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Quantum Programs as Kleisli Maps

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Furber and Jacobs have shown in their study of quantum computation that the category of commutative $C^*$-algebras and $PU$-maps (positive linear maps which preserve the unit) is isomorphic to the Kleisli category of a comonad on the category of commutative $C^*$-algebras with $MIU$-maps (linear maps which preserve multiplication, involution and unit).\[^3\]

In this paper, we prove a non-commutative variant of this result: the category of $C^*$-algebras and $PU$-maps is isomorphic to the Kleisli category of a comonad on the subcategory of $MIU$-maps.

A variation on this result has been used to construct a model of Selinger and Valiron’s quantum lambda calculus using von Neumann algebras.\[^1\]

The semantics of a non-deterministic program that takes two bits and returns three bits can be described as a multimap (= binary relation) from $\{0,1\}^2$ to $\{0,1\}^3$. Similarly, a program that takes two qubits and returns three qubits can be modelled as a positive linear unit-preserving map from $M_2 \otimes M_2 \otimes M_2$ to $M_2 \otimes M_2$, where $M_2$ is the $C^*$-algebra of $2 \times 2$-matrices over $\mathbb{C}$.

More generally, the category $\text{Set}_{\text{multi}}$ of multimaps between sets models non-deterministic programs (running on an ordinary computer), while the opposite of the category $C^*_{\text{MIU}}$ of $PU$-maps (positive linear unit-preserving maps) between $C^*$-algebras models programs running on a quantum computer. (When we write $"C^*$-algebra" we always mean $"C^*$-algebra with unit ".)

A multimap from $\{0,1\}^2$ to $\{0,1\}^3$ is simply a map from $\{0,1\}^2$ to $\mathcal{P}(\{0,1\}^3)$. In the same line $\text{Set}_{\text{multi}}$ is (isomorphic to) the Kleisli category of the powerset monad $\mathcal{P}$ on $\text{Set}$. What about $C^*_{\text{PU}}$?

We will show that there is a monad $\Omega$ on $(C^*_{\text{MIU}})^\text{op}$, the opposite of the category $C^*_{\text{MIU}}$ of $C^*$-algebras and $MIU$-maps (linear maps that preserve the multiplication, involution and unit), such that $(C^*_{\text{PU}})^\text{op}$ is isomorphic to the Kleisli category of $\Omega$. We say that $(C^*_{\text{PU}})^\text{op}$ is Kleislian over $(C^*_{\text{MIU}})^\text{op}$. So in the same way we add non-determinism to $\text{Set}$ by the powerset monad $\mathcal{P}$ yielding $\text{Set}_{\text{multi}}$, we can obtain $(C^*_{\text{PU}})^\text{op}$ from $(C^*_{\text{MIU}})^\text{op}$ by a monad $\Omega$.

Let us spend some words on how we obtain this monad $\Omega$. Note that since every positive element of a $C^*$-algebra $\mathcal{A}$ is of the form $a^*a$ for some $a \in \mathcal{A}$ any MIU-map will be positive. Thus $C^*_{\text{MIU}}$ is a subcategory of $C^*_{\text{PU}}$. Let $U: C^*_{\text{MIU}} \rightarrow C^*_{\text{PU}}$ be the embedding.

In Section\[^1\] we will prove that $U$ has a left adjoint $F: C^*_{\text{PU}} \rightarrow C^*_{\text{MIU}}$, see Theorem\[^5\]. This adjunction gives us a comonad $\Omega := FU$ on $C^*_{\text{MIU}}$ (which is a monad on $(C^*_{\text{MIU}})^\text{op}$) with the same counit as the adjunction. The comultiplication $\delta$ is given by $\delta_{\mathcal{A}} = F\eta_{\mathcal{A}}$ for every object $\mathcal{A}$ from $C^*_{\text{MIU}}$ where $\eta$ is the unit of the adjunction between $F$ and $U$.

In Section\[^2\] we will prove that $(C^*_{\text{PU}})^\text{op}$ is isomorphic to $\mathcal{H}l(FU)$ if $FU$ is considered a monad on $(C^*_{\text{MIU}})^\text{op}$. In fact, we will prove that the comparison functor $L: \mathcal{H}l(FU) \rightarrow (C^*_{\text{PU}})^\text{op}$ (which sends a MIU-map $f: FU\mathcal{A} \rightarrow B$ to $Uf \circ \eta_{\mathcal{A}}: U\mathcal{A} \rightarrow U(B)$) is an isomorphism, see Corollary\[^10\].

The method used to show that $(C^*_{\text{PU}})^\text{op}$ is Kleislian over $(C^*_{\text{MIU}})^\text{op}$ is quite general and it will be obvious that many variations on $(C^*_{\text{PU}})^\text{op}$ will be Kleislian over $(C^*_{\text{MIU}})^\text{op}$ as well, such as the opposite of the category of subunital completely positive linear maps between $C^*$-algebras. The flip-side of this generality is that we discover preciously little about the monad $\Omega$ which leaves room for future inquiry (see Section\[^3\]).
We will also see that the opposite \((W_{\text{NCPU}}^*)^{\text{op}}\) of the category of normal completely positive subunital maps between von Neumann algebras is Kleislian over the subcategory \((W_{\text{SMIU}}^*)^{\text{op}}\) of normal unital \(*\)-homomorphisms. This fact is used in [1] to construct an adequate model of Selinger and Valiron’s quantum lambda calculus using von Neumann algebras.

1 The Left Adjoint

In Theorem 5 we will show that \(U\) has a left adjoint, \(F: C_{\text{MIU}}^* \rightarrow C_{\text{PU}}^*\), using a quite general method. As a result we do not get any “concrete” information about \(F\) in the sense that while we will learn that for every \(C^*\)-algebra \(\mathcal{A}\) there exists an arrow \(\rho: \mathcal{A} \rightarrow UF\mathcal{A}\) which is initial from \(\mathcal{A}\) to \(U\) we will learn nothing more about \(\rho\) than this. Nevertheless, for some (very) basic \(C^*\)-algebras \(\mathcal{A}\) we can describe \(F \mathcal{A}\) directly, as is shown below in Example 1.

Example 1. Let us start easy: \(C\) will be mapped to itself by \(F\), that is:

the identity \(\rho : C \rightarrow UC\) is an initial arrow from \(C\) to \(U(-)\).

Indeed, let \(\mathcal{A}\) be a \(C^*\)-algebra and let \(\sigma : C \rightarrow U \mathcal{A}\) be a PU-map. Then \(\sigma\) must be given by \(\sigma(\lambda) = \lambda \cdot 1\) for \(\lambda \in C\), where \(1\) is the identity of \(\mathcal{A}\). Thus \(\sigma\) is a MIU-map as well. Hence there is a unique MIU-map \(\tilde{\sigma} : C \rightarrow \mathcal{A}\) (namely \(\tilde{\sigma} = \sigma\)) such that \(\tilde{\sigma} \circ \rho = \sigma\). (\(C\) is initial in both \(C_{\text{MIU}}\) and \(C_{\text{PU}}^*\).)

Example 2. The image of \(C^2\) under \(F\) will be the \(C^*\)-algebra \(C[0,1]\) of continuous functions from \([0,1]\) to \(C\). As will become clear below, this is very much related to the familiar functional calculus for \(C^*\)-algebras: given an element \(a\) of a \(C^*\)-algebra \(\mathcal{A}\) with \(0 \leq a \leq 1\) and \(f \in C[0,1]\) we can make sense of \(f(a)\), as an element of \(\mathcal{A}\).

The map \(\rho : C^2 \rightarrow UC[0,1]\) given by, for \(\lambda , \mu \in C\), \(x \in [0,1]\),

\[\rho(\lambda , \mu)(x) = \lambda x + \mu (1-x)\]

is an initial arrow from \(C^2\) to \(U\).

Let \(\sigma : C^2 \rightarrow U \mathcal{A}\) be a PU-map. We must show that there is a unique MIU-map \(\bar{\sigma} : C[0,1] \rightarrow \mathcal{A}\) such that \(\bar{\sigma} = \tilde{\sigma} \circ \rho\).

