BRANCHING RULES FOR FINITE-DIMENSIONAL $\mathcal{U}_q(\mathfrak{su}(3))$-REPRESENTATIONS WITH RESPECT TO A RIGHT COIDEAL SUBALGEBRA

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ABSTRACT. We consider the quantum symmetric pair $(\mathcal{U}_q(\mathfrak{su}(3)), \mathcal{B})$ where $\mathcal{B}$ is a right coideal subalgebra. We prove that all finite-dimensional irreducible representations of $\mathcal{B}$ are weight representations and are characterised by their highest weight and dimension.

We show that the restriction of a finite-dimensional irreducible representation of $\mathcal{U}_q(\mathfrak{su}(3))$ to $\mathcal{B}$ decomposes multiplicity free into irreducible representations of $\mathcal{B}$. Furthermore we give explicit expressions for the highest weight vectors in this decomposition in terms of dual q-Krawtchouk polynomials.

1. INTRODUCTION

The theory of quantum symmetric pairs of Lie groups has been developed by Koornwinder, Dijkhuizen, Noumi and Sugitani and others [2, 3, 22, 20, 21, 24] for classical Lie groups and by G. Letzter [13, 15, 16, 17, 18] for all semisimple Lie algebras, see also [10]. The motivating example for the development for this theory was given by Koornwinder [11], who studied scalar-valued spherical functions on the quantum analogue of $(\mathrm{SU}(2), \mathrm{U}(1))$ considering twisted primitive elements in the quantised universal enveloping algebra of $\mathcal{U}_q(\mathfrak{sl}(2))$. Koornwinder identified all scalar-valued spherical functions with Askey-Wilson polynomials in two free parameters. Dijkhuizen and Noumi [2] extended the work of Koornwinder to quantum analogues of $(\mathrm{SU}(n+1), \mathrm{U}(n))$ considering two sided coideals of the quantised universal enveloping algebra of $\mathcal{U}_q(\mathfrak{gl}(n+1))$. More generally, Letzter considered the quantised universal enveloping algrebra $\mathcal{U}_q(\mathfrak{g})$ with a right coideal subalgebra $\mathcal{B}$, which is the quantum analogue of $\mathcal{U}(\mathfrak{k})$ for a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. In [17] all scalar-valued spherical functions for quantum symmetric pairs with reduced restricted root systems are identified with Macdonald polynomials. However, the requirement of having a reduced restricted root system excludes the quantum analogue of $(\mathrm{SU}(3), \mathrm{U}(2))$.

One recent extension of this situation [1] arises with the study of matrix-valued spherical functions of the quantum analogue of $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{diag})$ where higher-dimensional representations of coideal subalgebra $\mathcal{B}$ are involved. The quantum symmetric pair is given by the quantised universal enveloping algebra of $\mathcal{U}_q(\mathfrak{g})$, where $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, and a right coideal subalgebra $\mathcal{B}$ than can be identified with $\mathcal{U}_q(\mathfrak{su}(2))$. As in the Lie group setting [8, 9, 25, 5], the explicit knowledge of the branching rules plays a fundamental role in the explicit determination of the matrix-valued spherical functions. In this first case, the branching rules for the irreducible representations of $\mathcal{U}_q(\mathfrak{g})$ with respect to $\mathcal{B}$ follow using the standard Clebsch-Gordan decomposition.

One of the first technical difficulties that one finds to extend the results of [1] to more general quantum symmetric pairs is the lack of the explicit branching rules for finite-dimensional

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$\mathcal{U}_q(\mathfrak{g})$-representations with respect to a right coideal subalgebra. In this paper we deal with this problem for the quantised universal enveloping algebra $\mathcal{U}_q(\mathfrak{su}(3))$ with a right coideal subalgebra $\mathcal{B}$ as in Kolb [10]. We study the problem of describing all irreducible representations that occur in the restriction to $\mathcal{B}$ of finite-dimensional irreducible representations of $\mathcal{U}_q(\mathfrak{su}(3))$. In general, information about branching rules for quantum symmetric pairs $(\mathcal{U}_q(\mathfrak{g}), \mathcal{B})$ as in Kolb [10] and Letzter [13, 15] is relatively scarce in particular in case the coideal subalgebra depends on an additional parameter as in this paper. However see Oblomkov and Stokman [23] for partial information on the branching rules for the quantum analogue of $(\mathfrak{gl}(2n), \mathfrak{gl}(n) \oplus \mathfrak{gl}(n))$.

This paper is organised as follows. In Section 2 we review the construction of the quantised universal enveloping algebra $\mathcal{U}_q(\mathfrak{su}(3))$ and its finite dimensional irreducible representations. Then we collect a series of commutation identities for the generators of $\mathcal{U}_q(\mathfrak{su}(3))$ and we introduce a an orthogonal basis for finite-dimensional $\mathcal{U}_q(\mathfrak{su}(3))$-representations which is an analogue of Mudrov [19]. We also describe the action of the generators of $\mathcal{U}_q(\mathfrak{su}(3))$ on this basis. In Section 3 we fix a right coideal subalgebra $\mathcal{B}$ of the quantised universal enveloping algebra which depends on two complex parameters c_1, c_2. We describe the generators of the Cartan subalgebra of $\mathcal{B}$ and we use them to classify all finite-dimensional irreducible representations of $\mathcal{B}$ under a mild genericity condition on the parameters. More precisely we prove that every finite-dimensional irreducible representation of $\mathcal{B}$ is completely characterised by its highest weight and its dimension. In Section 4 we prove the main theorem of the paper. We show that any irreducible finite-dimensional representation of $\mathcal{U}_q(\mathfrak{su}(3))$ decomposes multiplicity free into irreducible representations of the $\mathcal{B}$ and we characterise the representations that occur in the decomposition by their highest weight and dimension. The highest weight vectors of the coideal subalgebra $\mathcal{B}$-representations are obtained by diagonalising an element of the Cartan subalgebra of $\mathcal{B}$ restricted to a certain subspace where it acts tridiagonally. The eigenvectors can be then identified explicitly in terms of dual q-Krawtchouk polynomials.

2. The quantised universal enveloping algebra $\mathcal{U}_q(\mathfrak{su}(3))$

Let $\mathfrak{g} = \mathfrak{sl}(3) = \{X \in \mathfrak{gl}(3, \mathbb{C}) : \mathrm{tr}(X) = 0\}$. We fix the Cartan subalgebra $\mathfrak{h}$ of diagonal matrices. Let $A = (a_{i,j})_{i,j}$ be the Cartan matrix for $\mathfrak{g}$, i.e. $a_{i,i} = 2$, $i = 1, 2$, and $a_{i,j}$=-1 for $i \neq j$. Let $R \subset \mathfrak{h}$ denote the root system of $\mathfrak{g}$. We denote by R^+ the subset of positive roots, so that we have the decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. We denote by $(\cdot, \cdot)$ the canonical inner product on $\mathfrak{h}$ and by $\Pi = \{\alpha_1, \alpha_2\}$ the simple roots so that $(\alpha_i, \alpha_j) = a_{i,j}$. The fundamental weights are given by $\varpi_1 = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2$ and $\varpi_2 = \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2$.

The quantised universal enveloping algebra $\mathcal{U}_q(\mathfrak{su}(3))$ is the unital associative algebra generated by E_i, F_i and $K_i^{\pm 1}$, where $i = 1, 2$, subject to the relations

$$
\begin{gathered}
K_i^{\pm 1} K_j^{\pm 1} = K_j^{\pm 1} K_i^{\pm 1}, \quad K_i^{\pm 1} K_j^{\mp 1} = K_j^{\mp 1} K_i^{\pm 1}, \quad K_i K_i^{-1} = 1 = K_i^{-1} K_i, \\
K_i E_j = q^{(\alpha_i, \alpha_j)} E_j K_i, \quad K_i F_j = q^{-(\alpha_i, \alpha_j)} F_j K_i, \quad [E_i, F_j] = \frac{K_i - K_i^{-1}}{q - q^{-1}} \delta_{i,j},
\end{gathered}
\tag{2.1}
$$

for $i, j = 1, 2$ and, for $i \neq j$, the quantum Serre's relations

$$
E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 = 0 = F_i^2 F_j - (q + q^{-1}) F_i F_j F_i + F_j F_i^2. \tag{2.2}
$$

We assume that $q \in [0,1]$. The quantised universal enveloping algebra $\mathcal{U}_q(\mathfrak{su}(3))$ has a Hopf algebra structure with comultiplication Δ, counit ϵ and antipode S defined by

$$\Delta : E_i, F_i, K_i^{\pm 1} \mapsto E_i \otimes 1 + K_i \otimes E_i, F_i \otimes K_i^{-1} + 1 \otimes F_i, K_i^{\pm 1} \otimes K_i^{\pm 1},$$
$$\epsilon : E_i, F_i, K_i^{\pm 1} \mapsto 0, 0, 1, \quad S : E_i, F_i, K_i^{\pm 1} \mapsto -K_i^{-1}E_i, -F_iK_i, K_i^{\mp 1},$$

with $i = 1, 2$. The $*$-structure on $\mathcal{U}_q(\mathfrak{su}(3))$ is given by

$$E_i^* = K_iF_i, \quad F_i^* = E_iK_i^{-1}, \quad (K_i^{\pm 1})^* = K_i^{\pm 1}, \quad i = 1, 2, \tag{2.3}$$

so that $\mathcal{U}_q(\mathfrak{su}(3))$ is a Hopf $*$-algebra. Following Mudrov [19] we define for $a \in \mathbb{R}$

$$F_3 = [F_1, F_2]_q = F_1F_2 - qF_2F_1, \quad E_3 = [E_2, E_1]_q = E_2E_1 - qE_1E_2,$$
$$\hat{F}_3[a] = F_1F_2 \left(\frac{q^{a+1}K_2 - q^{-a-1}K_2^{-1}}{q - q^{-1}} \right) - F_2F_1 \left(\frac{q^{a}K_2 - q^{-a}K_2^{-1}}{q - q^{-1}} \right),$$
$$\hat{E}_3[a] = \left(\frac{q^{a+1}K_2 - q^{-a-1}K_2^{-1}}{q - q^{-1}} \right) E_2E_1 - \left(\frac{q^{a}K_2 - q^{-a}K_2^{-1}}{q - q^{-1}} \right) E_1E_2,$$

and $\hat{F}_3 = \hat{F}_3[0]$, $\hat{E}_3 = \hat{E}_3[0]$.

Lemma 2.1. *The following relations hold in* $\mathcal{U}_q(\mathfrak{su}(3))$*:*

(i) $F_1\hat{F}_3[a] = \hat{F}_3[a]F_1$,

(ii) $E_2\hat{F}_3[a] = \hat{F}_3[a-2]E_2 - \frac{(q^a - q^{-a})}{(q - q^{-1})}F_1$,

(iii) $K_i\hat{F}_3[a] = q^{-1}\hat{F}_3[a]K_i$, $K_i\hat{E}_3[a] = q\hat{E}_3[a]K_i$, $i = 1, 2$.

