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Quantum Programs as Kleisli Maps

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Furber and Jacobs have shown in their study of quantum computation that the category of commutative $C^*$-algebras and $PU$-maps (positive linear maps which preserve the unit) is isomorphic to the Kleisli category of a comonad on the category of commutative $C^*$-algebras with $MIU$-maps (linear maps which preserve multiplication, involution and unit). [3]

In this paper, we prove a non-commutative variant of this result: the category of $C^*$-algebras and $PU$-maps is isomorphic to the Kleisli category of a comonad on the subcategory of $MIU$-maps.

A variation on this result has been used to construct a model of Selinger and Valiron’s quantum lambda calculus using von Neumann algebras. [11]

The semantics of a non-deterministic program that takes two bits and returns three bits can be described as a multimap (= binary relation) from $\{0,1\}^2$ to $\{0,1\}^3$. Similarly, a program that takes two qubits and returns three qubits can be modelled as a positive linear unit-preserving map from $M_2 \otimes M_2 \otimes M_2$ to $M_2 \otimes M_2$, where $M_2$ is the $C^*$-algebra of $2 \times 2$-matrices over $\mathbb{C}$.

More generally, the category $Set_{\text{multi}}$ of multimeads between sets models non-deterministic programs (running on an ordinary computer), while the opposite of the category $C_{\text{multi}}^*$ of $PU$-maps (positive linear unit-preserving maps) between $C^*$-algebras models programs running on a quantum computer. (When we write “$C^*$-algebra” we always mean “$C^*$-algebra with unit”.)

A multimap from $\{0,1\}^2$ to $\{0,1\}^3$ is simply a map from $\{0,1\}^2$ to $\mathcal{P}(\{0,1\}^3)$. In the same line $Set_{\text{multi}}$ is (isomorphic to) the Kleisli category of the powerset monad $\mathcal{P}$ on $Set$. What about $C_{\text{multi}}^*$?

We will show that there is a monad $\Omega$ on $(C_{\text{MIU}}^*)^{\text{op}}$, the opposite of the category $C_{\text{MIU}}^*$ of $C^*$-algebras and $MIU$-maps (linear maps that preserve the multiplication, involution and unit), such that $(C_{\text{PU}}^*)^{\text{op}}$ is isomorphic to the Kleisli category of $\Omega$. We say that $(C_{\text{PU}}^*)^{\text{op}}$ is Kleislian over $(C_{\text{MIU}}^*)^{\text{op}}$. So in the same way we add non-determinism to $Set$ by the powerset monad $\mathcal{P}$ yielding $Set_{\text{multi}}$, we can obtain $(C_{\text{PU}}^*)^{\text{op}}$ from $(C_{\text{MIU}}^*)^{\text{op}}$ by a monad $\Omega$.

Let us spend some words on how we obtain this monad $\Omega$. Note that since every positive element of a $C^*$-algebra $\mathcal{A}$ is of the form $a^*a$ for some $a \in \mathcal{A}$ any MIU-map will be positive. Thus $C_{\text{MIU}}^*$ is a subcategory of $C_{\text{PU}}^*$. Let $U: C_{\text{MIU}}^* \to C_{\text{PU}}^*$ be the embedding.

In Section [1] we will prove that $U$ has a left adjoint $F: C_{\text{PU}}^* \to C_{\text{MIU}}^*$, see Theorem [5]. This adjunction gives us a comonad $\Omega := FU$ on $C_{\text{MIU}}^*$ (which is a monad on $(C_{\text{MIU}}^*)^{\text{op}}$) with the same counit as the adjunction. The comultiplication $\delta$ is given by $\delta_{\mathcal{A}} = F\eta_{U_{\mathcal{A}}}$ for every object $\mathcal{A}$ from $C_{\text{MIU}}^*$ where $\eta$ is the unit of the adjunction between $F$ and $U$.