Writing \(a := \sigma(1,0)\), we have \(\sigma(\lambda , \mu) = \lambda a + \mu (1-a)\) for all \(\lambda , \mu \in C\). Note that \((0,0) \leq (1,0) \leq (1,1)\) and thus \(0 \leq a \leq 1\). Let \(C^a\) be the \(C^*\)-subalgebra of \(\mathcal{A}\) generated by \(a\). Then \(C^a\) is commutative since \(a\) is positive (and thus normal). Given a MIU-map \(\omega : C^a \rightarrow C\) we have \(\omega(a) \in [0,1]\) since \(0 \leq a \leq 1\). Thus \(\omega \circ (\omega(a)\cdot 1)\) gives a map \(j : \Sigma C^a \rightarrow [0,1]\), where \(\Sigma C^a\) is the spectrum of \(C^a\), that is, \(\Sigma C^a\) is the set of MIU-maps from \(C^a\) to \(C\) with the topology of pointwise convergence. (By the way, the map \(j\) is the spectrum of the element \(a\).) The map \(j\) is continuous since the topology on \(\Sigma C^a\) is induced by the product topology on \(C C^a\). Thus the assignment \(h \mapsto h \cdot 1\) gives a MIU-map \(C_j : C[0,1] \rightarrow C C^a\). By Gelfand’s representation theorem there is a MIU-isomorphism

\[\gamma : C^a \rightarrow C C^a\]

given by \(\gamma(b) = \omega(b)\) for all \(b \in C^a\) and \(\omega \in \Sigma C^a\). Now, define

\[\bar{\sigma} := \gamma^{-1} \circ C_j : C[0,1] \rightarrow C^a \hookrightarrow \mathcal{A}\]

(In the language of the functional calculus, \(\bar{\sigma}\) maps \(f\) to \(f(a)\).) We claim that \(\bar{\sigma} \circ \rho = \sigma\). It suffices to
show that $C_j \circ \rho \equiv \gamma \circ \sigma \circ \rho = \gamma \circ \sigma$. Let $\lambda, \mu \in \mathbb{C}$ and $\omega \in \Sigma C^*(a)$ be given. We have

\[
(C_j \circ \rho)(\lambda, \mu)(\omega) = (C_j)(\rho(\lambda, \mu))(\omega) \\
= \rho(\lambda, \mu)(j(\omega)) \quad \text{by def. of } C_j \\
= \lambda j(\omega) + \mu (1 - j(\omega)) \quad \text{by def. of } \rho \\
= \lambda \omega(a) + \mu (1 - \omega(a)) \quad \text{by def. of } j \\
= \omega(\lambda a + \mu (1 - a)) \quad \text{as } \omega \text{ is a MIU-map} \\
= \omega(\sigma(\lambda, \mu)) \quad \text{by choice of } a \\
= \gamma(\sigma(\lambda, \mu))(\omega) \quad \text{by def. of } \gamma \\
= (\gamma \circ \sigma)(\lambda, \mu)(\omega).
\]

It remains to be shown that $\overline{\sigma}$ is the only MIU-map $\tau : C[0, 1] \to \mathcal{A}$ such that $U \tau \circ \rho = \sigma$. Let $\tau$ be such a map; we prove that $\tau = \overline{\sigma}$. By assumption $\tau$ and $\overline{\sigma}$ agree on the elements $f \in C[0, 1]$ of the form

\[
f(x) = \lambda x + \mu (1 - x).
\]

In particular, $\overline{\sigma}$ and $\tau$ agree on the map $h : [0, 1] \to \mathbb{C}$ given by $h(x) = x$.

Now, since $\overline{\sigma}$ and $\tau$ are MIU-maps and $h$ generates the $C^*$-algebra $C[0, 1]$ (this is Weierstrass’s theorem), it follows that $\overline{\sigma} = \tau$.

**Example 3.** The image of $\mathbb{C}^3$ under $F$ will not be commutative, or more formally:

*If $\rho : \mathbb{C}^3 \to \mathcal{B}$ is an initial map from $\mathbb{C}^3$ to $U$, then $\mathcal{B}$ is not commutative.*

Suppose that $\mathcal{B}$ is commutative towards contradiction. Let $\mathcal{A}$ be a $C^*$-algebra in which there are positive $a_1, a_2, a_3$ such that $a_1 a_2 \neq a_2 a_1$ and $a_1 + a_2 + a_3 = 1$.

(For example, we can take $\mathcal{A}$ to be the set of linear operators on $\mathbb{C}^2$ and let

\[
a_1 := 1/2 P_1 \\n= 1/2 P_+ \\n= a_3 := I - 1/2 P_1 - 1/2 P_+
\]

where $P_1$ denotes the orthogonal projection onto $\{(0, x) : x \in \mathbb{C}\}$ and $P_+$ is the orthogonal projection onto $\{(x, x) : x \in \mathbb{C}\}$.)

Define $f : \mathbb{C}^3 \to \mathcal{A}$ by, for all $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$,

\[
f(\lambda_1, \lambda_2, \lambda_3) = \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3.
\]

Then it is not hard to see that $f$ a PU-map. So as $\mathcal{B}$ is the initial arrow from $\mathbb{C}^3$ to $U$ there is a (unique) MIU-map $\overline{f} : \mathcal{B} \to \mathcal{A}$ such that $\overline{f} \circ \rho = f$. We have

\[
a_1 \cdot a_2 = f(1, 0, 0) \cdot f(0, 1, 0) \\
= \overline{f}(\rho(1, 0, 0)) \cdot \overline{f}(\rho(0, 1, 0)) \\
= \overline{f}(\rho(1, 0, 0) \cdot \rho(0, 1, 0)) \quad \text{because } \mathcal{B} \text{ is commutative} \\
= \overline{f}(\rho(0, 1, 0) \cdot \rho(1, 0, 0)) \\
= a_2 \cdot a_1.
\]

This contradicts $a_1 \cdot a_2 \neq a_2 \cdot a_1$. Hence $\mathcal{B}$ is not commutative.
Remark 4. Before we prove that the embedding \( C^*_{\text{MIU}} \to C^*_{\text{PU}} \) has a left adjoint \( F \) (see Theorem 5) let us compare what we already know about \( F \) with the commutative case. Let \( \text{CC}_{\text{MIU}}^* \) denote the category of MIU-maps between commutative \( C^* \)-algebras and let \( \text{CC}_{\text{PU}}^* \) denote the category of PU-maps between commutative \( C^* \)-algebras. From the work in [3] it follows that the embedding \( \text{CC}_{\text{MIU}}^* \to \text{CC}_{\text{PU}}^* \) has a left adjoint \( F' \) and moreover that \( F' \sigma = \text{Stat} \sigma \), where \( \text{Stat} \sigma \) is the topological space of PU-maps from \( \sigma \) to \( C \) with pointwise convergence and \( \text{Stat} \sigma \) is the \( C^* \)-algebra of continuous functions from \( \text{Stat} \sigma \) to \( C \).

Let \( x \in [0, 1] \). Then the assignment \((\lambda, \mu) \mapsto x \lambda + (1 - x) \mu \) gives a PU-map \( \tau: C^2 \to C \). It is not hard to see that \( x \mapsto \tau \) gives an isomorphism from \([0, 1]\) to \( \text{Stat} C^2 \). Thus \( F'C^2 \cong C[0, 1] \). Hence on \( C^2 \) the functor \( F \) and its commutative variant \( F' \) agree (see Example 2). However, on \( C^3 \) the functors \( F \) and \( F' \) differ. Indeed, \( F'C^3 \) is commutative while \( FC^3 \) is not (see Example 3).

\[
\begin{array}{ccc}
\text{CC}_{\text{MIU}}^* & \xrightarrow{F} & \text{CC}_{\text{PU}}^* \\
\downarrow & & \downarrow \\
\text{C}_{\text{MIU}}^* & \xrightarrow{F'} & \text{C}_{\text{PU}}^*
\end{array}
\]

Roughly summarised: while in the diagram above the right adjoints commute with the vertical embeddings, the left adjoints do not.

**Theorem 5.** The embedding \( U: C^*_{\text{MIU}} \to C^*_{\text{PU}} \) has a left adjoint.

**Proof.** By Freyd’s Adjoint Functor Theorem (see Theorem V.6.1 of [6]) and the fact that all limits can be formed using only products and equalisers (see Theorem V.2.1 and Exercise V.4.2 of [6]) it suffices to prove the following.

(i) The category \( C^*_{\text{MIU}} \) has all small products and equalisers.

(ii) The functor \( U: C^*_{\text{MIU}} \to C^*_{\text{PU}} \) preserves small products and equalisers.