Proof. Straightforward verifications using (2.1) and (2.3). □

Lemma 2.2. *For* $i = 1, 2$*:*

(i)

$$E_iF_i^k = F_i^kE_i + \frac{q^k - q^{-k}}{q - q^{-1}} F_i^{k-1} \frac{q^{1-k}K_i - q^{k-1}K_i^{-1}}{q - q^{-1}}$$

(ii)

$$\begin{aligned} E_i^kF_i^k &= \frac{q^k\,(q^2;q^2)_k}{(1-q^2)^{2k}} (q^{2-2k}K_i^2; q^2)_k K_i^{-k} + \mathcal{U}_q(\mathfrak{su}(3))\,E_i \\ &= \frac{(q^2;q^2)_k}{(1-q^2)^{2k}} (-1)^k q^{-k(k-2)} (K_i^{-2}; q^2)_k K_i^k + \mathcal{U}_q(\mathfrak{su}(3))\,E_i. \end{aligned}$$

Proof. Straightforward verifications using (2.1) and (2.3) and induction. □

2.1. **The finite-dimensional representations of $\mathcal{U}_q(\mathfrak{su}(3))$.** Finite-dimensional representations of $\mathcal{U}_q(\mathfrak{su}(3))$ are weight representations and are uniquely determined, up to equivalence, by their highest weights. Let (π_λ, V_λ) be an irreducible finite-dimensional representation with highest weight $\lambda = \lambda_1\varpi_1 + \lambda_2\varpi_2$, $\lambda_1, \lambda_2 \in \mathbb{N}$, and v_λ a highest weight vector such that

$$E_i\,v_\lambda = 0, \quad K_i\,v_\lambda = q^{(\lambda,\alpha_i)}v_\lambda = q^{\lambda_i}v_\lambda. \tag{2.4}$$

Then the dimension of V_λ is the same as the dimension of the corresponding irreducible representation π_λ of $\mathfrak{su}(3)$, namely

$$\dim(V_\lambda) = \frac{1}{2}(\lambda_1 + 1)(\lambda_2 + 1)(\lambda_1 + \lambda_2 + 2).$$

Furthermore for a weight $\nu = \nu_1 \varpi_1 + \nu_2 \varpi_2$, the dimension of the weight space

$$V_\lambda(\nu) = \{v \in V_\lambda : \ K_i \, v = q^{(\nu,\alpha_i)} v, \ i = 1,2\},$$

and the dimension of the weight space corresponding to the weight ν in the representation of $\mathfrak{su}(3)$ coincide, see [6, Ch. 7]. In particular, $\dim(V_\lambda(\lambda)) = 1$. The vector space V_λ is generated by the vectors v_λ and $F_{i_1} F_{i_2} \dots F_{i_m} v_\lambda$, $i_j \in \{1, 2\}$ and is equipped with an inner product $\langle \cdot, \cdot \rangle$ that satisfies

$$\langle v_\lambda, v_\lambda \rangle = 1, \qquad \langle X \, v, w \rangle = \langle v, X^* \, w \rangle, \qquad \forall \, X \in \mathcal{U}_q(\mathfrak{su}(3)), \quad \forall \, v, w \in V_\lambda.$$

Mudrov [19] describes the Shapovalov basis for the Verma modules of $\mathcal{U}_q(\mathfrak{su}(3))$, and we have adapted his proof and construction to an orthonormal basis for the finite-dimensional unitary representations of $\mathcal{U}_q(\mathfrak{su}(3))$. For completeness, we have sketched the proof in Appendix A. It is essentially due to Mudrov [19, §8].

Theorem 2.3. *The set of vectors*

$$\mathscr{B} = \{F_2^k \hat{F}_3^l F_1^m \, v_\lambda \mid 0 \le m \le \lambda_1, \ 0 \le l \le \lambda_2, \ 0 \le k \le \lambda_2 + m - l\}$$

forms an orthogonal basis for V_λ. Explicitly,

$$\langle F_2^k \hat{F}_3^l F_1^m \, v_\lambda, F_2^{k'} \hat{F}_3^{l'} F_1^{m'} \, v_\lambda \rangle = \delta_{k,k'} \delta_{l,l'} \delta_{m,m'} H_{k,l,m},$$

where

$$H_{k,l,m} = (q^2, q^{-2(\lambda_2 - l + m)}; q^2)_k (q^2, q^{-2\lambda_1}; q^2)_m (q^2, q^{-2\lambda_2}, q^{-2(\lambda_2+1+m)}, q^{-2(\lambda_1+\lambda_2+1)}; q^2)_l \\ \times (1-q^2)^{-2(k+2l+m)} (-1)^{k+l+m} q^{3(k+3l+m)} q^{-l(l-2m)} q^{-2l\lambda_2}.$$

In Theorem 2.3 we use the standard notation in [4] for q-shifted factorials

$$(q^a; q)_n = (1-q^a)(1-q^{a+1}) \dots (1-q^{a+n-1}),$$
$$(q^{a_1}, q^{a_2}, \dots, q^{a_j}; q)_n = (q^{a_1}, q)_n (q^{a_2}, q)_n \dots (q^{a_j}, q)_n.$$

Note that $H_{k,l,m}$ is indeed positive. In the following proposition we calculate the action of the generators of $\mathcal{U}_q(\mathfrak{su}(3))$ in the basis $\mathscr{B}$ of Theorem 2.3.

Proposition 2.4. *In the basis $\mathscr{B}$ of V_λ as in Theorem 2.3 we have*

(i) $K_1 F_2^k \hat{F}_3^l F_1^m \, v_\lambda = q^{\lambda_1 + k - l - 2m} \, F_2^k \hat{F}_3^l F_1^m \, v_\lambda$,

(ii) $K_2 F_2^k \hat{F}_3^l F_1^m \, v_\lambda = q^{\lambda_2 - 2k - l + m} \, F_2^k \hat{F}_3^l F_1^m \, v_\lambda$,

(iii) $F_1 F_2^k \hat{F}_3^l F_1^m \, v_\lambda = a_k(l,m) \, F_2^k \hat{F}_3^l F_1^{m+1} \, v_\lambda + b_k(l,m) \, F_2^{k-1} \hat{F}_3^{l+1} F_1^m \, v_\lambda$,

(iv) $E_1 F_2^k \hat{F}_3^l F_1^m \, v_\lambda = \alpha_k(l,m) \, F_2^k \hat{F}_3^l F_1^{m-1} \, v_\lambda + \beta_k(l,m) \, F_2^{k+1} \hat{F}_3^{l-1} F_1^m \, v_\lambda$,

(v) $F_2 F_2^k \hat{F}_3^l F_1^m \, v_\lambda = F_2^{k+1} \hat{F}_3^l F_1^m \, v_\lambda$,

(vi) $E_2 F_2^k \hat{F}_3^l F_1^m \, v_\lambda = \eta_k(l,m) \, F_2^{k-1} \hat{F}_3^l F_1^m \, v_\lambda$,

with coefficients

$$a_k(l,m) = \frac{(q^{\lambda_2+m+1-k-l} - q^{-\lambda_2-m-1+k+l})}{(q^{\lambda_2+m+1-l} - q^{-\lambda_2-m-1+l})}, \qquad b_k(l,m) = \frac{(q^k - q^{-k})}{(q^{\lambda_2+m+1-l} - q^{-\lambda_2-m-1+l})},$$

$$\eta_k(l,m) = \frac{q^k - q^{-k}}{q - q^{-1}} \frac{q^{1-k+\lambda_2-l+m} - q^{k-1-\lambda_2+l-m}}{q - q^{-1}},$$

$$\alpha_k(l,m) = \frac{(q^m - q^{-m})(q^{\lambda_1-m+1} - q^{-\lambda_1+m-1})(q^{\lambda_2+m+1} - q^{-\lambda_2-m-1})}{(q - q^{-1})^2(q^{\lambda_2+m-l+1} - q^{-\lambda_2-m+l-1})},$$

$$\beta_k(l,m) = \frac{(q^l - q^{-l})(q^{\lambda_2-l+1} - q^{-\lambda_2+l-1})(q^{\lambda_1+\lambda_2-l+2} - q^{-\lambda_1-\lambda_2+l-2})}{(q - q^{-1})^2(q^{\lambda_2+m-l+1} - q^{-\lambda_2-m+l-1})}.$$

Remark 2.5. Note that the denominators in $a_k(l,m)$, $b_k(l,m)$, $\eta_k(l,m)$, $\alpha_k(l,m)$ and $\beta_k(l,m)$ are non-zero by the ranges of k,l,m as in Theorem 2.3.

Proof. The action of K_i, $i = 1,2$, follows from (2.4), (2.1) and Lemma 2.1(iii). The action of F_2 is trivial. The action of E_2 follows from Lemma 2.2(i), Lemma 2.1(ii) and (2.4) and the established actions of K_2. This completes the proof of (i), (ii), (v) and (vi).

In order to establish the action of F_1, we first show that there exist constants a_k and b_k so that

$$F_1 F_2^k \hat{F}_3^l F_1^m v_\lambda = a_k F_2^k \hat{F}_3^l F_1^{m+1} v_\lambda + b_k F_2^{k-1} \hat{F}_3^{l+1} F_1^m v_\lambda$$

by induction with respect to k. The case $k = 0$ with $a_0 = 1$, $b_0 = 0$ is immediate from Lemma 2.1(i). In case $k = 1$, we write

$$\begin{aligned}
F_1 F_2 \hat{F}_3^l F_1^m v_\lambda &= F_1 F_2 \frac{qK_2 - q^{-1}K_2^{-1}}{q - q^{-1}} \frac{q - q^{-1}}{q^{1+\lambda_2-l+m} - q^{-1-\lambda_2+l-m}} \hat{F}_3^l F_1^m v_\lambda \\
&= \frac{q - q^{-1}}{q^{1+\lambda_2-l+m} - q^{-1-\lambda_2+l-m}} \left(\hat{F}_3 + F_2 F_1 \frac{K_2 - K_2^{-1}}{q - q^{-1}} \right) \hat{F}_3^l F_1^m v_\lambda \\
&= \frac{q^{\lambda_2-l+m} - q^{-\lambda_2+l-m}}{q^{1+\lambda_2-l+m} - q^{-1-\lambda_2+l-m}} F_2 \hat{F}_3^l F_1^{m+1} v_\lambda \\
&\qquad + \frac{q - q^{-1}}{q^{1+\lambda_2-l+m} - q^{-1-\lambda_2+l-m}} \hat{F}_3^{l+1} F_1^m v_\lambda
\end{aligned}$$

again using Lemma 2.1(i). So the case $k = 1$ is proved with

$$a_1 = \frac{q^{\lambda_2-l+m} - q^{-\lambda_2+l-m}}{q^{1+\lambda_2-l+m} - q^{-1-\lambda_2+l-m}}, \quad b_1 = \frac{q - q^{-1}}{q^{1+\lambda_2-l+m} - q^{-1-\lambda_2+l-m}}.$$

For the induction we assume $k \geq 2$, so that

$$F_1 F_2^k \hat{F}_3^l F_1^m v_\lambda = F_1 F_2^2 F_2^{k-2} \hat{F}_3^l F_1^m v_\lambda = \left(-F_2^2 F_1 + (q + q^{-1}) F_2 F_1 F_2\right) F_2^{k-2} \hat{F}_3^l F_1^m v_\lambda$$

by the q-Serre relation (2.2). Using the induction hypothesis, we find

$$\begin{aligned}
F_1 F_2^k \hat{F}_3^l F_1^m v_\lambda &= -F_2^2 \left(a_{k-2} F_2^{k-2} \hat{F}_3^l F_1^{m+1} v_\lambda + b_{k-2} F_2^{k-3} \hat{F}_3^{l+1} F_1^m v_\lambda\right) \\
&\quad + (q + q^{-1}) F_2 \left(a_{k-1} F_2^{k-1} \hat{F}_3^l F_1^{m+1} v_\lambda + b_{k-1} F_2^{k-2} \hat{F}_3^{l+1} F_1^m v_\lambda\right) \\
&= (-a_{k-2} + (q + q^{-1}) a_{k-1}) F_2^k \hat{F}_3^l F_1^{m+1} v_\lambda + (-b_{k-2} + (q + q^{-1}) b_{k-1}) F_2^{k-1} \hat{F}_3^{l+1} F_1^m v_\lambda
\end{aligned}$$

which proves the induction step as well as the recurrence

$$a_k + a_{k-2} = (q + q^{-1}) a_{k-1}, \qquad b_k + b_{k-2} = (q + q^{-1}) b_{k-1}, \quad k \geq 2.$$

This recursion is solved by the Chebyshev polynomials (of the second kind) at $\frac{1}{2}(q+q^{-1})$ as well as by the associated Chebyshev polynomials. This gives the solution for the recurrences and proves (iii)

The action of E_1 follows from that of F_1, considering the adjoint. Note that

$$\begin{aligned}\langle E_1 F_2^k \hat{F}_3^l F_1^m\, v_\lambda, F_2^{k'} \hat{F}_3^{l'} F_1^{m'}\, v_\lambda\rangle &= \langle F_2^k \hat{F}_3^l F_1^m\, v_\lambda, E_1^* F_2^{k'} \hat{F}_3^{l'} F_1^{m'}\, v_\lambda\rangle \\ &= \langle F_2^k \hat{F}_3^l F_1^m\, v_\lambda, K_1 F_1 F_2^{k'} \hat{F}_3^{l'} F_1^{m'}\, v_\lambda\rangle,\end{aligned}$$

equals zero if $(k',l',m') \neq (k,l,m+1),(k+1,l-1,m)$. Moreover we have

$$\begin{aligned}\alpha_k(l,m) H_{k,l,m-1} &= \langle E_1 F_2^k \hat{F}_3^l F_1^m\, v_\lambda, F_2^k \hat{F}_3^l F_1^{m-1}\, v_\lambda\rangle \\ &= \langle F_2^k \hat{F}_3^l F_1^m\, v_\lambda, K_1 F_1 F_2^k \hat{F}_3^l F_1^{m-1}\, v_\lambda\rangle \\ &= q^{k-l-2m+\lambda_1}\, a_k(l,m-1)\, H_{k,l,m},\end{aligned}$$

and

$$\begin{aligned}\beta_k(l,m) H_{k+1,l-1,m} &= \langle E_1 F_2^k \hat{F}_3^l F_1^m\, v_\lambda, F_2^{k+1} \hat{F}_3^{l-1} F_1^{m}\, v_\lambda\rangle \\ &= \langle F_2^k \hat{F}_3^l F_1^m\, v_\lambda, K_1 F_1 F_2^{k-1} \hat{F}_3^{l-1} F_1^{m}\, v_\lambda\rangle \\ &= q^{k-l-2m+\lambda_1+2}\, b_{k+1}(l-1,m)\, H_{k,l,m}.\end{aligned}$$