In Section [2] we will prove that $(C_{\text{PU}}^*)^{\text{op}}$ is isomorphic to $\mathcal{K}(FU)$ if $FU$ is considered a monad on $(C_{\text{MIU}}^*)^{\text{op}}$. In fact, we will prove that the comparison functor $L: \mathcal{K}(FU) \to (C_{\text{PU}}^*)^{\text{op}}$ (which sends a MIU-map $f: FU_{\mathcal{A}} \to \mathcal{B}$ to $Uf \circ \eta_{U_{\mathcal{A}}}: U_{\mathcal{A}} \to U_{\mathcal{B}}$) is an isomorphism, see Corollary [10].

The method used to show that $(C_{\text{PU}}^*)^{\text{op}}$ is Kleislian over $(C_{\text{MIU}}^*)^{\text{op}}$ is quite general and it will be obvious that many variations on $(C_{\text{PU}}^*)^{\text{op}}$ will be Kleislian over $(C_{\text{MIU}}^*)^{\text{op}}$ as well, such as the opposite of the category of subunitarily completely positive linear maps between $C^*$-algebras. The flip-side of this generality is that we discover precisely little about the monad $\Omega$ which leaves room for future inquiry (see Section [3]).
We will also see that the opposite \( (W_{\text{NCPU}}^*)^{\text{op}} \) of the category of normal completely positive subunital maps between von Neumann algebras is Kleislian over the subcategory \( (W_{\text{SMU}}^*)^{\text{op}} \) of normal unital *-homomorphisms. This fact is used in [1] to construct an adequate model of Selinger and Valiron’s quantum lambda calculus using von Neumann algebras.

1 The Left Adjoint

In Theorem[5] we will show that \( U \) has a left adjoint, \( F: C_{\text{MIU}} \to C_{\text{PU}}^* \), using a quite general method. As a result we do not get any “concrete” information about \( F \) in the sense that while we will learn that for every \( C^* \)-algebra \( \mathcal{A} \) there exists an arrow \( \rho: \mathcal{A} \to UF \mathcal{A} \) which is initial from \( \mathcal{A} \) to \( U \) we will learn nothing more about \( \rho \) than this. Nevertheless, for some (very) basic \( C^* \)-algebras \( \mathcal{A} \) we can describe \( F \mathcal{A} \) directly, as is shown below in Example[3]

Example 1. Let us start easy: \( \mathbb{C} \) will be mapped to itself by \( F \), that is:

the identity \( \rho: \mathbb{C} \to UC \) is an initial arrow from \( \mathbb{C} \) to \( U(-) \).

Indeed, let \( \mathcal{A} \) be a \( C^* \)-algebra and let \( \sigma: \mathbb{C} \to U \mathcal{A} \) be a PU-map. Then \( \sigma \) must be given by \( \sigma(\lambda) = \lambda \cdot 1 \) for \( \lambda \in \mathbb{C} \), where 1 is the identity of \( \mathcal{A} \). Thus \( \sigma \) is a MIU-map as well. Hence there is a unique MIU-map \( \tilde{\sigma}: \mathbb{C} \to \mathcal{A} \) (namely \( \tilde{\sigma} = \sigma \)) such that \( \tilde{\sigma} \circ \rho = \sigma \). (\( \mathbb{C} \) is initial in both \( C_{\text{MIU}} \) and \( C_{\text{PU}}^* \).)

Example 2. The image of \( \mathbb{C}^2 \) under \( F \) will be the \( C^* \)-algebra \( C[0,1] \) of continuous functions from \( [0,1] \) to \( \mathbb{C} \). As will become clear below, this is very much related to the familiar functional calculus for \( C^* \)-algebras: given an element \( a \) of a \( C^* \)-algebra \( \mathcal{A} \) with \( 0 \leq a \leq 1 \) and \( f \in C[0,1] \) we can make sense of “\( f(a) \)”, as an element of \( \mathcal{A} \).

The map \( \rho: \mathbb{C}^2 \to UC[0,1] \) given by, for \( \lambda, \mu \in \mathbb{C} \), \( x \in [0,1] \),

\[
\rho(\lambda, \mu)(x) = \lambda x + \mu(1-x)
\]

is an initial arrow from \( \mathbb{C}^2 \) to \( U \).