(iii) **Solution Set Condition.** For every \( C^* \)-algebra \( \mathcal{A} \) there is a set \( I \) and for each \( i \in I \) a PU-map \( f_i: \mathcal{A} \to \mathcal{A}_i \) such that for any PU-map \( f: \mathcal{A} \to \mathcal{B} \) there is an \( i \in I \) and a MIU-map \( h: \mathcal{A}_i \to \mathcal{B} \) such that \( h \circ f_i = f \).

Conditions (i) and (ii) can be verified with routine so we will spend only a few words on them (and leave the details to the reader). To see that Condition (iii) holds requires a little more ingenuity and so we will give the proof in detail.

(Conditions (i) and (ii)) Let us first think about small products in \( C^*_{\text{MIU}} \) and \( C^*_{\text{PU}} \).

Let \( I \) be a set, and for each \( i \in I \) let \( \mathcal{A}_i \) be a \( C^* \)-algebra.

It is not hard to see that cartesian product \( \prod_{i \in I} \mathcal{A}_i \) is a \(*\)-algebra when endowed with coordinate-wise operations (and it is in fact the product of the \( \mathcal{A}_i \) in the category of \(*\)-algebras with MIU-maps, and with PU-maps).

However, \( \prod_{i \in I} \mathcal{A}_i \) cannot be the product of the \( \mathcal{A}_i \) as \( C^* \)-algebras: there is not even a \( C^* \)-norm on \( \prod_{i \in I} \mathcal{A}_i \) unless \( \mathcal{A}_i \) is trivial for all but finitely many \( i \in I \). Indeed, if \( \| - \| \) were a \( C^* \)-norm on \( \prod_{i \in I} \mathcal{A}_i \), then we must have \( \| \sigma(i) \| \leq \| \sigma \| \) for all \( \sigma \in \prod_{i \in I} \mathcal{A}_i \) and \( i \in I \), and so for any sequence \( i_0, i_1, \ldots \) of distinct elements of \( I \) for which \( \mathcal{A}_{i_0}, \mathcal{A}_{i_1}, \ldots \) are non-trivial, and for every \( \sigma \in \prod_{i \in I} \mathcal{A}_i \) with \( \sigma(i_n) = 0 \) for all \( n \), we have \( n = \| \sigma(i_0) \| \leq \| \sigma \| \) for all \( n \), so \( \| \sigma \| = \infty \), which is not allowed.

Nevertheless, the \(*\)-subalgebra of \( \prod_{i \in I} \mathcal{A}_i \) given by

\[
\bigoplus_{i \in I} \mathcal{A}_i := \{ \sigma \in \prod_{i \in I} \mathcal{A}_i : \sup_{i \in I} \| \sigma(i) \| < +\infty \}
\]
is a $C^*$-algebra with norm given by, for $\sigma \in \bigoplus_{i \in I} \mathcal{A}_i$,
\[ \|\sigma\| = \sup_{i \in I} \|\sigma(i)\|. \]

We claim that $\bigoplus_{i \in I} \mathcal{A}_i$ is the product of the $\mathcal{A}_i$ in $C_{\text{PU}}^*$ (and in $C_{\text{MIU}}^*$).

Let $\mathcal{C}$ be a $C^*$-algebra, and for each $i \in I$, let $f_i: \mathcal{C} \rightarrow \mathcal{A}_i$ be a PU-map. We must show that there is a unique PU-map $f: \mathcal{C} \rightarrow \bigoplus_{i \in I} \mathcal{A}_i$ such that $\pi_i \circ f = f_i$ for all $i \in I$ where $\pi_i: \bigoplus_{i \in I} \mathcal{A}_i \rightarrow \mathcal{A}_i$ is the $i$-th projection. It is clear that there is at most one such $f$, and it would satisfy for all $i \in I$, and $c \in \mathcal{C}$, $f(c)(i) = f_i(c)$.

To see that such map $f$ exists is easy if we are able to prove that, for all $c \in \mathcal{C}$,
\[ \sup_{i \in I} \|f_i(c)\| < +\infty. \]  \hspace{1cm} (1)

Let $i \in I$ be given. We claim that that $\|f_i(c)\| \leq \|c\|$ for any positive $c \in \mathcal{C}$. Indeed, we have $c \leq \|c\| \cdot 1$, and thus $f_i(c) \leq \|c\| \cdot f(1) = \|c\| \cdot 1$, and so $\|f_i(c)\| \leq \|c\|$. It follows that $\|f_i(c)\| \leq 4 \cdot \|c\|$ for any $c \in \mathcal{A}$ by writing $c = c_1 - c_2 + i c_3 - i c_4$ where $c_1, c_2, c_3, c_4 \in \mathcal{C}$ are all positive. (We even have $\|f(c)\| \leq \|c\|$ for all $c \in \mathcal{C}$, but this requires a bit more effort[1].) Thus, we have $\sup_{i \in I} \|f_i(c)\| \leq 4 \|c\| < +\infty$. Hence Statement (1) holds.

Thus $\bigoplus_{i \in I} \mathcal{A}_i$ is the product of the $\mathcal{A}_i$ in $C_{\text{PU}}^*$. It is easy to see that $\bigoplus_{i \in I} \mathcal{A}_i$ is the product of the $\mathcal{A}_i$ in $C_{\text{MIU}}^*$ as well. Hence $C_{\text{MIU}}^*$ has all small products (as does $C_{\text{PU}}^*$) and $U: C_{\text{MIU}}^* \rightarrow C_{\text{PU}}^*$ preserves small products.

Let us think about equalisers in $C_{\text{MIU}}^*$ and $C_{\text{PU}}^*$. Let $\mathcal{A}$ and $\mathcal{B}$ be $C^*$-algebras and let $f, g: \mathcal{A} \rightarrow \mathcal{B}$ be MIU-maps. We must prove that $f$ and $g$ have an equaliser $e: \mathcal{E} \rightarrow \mathcal{A}$ in $C_{\text{MIU}}^*$, and that $e$ is the equaliser of $f$ and $g$ in $C_{\text{PU}}^*$ as well.

Since $f$ and $g$ are MIU-maps (and hence continuous), it is not hard to see that
\[ \mathcal{E} := \{ a \in \mathcal{A}: f(a) = g(a) \} \]
is a $C^*$-subalgebra of $\mathcal{A}$. We claim that the inclusion $e: \mathcal{E} \rightarrow \mathcal{A}$ is the equaliser of $f, g$ in $C_{\text{PU}}^*$. Let $\mathcal{D}$ be a $C^*$-algebra and let $d: \mathcal{D} \rightarrow \mathcal{A}$ be a PU-map such that $f \circ d = g \circ d$. We must show that there is a unique PU-map $h: \mathcal{D} \rightarrow \mathcal{E}$ such that $d = e \circ h$. Note that $d$ maps $\mathcal{A}$ into $\mathcal{E}$. The map $h: \mathcal{D} \rightarrow \mathcal{E}$ is simply the restriction of $d: \mathcal{D} \rightarrow \mathcal{A}$ in the codomain. Hence $e$ is the equaliser of $f, g$ in $C_{\text{PU}}^*$.

Note that in the argument above $h$ is a PU-map since $d$ is a PU-map. If $d$ were a MIU-map, then $h$ would be a MIU-map too. Hence $e$ is the equaliser of $f, g$ in the category $C_{\text{MIU}}^*$ as well.

Hence $C_{\text{MIU}}^*$ has all equalisers and $U: C_{\text{MIU}}^* \rightarrow C_{\text{PU}}^*$ preserves equalisers. Hence $C_{\text{MIU}}^*$ has all small limits and $U: C_{\text{MIU}}^* \rightarrow C_{\text{PU}}^*$ preserves all small limits.

(Condition [iii]). Let $\mathcal{A}$ be a $C^*$-algebra. We must find a set $I$ and for each $i \in I$ a PU-map $f_i: \mathcal{A} \rightarrow \mathcal{A}_i$ such that for every PU-map $f: \mathcal{A} \rightarrow \mathcal{B}$ there is a (not necessarily unique) $i \in I$ and $h: \mathcal{A} \rightarrow \mathcal{B}$ such that $f = h \circ f_i$.

Note that if $f: \mathcal{A} \rightarrow \mathcal{B}$ is a PU-map, then the range of the PU-map $f$ need not be a $C^*$-subalgebra of $\mathcal{B}$. (If the range of PU-maps would have been $C^*$-algebras, then we could have taken $I$ to be the set of all ideals of $\mathcal{A}$, and $f_j: \mathcal{A} \rightarrow \mathcal{A} / J$ to be the quotient map for any ideal $J$ of $\mathcal{A}$.)