Now the expressions of $\alpha_k(l,m)$ and $\beta_k(l,m)$ follow from the explicit expression of $H_{k,l,m}$ Theorem 2.3 by a straightforward computation. □

3. The coideal subalgebra

In this section we follow Kolb [10] and introduce a right coideal subalgebra $\mathcal{B}$ of $\mathcal{U}_q(\mathfrak{su}(3))$ which is the quantum analogue of $\mathcal{U}(\mathfrak{k})$ with $\mathfrak{k} = \mathfrak{u}(2)$ embedded in $\mathfrak{g} = \mathfrak{su}(3)$. Let $c_1, c_2 \in \mathbb{C}^\times$ and write $c = (c_1, c_2)$. Following [10, Example 9.4], $\mathcal{B}_c = \mathcal{B}$ is the right coideal subalgebra of $\mathcal{U}_q(\mathfrak{su}(3))$, i.e. $\Delta(\mathcal{B}) \subset \mathcal{B} \otimes \mathcal{U}_q(\mathfrak{su}(3))$, generated by

$$K^{\pm 1} = \left(K_1 K_2^{-1}\right)^{\pm 1}, \quad B_1^c = B_1 = F_1 - c_1 E_2 K_1^{-1}, \quad B_2^c = B_2 = F_2 - c_2 E_1 K_2^{-1}. \tag{3.1}$$

Throughout Sections 3 and 4 we omit the subscript and superscript c in $\mathcal{B}_c$ and B_i^c since the coideal subalgebra $\mathcal{B}$ will be fixed.

If we assume $c_1 \overline{c_2} = q^3 = \overline{c_1} c_2$ then it follows that $B_1^* = -\overline{c_1} K^{-1} B_2$, $B_2^* = -\overline{c_2} K B_1$ and $K^* = K$, so that $\mathcal{B}^* = \mathcal{B}$. By a straightforward computation we have

$$\begin{aligned}\Delta(B_1) &= B_1 \otimes K_1^{-1} + 1 \otimes F_1 - c_1 K^{-1} \otimes E_2 K_1^{-1}, \\ \Delta(B_2) &= B_2 \otimes K_2^{-1} + 1 \otimes F_2 - c_2 K \otimes E_1 K_2^{-1}.\end{aligned}$$

The Serre relations for $\mathcal{B}$ follow from from [10, Lemma 7.2, Theorem 7.4] taking $\mathcal{Z}_1 = -K^{-1}$ and $\mathcal{Z}_2 = -K$

$$\begin{aligned}B_1^2 B_2 - [2]_q B_1 B_2 B_1 + B_2 B_1^2 &= [2]_q (q c_2 K + q^{-2} c_1 K^{-1}) B_1, \\ B_2^2 B_1 - [2]_q B_2 B_1 B_2 + B_1 B_2^2 &= [2]_q (q c_1 K^{-1} + q^{-2} c_2 K) B_2.\end{aligned} \tag{3.2}$$

Alternatively (3.2) can be verified directly from the definitions of B_1, B_2 and K.

The Cartan subalgebra of $\mathcal{B}$ is generated by $K^{\pm 1}$, C_1 and C_2, where

$$\begin{aligned} C_1 &= B_1B_2 - qB_2B_1 - \frac{1}{q-q^{-1}}c_2K + \frac{q+q^{-1}}{q-q^{-1}}c_1K^{-1}, \\ C_2 &= B_2B_1 - qB_1B_2 - \frac{1}{q-q^{-1}}c_1K^{-1} + \frac{q+q^{-1}}{q-q^{-1}}c_2K. \end{aligned} \tag{3.3}$$

Moreover if $c_1, c_2 \in \mathbb{R}^\times$, then C_1 and C_2 are self-adjoint. The generators of the Cartan subalgebra of $\mathcal{B}$ satisfy the relations $[K, C_i] = 0$ for $i = 1, 2$, $[C_1, C_2] = 0$ and

$$\begin{aligned} KB_1 &= q^{-3}B_1K, \quad C_1B_1 = qB_1C_1, \quad C_2B_1 = q^{-1}B_1C_2, \\ KB_2 &= q^{3}B_2K, \quad C_1B_2 = q^{-1}B_2C_1, \quad C_2B_2 = qB_2C_2. \end{aligned} \tag{3.4}$$

Note that by [12, Theorem 8.5] the center of $\mathcal{B}$ is of rank 2. Hence the center of $\mathcal{B}$ is generated by $K^{\frac{1}{3}}C_1$ and $K^{-\frac{1}{3}}C_2$, extending $\mathcal{B}$ by cube roots of K. Then the central elements are self-adjoint for $c_1, c_2 \in \mathbb{R}^\times$.

3.1. **Representation theory of $\mathcal{B}$.** Let (τ, W) be a finite-dimensional representation of $\mathcal{B}$. Since W is a finite-dimensional complex vector space, there exists a vector $w \in W$ such that $\tau(K)w = \nu\, w$ for some $\nu \in \mathbb{C}$. Then it follows from (3.4) that

$$\tau(K)\tau(B_1)^i\, w = q^{-3i}\,\tau(B_1)^i\tau(K)w = q^{-3i}\nu\tau(B_1)^i\, w, \qquad i \in \mathbb{N},$$

so that the vectors $(\tau(B_1)^i\, w)_i$ are eigenvectors of $\tau(K)$ with different eigenvalues. Since W is finite-dimensional, there exists $j \in \mathbb{N}$ such that $\tau(B_1^{j+1})w = 0$ and $\tau(B_1^j)w \neq 0$. Therefore $w_0 = \tau(B_1^j)w$ is a highest weight vector, i.e.

$$\tau(B_1)\, w_0 = 0, \qquad \tau(K)\, w_0 = \kappa\, w_0, \qquad q^{-3}\kappa \notin \sigma(K),$$

where κ is the weight of w_0 and $\sigma(K)$ is the spectrum of K. Note that $\kappa \in \mathbb{C}^\times$ since it is the eigenvalue of an invertible operator.

Proposition 3.1. *Let τ be a finite-dimensional irreducible representation of $\mathcal{B}$ on the vector space W. Then τ is determined by the dimension of W and the action of K on a highest weight vector.*

Proof. Let $\kappa \in \mathbb{C}^\times$ be the highest weight of τ and let w_0 be a highest weight vector, i.e. $\tau(K)\, w_0 = \kappa\, w_0$ and $\tau(B_1)w_0 = 0$. Since $\tau(K), \tau(C_1)$ and $\tau(C_2)$ form a commuting family of operators, we can assume that $\tau(C_1)\, w_0 = \eta_1\, w_0$ and $\tau(C_2)\, w_0 = \eta_2\, w_0$. For every $i \in \mathbb{N}$, we define the vector $w_i = \tau(B_2)^i w_0 \in W$. Since W is finite-dimensional, there exists $n \in \mathbb{N}$ such that $w_i \neq 0$ for $0 \leq i \leq n$ and $w_{n+1} = 0$. It follows from (3.4) that $\tau(K)w_i = q^{3i}\,\kappa\, w_i$, so that $(w_i)_{i=0}^n$ is a set of linearly independent vectors since they are eigenvectors of $\tau(K)$ for different eigenvalues. Moreover (3.4) implies

$$\tau(C_1)\, w_i = \tau(C_1)\tau(B_2)^i\, w_0 = q^{-i}\,\tau(B_2)^i\tau(C_1)\, w_0 = \eta_1 q^{-i}\, w_i,$$

and similarly $\tau(C_2)\, w_i = \eta_2 q^i\, w_i$. We will show that it is indeed a basis of W.

We prove by induction in i that there exist $b_i \in \mathbb{C}$ such that $\tau(B_1)w_i = b_i w_{i-1}$ for $i = 0, \ldots, n$. The statement holds for $i = 0$ taking $b_0 = 0$ since w_0 is a highest weight vector. Let $i > 0$ and assume that $\tau(B_1)w_j = b_j w_{j-1}$ for for all $j < i$. Using (3.3) we find the recurrence relation

$$\begin{aligned}\tau(B_1)w_i &= \tau(B_1)\tau(B_2)^i w_0 = \tau(B_1B_2)w_{i+1}\\ &= \tau\left(C_1 + qB_2B_1 + \frac{c_2}{(q-q^{-1})}K - \frac{(q+q^{-1})}{(q-q^{-1})}c_1K^{-1}\right)w_{i-1}\\ &= q\tau(B_2B_1)w_{i-1} + \tau\left(C_1 + \frac{c_2}{(q-q^{-1})}K - \frac{(q+q^{-1})}{(q-q^{-1})}c_1K^{-1}\right)w_{i-1},\end{aligned}$$

By the inductive hypothesis, $\tau(B_2B_1)w_{i-1} = b_{i-1}\tau(B_2)w_{i-2} = b_{i-1}w_{i-1}$, so that

$$\tau(B_1)w_i = \left(qb_{i-1} + q^{1-i}\eta_1 + \frac{q^{3i-3}\kappa\, c_2}{(q-q^{-1})} - \frac{(q+q^{-1})}{(q-q^{-1})}q^{3-3i}\kappa^{-1}\, c_1\right)w_{i-1}. \tag{3.5}$$

Hence $\tau(B_1)w_i = b_iw_{i-1}$. Since τ is an irreducible representation we have that $W = \tau(\mathcal{B})\, w_0 = \langle\{w_0, w_1, \ldots, w_n\}\rangle$, and therefore $(w_i)_{i=0}^n$ is a basis of W. This completes the proof of the proposition. ☐

Remark 3.2. Since we assume (τ, W) irreducible, the coefficients b_i in the proof of Proposition 3.1 are non-zero for $i = 1, \ldots, n$. This follows from the fact that if $b_{i_0} = 0$ for some $1 \leq i_0 \leq n$, then $\langle\{w_{i_0}, w_{i_0+1}, \ldots, w_n\}\rangle$ is an invariant subspace and this contradicts the irreducibility of τ.

Corollary 3.3. *Let (τ, W) be a finite-dimensional irreducible representation of $\mathcal{B}$ of dimension $n+1$ and highest weight κ. let w_0 be a highest weight vector and let $w_i = (B_2)^i w_0$ for $i = 1, \ldots, n$. Then $(w_i)_{i=0}^n$ is a basis of W. The action of the generators of $\mathcal{B}$ on this basis is given by*

$$\tau(K)\, w_j = q^{3j}\,\kappa\, w_j, \qquad \tau(B_2)\, w_j = w_{j+1}, \qquad \tau(B_1)\, w_j = b_j w_{j-1}$$

where

$$b_0 = 0, \qquad b_j = c_1\,\kappa^{-1}\, q^{-2n-1}\,[j]_q\, \frac{(1-q^{2n-2j+2})(1+c_2c_1^{-1}\kappa^2q^{2j+2n-1})}{(q-q^{-1})}.$$

Moreover, $\tau(C_1)\, w_j = q^{-j}\,\eta_1\, w_j$ and $\tau(C_2)\, w_j = q^j\,\eta_2\, w_j$, where

$$\eta_1 = \frac{c_1\,\kappa^{-1}\, q(1+q^{-2n-2}) - c_2\,\kappa\, q^{2n}}{q-q^{-1}}, \qquad \eta_2 = \frac{c_2\,\kappa\, q^{-1}(1+q^{2n+2}) - c_1\,\kappa^{-1}\, q^{-2n}}{q-q^{-1}}.$$

Proof. The fact that $(w_i)_{i=0}^n$ is a basis of W and the action of $\tau(K)$ on w_j follow directly from the proof of Proposition 3.1. It is clear that $b_0 = 0$. We now show that

$$b_j = [j]_q\left(\eta_1 + \frac{c_2\,\kappa\, q^{2j-2} - c_1\,\kappa^{-1}\, q^{1-2j}(1+q^{2j})}{(q-q^{-1})}\right), \tag{3.6}$$

for all $j = 1, \ldots, n$. We proceed by induction on i. If $i = 1$, then the statement follows directly from (3.5). Now we assume that (3.6) is true for some j, $1 < j \leq n$. Then it follows from (3.5) and the inductive hypothesis that

$$\begin{aligned}b_j = q\,[j-1]_q &\left(\eta_1 + \frac{c_2\,\kappa\, q^{2j-4} - c_1\,\kappa^{-1}\, q^{3-2j}(1+q^{2j-2})}{(q-q^{-1})}\right)\\ &\qquad + q^{1-j}\eta_1 + \frac{q^{3j-3}\kappa\, c_2}{(q-q^{-1})} - \frac{(q+q^{-1})}{(q-q^{-1})}q^{3-3j}\kappa^{-1}\, c_1.\end{aligned}$$

Now (3.6) follows by a straightforward computation.