Let \( \sigma: \mathbb{C}^2 \to U \mathcal{A} \) be a PU-map. We must show that there is a unique MIU-map \( \bar{\sigma}: C[0,1] \to \mathcal{A} \) such that \( \sigma = \bar{\sigma} \circ \rho \).

Writing \( a := \sigma(1,0) \), we have \( \sigma(\lambda, \mu) = \lambda a + \mu(1-a) \) for all \( \lambda, \mu \in \mathbb{C} \). Note that \( (0,0) \leq (1,0) \leq (1,1) \) and thus \( 0 \leq a \leq 1 \). Let \( C^+(a) \) be the \( C^* \)-subalgebra of \( \mathcal{A} \) generated by \( a \). Then \( C^+(a) \) is commutative since \( a \) is positive (and thus normal). Given a MIU-map \( \omega: C^+(a) \to \mathbb{C} \) we have \( \omega(a) \in [0,1] \) since \( 0 \leq a \leq 1 \). Thus \( \omega \to \omega(a) \) gives a map \( j: \Sigma C^+(a) \to [0,1] \), where \( \Sigma C^+(a) \) is the spectrum of \( C^+(a) \), that is, \( \Sigma C^+(a) \) is the set of MIU-maps from \( C^+(a) \) to \( \mathbb{C} \) with the topology of pointwise convergence. (By the way, the image of \( j \) is the spectrum of the element \( a \).) The map \( j \) is continuous since the topology on \( \Sigma C^+(a) \) is induced by the product topology on \( C\Sigma C^+(a) \). Thus the assignment \( h \mapsto h \circ j \) gives a MIU-map \( C_j: C[0,1] \to C\Sigma C^+(a) \). By Gelfand’s representation theorem there is a MIU-isomorphism

\[
\gamma: C^+(a) \to C\Sigma C^+(a)
\]

given by \( \gamma(b)(\omega) = \omega(b) \) for all \( b \in C^+(a) \) and \( \omega \in \Sigma C^+(a) \). Now, define

\[
\bar{\sigma} := \gamma^{-1} \circ C_j: C[0,1] \to C^+(a) \leftrightarrow \mathcal{A}.
\]

(In the language of the functional calculus, \( \bar{\sigma} \) maps \( f \) to \( f(a) \).) We claim that \( \bar{\sigma} \circ \rho = \sigma \). It suffices to
show that $C_j \circ \rho \equiv \gamma \circ \sigma \circ \rho = \gamma \circ \sigma$. Let $\lambda, \mu \in \mathbb{C}$ and $\omega \in \Sigma C^\gamma(a)$ be given. We have

\[(C_j \circ \rho)(\lambda, \mu)(\omega) = (C_j)(\rho(\lambda, \mu))(\omega)\]
\[= \rho(\lambda, \mu)(j(\omega)) \quad \text{by def. of } C_j\]
\[= \lambda j(\omega) + \mu(1 - j(\omega)) \quad \text{by def. of } \rho\]
\[= \omega(\lambda a + \mu(1 - a)) \quad \text{as } \omega \text{ is a MIU-map}\]
\[= \omega(\sigma(\lambda, \mu)) \quad \text{by choice of } a\]
\[= \gamma(\sigma(\lambda, \mu))(\omega). \quad \text{by def. of } \gamma\]
\[= (\gamma \circ \sigma)(\lambda, \mu)(\omega).\]

It remains to be shown that $\overline{\sigma}$ is the only MIU-map $\tau: C[0,1] \to \mathcal{A}$ such that $U \tau \circ \rho = \sigma$. Let $\tau$ be such a map; we prove that $\tau = \overline{\sigma}$. By assumption $\tau$ and $\overline{\sigma}$ agree on the elements $f \in C[0,1]$ of the form

\[f(x) = \lambda x + \mu(1 - x).\]

In particular, $\overline{\sigma}$ and $\tau$ agree on the map $h: [0,1] \to \mathbb{C}$ given by $h(x) = x$.

