6.** Let $F(x) \in \mathbb{C}[x_1, x_2, ..., x_n]$ such that $F(0) \neq 0$. Then $\mathbb{E}(F^m(X)) \neq 0$ for infinitely many $m \geq 1$.

**Proof.** Let $\Phi : \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}$ be given by $\Phi(P(x)) = \mathbb{E}(P(X))$. Set $(-1)!! := 1$ and $(2k-1)!! := (2k-1)(2k-3) \cdots 3 \cdot 1$ for all $k \geq 1$. Furthermore, for each $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in 2\mathbb{N}$, we set $(\alpha-1)!! := \prod_{i=1}^{n} (\alpha_i - 1)!!$. Then for each $\alpha \in \mathbb{N}^n$, we have

$$\Phi(x^\alpha) = \begin{cases} 
(\alpha-1)!! & \text{if } \alpha \in 2\mathbb{N}^n; \\
0 & \text{otherwise.}
\end{cases}$$

(4.2)

Now assume that the proposition fails, i.e., there exists $N \geq 1$ such that $\Phi(F^m) = 0$ for all $m \geq N$. Since $F(0) \neq 0$, replacing $F$ by $F/F(0)$ we may assume $F(0) = 1$. Write $F(x) = 1 - \sum_{i=1}^{k} c_i x^{\beta_i}$ with $c_i \in \mathbb{C}$ and $0 \neq \beta_i \in \mathbb{N}^n$ for all $1 \leq i \leq k$. Note that if $c_i = 0$ for all $1 \leq i \leq k$, i.e., $F(x) = 1$, the proposition obviously holds. So we assume $c_i \neq 0$ for all $1 \leq i \leq k$. Replacing $F$ by $F^2$ we may also assume that $0 \neq \beta_i \in 2\mathbb{N}$ for at least one $1 \leq i \leq k$. 


Furthermore, by a reduction due to Mitya Boyarchenko (see the proof of [9, Theorem 4.1] or [10, Remarks 4.5 and 4.6]), we may also assume that \( c_i \in \mathbb{Q} \) for all \( 1 \leq i \leq k \).

Let \( B = \mathbb{Z}[c_1, c_2, \ldots, c_k] \) and \( p \) be an odd prime such that \( p \geq N \) and \( \nu_p(c_i) = 0 \) for all \( 1 \leq i \leq k \), where \( \nu_p \) denotes an extension of the \( p \)-valuation of \( \mathbb{Z} \) to \( B \).

Since \( p \geq N \) and \( F^p = 1 - \sum_{i=1}^{k} c_i^p x^{p \beta_i} \mod pB \), we have \( \Phi(F^p) = 0 \) and

\[
1 \equiv \sum_{1 \leq i \leq k} c_i^p (p \beta_i - 1)!! \mod pB.
\]

Since each \( 0 \neq \beta_i \in 2\mathbb{N} \) in the sum above has at least one nonzero (and even) component, so \((p \beta_i - 1)!! \) is divisible by \( p \). Then applying \( \nu_p \) to Eq. (4.3) we get \( \nu_p(1) = 0 \), which is a contradiction. \( \square \)

The next proposition is parallel to [5, Proposition 4.10].

**Proposition 4.7.** Let \( F(x) = c_0 M_0 + \sum_{i=1}^{d} c_i M_i \) with \( M_0 = x_1^{k_1} \cdots x_n^{k_n} \) such that \( k_1 \geq 1 \) and \( k_1 \geq k_j \) for all \( 2 \leq j \leq n \); \( c_i \in \mathbb{C} \) (\( 0 \leq i \leq d \)) with \( c_0 \neq 0 \); and \( M_i \) (\( 1 \leq i \leq d \)) are monomials in \( x \) that are divisible by \( x_1^{k_1+1} \). Then \( \mathbb{E}(F^m(X)) \neq 0 \) for infinitely many \( m \geq 1 \).