It follows from the proof of Proposition 3.1 that $\tau(C_1)\,w_j = q^{-j}\,\eta_1\,w_j$ where η_1 is the eigenvalue for the highest weight vector w_0. From the construction of the vectors w_i in Proposition (3.1), it follows that $\tau(B_2)\,w_n = 0$. Hence (3.3) and (3.6) yield

$$\begin{aligned} q^{-n}\eta_1\,w_n = \tau(C_1)\,w_n &= q\tau(B_2B_1)\,w_n - \frac{1}{q-q^{-1}}c_2\,\tau(K)w_n + \frac{q+q^{-1}}{q-q^{-1}}\,c_1\,\tau(K^{-1})w_n \\ &= -\frac{q^{n+1}-q^{-n+1}}{q-q^{-1}}\eta_1 - \frac{q^{n+1}-q^{-n+1}}{q-q^{-1}}\left(\frac{c_2\,\kappa\,q^{2n-2} - c_2\,\kappa^{-1}\,q^{1-2n}(1+q^{2n})}{q-q^{-1}}\right) \\ &\quad - \frac{c_2\,\kappa\,q^{3n}}{q-q^{-1}} + \frac{(q+q^{-1})\,c_1\,\kappa^{-1}\,q^{-3n}}{q-q^{-1}}. \end{aligned}$$

Now the expression of η_1 follows by a straightforward computation. The expression of η_2 can be obtained similarly from the action of C_2 on w_n. □

Remark 3.4. If τ is an irreducible representation with highest weight κ and dimension $n+1$, it follows from Remark 3.2 and the explicit expression of the coefficient b_i in Corollary 3.3 that $c_2c_1^{-1}\kappa^2 \neq -q^{-2j-2n+1}$ for all $j = 1, \dots, n$.

Remark 3.5. It follows from Proposition 3.1 and Corollary 3.3 that a finite-dimensional irreducible representation (τ, W) of $\mathcal{B}$ is completely determined by the highest weight κ and the eigenvalue of η_1 of the highest weight vector as eigenvector of $\tau(C_1)$.

Corollary 3.6. *Every irreducible finite-dimensional representation of $\mathcal{B}$ is determined by a pair (κ, n) where κ is the highest weight and the dimension is $n+1$. Conversely, to each pair (κ, n) with $\kappa \in \mathbb{C}^\times$, $n \in \mathbb{N}$ and $\kappa^2 \notin -c_1c_2^{-1}q^{1-\mathbb{N}}$, there corresponds an irreducible representation $(\tau_{(\kappa,n)}, W_{(\kappa,n)})$ with highest weight κ and dimension $n+1$.*

Proof. It follows directly from Proposition 3.1, Corollary 3.3 and Remark 3.4. □

Proposition 3.7. *Assume that $\kappa \in \mathbb{R}^\times$ and $c_1\overline{c}_2 = q^3$. Let (τ, W) be an irreducible finite-dimensional representation of $\mathcal{B}$. Then τ is unitarizable.*

Proof. Since $c_1\overline{c_2} = q^3$, we have that $\mathcal{B}^* = \mathcal{B}$. More precisely $B_1^* = -\overline{c_1}K^{-1}B_2$, $B_2^* = -\overline{c_2}KB_1$ and $K^* = K$. Let $(w_i)_{i=0}^n$ be the basis of W given in Corollary 3.3 and let $\langle\cdot,\cdot\rangle$ be the hermitian bilinear form defined on the basis elements by $\langle w_0, w_0\rangle = 1$,

$$\langle w_k, w_k\rangle = \langle\tau(((B_2)^k)^*(B_2)^k)w_0, w_0\rangle, \qquad \langle w_i, w_j\rangle = 0, \quad i \neq j.$$

Observe that

$$\begin{aligned} \langle w_k, w_k\rangle &= \langle\tau(((B_2)^k)^*(B_2)^k)w_0, w_0\rangle \\ &= (-1)^k\overline{c_2}^k q^{3\binom{k}{2}}\langle\tau(K^k(B_1)^k(B_2)^k)w_0, w_0\rangle = (-1)^k\overline{c_2}^k q^{3\binom{k}{2}}\,\langle\tau(K^k)w_0, w_0\rangle\prod_{i=1}^k b_i \\ &= \frac{q^{3\binom{k}{2}-k(2n-1)}}{(1-q^2)^k}\,[k]_q!\,(q^{2n};q^{-2})_k\,(-c_2c_1^{-1}\,\kappa^2\,q^{2n-1};q^2)_k\,\langle w_0, w_0\rangle. \end{aligned} \tag{3.7}$$

Since $q^3c_2c_1^{-1} = c_1\overline{c_2}c_2c_1^{-1} = |c_2|^2 > 0$, it follows that $c_2c_1^{-1} > 0$ and thus (3.7) is positive. Therefore $\langle\cdot,\cdot\rangle$ is a positive definite bilinear form. Moreover, $\langle\tau(X)\,w_i, w_j\rangle = \langle w_i, \tau(X^*)\,w_j\rangle$ for all $X \in \mathcal{B}$. This follows from a straightforward verification on the generators of $\mathcal{B}$. □

Remark 3.8. Let $\kappa \in \mathbb{R}^\times$ and $n \in \mathbb{N}$. Let $(w_i)_{i=0}^n$ be the orthogonal basis for $W^{(\mu,n)}$ as in Corollary 3.3. We define an orthonormal basis $(\widetilde{w}_i)_{i=0}^n$ by $\widetilde{w}_i = w_i/||w_i||$. The actions of C_1, C_2 and K on the orthonormal basis are the same. For B_1 and B_2 we have

$$\tau_{(\kappa,n)}(B_1)\widetilde{w}_i = -c_1\,\kappa^{-1}\,q^{-2i-n+1}\sqrt{\frac{(1-q^{2i})}{(1-q^2)}\frac{(1-q^{2n-2i+2})}{(1-q^2)}(q+c_2c_1^{-1}\,\kappa^2\,q^{2n+2i})}\widetilde{w}_{i-1},$$
$$\tau_{(\kappa,n)}(B_2)\widetilde{w}_i = q^{i-n+1}\sqrt{\frac{(1-q^{2i+2})}{(1-q^2)}\frac{(1-q^{2n-2i})}{(1-q^2)}(q+c_2c_1^{-1}\,\kappa^2\,q^{2n+2i+2})}\widetilde{w}_{i+1}.$$

4. The branching rule

In this section we prove the main theorem of the paper. We fix a coideal subalgebra $\mathcal{B}$ and show that any finite-dimensional representation of $\mathcal{U}_q(\mathfrak{su}(3))$ restricted to $\mathcal{B}$ decomposes multiplicity free as finite-dimensional representations of $\mathcal{B}$ and we characterise the $\mathcal{B}$-representations that occur in this decomposition. In case $\mathcal{B}$ is $*$-invariant, every finite-dimensional irreducible representation of $\mathcal{U}_q(\mathfrak{su}(3))$ restricted to $\mathcal{B}$ obviously decomposes into finite-dimensional irreducible representations. This fact is also noted by Letzter [14, Theorem 3.3].

Theorem 4.1. *Let $\lambda \in P^+$ such that $\lambda = \lambda_1\varpi_1 + \lambda_2\varpi_2$ and fix the finite-dimensional irreducible representation π_λ of $\mathcal{U}_q(\mathfrak{su}(3))$ on the vector space V_λ. Let $\mathcal{B}$ be a coideal subalgebra with $c_2c_1^{-1} \notin -q^{2\lambda_1+2\lambda_2+1-\mathbb{N}}$. The representation π_λ restricted to $\mathcal{B}$ decomposes multiplicity free into irreducible representations;*

$$\pi_\lambda|_{\mathcal{B}} \simeq \bigoplus_{(\kappa,n)} \tau_{(\kappa,n)}, \qquad V_\lambda = \bigoplus_{(\kappa,n)} W_{(\kappa,n)},$$

where the sum is taken over $(\kappa,n) = (q^{\lambda_1-\lambda_2-3i}, i+x)$, with $0 \le i \le \lambda_1$ and $0 \le x \le \lambda_2$.

The proof of Theorem 4.1 will be carried out in the next subsections. If $(\tau_{(\kappa,n)}, W_{(\kappa,n)})$ is a representation of $\mathcal{B}$ that occurs in the representation π_λ upon restriction to $\mathcal{B}$ then a highest weight vector $w_0^{(\mu,n)}$ for $\tau_{(\kappa,n)}$ is completely determined by the highest weight κ and the eigenvalue η_1, see Remark 3.5. Hence, highest weight vectors for $\mathcal{B}$-representations in V_λ are the eigenvectors of $\pi_\lambda(C_1)$ that belong to the kernel of $\pi_\lambda(B_1)$. In Subsection 4.1 we determine the kernel of $\pi_\lambda(B_1)$.

Remark 4.2. Observe that the Serre relations (3.2) for $\mathcal{B}$ imply that the kernel of $\pi_\lambda(B_1)$ is invariant under the action of B_1B_2 and thus under the action of C_1.

In Subsection 4.2 we diagonalize the restriction of $\pi_\lambda(C_1)$ to $\ker(\pi_\lambda(B_1))$. In most of the proofs we identify $\pi_\lambda(X)$, $X \in \mathcal{U}_q(\mathfrak{su}(3))$, with X.

Remark 4.3. The restriction on c_1 and c_2 in Theorem 4.1 is assumed in order to ensure the complete reducibility of π_λ upon restriction to $\mathcal{B}$. This is not always true for the excluded values of c_1 and c_2. For example let $\lambda = \varpi_1$. Then V_λ is a three dimensional vector space. Mudrov's basis in Theorem 2.3 is given by

$$\mathscr{B} = \{v_\lambda,\; F_1\,v_\lambda,\; F_2F_1\,v_\lambda\}.$$

In this basis, the operator C_1 is given by the 3×3 matrix

$$C_1 = \begin{pmatrix} \frac{c_1 q^2 + c_1 - q c_2)}{q(q^2-1)} & 0 & -c_1 c_2 \\ 0 & \frac{x_1 q^4 + c_1 - q c_2}{q^2-1} & 0 \\ -q & 0 & \frac{c_1 q^4 + c_1 - q^3 c_2}{q(q^2-1)} \end{pmatrix}.$$

The eigenvectors of C_1 are (multiples of) the vectors

$$\rho_1 = \begin{pmatrix} c_1 \\ 0 \\ 1 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} -c_2/q \\ 0 \\ 1 \end{pmatrix}.$$

If $c_1 \neq -c_2/q$, then V_λ decomposes as a sum of a two-dimensional and a one-dimensional irreducible representations of W:

$$V_\lambda = W_{(q,0)} \oplus W_{(q^{-2},1)},$$

where $W_{(q,0)} = \langle \{\rho_1\} \rangle$ and $W_{(q^{-2},1)} = \langle \{\rho_2, \rho_3\} \rangle$. Moreover, the highest weight vectors of $W_{(q,0)}$ and $W_{(q^{-2},1)}$ are ρ_1 and ρ_2 respectively. If we let $c_1 = -c_2/q$ then the matrix C_1 degenerates into a non-diagonalizable matrix. The only eigenvectors are the multiples of ρ_2 and ρ_3 and therefore, although $W_{(q^{-2},1)}$ is a $\mathcal{B}$-invariant subspace of V_λ, there is no one-dimensional $\mathcal{B}$-invariant subspace in V_λ.