Now, since $\overline{\sigma}$ and $\tau$ are MIU-maps and $h$ generates the $C^\gamma$-algebra $C[0,1]$ (this is Weierstrass’s theorem), it follows that $\overline{\sigma} = \tau$.

**Example 3.** The image of $\mathbb{C}^3$ under $F$ will not be commutative, or more formally:

If $\rho: \mathbb{C}^3 \longrightarrow U \mathcal{B}$ is an initial map from $\mathbb{C}^3$ to $U$, then $\mathcal{B}$ is not commutative.

Suppose that $\mathcal{B}$ is commutative towards contradiction. Let $\mathcal{A}$ be a $C^\gamma$-algebra in which there are positive $a_1, a_2, a_3$ such that $a_1 a_2 \neq a_2 a_1$ and $a_1 + a_2 + a_3 = 1$.

(For example, we can take $\mathcal{A}$ to be the set of linear operators on $\mathbb{C}^2$ and let

\[a_1 := \frac{1}{2} P_1, \quad a_2 := \frac{1}{2} P_+ \quad a_3 := I - \frac{1}{2} P_1 - \frac{1}{2} P_+\]

where $P_1$ denotes the orthogonal projection onto $\{(0, x): x \in \mathbb{C}\}$ and $P_+$ is the orthogonal projection onto $\{(x, x): x \in \mathbb{C}\}$.)

Define $f: \mathbb{C}^3 \to \mathcal{A}$ by, for all $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$,

\[f(\lambda_1, \lambda_2, \lambda_3) = \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3.\]

Then it is not hard to see that $f$ a PU-map. So as $\mathcal{B}$ is the initial arrow from $\mathbb{C}^3$ to $U$ there is a (unique) MIU-map $\overline{f}: \mathcal{B} \to \mathcal{A}$ such that $\overline{f} \circ \rho = f$. We have

\[a_1 \cdot a_2 = f(1,0,0) \cdot f(0,1,0)\]
\[= \overline{f}(\rho(1,0,0)) \cdot \overline{f}(\rho(0,1,0))\]
\[= \overline{f}(\rho(1,0,0) \cdot \rho(0,1,0)) \quad \text{because } \mathcal{B} \text{ is commutative}\]
\[= \overline{f}(\rho(0,1,0) \cdot \rho(1,0,0))\]
\[= a_2 \cdot a_1.\]

This contradicts $a_1 \cdot a_2 \neq a_2 \cdot a_1$. Hence $\mathcal{B}$ is not commutative.
Remark 4. Before we prove that the embedding $C^*_{\text{MIU}} \to C^*_{\text{PU}}$ has a left adjoint $F$ (see Theorem 5) let us compare what we already know about $F$ with the commutative case. Let $C^*_{\text{MIU}}$ denote the category of MIU-maps between commutative $C^*$-algebras and let $C^*_{\text{PU}}$ denote the category of PU-maps between commutative $C^*$-algebras. From the work in [3] it follows that the embedding $C^*_{\text{MIU}} \to C^*_{\text{PU}}$ has a left adjoint $F'$ and moreover that $F':\mathcal{A} \to \text{CStat}\mathcal{A}$, where $\text{Stat}\mathcal{A}$ is the topological space of PU-maps from $\mathcal{A}$ to $\mathbb{C}$ with pointwise convergence and $C^*\text{Stat}\mathcal{A}$ is the $C^*$-algebra of continuous functions from $\text{Stat}\mathcal{A}$ to $\mathbb{C}$.