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[1] See Corollary 1 of [7].
Nevertheless, given a PU-map $f: \mathcal{A} \to \mathcal{B}$ there is a smallest $C^*$-subalgebra, say $\mathcal{B}'$, of $\mathcal{B}$ that contains the range of $f$. We claim that $\#\mathcal{B}' \leq \#(\mathcal{A}^N)$ where $\#\mathcal{B}'$ is the cardinality of $\mathcal{B}'$ and $\#(\mathcal{A}^N)$ is the cardinality of $\mathcal{A}^N$.\footnote{Although it has no bearing on the validity of the proof one might wonder if the simpler statement $\#\mathcal{B}' \leq \#\mathcal{A}$ holds as well. Indeed, if $\#\mathcal{A} = \#\mathbb{C}$ or $\#\mathcal{A} = \#(2^X)$ for some infinite set $X$, then we have $\#\mathcal{A} = \#(\mathcal{A}^N)$, and so $\#\mathcal{B}' \leq \#\mathcal{A}$. However, not every uncountable set is of the form $2^X$ for some infinite set $X$, and in fact, if $\#\mathcal{A} = \aleph_0$, then $\#(\mathcal{A}^N) > \#\mathcal{A}$ by Corollary 3.9.6 of \cite{2}.}

If we can find proof for our claim, the rest is easy. Indeed, to begin note that the collection of all $C^*$-algebras is not a small set. However, given a set $U$, the collection of all $C^*$-algebras $\mathcal{C}$ whose elements come from $U$ (so $\mathcal{C} \subseteq U$) is a small set. Now, let $\kappa := \#(\mathcal{A}^N)$ be the cardinality of $\mathcal{A}^N$ (so $\kappa$ is itself a set) and take

$$I := \{ (\mathcal{C}, c): \mathcal{C} \text{ is a } C^*-\text{algebra on a subset of } \kappa \text{ and } c: \mathcal{A} \to \mathcal{C} \text{ is a PU-map} \}.$$

Since the collection of $C^*$-algebras $\mathcal{C}$ with $\mathcal{C} \subseteq \kappa$ is small, and since the collection of PU-maps from $\mathcal{A}$ to $\mathcal{C}$ is small for any $C^*$-algebra $\mathcal{C}$, it follows that $I$ is small.

Let us first take care of pathological case. Note that if $\mathcal{A}$ is trivial, i.e. $\mathcal{A} = \{0\}$, then $\mathcal{B}' = \{0\}$, so $\#(\mathcal{A}^N) = 1 = \#\mathcal{B}'$. Now, let us assume that $\mathcal{A}$ is not trivial. Then we have an injection $\mathbb{C} \to \mathcal{A}$ given by $\lambda \mapsto \lambda \cdot 1$, and thus $\#\mathcal{C} \leq \#\mathcal{A}$.

The trick to prove $\#\mathcal{B}' \leq \#(\mathcal{A}^N)$ is to find a more explicit description of $\mathcal{B}'$. Let $T$ be the set of terms formed using a unary operation $(\cdot)^*$ (involution) and two binary operations, $\cdot$ (multiplication) and $+$ (addition), starting from the elements of $\mathcal{A}$. Let $f_T: T \to \mathcal{B}'$ be the map (recursively) given by, for $a \in \mathcal{A}$, and $s, t \in T$,

$$f_T(a) = f(a)$$
$$f_T(s^*) = (f_T(s))^*$$
$$f_T(s \cdot t) = f_T(s) \cdot f_T(t)$$
$$f_T(s + t) = f_T(s) + f_T(t).$$

Let us proof our claim. Let $\mathcal{A}$ and $\mathcal{B}$ be $C^*$-algebras and let $f: \mathcal{A} \to \mathcal{B}$ be a PU-map. Let $\mathcal{B}'$ be the smallest $C^*$-subalgebra that contains the range of $f$. By our claim we have $\#\mathcal{B}' \leq \#(\mathcal{A}^N)$. Let $\varphi: \mathcal{C} \to \mathcal{B}'$ be the isomorphism.

Note that $c := \varphi^{-1} \circ f: \mathcal{C} \to \mathcal{C}'$ is a PU-map. So we have $i := (\mathcal{C}, c) \in I$. Further, the inclusion $e: \mathcal{B}' \to \mathcal{B}$ is a MIU-map, as is $\varphi$. So we have:

$$\begin{align*}
\mathcal{A} & \xrightarrow{\text{PU}} \mathcal{B} \\
\mathcal{C} & \xleftarrow{\text{MIU}} \mathcal{B}'
\end{align*}$$

Now, $h := e \circ \varphi: \mathcal{C} \to \mathcal{B}$ is a MIU-map with $f = h \circ f_i$. Hence Cond. \text{(iii)} holds.

Let us proof our claim. Let $\mathcal{A}$ be a $C^*$-algebra and let $f: \mathcal{A} \to \mathcal{B}$ be a PU-map. Let $\mathcal{B}'$ be the smallest $C^*$-subalgebra that contains the range of $f$. We must show that $\#\mathcal{B}' \leq \#(\mathcal{A}^N)$.

Let us first take care of pathological case. Note that if $\mathcal{A}$ is trivial, i.e. $\mathcal{A} = \{0\}$, then $\mathcal{B}' = \{0\}$, so $\#(\mathcal{A}^N) = 1 = \#\mathcal{B}'$. Now, let us assume that $\mathcal{A}$ is not trivial. Then we have an injection $\mathbb{C} \to \mathcal{A}$ given by $\lambda \mapsto \lambda \cdot 1$, and thus $\#\mathcal{C} \leq \#\mathcal{A}$.

The trick to prove $\#\mathcal{B}' \leq \#(\mathcal{A}^N)$ is to find a more explicit description of $\mathcal{B}'$. Let $T$ be the set of terms formed using a unary operation $(\cdot)^*$ (involution) and two binary operations, $\cdot$ (multiplication) and $+$ (addition), starting from the elements of $\mathcal{A}$. Let $f_T: T \to \mathcal{B}'$ be the map (recursively) given by, for $a \in \mathcal{A}$, and $s, t \in T$,
Note that the range of \( f_B \), let us call it \( \text{Ran}_B \), is a \( \ast \)-subalgebra of \( \mathcal{B}' \). We will prove that \( \#\text{Ran}_B \leq \#\mathcal{A} \). Since \( f_B \) is a surjection of \( T \) onto \( \text{Ran}_B \) it suffices to prove that \( \#T \leq \#\mathcal{A} \). In fact, we will show that \( \#T = \#\mathcal{A} \).

First note that \( \mathcal{A} \) is infinite, and \( \mathcal{A} \subseteq T \), so \( T \) is infinite as well. To prove that \( \#T = \#\mathcal{A} \) we write the elements of \( T \) as words (with the use of brackets). Indeed, with \( Q := \mathcal{A} \cup \{ \cdot, +, \ast, ) \} \), there is an obvious injection from \( T \) into the set \( Q^* \) of words over \( Q \). Since \( \mathcal{A} \) is infinite, and \( Q \setminus \mathcal{A} \) is finite we have \( \#Q = \#\mathcal{A} \) by Hilbert’s hotel. Recall that \( Q^* = \bigcup_{n=0}^{\infty} Q^n \). Since \( Q \) is infinite, we also have \( \#(\mathbb{N} \times Q) = \#Q \) and even \( \#(Q \times Q) = \#Q \) (see Theorem 3.7.7 of [2]), so \( \#Q = \#(Q^n) \) for all \( n > 0 \). It follows that

\[
\#(Q^*) = \#(\bigcup_{n=0}^{\infty} Q^n) = \#(1 + \bigcup_{n=1}^{\infty} Q) = \#(1 + \mathbb{N} \times Q) = \#Q.
\]

Since there is an injection from \( T \) to \( Q^* \) we have \( \#\mathcal{A} \leq \#T \leq \#(Q^*) = \#Q = \#\mathcal{A} \) and so \( \#T = \#\mathcal{A} \).

Hence \( \#\text{Ran}_B \leq \#\mathcal{A} \).