4.1. **The kernel of B_1.** The goal of this subsection is to describe the structure of the kernel of $\pi_\lambda(B_1)$ by introducing a particular basis. For each $i = 0, \ldots, \lambda_1$, we introduce the following subspaces of V_λ:

$$U_i = \langle \mathscr{B}_i \rangle, \qquad \mathscr{B}_i = \{F_2^k \hat{F}_3^l F_1^{i+k} v_\lambda : 0 \le l \le \lambda_2, \ 0 \le k \le \lambda_1 - i\}. \tag{4.1}$$

It follows from weight space considerations, that $F_1, E_2 : U_i \to U_{i+1}$ and $F_2, E_1 : U_{i+1} \to U_i$ so that $B_1 : U_i \to U_{i+1}$ and $B_2 : U_{i+1} \to U_i$. This is shown in Figure 1 for the highest weight $\lambda = 2\varpi_1 + 5\varpi_2$.

Remark 4.4. For each $i = 0, \ldots, \lambda_1$, the basis $\mathscr{B}_i$ consists on $\lambda_1 - i + 1$ layers of $\lambda_2 + 1$ vectors. More precisely, for $k = 0, \ldots, \lambda_1 - i$, the k-th layer is given by the vectors

$$F_2^k \hat{F}_3^l F_1^{k+i} v_\lambda, \qquad l = 0, \ldots, \lambda_2.$$

This structure is indicated in the Figure 2 for the representation $\lambda = 2\varpi_1 + 5\varpi_2$. The layers appear as circled numbers.

Remark 4.5. The dimension of U_i is $(\lambda_2 + 1)(\lambda_1 - i + 1)$. Therefore, the dimension of $\ker(B_1)|_{U_i}$ is, at least, $\lambda_2 + 1$. In particular, $U_{\lambda_1} \subset \ker(B_1)$.

Proposition 4.6. *The kernel of $\pi_\lambda(B_1)|_{U_i}$ has dimension $\lambda_2 + 1$. Moreover, a basis of $\ker \pi_\lambda(B_1)|_{U_i}$ is given by $(u_n^i)_{n=0}^{\lambda_2}$, where*

$$u_n^i = \sum_{k=0}^{\lambda_1 - i} \sum_{l=0}^{\lambda_2} \gamma_{k,l}^n \, F_2^k \hat{F}_3^l F_1^{k+i} \, v_\lambda,$$

and the coefficients $\gamma_{k,l}^n$ are given by the recurrence relation

$$a_k(l, k+i) \, \gamma_{k,l}^n + b_{k+1}(l-1, k+i+1) \, \gamma_{k+1,l-1}^n \\ - c_1 \, q^{l+2i+k+1-\lambda_1} \, \eta_{k+1}(l, k+i+1) \, \gamma_{k+1,l}^n = 0,$$

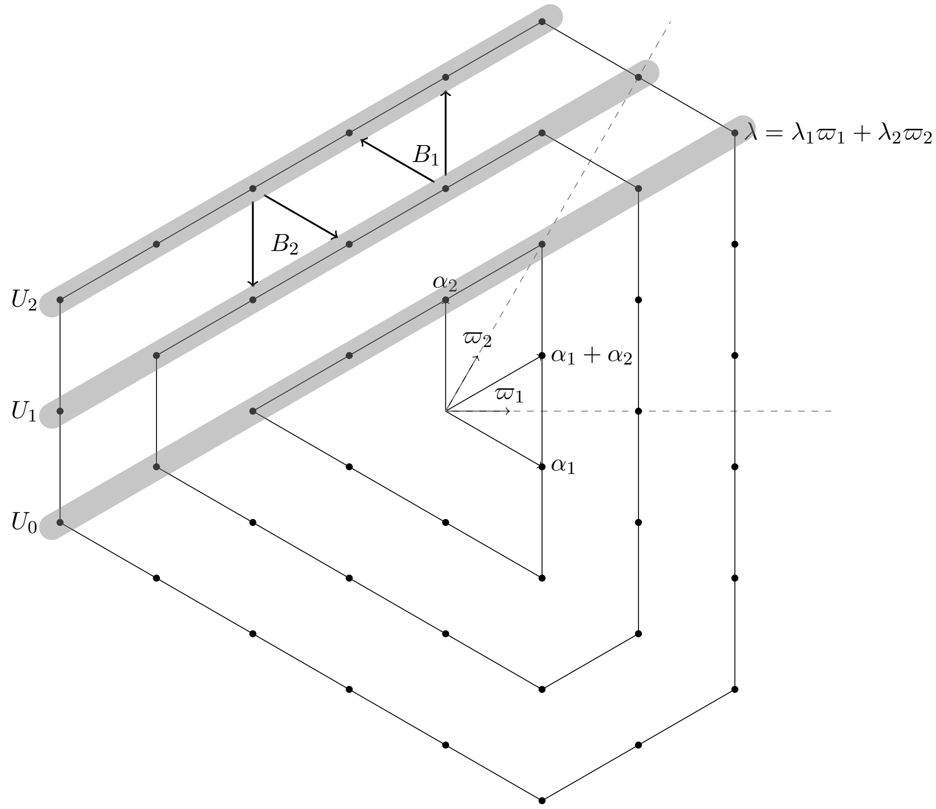

FIGURE 1. Weight diagram for the weight $\lambda = 2\varpi_1 + 5\varpi_2$. The subspaces U_i defined in (4.1) are spanned by the basis vectors indicated in gray.

for $k = 1, \dots, \lambda_1 - i - 1$, $l = 0, \dots, \lambda_2$, *with initial values* $\gamma^n_{\lambda_1 - i, l} = \delta_{n,l}$.

Proof. Let $u = \sum_{k=0}^{\lambda_1-i} \sum_{l=0}^{\lambda_2} \gamma_{k,l}\, F_2^k \hat{F}_3^l F_1^{k+i}\, v_\lambda$ be a vector in the kernel of B_1. Then

$$
\begin{aligned}
B_1 u &= \sum_{k=0}^{\lambda_1-i} \sum_{l=0}^{\lambda_2} \gamma_{k,l}\, (F_1 F_2^k \hat{F}_3^l F_1^{k+i}\, v_\lambda - c_1 E_2 K_1^{-1} F_2^k \hat{F}_3^l F_1^{k+i}\, v_\lambda) \\
&= \sum_{k=0}^{\lambda_1-i} \sum_{l=0}^{\lambda_2} \gamma_{k,l}\, (a_k(l, k+i)\, F_2^k \hat{F}_3^l F_1^{k+i+1}\, v_\lambda + b_k(l, k+i)\, F_2^{k-1} \hat{F}_3^{l+1} F_1^{k+i}\, v_\lambda \\
&\qquad\qquad - c_1 q^{l+2i+k-\lambda_1} \eta_k(l, k+i)\, F_2^{k-1} \hat{F}_3^l F_1^{k+i}\, v_\lambda) \\
&= \sum_{k=0}^{\lambda_1-i} \sum_{l=0}^{\lambda_2} (a_k(l, k+i)\, \gamma_{k,l} + b_{k+1}(l-1, k+i+1)\, \gamma_{k+1,l-1} \\
&\qquad\qquad - c_1\, q^{l+2i+k+1-\lambda_1}\, \eta_{k+1}(l, k+i+1) \gamma_{k+1,l})\, F_2^k \hat{F}_3^l F_1^{k+i+1}\, v_\lambda.
\end{aligned}
$$

Since the elements $F_2^k \hat{F}_3^l F_1^{k+i+1}$, $0 \le k \le \lambda_1 - i$, $0 \le l \le \lambda_2$, are linearly independent it follows that the coefficients $\gamma_{k,l}$ satisfy the following recurrence relation.

$$
a_k(l, k+i)\, \gamma_{k,l} + b_{k+1}(l-1, k+i+1)\, \gamma_{k+1,l-1} - c_1\, q^{l+2i+k+1-\lambda_1}\, \eta_{k+1}(l, k+i+1)\, \gamma_{k+1,l} = 0. \tag{4.2}
$$

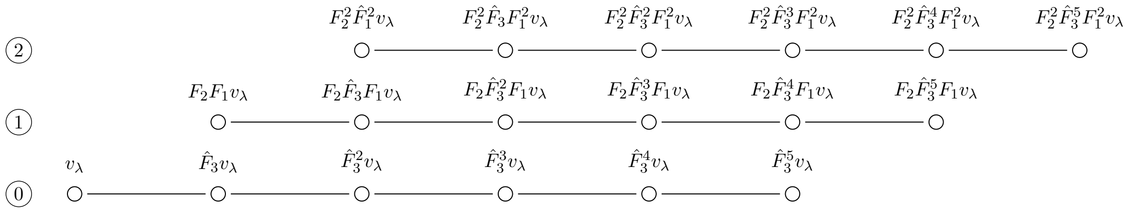

FIGURE 2. Structure of the basis of U_0 for the representation π_λ with $\lambda = 2\varpi_1 + 5\varpi_2$ as in Figure 1. The circled numbers indicate the layers of the basis.

For each $n = 0, 1, \ldots, \lambda_2$, if we set $\gamma^n_{\lambda_1-i,l} = \delta_{n,l}$, then (4.2) determines uniquely a vector u_n in the kernel of B_1. The vectors u_n are clearly linearly independent and span the kernel of B_1 restricted to U_i. This completes the proof of the proposition. □

Remark 4.7. According to the layer structure of $\mathscr{B}_i$ described in Remark 4.4, the vector u^i_n has a single non-zero contribution from the vectors of the upper layer, namely from $F_2^{\lambda_1-i}\hat{F}_3^n F_1^{\lambda_1}$, and two contributions from the one but upper layer. Therefore, we have

(4.3)
$$u^i_n = F_2^{\lambda_1-i}\hat{F}_3^n F_1^{\lambda_1}\, v_\lambda + \gamma^n_{\lambda_1-i-1,n}\, F_2^{\lambda_1-i-1}\hat{F}_3^n F_1^{\lambda_1-1}\, v_\lambda + \gamma^n_{\lambda_1-i-1,n+1}\, F_2^{\lambda_1-i-1}\hat{F}_3^{n-1} F_1^{\lambda_1-1}\, v_\lambda + \sum_{k=0}^{\lambda_1-i-2}\sum_{l=0}^{\lambda_2} \gamma^n_{k,l}\, F_2^k \hat{F}_3^l F_1^{i+k}\, v_\lambda.$$

The coefficients $\gamma^n_{\lambda_1-i-1,n}$ and $\gamma^n_{\lambda_1-i-1,n+1}$ corresponding to the vectors of the one but last layer are given by

(4.4)
$$\begin{aligned}\gamma^n_{\lambda_1-i-1,n} &= \frac{c_1\, q^{n+i}\,(q^{\lambda_1-i}-q^{-\lambda_1+i})(q^{\lambda_2+\lambda_1-n}-q^{-\lambda_2-\lambda_1+n})}{(q-q^{-1})^2},\\ \gamma^n_{\lambda_1-i-1,n+1} &= -\frac{(q^{\lambda_1-i}-q^{-\lambda_1+i})(q^{\lambda_2+\lambda_1-n-1}-q^{-\lambda_2-\lambda_1+n+1})}{(q^{\lambda_2+\lambda_1+1-n}-q^{-\lambda_2-\lambda_1-1+n})(q^{\lambda_2+i-n}-q^{-\lambda_2-i+n})}.\end{aligned}$$

The structure of the vectors u^i_n for U_{λ_1-2} is depicted in Figure 3.

Remark 4.8. The basis $\{u^i_n\}^i_n$ of the kernel of $\pi_\lambda(B_1)$ is not an orthogonal basis. In fact, it follows from Remark 4.7 that

$$\begin{aligned}u_0^{\lambda_1-1} &= F_2^{\lambda_1-1}F_1^{\lambda_1}\, v_\lambda + \gamma^0_{\lambda_1-2,0}\, F_2^{\lambda_1-2}F_1^{\lambda_1-1}\, v_\lambda,\\ u_1^{\lambda_1-1} &= F_2^{\lambda_1-1}\hat{F}_3 F_1^{\lambda_1}\, v_\lambda + \gamma^1_{\lambda_1-2,1}\, F_2^{\lambda_1-2}\hat{F}_3 F_1^{\lambda_1-1}\, v_\lambda + \gamma^1_{\lambda_1-2,2}\, F_2^{\lambda_1-2}F_1^{\lambda_1-1}\, v_\lambda,\end{aligned}$$

and therefore

$$\langle u_0^{\lambda_1-1}, u_1^{\lambda_1-1}\rangle = \gamma^0_{\lambda_1-2,0}\gamma^1_{\lambda_1-2,2} H_{\lambda_1-2,0,\lambda_1-1} \neq 0,$$

using the explicit expressions (4.4).