Let $x \in [0, 1]$. Then the assignment $(\lambda, \mu) \mapsto x\lambda + (1 - x)\mu$ gives a PU-map $\pi: \mathbb{C}^2 \to \mathbb{C}$. It is not hard to see that $x \mapsto \pi$ gives an isomorphism from $[0, 1]$ to $\text{Stat}\mathbb{C}^2$. Thus $F'\mathbb{C}^2 \cong \mathbb{C}[0, 1]$. Hence on $\mathbb{C}^2$ the functor $F$ and its commutative variant $F'$ agree (see Example 2). However, on $\mathbb{C}^3$ the functors $F$ and $F'$ differ. Indeed, $F'\mathbb{C}^3$ is commutative while $FC^3$ is not (see Example 3).

\[
\begin{array}{c}
C^*_{\text{MIU}} \quad \text{adjoint} \quad C^*_{\text{PU}} \\
\downarrow F' \quad \quad \quad \quad \quad \downarrow F \\
C^*_\text{MIU} \quad \text{commutative} \quad C^*_\text{PU}
\end{array}
\]

Roughly summarised: while in the diagram above the right adjoints commute with the vertical embeddings, the left adjoints do not.

**Theorem 5.** The embedding $U: C^*_{\text{MIU}} \to C^*_{\text{PU}}$ has a left adjoint.

**Proof.** By Freyd’s Adjoint Functor Theorem (see Theorem V.6.1 of [6]) and the fact that all limits can be formed using only products and equalisers (see Theorem V.2.1 and Exercise V.4.2 of [6]) it suffices to prove the following.

(i) The category $C^*_{\text{MIU}}$ has all small products and equalisers.

(ii) The functor $U: C^*_{\text{MIU}} \to C^*_{\text{PU}}$ preserves small products and equalisers.

(iii) **Solution Set Condition.** For every $C^*$-algebra $\mathcal{A}$ there is a set $I$ and for each $i \in I$ a PU-map $f_i: \mathcal{A} \to \mathcal{A}_i$ such that for any PU-map $f: \mathcal{A} \to \mathcal{B}$ there is an $i \in I$ and a MIU-map $h: \mathcal{A}_i \to \mathcal{B}$ such that $h \circ f_i = f$.

Conditions (i) and (ii) can be verified with routine so we will spend only a few words on them (and leave the details to the reader). To see that Condition (iii) holds requires a little more ingenuity and so we will give the proof in detail.

(Conditions (i) and (ii)) Let us first think about small products in $C^*_{\text{MIU}}$ and $C^*_{\text{PU}}$.

Let $I$ be a set, and for each $i \in I$ let $\mathcal{A}_i$ be a $C^*$-algebra.

It is not hard to see that cartesian product $\prod_{i \in I} \mathcal{A}_i$ is a $*$-algebra when endowed with coordinate-wise operations (and it is in fact the product of the $\mathcal{A}_i$ in the category of $*$-algebras with MIU-maps, and with PU-maps).

However, $\prod_{i \in I} \mathcal{A}_i$ cannot be the product of the $\mathcal{A}_i$ as $C^*$-algebras: there is not even a $C^*$-norm on $\prod_{i \in I} \mathcal{A}_i$ unless $\mathcal{A}_i$ is trivial for all but finitely many $i \in I$. Indeed, if $\| - \|$ were a $C^*$-norm on $\prod_{i \in I} \mathcal{A}_i$, then we must have $\|\sigma(i)\| \leq \|\sigma\|$ for all $\sigma \in \prod_{i \in I} \mathcal{A}_i$ and $i \in I$, and so for any sequence $i_0, i_1, \ldots$ of distinct elements of $I$ for which $\mathcal{A}_{i_0}, \mathcal{A}_{i_1}, \ldots$ are non-trivial, and for every $\sigma \in \prod_{i \in I} \mathcal{A}_i$ with $\sigma(i_n) = n \cdot 1$ for all $n$, we have $n = \|\sigma(i_n)\| \leq \|\sigma\|$ for all $n$, so $\|\sigma\| = \infty$, which is not allowed.