Since \( \text{Ran}_B \) is a \( \ast \)-algebra that contains \( \text{Ran}_f \), the closure \( \overline{\text{Ran}_B} \) of \( \text{Ran}_B \) with respect to the norm on \( \mathcal{B}' \) is a \( C^* \)-algebra that contains \( \text{Ran}_f \). As \( \mathcal{B}' \) is the smallest \( C^* \)-subalgebra that contains \( \text{Ran}_f \), we see that \( \mathcal{B}' = \overline{\text{Ran}_B} \).

Let \( S \) be the set of all Cauchy sequences in \( \text{Ran}_B \). As every point in \( \mathcal{B}' \) is the limit of a Cauchy sequence in \( \text{Ran}_B \), we get \( \#\mathcal{B}' \leq \#S \). Thus:

\[
\#\mathcal{B}' \leq \#S \\
\leq \#(\text{Ran}_B)^N \quad \text{as } S \subseteq (\text{Ran}_B)^N \\
\leq \#(\mathcal{A})^N \quad \text{as } \#\text{Ran}_B \leq \#\mathcal{A}.
\]

Thus we have proven our claim.

Hence Conditions (i)\((i)\) and (iii)\((i)\) hold and \( U : \text{C}_{\text{MUI}U}^{*} \rightarrow \text{C}_{\text{PU}}^{*} \) has a left adjoint. \( \square \)

We have seen that \( U : \text{C}_{\text{MUI}U}^{*} \rightarrow \text{C}_{\text{PU}}^{*} \) has a left adjoint \( F : \text{C}_{\text{PU}}^{*} \rightarrow \text{C}_{\text{MUI}U}^{*} \). This adjunction gives a comonad \( FU \) on \( \text{C}_{\text{MUI}U}^{*} \), which in turns gives us two categories: the Eilenberg–Moore category \( \mathbb{EM}(FU) \) of \( FU \)-coalgebras and the Kleisli category \( \mathbb{K}l(FU) \). We claim that \( \text{C}_{\text{PU}}^{*} \) is isomorphic to \( \mathbb{K}l(FU) \) since \( \text{C}_{\text{MUI}U}^{*} \) is a subcategory of \( \text{C}_{\text{PU}}^{*} \) with the same objects.

This is a special case of a more general phenomenon which we discuss in the next section (in terms of monads instead of comonads), see Theorem 9.

2 Kleislian Adjunctions

Beck’s Theorem (see [6], VI.7) gives a criterion for when an adjunction \( F \dashv U \) “is” an adjunction between \( \text{C} \) and \( \mathbb{EM}(FU) \). We give a similar (but easier) criterion for when an adjunction “is” an adjunction between \( \text{C} \) and \( \mathbb{K}l(UF) \). The criterion is not new; e.g., it is mentioned in [5] (paragraph 8.6) without proof or reference, and it can be seen as a consequence of Exercise VI.5.2 of [6] (if one realises that an equivalence which is bijective on objects is an isomorphism). Proofs can be found in the appendix.
Notation 6. Let $F: C \to D$ be a functor with right adjoint $U$. Denote the unit of the adjunction by $\eta: \text{id}_D \to U F$, and the counit by $\varepsilon: U F \to \text{id}_C$.

Recall that $U F$ is a monad with unit $\eta$ and as multiplication, for $C$ from $C$,

$$\mu_C := U \varepsilon_F: U F U F C \to U F C.$$ 

Let $\mathcal{K}(U F)$ be the Kleisli category of the monad $U F$. So $\mathcal{K}(U F)$ has the same objects as $C$, and the morphisms in $\mathcal{K}(U F)$ from $C_1$ to $C_2$ are the morphism in $C$ from $C_1$ to $U F C_2$. Given $C$ from $C$ the identity in $\mathcal{K}(U F)$ on $C$ is $\eta_C$.

Let $G: \mathcal{K}(U F) \to C$ be given by, for $f: C_1 \to U F C_2$ from $C$,

$$G f := \mu_{C_2} \circ U F g \circ f: U F C_1 \to U F C_2.$$ 

Let $V: C \to \mathcal{K}(U F)$ be given by, for $f: C_1 \to C_2$ from $C$,

$$V f := \eta_{C_2} \circ f: C_1 \to U F C_2.$$ 

The following is Exercise VI.5.1 of [6].

Lemma 7. Let $F: C \to D$ be a functor with a right adjoint $U$.

Then there is a unique functor $L: \mathcal{K}(U F) \to D$ (called the comparison functor) such that $U \circ L = G$ and $L \circ V = F$ (see Notation 6).

Definition 8. Let $C$ and $D$ be categories.

(i) A functor $F: C \to D$ is called Kleislian when it has a right adjoint $U: D \to C$, and the functor $L: \mathcal{K}(U F) \to D$ from Lemma 7 is an isomorphism.

(ii) We say that $D$ is Kleislian over $C$ when there is a Kleislian functor $F: C \to D$.

Theorem 9. Let $F: C \to D$ be a functor with a right adjoint $U$.

The following are equivalent.

(i) $F$ is Kleislian (see Definition 8).

(ii) $F$ is bijective on objects (i.e. for every object $D$ from $D$ there is a unique object $C$ from $C$ such that $F C = D$).

Corollary 10. The embedding $U^{op}: (C^*_{\text{MIU}})^{op} \to (C^*_{\text{PU}})^{op}$ is Kleislian (see Def. 8).

Proof. By Theorem 9 we must show that $U^{op}$ has a left adjoint and is bijective on objects. Since the embedding $U: C^*_{\text{MIU}} \to C^*_{\text{PU}}$ has a left adjoint $F: C^*_{\text{PU}} \to C^*_{\text{MIU}}$; it follows that $F^{op}: (C^*_{\text{PU}})^{op} \to (C^*_{\text{MIU}})^{op}$ is the right adjoint of $U^{op}$. Thus $U^{op}$ has a left adjoint. Further, as $C^*_{\text{MIU}}$ and $C^*_{\text{PU}}$ have the same objects, $U$ is bijective on objects, and so is $U^{op}$. Hence $U^{op}$ is Kleislian. $\square$
In summary, the embedding \( U : \mathcal{C}_{\text{MIU}}^* \to \mathcal{C}_{\text{PU}}^* \) has a left adjoint \( F \) and so \( F^{\text{op}} : (\mathcal{C}_{\text{MIU}}^*)^{\text{op}} \to (\mathcal{C}_{\text{PU}}^*)^{\text{op}} \) is right adjoint to \( U^{\text{op}} \), and the unique functor from the Kleisli category \( \mathcal{K}(FU) \) of the monad \( FU \) on \( (\mathcal{C}_{\text{MIU}}^*)^{\text{op}} \) to \( (\mathcal{C}_{\text{PU}}^*)^{\text{op}} \) that makes the two triangles in the diagram below on the left commute is an isomorphism.

\[
\begin{array}{ccc}
\mathcal{K}(FU) & \cong & (\mathcal{C}_{\text{PU}}^*)^{\text{op}} \\
\downarrow & & \downarrow \\
(\mathcal{C}_{\text{MIU}}^*)^{\text{op}} & \xrightarrow{F^{\text{op}}} & (\mathcal{C}_{\text{PU}}^*)^{\text{op}}
\end{array}
\]

For the category \( \text{Set}_{\text{multi}} \) of multimaps between sets used in the introduction to describe the semantics of non-deterministic programs the situation is the same, see the diagram above to the right.

(The functor \( V \) is the obvious embedding. The right adjoint \( G \) of \( V \) sends a multimap \( f \) from \( X \) to \( Y \) to the function \( Gf : \mathcal{P}(X) \to \mathcal{P}(Y) \) that assigns to a subset \( A \in \mathcal{P}(X) \) the image of \( A \) under \( f \). Note that \( GV = \mathcal{P} \).

3 Discussion

3.1 Variations

Example 11 (Subunital maps). Let \( \mathcal{C}_{\text{PU}}^* \) be the category of \( C^* \)-algebras and the positive linear maps \( f \) between them that are subunital, i.e. \( f(1) \leq 1 \). The morphisms of \( \mathcal{C}_{\text{PU}}^* \) are called PsU-maps.