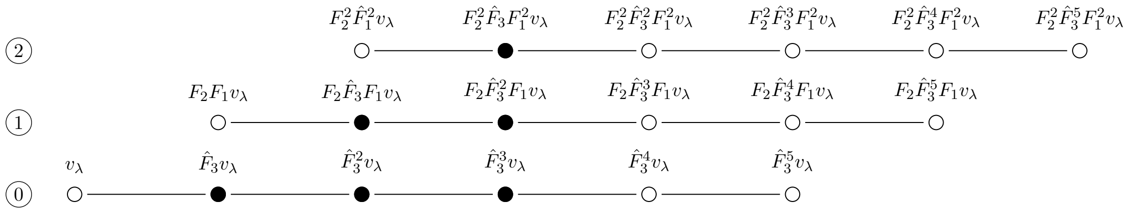

FIGURE 3. Structure of the basis $(u^0_n)_n$ of $\ker(B_1)|_{U_0}$ for the representation $\lambda = 2\varpi_1 + 5\varpi_2$ as in Figure 1. The black circles indicate the terms that contribute to the expression of the element $u^0_1 = F_2^2\hat{F}_3F_1^2\, v_\lambda + \cdots$.

4.2. **The action of C_1.** In Remark 4.2 we observed that the kernel of B_1 is stable under the action of C_1. Furthermore for each $i = 0, \ldots, \lambda_1$, U_i is stable under C_1. The goal of this subsection is to compute the action of C_1 in the basis of $\ker \pi_\lambda(B_1)$ given in Proposition 4.6.

Lemma 4.9. *In the basis $\mathscr{B}$ of Theorem 2.3 we have*

$$\begin{aligned}
F_1F_2F_2^k\hat{F}_3^lF_1^{k+i}\, v_\lambda &= a_{k+1}(l,k+i)\, F_2^{k+1}\hat{F}_3^lF^{k+i+1}\, v_\lambda + b_{k+1}(l,k+i)\, F_2^k\hat{F}_3^{l+1}F_1^{k+i}\, v_\lambda,\\
E_2F_2F_2^k\hat{F}_3^lF_1^{k+i}\, v_\lambda &= \eta_{k+1}(l,k+i)\, F_2^k\hat{F}_3^l\, F_1^{k+i}\, v_\lambda,\\
F_1E_1F_2^k\hat{F}_3^lF_1^{k+i}\, v_\lambda &= \alpha_k(l,k+i)\, a_k(l,k+i-1)\, F_2^k\hat{F}_3^lF_1^{k+i}\, v_\lambda\\
&\quad + \alpha_k(l,k+i)\, b_k(l,k+i-1)\, F_2^{k-1}\hat{F}_3^{l+1}F_1^{k+i-1}\, v_\lambda\\
&\quad + \beta_k(l,k+i)\, a_{k+1}(l-1,k+i)F_2^{k+1}\hat{F}_3^{l-1}F_1^{k+i+1}\, v_\lambda\\
&\quad + \beta_k(l,k+i)\, b_{k+1}(l-1,k+i)F_2^k\hat{F}_3^lF_1^{k+i}\, v_\lambda,\\
E_2E_1F_2^k\hat{F}_3^lF_1^{k+i} &= \alpha_k(l,k+i)\, \eta_k(l,k+i-1)\, F_2^{k-1}\hat{F}_3^lF_1^{k+i-1}\, v_\lambda\\
&\quad + \beta_k(l,k+i)\, \eta_{k+1}(l-1,k+i)\, F_2^k\hat{F}_3^{l-1}F_1^{k+i}\, v_\lambda,\\
KF_2^{\lambda_1-i}\hat{F}_3^lF_1^{\lambda_1}\, v_\lambda &= q^{\lambda_1-\lambda_2-3i}\, F_2^{\lambda_1-i}\hat{F}_3^lF_1^{\lambda_1}\, v_\lambda,\\
K^{-1}F_2^{\lambda_1-i}\hat{F}_3^lF_1^{\lambda_1}\, v_\lambda &= q^{\lambda_2-\lambda_1+3i}\, F_2^{\lambda_1-i}\hat{F}_3^lF_1^{\lambda_1}\, v_\lambda.
\end{aligned}$$

Proof. The lemma is a direct consequence of Proposition 2.4. □

Since K acts as a multiple of the identity on each U_i, it suffices to determine the action of B_1B_2 on U_i.

Lemma 4.10. *For $i \in 0, \ldots, \lambda_1$, in the basis $(u^i_n)_n$ of* $\ker(B_1)$, *we have*

$$B_1B_2\, u^i_n = A(n)u^i_{n+1} + B(n)u^i_n + C(n)u^i_{n-1}, \qquad n = 0, \ldots, \lambda_2,$$

where

$$\begin{aligned}
A(n) &= \frac{q^{\lambda_2+i-n}(1-q^2)(1-q^{2\lambda_1+2\lambda_2-2n})}{(1-q^{2\lambda_2+2\lambda_1-2n+2})(1-q^{2\lambda_2+2i-2n})},\\
B(n) &= -c_1\frac{q^{2n+i-\lambda_1-\lambda_2}\,(1-q^{2\lambda_2-2n+2i})}{(1-q^2)} + \frac{c_2\, q^{\lambda_1-\lambda_2+2n-i+1}(1-q^{-2n-2i})}{(1-q^2)},\\
C(n) &= \frac{c_1c_2\, q^{3n-3\lambda_2-i-2}(1-q^{2n})(1-q^{2\lambda_2-2n+2})(1-q^{2\lambda_1+2\lambda_2-2n+4})(1-q^{2\lambda_2+2i+2-2n})}{(1-q^2)^3(1-q^{2\lambda_2+2\lambda_1+2-2n})}.
\end{aligned}$$

Proof. Since U_i is stable under B_1B_2 and $(u^i_n)_n$ is a basis of U_i, we have

$$B_1B_2\, u^i_n = \sum_{j=0}^{\lambda_2} \nu_j\, u^i_j,$$

for certain coefficients ν_j. Since $\mathscr{B}_i$ is an orthogonal basis and u^i_n has a single contribution from the vectors in the upper layer of $\mathscr{B}_i$, see Remark 4.7, we obtain that

$$\langle B_1B_2\, u^i_n,\, F_2^{\lambda_1-i}\, \hat{F}_3^s\, F_1^{\lambda_1}\, v_\lambda\rangle = \sum_{j=0}^{\lambda_2} \nu_j \langle u^i_j, F_2^{\lambda_1-i}\, \hat{F}_3^s\, F_1^{\lambda_1}\, v_\lambda\rangle = \nu_s H^2_{\lambda_1-i,s,\lambda_1}.$$

On the other hand, from (3.1) we have

$$\begin{aligned}(4.5)\quad B_1B_2F_2^k\hat{F}_3^lF_1^{k+i}\, v_\lambda &= F_1F_2F_2^k\hat{F}_3^lF_1^{k+i}\, v_\lambda - c_1q^{l+k+2i-1-\lambda_1}E_2F_2F_2^k\hat{F}_3^lF_1^{k+i}\, v_\lambda\\ &- c_2q^{k+l-i-\lambda_2}F_1E_1F_2^k\hat{F}_3^lF_1^{k+i}\, v_\lambda + c_1c_2\, q^{2l+2k+i-\lambda_1-\lambda_2-2}E_2E_1F_2^k\hat{F}_3^lF_1^{k+i}\, v_\lambda.\end{aligned}$$

Applying Lemma 4.9 to (4.5), we verify that the action of B_1B_2 on the vector of the k-th layer $F_2^k\hat{F}_3^lF_1^{k+i}\, v_\lambda$ has contributions from the $(k-1)$-th, k-th and $(k+1)$-th layer. Hence, Remark 4.7 implies

(4.6)

$$\begin{aligned}\langle B_1B_2\, u^i_n,\, F_2^{\lambda_1-i}\, \hat{F}_3^s\, F_1^{\lambda_1}\, v_\lambda\rangle &= \langle B_1B_2\, F_2^{\lambda_1-i}\hat{F}_3^nF_1^{\lambda_1}\, v_\lambda,\, F_2^{\lambda_1-i}\, \hat{F}_3^s\, F_1^{\lambda_1}\, v_\lambda\rangle\\ &+ \gamma^n_{\lambda_1-i-1,n}\, \langle B_1B_2\, F_2^{\lambda_1-i-1}\hat{F}_3^nF_1^{\lambda_1-1}\, v_\lambda,\, F_2^{\lambda_1-i}\, \hat{F}_3^s\, F_1^{\lambda_1}\, v_\lambda\rangle\\ &+ \gamma^n_{\lambda_1-i-1,n+1}\, \langle B_1B_2\, F_2^{\lambda_1-i-1}\hat{F}_3^{n-1}F_1^{\lambda_1-1}\, v_\lambda,\, F_2^{\lambda_1-i}\, \hat{F}_3^s\, F_1^{\lambda_1}\, v_\lambda\rangle.\end{aligned}$$

From Lemma 4.9 we obtain that (4.6) is zero unless $s = n-1, n, n+1$. Moreover, we have

$$\begin{aligned}&\langle B_1B_2\, u^i_n,\, F_2^{\lambda_1-i}\, \hat{F}_3^{n+1}\, F_1^{\lambda_1}\, v_\lambda\rangle = [b_{\lambda_1-i+1}(n,\lambda_1) + \gamma^n_{\lambda_1-i-1,n+1}\, a_{\lambda_1-i}(n+1,\lambda_1-1)]H^2_{\lambda_1-i,n+1,\lambda_1},\\ &\langle B_1B_2\, u^i_n,\, F_2^{\lambda_1-i}\, \hat{F}_3^{n}\, F_1^{\lambda_1}\, v_\lambda\rangle = [-c_1\, q^{n+i-1}\, \eta_{\lambda_1-i+1}(l,\lambda_1)\\ &\quad - c_2\, q^{\lambda_1+n-2i-\lambda_2}\, \alpha_{\lambda_1-i}(n,\lambda_1)\, a_{\lambda_1-i}(n,\lambda_1-1)\\ &\qquad - c_2\, q^{\lambda_1-2i+n-\lambda_2}\, \beta_{\lambda_1-i}(n,\lambda_1)\, b_{\lambda_1-i+1}(n-1,\lambda_1) + \gamma^n_{\lambda_1-i-1,n}\, a_{\lambda_1-i}(n,\lambda_1-1)\\ &\qquad\quad - c_2\, q^{\lambda_1+n-2i-\lambda_2}\gamma^n_{\lambda_1-i-1,n+1}\, \beta_{\lambda_1-i-1}(n+1,\lambda_1-1)\, a_{\lambda_1-i}(n,\lambda_1-1)]H^2_{\lambda_1-i,n,\lambda_1},\\ &\langle B_1B_2\, u^i_n,\, F_2^{\lambda_1-i}\, \hat{F}_3^{n-1}\, F_1^{\lambda_1}\, v_\lambda\rangle = [c_1c_2\, q^{\lambda_1-\lambda_2+2n-i-2}\, \beta_{\lambda_1-i}(n,\lambda_1)\, \eta_{\lambda_1-i+1}(n-1,\lambda_1)\\ &\quad - c_2q^{\lambda_1-\lambda_2+n-2i-1}\, \gamma^n_{\lambda_1-i-1,n}\, \beta_{\lambda_1-i-1}(n,\lambda_1-1)\, a_{\lambda_1-i}(n-1,\lambda_1-1)]H^2_{\lambda_1-i,n-1,\lambda_1}.\end{aligned}$$

Now the lemma follows from Proposition 2.4 and (4.4). □

Lemma 4.11. *For $i \in 0,\dots,\lambda_1$, in the basis $(u^i_n)_n$ of $\ker(B_1)$, we have*

$$C_1\, u^i_n = A(n)u^i_{n+1} + (B(n)+D)\, u^i_n + C(n)u^i_{n-1}, \quad D = -c_2\frac{q^{\lambda_1-\lambda_2-3i}}{q-q^{-1}} + c_1\frac{q^{\lambda_2-\lambda_1+3i}(q+q^{-1})}{q-q^{-1}}.$$

Proof. Lemma 4.10, (3.3) and K acting as a multiple of the identity give the result. □