Nevertheless, the $*$-subalgebra of $\prod_{i \in I} \mathcal{A}_i$ given by

\[
\bigoplus_{i \in I} \mathcal{A}_i := \{ \sigma \in \prod_{i \in I} \mathcal{A}_i : \sup_{i \in I} \|\sigma(i)\| < +\infty \}
\]
is a $C^*$-algebra with norm given by, for $\sigma \in \bigoplus_{i \in I} \mathcal{A}_i$,

\[ \|\sigma\| = \sup_{i \in I} \|\sigma(i)\| . \]

We claim that $\bigoplus_{i \in I} \mathcal{A}_i$ is the product of the $\mathcal{A}_i$ in $C_{PU}^*$ (and in $C_{MIU}^*$).

Let $\mathcal{C}$ be a $C^*$-algebra, and for each $i \in I$, let $f_i : \mathcal{C} \rightarrow \mathcal{A}_i$ be a PU-map. We must show that there is a unique PU-map $f : \mathcal{C} \rightarrow \bigoplus_{i \in I} \mathcal{A}_i$ such that $\pi_i \circ f = f_i$ for all $i \in I$ where $\pi_i : \bigoplus_{i \in I} \mathcal{A}_i \rightarrow \mathcal{A}_i$ is the $i$-th projection. It is clear that there is at most one such $f$, and it would satisfy for all $i \in I$, and $c \in \mathcal{C}$, $f(c)(i) = f_i(c)$.

To see that such map $f$ exists is easy if we are able to prove that, for all $c \in \mathcal{C}$,

\[ \sup_{i \in I} \|f_i(c)\| < +\infty. \] (1)

Let $i \in I$ be given. We claim that that $\|f_i(c)\| \leq \|c\|$ for any positive $c \in \mathcal{C}$. Indeed, we have $c \leq \|c\| \cdot 1$, and thus $f_i(c) \leq \|c\| \cdot 1 = \|c\| \cdot 1$, and so $\|f_i(c)\| \leq \|c\|$. It follows that $\|f_i(c)\| \leq 4 \cdot \|c\|$ for any $c \in \mathcal{A}_i$ by writing $c = c_1 - c_2 + i c_3 - i c_4$ where $c_1, c_2, c_3, c_4 \in \mathcal{C}$ are all positive. (We even have $\|f_i(c)\| \leq \|c\|$ for all $c \in \mathcal{C}$, but this requires a bit more effort.) Thus, we have $\sup_{i \in I} \|f_i(c)\| \leq 4 \|c\| < +\infty$. Hence Statement (1) holds.

Thus $\bigoplus_{i \in I} \mathcal{A}_i$ is the product of the $\mathcal{A}_i$ in $C_{PU}^*$ as well. It is easy to see that $\bigoplus_{i \in I} \mathcal{A}_i$ is the product of the $\mathcal{A}_i$ in $C_{MIU}^*$ as well. Hence $C_{MIU}^*$ has all small products (as does $C_{PU}^*$) and $U : C_{MIU}^* \rightarrow C_{PU}^*$ preserves small products.

Let us think about equalisers in $C_{MIU}^*$ and $C_{PU}^*$. Let $\mathcal{A}$ and $\mathcal{B}$ be $C^*$-algebras and let $f, g : \mathcal{A} \rightarrow \mathcal{B}$ be MIU-maps. We must prove that $f$ and $g$ have an equaliser $e : \mathcal{E} \rightarrow \mathcal{A}$ in $C_{MIU}^*$, and that $e$ is the equaliser of $f$ and $g$ in $C_{PU}^*$ as well.

Since $f$ and $g$ are MIU-maps (and hence continuous), it is not hard to see that

\[ \mathcal{E} := \{ a \in \mathcal{A} : f(a) = g(a) \} \]

is a $C^*$-subalgebra of $\mathcal{A}$. We claim that the inclusion $e : \mathcal{E} \rightarrow \mathcal{A}$ is the equaliser of $f, g$ in $C_{PU}^*$. Let $\mathcal{D}$ be a $C^*$-algebra and let $d : \mathcal{D} \rightarrow \mathcal{A}$ be a PU-map such that $f \circ d = g \circ d$. We must show that there is a unique PU-map $h : \mathcal{D} \rightarrow \mathcal{E}$ such that $d = e \circ h$. Note that $d$ maps $\mathcal{A}$ into $\mathcal{E}$. The map $h : \mathcal{D} \rightarrow \mathcal{E}$ is simply the restriction of $d : \mathcal{D} \rightarrow \mathcal{A}$ in the codomain. Hence $e$ is the equaliser of $f, g$ in $C_{PU}^*$.