It is not hard to see that the products in \( \mathcal{C}_{\text{PU}}^* \) are the same as in \( \mathcal{C}_{\text{MIU}}^* \), and that the equaliser in \( \mathcal{C}_{\text{MIU}}^* \) of a pair \( f, g \) of MIU-maps is the equaliser of \( f, g \) in \( \mathcal{C}_{\text{PU}}^* \) as well. Thus the embedding \( U : \mathcal{C}_{\text{MIU}}^* \to \mathcal{C}_{\text{PU}}^* \) preserves limits. Using the same argument as in Theorem 5 but with “PU-map” replaced by “PsU-map” one can show that \( U \) satisfies the Solution Set Condition. Hence \( U \) has a left adjoint by Freyd’s Adjoint Function Theorem, say \( F : \mathcal{C}_{\text{PU}}^* \to \mathcal{C}_{\text{MIU}}^* \).

Since \( \mathcal{C}_{\text{PU}}^* \) has the same objects as \( \mathcal{C}_{\text{MIU}}^* \) (namely the \( C^* \)-algebras) the functor \( U^{\text{op}} : (\mathcal{C}_{\text{MIU}}^*)^{\text{op}} \to (\mathcal{C}_{\text{PU}}^*)^{\text{op}} \) is bijective on objects and thus Kleislian (by Th. \( \ref{thm:kleisli} \)).

Hence \( (\mathcal{C}_{\text{PU}}^*)^{\text{op}} \) is Kleislian over \( (\mathcal{C}_{\text{MIU}}^*)^{\text{op}} \).

Example 12 (Bounded linear maps). Let \( \mathcal{C}_{\mathcal{P}}^* \) be the category of positive bounded linear maps between \( C^* \)-algebras. We will show that \((\mathcal{C}_{\mathcal{P}}^*)^{\text{op}} \) is not Kleislian over \((\mathcal{C}_{\text{MIU}}^*)^{\text{op}} \). Indeed, if it were then \((\mathcal{C}_{\mathcal{P}}^*)^{\text{op}} \) would be cocomplete, but it is not: there is no \( \omega \)-fold product of \( \mathcal{C} \) in \( \mathcal{C}_{\mathcal{P}}^* \). To see this, suppose that there is a \( \omega \)-fold product \( \mathcal{P} \) in \( \mathcal{C}_{\mathcal{P}}^* \) with projections \( \pi_i : \mathcal{P} \to \mathcal{C} \) for \( i \in \omega \). Since \( \pi_i \) is a bounded linear map for \( i \in \omega \), it has finite operator norm, say \( \| \pi_i \| \). By symmetry, \( \| \pi_i \| = \| \pi_j \| \) for all \( i, j \in \omega \). Write \( K := \| \pi_0 \| = \| \pi_1 \| = \| \pi_2 \| = \cdots \). Define \( f_i : \mathcal{C} \to \mathcal{C} \) by \( f_i(z) = iz \) for all \( z \in \mathcal{C} \) and \( i \in \omega \). Then \( f_i \) is a positive bounded linear map for each \( i \in \omega \). Since \( \mathcal{P} \) is the \( \omega \)-fold product of \( \mathcal{C} \), there is a (unique positive) bounded linear map \( f : \mathcal{C} \to \mathcal{P} \) such that \( \pi_i \circ f = f_i \) for all \( i \in \omega \). For each \( N \in \omega \) we have

\[
N = \| f_N(1) \| \leq \| f_N \| = \| \pi_N \circ f \| \leq \| \pi_N \| \| f \| = K \| f \|.
\]

Thus \( K \| f \| \) is greater than any number, which is absurd.

Example 13 ( Completely positive maps). For clarity’s sake we recall what it means for a linear map \( f \) between \( C^* \)-algebras to be completely positive (see [8]). For this we need some notation. Given a \( C^* \)-algebra \( \mathcal{A} \), and \( n \in \mathbb{N} \) let \( M_n(\mathcal{A}) \) denote the set of \( n \times n \)-matrices with entries from \( \mathcal{A} \). We leave it to the
reader to check that $M_n(\mathcal{A})$ is a $*$-algebra with the obvious operations. In fact, it turns out that $M_n(\mathcal{A})$ is a $C^*$-algebra, but some care must be taken to define the norm on $M_n(\mathcal{A})$ as we will see below. Now, a linear map $f: \mathcal{A} \to B$ is called completely positive when $M_n f$ is positive for each $n \in \mathbb{N}$, where $M_n f: M_n(\mathcal{A}) \to M_n(B)$ is the map obtained by applying $f$ to each entry of a matrix in $M_n(\mathcal{A})$. Of course, “$M_n f$ is positive” only makes sense once we know that $M_n(\mathcal{A})$ and $M_n(B)$ are $C^*$-algebras.

Let $\mathcal{A}$ be a $C^*$-algebra. We will put a $C^*$-norm on $M_n(\mathcal{A})$. Let $\mathcal{H}$ be a Hilbert space and let $\pi: \mathcal{A} \to \mathcal{B}(\mathcal{H})$, be an isometric MIU-map. We get a norm $\| - \|_\pi$ on $M_n(\mathcal{A})$ given by for $A \in M_n(\mathcal{A})$,

$$\| A \|_\pi = \| \xi((M_n \pi)(A)) \|,$$

where $\xi((M_n \pi)(A)): \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ is the bounded linear map represented by the matrix $(M_n \pi)(A)$, and $\| \xi((M_n \pi)(A)) \|$ is the operator norm of $\xi((M_n \pi)(A))$ in $\mathcal{B}(\mathcal{H} \otimes \mathcal{H})$.

It is easy to see that $\| - \|_\pi$ satisfies the $C^*$-identity, $\| A^* A \|_\pi = \| A \|_\pi^2$ for all $A \in M_n(\mathcal{A})$. It is less obvious that $M_n(\mathcal{A})$ is complete with respect to $\| - \|_\pi$. To see this, first note that $\| A_{ij} \| \leq \| A \|_\pi$ for all $i, j$. So given a Cauchy sequence $A_1, A_2, \ldots$ in $M_n(\mathcal{A})$ we can form the entrywise limit $A$, that is, $A_{ij} = \lim_{m \to \infty} A_{ij}$. We leave it to the reader to check that $A$ is the limit of $A_1, A_2, \ldots$, and thus $M_n(\mathcal{A})$ is complete with respect to $\| - \|_\pi$. Hence $M_n(\mathcal{A})$ is a $C^*$-algebra with norm $\| - \|_\pi$.

The $C^*$-norm $\| - \|_\pi$ does not depend on $\pi$. Indeed, let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and let $\pi_1: \mathcal{A} \to \mathcal{B}(\mathcal{H}_1)$ and $\pi_2: \mathcal{A} \to \mathcal{B}(\mathcal{H}_2)$ be isometric MIU-maps; we will show that $\| - \|_{\pi_1} = \| - \|_{\pi_2}$. Recall that the norm $\| - \|_A$ induces an order $\leq_A$ on $M_n(\mathcal{A})$ given by $0 \leq_A A$ if $\| A - \| A \|_\pi \| A \|_\pi \leq \| A \|_\pi$, where $A \in M_n(\mathcal{A})$. Since $\| A \|_{\pi} = \inf \{ \lambda \in [0, \infty]: A^* A \leq \lambda \}$ for all $A \in M_n(\mathcal{A})$, to prove $\| - \|_{\pi_1} = \| - \|_{\pi_2}$ it suffices to show that the orders $\leq_{\pi_1}$ and $\leq_{\pi_2}$ coincide. But this is easy when one recalls that $A \in M_n(\mathcal{A})$ is positive iff $A$ is of the form $B^* B$ for some $B \in M_n(\mathcal{A})$.

The completely positive linear maps that preserve the unit are called CPU-maps. Let $C_{CPU}$ be the category of CPU-maps between $C^*$-algebras. Since $M_n(f)$ is a MIU-map when $f$ is a MIU-map and a MIU-map is positive, we see that any MIU-map is completely positive. Thus $C^*_MIU$ is a subcategory of $C^*CPU$. We claim that $(C^*CPU)^{op}$ is Kleislian over $(C^*MIU)^{op}$.