**Proof.** Replacing \( F \) by \( c_0^{-1}F \) we may assume \( c_0 = 1 \) and replacing \( F \) by \( F^2 \) we may assume that \( k_1 \) is an even positive integer. Then under these assumptions the proof of [5, Proposition 4.10] works through similarly for the linear functional \( \Phi \) of \( \mathbb{C}[x_1, \ldots, x_n] \) given in Eq. (4.2). \( \square \)

**Remark 4.8.** Note that when \( n = 1 \) the conditions of Proposition 4.7 hold automatically for all nonzero univariate polynomials \( F(x) \). Hence GMC(1) also follows directly from Proposition 4.7.

**Proposition 4.9.** Let \( d \geq 1 \) and \( P(x) = \sum_{i=1}^{n} c_i x_i^d \in \mathbb{C}[x_1, \ldots, x_n] \) for some \( c_i \in \mathbb{C} \) (\( 1 \leq i \leq n \)). Assume that \( \mathbb{E}(P^m(X)) = 0 \) for all \( m \gg 0 \). Then \( P = 0 \). In particular, GMC(n) holds for \( P(x) \).

This proposition can be proved similarly as Proposition 4.16 in [5] if we choose the integer \( m \) there to be even, and the prime \( p \) to be \((m + 2)d - 1\) or \((m + 1)d - 1\), depending \( d \) is odd or even, respectively. Note that the components \( k_i \)'s in the proof of Proposition 4.16 in [5] for our case must be even when \( m \) is chosen to be even.

5. **Moment Vanishing Polynomials**

Let again \( X = (X_1, \ldots, X_n) \) be a random vector with joint Gaussian distribution. For \( n \geq 2 \), there exist many polynomials \( P(x) \in \mathbb{C}[x] \)
for which $\mathbb{E}(P(X)^m) = 0$ for all $m \geq 1$: if 0 lies in the closure of the $O(n)$ orbit of $P(x)$, then $\mathbb{E}(P(x)^m) = 0$ for all $m \geq 1$. Indeed, if there exists a sequence of orthogonal matrices $A_1, A_2, \ldots$ such that $\lim_{k \to \infty} P(A_k(x)) = 0$, then we have $\mathbb{E}(P(X)) = \lim_{k \to \infty} \mathbb{E}(P(A_k(X))) = \mathbb{E}(\lim_{k \to \infty} P(A_k(X))) = \mathbb{E}(0) = 0$. A 1-parameter subgroup is a homomorphism $\lambda : \mathbb{C}^* \to O_n(\mathbb{C})$ of algebraic groups. We can view $\lambda$ as an orthogonal matrix with entries in $\mathbb{C}[t, t^{-1}]$. If $P(\lambda(t)(x))$ lies in $t\mathbb{C}[t][x]$, then $\lim_{t \to 0} P(\lambda(t)x) = 0$ and 0 lies in the closure of the $O_n(\mathbb{C})$ orbit of $P(x)$. Conversely, the Hilbert-Mumford criterion states that if 0 lies in the $O_n(\mathbb{C})$-orbit closure of $P(x)$, then there exists such a 1-parameter subgroup $\lambda : \mathbb{C}^* \to O_n(\mathbb{C})$ such that $P(\lambda(t)(x)) \in t\mathbb{C}[t][x]$. If $Q(x) \in \mathbb{C}[x]$, then for large $m$, $Q(\lambda(t)(x))P(\lambda(t)x)^m \in t\mathbb{C}[t][x]$ and

$$\mathbb{E}(Q(X)P(X)^{m}) = \mathbb{E}(\lim_{t \to 0} Q(\lambda(t)(X))P(\lambda(t)X) = \mathbb{E}(0) = 0.$$

We make the following conjecture:

**Conjecture 5.1.** If $\mathbb{E}(P(X)^m) = 0$ for all $m \geq 1$, then there exists a 1-parameter subgroup $\lambda : \mathbb{C}^* \to O_n(\mathbb{C})$ such that $P(\lambda(t)(x)) \in t\mathbb{C}[t][x]$.

**References**


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