Note that in the argument above $h$ is a PU-map since $d$ is a PU-map. If $d$ were a MIU-map, then $h$ would be a MIU-map too. Hence $e$ is the equaliser of $f, g$ in the category $C_{MIU}^*$ as well.

Hence $C_{MIU}^*$ has all equalisers and $U : C_{MIU}^* \rightarrow C_{PU}^*$ preserves equalisers. Hence $C_{MIU}^*$ has all small limits and $U : C_{MIU}^* \rightarrow C_{PU}^*$ preserves all small limits.

(Condition [iii]). Let $\mathcal{A}$ be a $C^*$-algebra. We must find a set $I$ and for each $i \in I$ a PU-map $f_i : \mathcal{A} \rightarrow \mathcal{A}_i$ such that for every PU-map $f : \mathcal{A} \rightarrow \mathcal{B}$ there is a (not necessarily unique) $i \in I$ and $h : \mathcal{A}_i \rightarrow \mathcal{B}$ such that $f = h \circ f_i$.

Note that if $f : \mathcal{A} \rightarrow \mathcal{B}$ is a PU-map, then the range of the PU-map $f$ need not be a $C^*$-subalgebra of $\mathcal{B}$. (If the range of PU-maps would have been $C^*$-algebras, then we could have taken $I$ to be the set of all ideals of $\mathcal{A}$, and $f_i : \mathcal{A} \rightarrow \mathcal{A}/J$ to be the quotient map for any ideal $J$ of $\mathcal{A}$.)

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1See Corollary 1 of [7].
Nevertheless, given a PU-map $f : \mathcal{A} \to \mathcal{B}$ there is a smallest $C^*$-subalgebra, say $\mathcal{B}'$, of $\mathcal{B}$ that contains the range of $f$. We claim that $\#(\mathcal{B}') \leq \#(\mathcal{A}^N)$ where $\#(\mathcal{B}')$ is the cardinality of $\mathcal{B}'$ and $\#(\mathcal{A}^N)$ is the cardinality of $\mathcal{A}^N$.\footnote{Although it has no bearing on the validity of the proof one might wonder if the simpler statement $\#(\mathcal{B}') \leq \#(\mathcal{A})$ holds as well. Indeed, if $\#(\mathcal{A}) = \#(\mathbb{C})$ or $\#(\mathcal{A}) = \#(2^X)$ for some infinite set $X$, then we have $\#(\mathcal{A}) = \#(\mathcal{A}^N)$, and so $\#(\mathcal{B}') \leq \#(\mathcal{A})$. However, not every uncountable set is of the form $2^X$ for some infinite set $X$, and in fact, if $\#(\mathcal{A}) = \aleph_0$, then $\#(\mathcal{A}^N) > \#(\mathcal{A})$ by Corollary 3.9.6 of [2].}

If we can find proof for our claim, the rest is easy. Indeed, to begin note that the collection of all $C^*$-algebras is not a small set. However, given a set $U$, the collection of all $C^*$-algebras $\mathcal{C}$ whose elements come from $U$ (so $\mathcal{C} \subseteq U$) is a small set. Now, let $\kappa := \#(\mathcal{A}^N)$ be the cardinality of $\mathcal{A}^N$ (so $\kappa$ is itself a set) and take

$$I := \{ (\mathcal{C}, c) \colon \mathcal{C} \text{ is a } C^* \text{-algebra on a subset of } \kappa \text{ and } c \colon \mathcal{A} \to \mathcal{C} \text{ is a PU-map} \}.\$$

Since the collection of $C^*$-algebras $\mathcal{C}$ with $\mathcal{C} \subseteq \kappa$ is small, and since the collection of PU-maps from $\mathcal{A}$ to $\mathcal{C}$ is small for any $C^*$-algebra $\mathcal{C}$, it follows that $I$ is small.