Let us show that $U$ preserves limits. To show that $U$ preserves equalisers, let $f, g: \mathcal{A} \to B$ be MIU-maps. Then $\mathcal{E} := \{ x \in \mathcal{A}: f(x) = g(x) \}$ is a $C^*$-subalgebra of $\mathcal{A}$ and the embedding $e: \mathcal{E} \to \mathcal{A}$ is an isometric MIU-map. Then $e$ is the equaliser of $f, g$ in $C^*_MIU$; we will show that $e$ is the equaliser of $f, g$ in $C^*_CPU$. Let $\mathcal{E}$ be a $C^*$-algebra, and let $c: \mathcal{E} \to \mathcal{A}$ be a CPU-map such that $f \circ c = g \circ c$. Let $d: \mathcal{E} \to \mathcal{E}$ be the restriction of $c$. It turns out we must prove that $d$ is completely positive. Let $n \in \mathbb{N}$ be given. We must show that $M_n d: M_n(\mathcal{E}) \to M_n(\mathcal{E})$ is positive. Note that $M_n e$ is an injective MIU-map and thus an isometry. So in order to prove that $M_n d$ is positive it suffices to show that $M_n e \circ M_n d = M_n(e \circ d) = M_n c$ is positive, which it is since $c$ is completely positive. Thus $e$ is the equaliser of $f, g$ in $C^*_CPU$. Hence $U$ preserves equalisers.

To show that $U$ preserves products, let $I$ be a set and for each $i \in I$ let $\mathcal{A}_i$ be a $C^*$-algebra. We will show that $\bigoplus_{i \in I} \mathcal{A}_i$ is the product of the $\mathcal{A}_i$ in $C^*_CPU$. Let $\mathcal{E}$ be a $C^*$-algebra, and for each $i \in I$, let $f_i: \mathcal{E} \to \mathcal{A}_i$ be a CPU-map. As before, let $f: \mathcal{E} \to \bigoplus_{i \in I} \mathcal{A}_i$ be the map given by $f(x)(i) = f_i(x)$ for all $i \in I$ and $x \in \mathcal{E}$. Leaving the details to the reader it turns out that it suffices to show that $f$ is completely positive. Let $n \in \mathbb{N}$ be given. We must prove that $M_n f: M_n(\mathcal{E}) \to M_n(\bigoplus_{i \in I} \mathcal{A}_i)$ is positive. Let $\mathcal{F}: M_n(\bigoplus_{i \in I} \mathcal{A}_i) \to \bigoplus_{i \in I} M_n(\mathcal{A}_i)$ be the unique MIU-map such that $\pi_i \circ \mathcal{F} = M_n(\pi_i)$ for all $i \in I$. Then $\mathcal{F}$ is a MIU-isomorphism and thus to prove that $M_n f$ is positive, it suffices to show that $\mathcal{F} \circ M_n f$ is positive. Let $i \in I$ be given. We must prove that $\pi_i \circ \mathcal{F} \circ M_n f$ is positive. But we have $\pi_i \circ \mathcal{F} \circ M_n f = M_n(\pi_i \circ f) = M_n f_i$, which is positive since $f$ is completely positive. Thus $\bigoplus_{i \in I} \mathcal{A}_i$ is the product of the $\mathcal{A}_i$ in $C^*_CPU$ and hence $U$ preserves limits.
With the same argument as in Theorem 9 the functor \( U \) satisfies the Solution Set Condition and thus \( U \) has a left adjoint. It follows that \( U^{\text{op}}: (C^*_{\text{MIU}})^{\text{op}} \rightarrow (C^*_{\text{CPU}})^{\text{op}} \) is Kleislian.

**Example 14** \((W^*\text{-algebras})\). Let \( W_{\text{NMIU}}^* \) be the category of von Neumann algebras (also called \( W^* \)-algebras) and the MIU-maps between them that are normal, i.e., preserve suprema of upwards directed sets of self-adjoint elements. Let \( W_{\text{NPU}}^* \) be the category of von Neumann and normal PU-maps. Note that \( W_{\text{NMIU}}^* \) is a subcategory of \( W_{\text{NPU}}^* \). We will prove that \( (W_{\text{NPU}}^*)^{\text{op}} \) is Kleislian over \( (W_{\text{NMIU}}^*)^{\text{op}} \).

It suffices to show that \( U \) has a left adjoint. Again we follow the lines of the proof of Theorem 5. Products and equalisers in \( W_{\text{NMIU}}^* \) are the same as in \( C_{\text{MIU}}^* \). It is not hard to see that the embedding \( U: W_{\text{NMIU}}^* \rightarrow W_{\text{NPU}}^* \) preserves limits. To see that \( U \) satisfies the Solution Set Condition we use the same method as before: given a von Neumann algebra \( \mathcal{A} \), find a suitable cardinal \( \kappa \) such that the following is a solution set.

\[
I := \{ (C, c): C \text{ is a von Neumann algebra on a subset of } \kappa \\
\text{ and } c: \mathcal{A} \rightarrow C \text{ is a normal PU-map } \},
\]

Only this time we take \( \kappa = \#(\phi(\mathcal{A})) \) instead of \( \kappa = \#(\mathcal{A}^\mathbb{N}) \). We leave the details to the reader, but it follows from the fact that given a subset \( X \) of a von Neumann algebra \( \mathcal{B} \) the smallest von Neumann subalgebra \( \mathcal{B}' \) that contains \( X \) has cardinality at most \( \#(\phi(\mathcal{B}(X))) \). Indeed, if \( \mathcal{H} \) is a Hilbert space such that \( \mathcal{B} \subseteq \mathcal{B}(\mathcal{H}) \) (perhaps after renaming the elements of \( \mathcal{B} \)), then \( \mathcal{B}' \) is the closure (in the weak operator topology on \( \mathcal{B}(\mathcal{H}) \)) of the smallest \(*\)-subalgebra containing \( X \). Thus any element of \( \mathcal{B}' \) is the limit of a filter — a special type of net, see paragraph 12 of [9] — of \(*\)-algebra terms over \( X \), of which there are no more than \( \#(\phi(\mathcal{B}(X))) \).

By a similar reasoning one sees that the opposite \( (W_{\text{NCPU}}^*)^{\text{op}} \) of the category of normal completely positive subunital linear maps between von Neumann algebras is Kleislian over \( (W_{\text{NMIU}}^*)^{\text{op}} \). The existence of the adjoint to the inclusion \( W_{\text{NMIU}}^* \rightarrow W_{\text{NCPU}}^* \) is key in our construction of a model of Selinger and Valiron’s quantum lambda calculus by von Neumann algebras, see [11].

### 3.2 Concrete description

In this note we have shown that the embedding \( U: C_{\text{MIU}}^* \rightarrow C_{\text{CPU}}^* \) has a left adjoint \( F \), but we miss a concrete description of \( F \) for all but the simplest \( C^* \)-algebras \( \mathcal{A} \). What constitutes a “concrete description” is perhaps a matter of taste or occasion, but let us pose that it should at least enable us to describe the Eilenberg–Moore category \( \mathcal{E}/\mathcal{M}(FU) \) of the comonad \( FU \). More concretely, it should settle the following problem.

**Problem 15.** Writing \( \text{BOUS} \) for the category of positive linear maps that preserve the unit between Banach order unit spaces, determine whether \( \mathcal{E}/\mathcal{M}(FU) \cong \text{BOUS} \).

(An order unit space is an ordered vector space \( V \) over \( \mathbb{R} \) with an element \( 1 \), the order unit, such that for all \( v \in V \) there is \( \lambda \in [0, \infty) \) such that \( -\lambda \cdot 1 \leq v \leq \lambda \cdot 1 \). The smallest such \( \lambda \) is denoted by \( \|v\| \). See [4] for more details. If \( v \mapsto \|v\| \) gives a complete norm, \( V \) is called a Banach order unit space.)

### 3.3 MIU versus PU

A second “problem” is to give a physical description (if there is any) of what it means for a quantum program’s semantics to be a MIU-map (and not just a PU-map). A step in this direction might be to define for a \( C^* \)-algebra \( \mathcal{A} \), a PU-map \( \varphi: \mathcal{A} \rightarrow \mathbb{C} \), and \( a, b \in \mathcal{A} \) the quantity

\[
\text{Cov}_\varphi(a, b) := \varphi(a^* b) - \varphi(a)^* \varphi(b)
\]
and interpret it as the covariance between the observables $a$ and $b$ in state $\varphi$ of the quantum system $\mathcal{A}$. Let $T : \mathcal{A} \rightarrow \mathcal{B}$ be a PU-map between $C^*$-algebras (so perhaps $T$ is the semantics of a quantum program). Then it is not hard to verify that $T$ is a MIU-map if and only if $T$ preserves covariance, that is,

$$\text{Cov}_{\varphi}(Ta, Tb) = \text{Cov}_{\varphi \circ T}(a, b) \quad \text{for all } a, b \in \mathcal{A}.$$ 

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\section*{References}


\section*{A Additional Proofs}

\textit{Proof of Lemma}\textsuperscript{7} Define $L C := FC$ for all objects $C$ of $\mathcal{K}\ell(UF)$ and

$$Lf := \varepsilon_{FC_2} \circ Ff$$

for $f : C_1 \rightarrow UFC_2$ from $C$. We claim this gives a functor $L : \mathcal{K}\ell(UF) \rightarrow D$.