We are now ready to find the eigenvectors of C_1 restricted to $\ker(B_1)|_{U_i}$. We will describe these eigenvectors as a linear combination of the vectors u^i_n with explicit coefficients given in terms of dual q-Krawtchouk polynomials. For $N \in \mathbb{N}$ and $n = 0, 1, \ldots, N$, the dual q-Krawtchouk polynomials are given explicitly by

$$K_n(\lambda(x); c, N|q) = \frac{(q^{x-N}; q)_n}{(q^{-N}; q)_n q^{nx}} \, {}_2\phi_1\left(\begin{matrix} q^{-n}, q^{-x} \\ q^{N-x-n+1} \end{matrix} \middle| q, cq^{x+1} \right),$$

where $\lambda(x) = q^{-x} + cq^{x-N}$, see [7, (3.17.1)]. We follow the standard notation of [4] for basic hypergeometric series. The polynomials

$$r_l(\lambda(x)) = (q^{-2N}; q)_l \, K_l(\lambda(x); c, N|q^2), \tag{4.7}$$

satisfy the three term recurrence relation

$$x \, r_l(x) = r_{l+1}(x) + (1 + c)q^{2l-2N} \, r_l(x) + c \, q^{-2N}(1 - q^{2l})(1 - q^{2l-2N-2}) \, r_{l-1}(x). \tag{4.8}$$

Proposition 4.12. *For $i = 0, \ldots, \lambda_1$, the set $\{\psi^i_x\}_{x=0}^{\lambda_2}$ where*

$$\psi^i_x = \sum_{l=0}^{\lambda_2} \frac{c_1^l \, q^{-l(\lambda_1+2)+l(l-1)/2} \, (q^{-2\lambda_2}, q^{-2\lambda_2-2\lambda_1}; q^2)_l}{(q^{-2\lambda_2-2\lambda_1-2}, q^{-2\lambda_2-2i}; q^2)_l} \, K_l(\lambda(x), -c_1^{-1}c_2q^{2\lambda_1-2i+1}, \lambda_2, q^2) \, u^i_l,$$

is a basis of eigenvectors of C_1 restricted to $\ker(B_1)|_{U_i}$. The eigenvalue of ψ^i_x is

$$\eta_1 = \frac{c_1 \, \kappa^{-1} \, q(1 + q^{-2n-2}) - c_2 \, \kappa \, q^{2n}}{q - q^{-1}},$$

for $\kappa = q^{\lambda_1 - 3i - \lambda_2}$ and $n = x + i$.

Remark 4.13. As we pointed out in Remark 4.8, the basis $(u^i_n)_n$ is not orthogonal. Still the operator C_1 acts tridiagonally. Moreover, if $\mathcal{B}$ is $*$-invariant then the basis $\{\psi^i_x\}_{x=0}^{\lambda_2}$ in Proposition 4.12 is orthogonal although, because of the non-orthogonality of $(u^i_n)_n$, this does not follow directly from the orthogonality of the dual q-Krawtchouk polynomials.

Proof. Assume there exist polynomials $p_n(x)$ such that $v = \sum_{l=0}^{\lambda_2} p_l(x) \, u^i_l$ is an eigenvector of C_1 with eigenvalue η_1, i.e. $C_1 \, v = \eta_1 \, v$. From Lemma 4.11 we have

$$C_1 \, v = \sum_{l=0}^{\lambda_2} p_l(x)(A(l)u^i_{l+1} + (B(l) + D) \, u^i_l + C(l)u^i_{l-1}) = \sum_{l=0}^{\lambda_2} \eta_1 \, p_l(x) \, u^i_l.$$

Since $(u^i_l)_l$ is a basis of $\ker(B_1)|_{U_i}$ the vectors u^i_l are linearly independent and hence the polynomials p_l satisfy the following three term recurrence relation

$$\eta_1 \, p_l(x) = C(l + 1)p_{l+1}(x) + (B(l) + D) \, p_l(x) + A(l - 1)p_{l-1}(x).$$

If k_l is the leading coefficient of p_l, then $P_l = k_l^{-1} p_l$ is a sequence of monic polynomials satisfying the recurrence relation

$$\eta_1 \, P_l(x) = P_{l+1}(x) + (B(l) + D) \, P_l(x) + C(l)A(l - 1)P_{l-1}(x), \tag{4.9}$$

where

$$B(l) + D = -\frac{c_1 \, q^{2l+i-\lambda_1-\lambda_2}(1 - c_1^{-1} c_2 \, q^{2\lambda_1-2i+1})}{(1 - q^2)} - \frac{c_1 \, q^{3i-\lambda_1+\lambda_2+2}}{(1 - q^2)},$$

$$C(l)A(l - 1) = -\frac{c_1 c_2 \, q \, (1 - q^{2l})(1 - q^{2l-2\lambda_2-2})}{(1 - q^2)^2},$$

using Lemma 4.10 and Lemma 4.11. We will identify the polynomials P_l with the dual q-Krawtchouk polynomials. If we let

$$c = -c_1^{-1} c_2\, q^{2\lambda_1 - 2i + 1}, \quad N = \lambda_2,$$

the recurrence relation (4.8) is given by

$$x\, r_l(x) = r_{l+1}(x) + (1 + c_1^{-1} c_2 q^{2\lambda_1 - 2i+1}) q^{2l - 2\lambda_2}\, r_l(x) \\ + c_1^{-1} c_2 q^{2\lambda_1 - 2\lambda_2 - 2i + 1}(1 - q^{2l})(1 - q^{2l - 2\lambda_2 - 2})\, r_{l-1}(x).$$

If we let $\tilde{r}_l(x) = a^{-l}\, r_l(ax)$ with $a = -c_1^{-1}\, q^{\lambda_1 - \lambda_2 - i}(1 - q^2)$, by a straightforward computation we obtain

$$\left(x - \frac{c_1\, q^{3i - \lambda_1 + \lambda_2 + 2}}{(1 - q^2)} \right) \tilde{r}_l(x) \;=\; \tilde{r}_{l+1}(x) + (B(l) + D)\, \tilde{r}_l(x) + C(l) A(l-1)\, \tilde{r}_{l-1}(x). \tag{4.10}$$

If we evaluate (4.10) in $\lambda(x) a^{-1}$, the eigenvalue is given by

$$\frac{\lambda(x)}{a} - \frac{c_1\, q^{3i - \lambda_1 + \lambda_2 + 2}}{(1 - q^2)} = \frac{c_1\, q^{3i - \lambda_1 + \lambda_2 + 2}(1 + q^{-2x - 2i - 2}) + c_2\, q^{\lambda_1 - \lambda_2 - i + 2x}}{q - q^{-1}}.$$

Therefore the polynomials $P_l(x) = \tilde{r}(\lambda(x) a^{-1}) = a^{-l}\, r_l(\lambda(x))$ satisfy the recurrence (4.9) with eigenvalue

$$\eta_1 = \frac{c_1\, \kappa^{-1}\, q(1 + q^{-2n-2}) - c_2\, \kappa\, q^{2n}}{q - q^{-1}},$$

with $\kappa = q^{\lambda_1 - 3i - \lambda_2}$ and $n = x + i$, for $x = 0, \ldots, \lambda_2$. Finally, $p_l(x) = k_l\, a^{-1} r_l(\lambda(x))$. The explicit expression of p_l follows from (4.7) and Lemma 4.10. □

Proof of Theorem 4.1. From Proposition 4.12 we obtain vectors ψ_x^i for $i = 0, \ldots, \lambda_1$, $x = 0, \ldots, \lambda_2$ such that

$$\pi_\lambda(B_1)\, \psi_x^i = 0, \quad \text{and} \quad C_1\, \psi_x^i = \frac{c_1\, \kappa^{-1}\, q(1 + q^{-2n-2}) - c_2\, \kappa\, q^{2n}}{q - q^{-1}}\, \psi_x^i = \eta_1\, \psi_x^i.$$

where $\kappa = q^{\lambda_1 - 3i - \lambda_2}$ and $n = x + i$, so that ψ_x^i is a highest weight vector. It follows from Corollary 3.6 that the highest weight vector ψ_x^i defines an irreducible representation of $\mathcal{B}$ of dimension $x + i + 1$

$$W_{q^{\lambda_1 - \lambda_2 - 3i}, x+i} = \langle \{\, \psi_x^i,\, \pi_\lambda(B_2)\, \psi_x^i,\, \pi_\lambda(B_2)^2\, \psi_x^i,\, \ldots\,,\, \pi_\lambda(B_2)^{x+i}\, \psi_x^i \,\} \rangle.$$

Let $W = \oplus_{(\kappa,n)} W_{(\kappa,n)}$ where the sum is taken over $(\kappa, n) = (q^{\lambda_1 - 3i - \lambda_2}, x + i)$ for $i = 0, \ldots, \lambda_1$, $x = 0, \ldots, \lambda_2$. We have that $W \subset V_\lambda$ and

$$\dim W = \sum_{i,x} \dim W_{q^{\lambda_1 - \lambda_2 - 3i}, x+i} = \frac{1}{2}(\lambda_1 + 1)(\lambda_2 + 1)(\lambda_1 + \lambda_2 + 2) = \dim V_\lambda.$$

Therefore $W = V_\lambda$ and this completes the proof of the theorem. □

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Appendix A. Proof of Theorem 2.3

Lemma A.1. *The following relations hold in $\mathcal{U}_q(\mathfrak{su}(3))$:*

(i) $F_2\hat{F}_3[a] = \hat{F}_3[a+1]F_2$,

(ii) $E_1\hat{F}_3[a] = \hat{F}_3[a+1]E_1 + F_2\frac{(q^{a+1}K_1K_2 - q^{-a-1}(K_1K_2)^{-1})}{(q-q^{-1})}$,

(iii) $F_2F_3 = qF_3F_2$,

(iv) $\big(\hat{F}_3[a]\big)^* = q\hat{E}_3[a](K_1K_2)^{-1}$, $F_3^* = qE_3(K_1K_2)^{-1}$,

(v) $\hat{F}_3 = F_3\frac{qK_2 - q^{-1}K_2^{-1}}{q-q^{-1}} + qF_2F_1K_2$,

(vi) $E_1F_3 = F_3E_1 + F_2K_1$,

Proof. Straightforward verifications using (2.1) and (2.3). □

Corollary A.2. *For $l \in \mathbb{N}$ and $a \in \mathbb{R}$ we have*

$$E_1\big(\hat{F}_3[a]\big)^l = \big(\hat{F}_3[a+1]\big)^l E_1 + \frac{q^l - q^{-l}}{q-q^{-1}} F_2\big(\hat{F}_3[a]\big)^{l-1}\frac{(q^{a+2-l}K_1K_2 - q^{-a-2+l}(K_1K_2)^{-1})}{(q-q^{-1})}$$

Proof. By induction on l using Lemma A.1(ii) and (i). □

Proof of Theorem 2.3. By the PBW-theorem, $F_2^k\hat{F}_3^lF_1^m\, v_\lambda$ for $k,l,m \in \mathbb{N}$ spans V_λ. By Proposition 2.4

$$\begin{aligned} K_1F_2^k\hat{F}_3^lF_1^m\, v_\lambda &= q^{\lambda_1+k-l-2m}F_2^k\hat{F}_3^lF_1^m\, v_\lambda, \\ K_2F_2^k\hat{F}_3^lF_1^m\, v_\lambda &= q^{\lambda_2-2k-l+m}F_2^k\hat{F}_3^lF_1^m\, v_\lambda. \end{aligned} \tag{A.1}$$

Since K_i, $i=1,2$, are self-adjoint, we find that $\langle F_2^k\hat{F}_3^lF_1^m\, v_\lambda, F_2^{k'}\hat{F}_3^{l'}F_1^{m'}\, v_\lambda\rangle = 0$ in case $k-l-2m \neq k'-l'-2m'$ or $-2k-l+m \neq -2k'-l'+m'$. For $k' > k$ we find

$$\begin{aligned} \langle F_2^k\hat{F}_3^lF_1^m\, v_\lambda, F_2^{k'}\hat{F}_3^{l'}F_1^{m'}\, v_\lambda\rangle &= \langle (E_2K_2^{-1})^{k'}F_2^k\hat{F}_3^lF_1^m\, v_\lambda, \hat{F}_3^{l'}F_1^{m'}\, v_\lambda\rangle \\ &= q^{k'(k'+1)}\langle K_2^{-k'}E_2^{k'}F_2^k\hat{F}_3^lF_1^m\, v_\lambda, \hat{F}_3^{l'}F_1^{m'}\, v_\lambda\rangle = 0, \end{aligned} \tag{A.2}$$

since $E_i^{k'}F_i^k \in \mathcal{U}_q(\mathfrak{su}(3))\, E_i^{k'-k}$ for $k,k' \in \mathbb{N}$, $k' > k$, using also Lemma 2.1(ii) for $a=0$, (2.1) and (2.4). Because of the symmetry between k and k', we see that the inner product (A.2) is 0 for $k \neq k'$. With the above remark, we find

$$\langle F_2^k\hat{F}_3^lF_1^m\, v_\lambda, F_2^{k'}\hat{F}_3^{l'}F_1^{m'}\, v_\lambda\rangle = 0$$

in case $k \neq k'$ or $l \neq l'$ or $m \neq m'$.

So it suffices to calculate the norm of the vectors, and see that this is non-zero precisely for the range mentioned. First, using the case $k = k'$ of the first part of (A.2) and that K_2 acts on $E_2^k F_2^k \hat{F}_3^l F_1^m v_\lambda$ by the scalar q^{λ_2-l+m}, we find

$$\langle F_2^k \hat{F}_3^l F_1^m v_\lambda, F_2^k \hat{F}_3^l F_1^m v_\lambda\rangle = q^{k(k+1)-k(\lambda_2-l+m)}\langle E_2^k F_2^k \hat{F}_3^l F_1^m v_\lambda, \hat{F}_3^l F_1^m v_\lambda\rangle.$$

Now use Lemma 2.2(ii) for $i = 2$ and next the commutation relations of Lemma 2.1(ii) and (2.1) to see that the $\mathcal{U}_q(\mathfrak{su}(3))E_2$-part of Lemma 2.2(ii) gives zero contribution. Because of the action of K_2 being diagonal, we find

$$\langle F_2^k \hat{F}_3^l F_1^m v_\lambda, F_2^k \hat{F}_3^l F_1^m v_\lambda\rangle = \frac{(q^2;q^2)_k}{(1-q^2)^{2k}}(q^{-2(\lambda_2-l+m)};q^2)_k(-1)^k q^{3k}\langle \hat{F}_3^l F_1^m v_\lambda, \hat{F}_3^l F_1^m v_\lambda\rangle$$

Next we write

$$\langle \hat{F}_3^l F_1^m v_\lambda, \hat{F}_3^l F_1^m v_\lambda\rangle = \langle \hat{F}_3^l F_1^m v_\lambda, F_1^m \hat{F}_3^l v_\lambda\rangle = q^{m(m+1)}q^{-m(\lambda_1-l)}\langle E_1^m \hat{F}_3^l F_1^m v_\lambda, \hat{F}_3^l v_\lambda\rangle$$

using Lemma 2.1(i), the $*$-structure (2.3), (2.1) and (A.1). Following Mudrov [19, §8] we replace $\hat{F}_3^l$ on the left by F_3^l. First use Lemma A.1(v)

$$\begin{aligned}\langle E_1^m \hat{F}_3^l F_1^m v_\lambda, \hat{F}_3^l v_\lambda\rangle = {} & \frac{q^{2+\lambda_2-l+m} - q^{-2-\lambda_2+l-m}}{q-q^{-1}}\langle E_1^m F_3 \hat{F}_3^{l-1} F_1^m v_\lambda, \hat{F}_3^l v_\lambda\rangle \\ & + q^{2+\lambda_2-l+m}\langle E_1^m F_2 F_1 \hat{F}_3^{l-1} F_1^m v_\lambda, \hat{F}_3^l v_\lambda\rangle\end{aligned}$$

In the second term, move F_2 to the left using (2.1), and then the other side so that is essentially an E_2 which we can move through, by Lemma 2.1(ii), to the highest weight vector, and hence gives zero. This we can repeat, since F_2 also q-commutes with F_3 by Lemma A.1(iii). This yields

$$\langle E_1^m \hat{F}_3^l F_1^m v_\lambda, \hat{F}_3^l v_\lambda\rangle = \frac{(-1)^l q^{l(2+\lambda_2+m)l} q^{-\frac{1}{2}l(l-1)}}{(1-q^2)^l}(q^{-\lambda_2-2-m};q^2)_l\langle E_1^m F_3^l F_1^m v_\lambda, \hat{F}_3^l v_\lambda\rangle.$$

Using Lemma A.1(vi), and moving F_2 to the other side, where F_2^* kills $\hat{F}_3^l v_\lambda$, we see

$$\langle E_1^m F_3^l F_1^m v_\lambda, \hat{F}_3^l v_\lambda\rangle = (-1)^m q^{-m(m-2)+m\lambda_1}\frac{(q^2;q^2)_m}{(1-q^2)^{2m}}(q^{-2\lambda_1};q^2)_m\langle F_3^l v_\lambda, \hat{F}_3^l v_\lambda\rangle$$

by Lemma 2.2(ii). Assume $l \geq 1$, so it remains to calculate

$$\begin{aligned}\langle F_3^l v_\lambda, \hat{F}_3^l v_\lambda\rangle = \langle F_3^{l-1} v_\lambda, (F_3)^* \hat{F}_3^l v_\lambda\rangle &= q^{1-(\lambda_1+\lambda_2-2l)}\langle F_3^{l-1} v_\lambda, (E_2E_1 - E_1E_2)\, \hat{F}_3^l v_\lambda\rangle \\ &= q^{1-(\lambda_1+\lambda_2-2l)}\langle F_3^{l-1} v_\lambda, E_2E_1\hat{F}_3^l v_\lambda\rangle\end{aligned}$$

where we use Lemma A.1(iv), the diagonal action of K_i and the fact that the action of E_1E_2 is zero by Lemma 2.1(ii) and (2.4). By Corollary A.2 for $a = 0$ and (2.4) we find

$$E_1\hat{F}_3^l v_\lambda = \frac{q^l - q^{-l}}{q-q^{-1}}\frac{q^{2+\lambda_1+\lambda_2-l} - q^{-2-\lambda_1-\lambda_2+l}}{q-q^{-1}}F_2\hat{F}_3^{l-1} v_\lambda$$

and next applying E_2, using (2.1), (2.4) and Lemma 2.1(ii) we find

$$E_2E_1\hat{F}_3^l v_\lambda = \frac{q^l - q^{-l}}{q-q^{-1}}\frac{q^{2+\lambda_1+\lambda_2-l} - q^{-2-\lambda_1-\lambda_2+l}}{q-q^{-1}}\frac{q^{\lambda_2-l+1} - q^{-\lambda_2+l-1}}{q-q^{-1}}\hat{F}_3^{l-1} v_\lambda,$$

so that

$$\langle F_3^l v_\lambda, \hat{F}_3^l v_\lambda\rangle = q^{1-(\lambda_1+\lambda_2-2l)}\frac{q^l-q^{-l}}{q-q^{-1}} \\ \times \frac{q^{2+\lambda_1+\lambda_2-l}-q^{-2-\lambda_1-\lambda_2+l}}{q-q^{-1}}\frac{q^{\lambda_2-l+1}-q^{-\lambda_2+l-1}}{q-q^{-1}}\langle F_3^{l-1} v_\lambda, \hat{F}_3^{l-1} v_\lambda\rangle.$$

Iterating, since we normalize $\langle v_\lambda, v_\lambda\rangle = 1$, we find

$$\langle F_3^l v_\lambda, \hat{F}_3^l v_\lambda\rangle = q^{l(\lambda_2+7)}q^{-\frac{1}{2}l(l+1)}\frac{(q^2;q^2)_l}{(1-q^2)^{3l}}(q^{-2\lambda_2};q^2)_l(q^{-2(\lambda_1+\lambda_2+1)};q^2)_l.$$

Note that this expression is positive for $0 \leq l \leq \lambda_2$ and equals zero for $l > \lambda_2$. Collecting all the intermediate results gives the explicit expression for the norm of the basis elements. □

References

[1] N. Aldenhoven, E. Koelink, P. Román, *Matrix valued orthogonal polynomials for the quantum analogue of* $(\mathrm{SU}(2)\times\mathrm{SU}(2),\mathrm{diag})$, preprint, arXiv:1507.03426.

[2] M.S. Dijkhuizen, M. Noumi, *A family of quantum projective spaces and related q-hypergeometric orthogonal polynomials*, Trans. Amer. Math. Soc., **350** (1998), 3269–3296.

[3] M.S. Dijkhuizen, J.V. Stokman, *Some limit transitions between BC type orthogonal polynomials interpreted on quantum complex Grassmannians*, Publ. Res. Inst. Math. Sci., **35** (1999), 451–500.

[4] G. Gasper, M. Rahman. *Basic Hypergeometric Series*, volume 96, Cambridge University Press, 2nd edition, 2004.

[5] G. Heckman, M. van Pruijssen, *Matrix valued orthogonal polynomials for Gelfand pairs of rank one*, Tohoku Math. J. (2), to appear, `arXiv:1310.5134`.

[6] A. Klimyk, K. Schmüdgen, *Quantum Groups and Their Representations*, Texts and Monographs in Physics, Springer-Verlag, Berlin, 1997.

[7] R. Koekoek, R.F. Swarttouw, The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue, online at `http://aw.twi.tudelft.nl/~koekoek/askey.html`, Report 98-17, Technical University Delft, 1998.

[8] E. Koelink, M. van Pruijssen, P. Román, *Matrix-valued orthogonal polynomials related to* $(\mathrm{SU}(2)\times\mathrm{SU}(2),\mathrm{diag})$. Int. Math. Res. Not. IMRN, **24** (2012), 5673–5730.

[9] E. Koelink, M. van Pruijssen, R. Román, *Matrix-valued orthogonal polynomials related to* $(\mathrm{SU}(2)\times\mathrm{SU}(2),\mathrm{diag})$, II, Publ. Res. Inst. Math. Sci., **49** (2013), 271–312.

[10] S. Kolb, *Quantum symmetric Kac-Moody pairs*, Adv. Math., **267** (2014), 395–469.

[11] T. Koornwinder, *Askey-Wilson polynomials as zonal spherical functions on the* SU(2) *quantum group*, SIAM J. Math. Anal., **24** (1993), 795–813.

[12] S. Kolb, G. Letzter, *The center of quantum symmetric pair coideal subalgebras*, Rep. Theory, **12** (2008), 294–326.

[13] G. Letzter, *Symmetric pairs for quantized enveloping algebras*, J. Alg., **220** (1999), 729–767.

[14] G. Letzter, *Harish-Chandra modules for quantum symmetric pairs*, Rep. Theory, **4** (2000), 64–96.

[15] G. Letzter, *Coideal subalgebras and quantum symmetric pairs*, New Directions in Hopf Algebras, Cambridge University Press, Cambridge, **43** (2002), 117–166.

[16] G. Letzter, *Quantum symmetric pairs and their zonal spherical functions*, Transform. Groups, **8** (2003),261–292.

[17] G. Letzter, *Quantum zonal spherical functions and Macdonald polynomials*, Adv. Math., **189** (2004), 88–147.

[18] G. Letzter, *Invariant differential operators for quantum symmetric spaces*, Mem. Amer. Math. Soc., **193** (2008).

[19] A. Mudrov, *Orthogonal basis for the Shapovalov form on* $U_q(\mathfrak{sl}(n+1))$, Rev. Math. Phys. **27** (2015), no. 2, 1550004, 23 pp.

[20] M. Noumi, M.S. Dijkhuizen, T. Sugitani, *Multivariable Askey-Wilson polynomials and quantum complex Grassmannians*, Fields Inst. Commun., **14** (1997), 167–177.

[21] M. Noumi, T. Sugitani, *Quantum symmetric spaces and related q-orthogonal polynomials*, Group Theoretical Methods in Physics, World Science Publishing, River Edge, (1994), 28–40.

[22] M. Noumi, *Macdonald's symmetric polynomials as zonal spherical functions on some quantum homogeneous spaces*, Adv. Math., **123** (1996), 16–77.

[23] A. A. Oblomkov, J. V. Stokman, *Vector valued spherical functions and Macdonald-Koornwinder polynomials*, Compos. Math. **141** (2005), no. 5, 1310–1350.

[24] T. Sugitani, *Zonal spherical functions on quantum Grassmann manifolds*, J. Math. Sci Univ. Tokyo, **6** (1999), 335–369.

[25] M. van Pruijssen, *Matrix valued orthogonal polynomials related to compact Gel'fand paris of rank one*, PhD Thesis, Radboud Universiteit, (2012).

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