\textit{(L preserves the identity)} Let $C$ be an object of $\mathcal{K}\ell(UF)$, that is, an object of $\mathcal{C}$. Then the identity on $C$ in $\mathcal{K}\ell(UF)$ is $\eta_C$. We have $L(\eta_C) = \varepsilon_{FC} \circ F\eta_C = \text{id}_{FC}$. 
To prove that \( L \) is a functor from \( \mathcal{K} \ell (UF) \) to \( \textbf{D} \).

Let us first prove that \( \mathcal{K} \ell (UF) \) is bijective on objects. Let \( f : C_1 \rightarrow C_2 \) be given, we have

\[
ULf = U(\varepsilon_{\mathcal{C}_2} \circ Ff)
\]

by def. of \( Uf \)

\[
= UF\varepsilon_{\mathcal{C}_2}
\]

by def. of \( \mu_{\mathcal{C}_2} \)

\[
= GFf
\]

by def. of \( Gf \).

Hence \( L \) is a functor from \( \mathcal{K} \ell (UF) \) to \( \textbf{D} \).

Let us prove that \( U \circ L = G \). For \( f : C_1 \rightarrow UFC_2 \) from \( \textbf{C} \) we have

\[
ULf = U(\varepsilon_{\mathcal{C}_2} \circ Ff)
\]

by def. of \( L \)

\[
= U\varepsilon_{\mathcal{C}_2} \circ UFf
\]

by def. of \( \mu_{\mathcal{C}_2} \)

\[
= Gf
\]

by def. of \( Gf \).

Let us prove that \( L \circ V = F \). For \( f : C_1 \rightarrow C_2 \) from \( \textbf{C} \) be given, we have

\[
LVf = L(\eta_{\mathcal{C}_2} \circ f)
\]

by def. of \( V \)

\[
= \varepsilon_{\mathcal{C}_2} \circ F\eta_{\mathcal{C}_2} \circ Ff
\]

by def. of \( L \)

\[
= Ff
\]

by counit–unit eq.

We have proven that there is a functor \( L : \mathcal{K} \ell (UF) \rightarrow \textbf{D} \) such that \( U \circ L = G \) and \( L \circ V = F \). We must still prove that it is as such unique.

Let \( L' : \mathcal{K} \ell (UF) \rightarrow \textbf{D} \) be a functor such that \( U \circ L' = G \) and \( L' \circ V = F \). We must show that \( L = L' \). Let us first prove that \( L' \) and \( L \) agree on objects. Let \( C \) be an object of \( \mathcal{K} \ell (UF) \), i.e., \( C \) is an object of \( \textbf{C} \). Since \( L' \circ V = F \) and \( VC = C \) we have \( L'C = L'VC = FC = LC \). Now, let \( f : C_1 \rightarrow UFC_2 \) from \( \textbf{C} \) be given (so \( f \) is a morphism in \( \mathcal{K} \ell (UF) \) from \( C_1 \) to \( C_2 \)). We must show that \( L'f = LU \varepsilon_{\mathcal{C}_2} \circ Ff \). Note that since \( F \) is the left adjoint of \( U \) there is a unique morphism \( \overline{f} : FC_1 \rightarrow FC_2 \) in \( \textbf{D} \) such that \( UF \circ \eta_{\mathcal{C}_1} = f \). To prove that \( L'f = Lf \), we show that both \( LF \) and \( L'f \) have this property. We have

\[
ULf \circ \eta_{\mathcal{C}_1} = GF \circ \eta_{\mathcal{C}_1}
\]

as \( U \circ L' = G \) by assump.

\[
= \mu_{\mathcal{C}_2} \circ UFf \circ \eta_{\mathcal{C}_1}
\]

by def. of \( G \)

\[
= \mu_{\mathcal{C}_2} \circ \eta_{UFC_2} \circ f
\]

by nat. of \( \eta \)

\[
= f
\]

as \( UF \) is a monad.

By a similar argument we get \( ULf \circ \eta_{\mathcal{C}_1} = f \). Hence \( LF = L'f \).

\( \square \)

**Proof of Theorem** 9. We use the symbols from Notation 5.

(i)⇒(ii) Suppose that \( L \) is an isomorphism. We must prove that \( F \) is bijective on objects. Note that \( F = L \circ V \), so it suffices to show that both \( L \) and \( V \) are bijective on objects. Clearly, \( L \) is bijective on objects as \( L \) is an isomorphism, and \( V : \textbf{C} \rightarrow \mathcal{K} \ell (UF) \) is bijective on objects since the objects of \( \mathcal{K} \ell (UF) \) are those of \( \textbf{C} \) and \( VC = C \) for all \( C \) from \( \textbf{C} \).
[1 (ii)] Suppose that (ii) holds. We prove that $L$ is an isomorphism by giving its inverse. Let $D$ be an object from $\mathbf{D}$. Note that since $F$ is bijective on objects there is a unique object $C$ from $\mathbf{C}$ such that $FD = C$. Define $KC := D$.

Let $g: D_1 \rightarrow D_2$ from $\mathbf{D}$ be given. Note that by definition of $K$ we have:

$$KD_1 \xrightarrow{\eta_{KD_1}} UFKD_1 \xrightarrow{Ug} UD_1 \rightarrow UD_2 \rightarrow UFKD_2$$

Now, define $Kg: KD_1 \rightarrow UFKD_2$ in $\mathbf{D}$ by $Kg := Ug \circ \eta_{KD_1}$.

(\textit{K preserves the identity}) For an object $D$ of $\mathbf{D}$ we have

$$\text{Kid}_D = U \text{id}_D \circ \eta_{KD} = \eta_{KD},$$

and $\eta_{KD}$ is the identity on $KD$ in $\mathcal{K}(\mathcal{U}F)$.

(\textit{K preserves composition}) Let $f: D_1 \rightarrow D_2$ and $g: D_2 \rightarrow D_3$ from $\mathbf{D}$ be given. We must prove that $K(g \circ f) = K(g) \circ K(f)$. We have

$$K(g) \circ K(f) = \mu_{KD_3} \circ UFKg \circ Kf$$

by def. of $\circ$

$$= \mu_{KD_3} \circ UFUg \circ UF \eta_{KD_2} \circ Uf \circ \eta_{KD_1}$$

by def. of $K$

$$= Ug \circ U \varepsilon_{D_2} \circ UF \eta_{KD_2} \circ Uf \circ \eta_{KD_1}$$

by def. of $\mu$

$$= Ug \circ Uf \circ \eta_{KD_1}$$

by nat. of $\varepsilon$

$$= K\cdot g \circ f$$

by def of $K$.

Hence $K$ is a functor from $\mathbf{D}$ to $\mathcal{K}(\mathcal{U}F)$. We will show that $K$ is the inverse of $L$. For this we must prove that $K \circ L = \text{id}_D$ and $L \circ K = \text{id}_{\mathcal{K}(\mathcal{U}F)}$.

For a morphism $g: D_1 \rightarrow D_2$ from $\mathbf{D}$, we have

$$LKg = L(Ug \circ \eta_{KD_1})$$

by def. of $L$

$$= \varepsilon_{FKD_2} \circ UFg \circ F \eta_{KD_1}$$

by def. of $L$

$$= g \circ \varepsilon_{FKD_1} \circ F \eta_{KD_1}$$

by nat. of $\varepsilon$

$$= g$$

by counit–unit eq.

For a morphism $f: C_1 \rightarrow UFC_2$ in $\mathbf{C}$ we have

$$KLf = K(\varepsilon_{FC_2} \circ Ff)$$

by def. of $L$

$$KLfd = U \varepsilon_{FC_2} \circ UFf \circ \eta_{KFC_1}$$

by def. of $K$

$$= U \varepsilon_{FC_2} \circ \eta_{UFC_2} \circ f$$

by nat. of $\eta$

$$= f$$

by counit–unit eq.

Hence $K$ is the inverse of $L$, so $L$ is an isomorphism.