For each $i \in I$ with $i \equiv (\mathcal{C}, c)$ define $\mathcal{A}_i := \mathcal{C}$ and $f_i := c$.

Let $f : \mathcal{A} \to \mathcal{B}$ be a PU-map. We must find $i \in I$ and a MIU-map $h : \mathcal{A}_i \to \mathcal{B}$ such that $h \circ f_i = f$. Let $\mathcal{B}'$ be the smallest $C^*$-subalgebra that contains the range of $f$. By our claim we have $\#(\mathcal{B}') \leq \#(\mathcal{A}^N) \equiv \kappa$. By renaming the elements of $\mathcal{B}'$ we can find a $C^*$-algebra $\mathcal{C}$ isomorphic to $\mathcal{B}'$ whose elements come from $\kappa$. Let $\varphi : \mathcal{C} \to \mathcal{B}'$ be the isomorphism.

Note that $c := \varphi^{-1} \circ f : \mathcal{A} \to \mathcal{C}$ is a PU-map. So we have $i := (\mathcal{C}, c) \in I$. Further, the inclusion $e : \mathcal{B}' \to \mathcal{B}$ is a MIU-map, as is $\varphi$. So we have:

$$\begin{array}{ccc}
\mathcal{A} & \xrightarrow{f} & \mathcal{B} \\
\downarrow{\mathcal{C}} & \hspace{1cm} & \downarrow{\mathcal{C}} \\
\\downarrow{\text{PU}} & & \downarrow{\text{MIU}} \\
\text{PU} & & \text{MIU} \\
\mathcal{C} & \xrightarrow{e} & \mathcal{B}' \\
\varphi & & \end{array}\$$

Now, $h := e \circ \varphi : \mathcal{C} \to \mathcal{B}$ is a MIU-map with $f = h \circ f_i$. Hence Cond.\[iii\] holds.

Let us proof our claim. Let $\mathcal{A}$ and $\mathcal{B}$ be $C^*$-algebras and let $f : \mathcal{A} \to \mathcal{B}$ be a PU-map. Let $\mathcal{B}'$ be the smallest $C^*$-subalgebra that contains the range of $f$.

We must show that $\#(\mathcal{B}') \leq \#(\mathcal{A}^N)$.

Let us first take care of pathological case. Note that if $\mathcal{A}$ is trivial, i.e. $\mathcal{A} = \{0\}$, then $\mathcal{B}' = \{0\}$, so $\#(\mathcal{A}^N) = 1 = \#(\mathcal{B}')$. Now, let us assume that $\mathcal{A}$ is not trivial. Then we have an injection $\mathbb{C} \to \mathcal{A}$ given by $\lambda \mapsto \lambda \cdot 1$, and thus $\#(\mathcal{A}) \leq \#(\mathcal{A}^N)$.

The trick to prove $\#(\mathcal{B}') \leq \#(\mathcal{A}^N)$ is to find a more explicit description of $\mathcal{B}'$. Let $T$ be the set of terms formed using a unary operation $(-)^*$ (involutions) and two binary operations, $\cdot$ (multiplication) and $+$ (addition), starting from the elements of $\mathcal{A}$. Let $f_T : T \to \mathcal{B}'$ be the map (recursively) given by, for $a \in \mathcal{A}$, and $s,t \in T$,

$$
\begin{align*}
f_T(a) & = f(a) \\
f_T(a^*) & = (f_T(a))^* \\
f_T(a \cdot b) & = f_T(a) \cdot f_T(b) \\
f_T(a + b) & = f_T(a) + f_T(b).
\end{align*}
$$
Note that the range of $f_B$, let us call it $\text{Ran} f_B$, is a $*$-subalgebra of $\mathcal{B}'$. We will prove that $\# \text{Ran} f_B \leq \# \mathcal{A}$. Since $f_B$ is a surjection of $T$ onto $\text{Ran} f_B$ it suffices to prove that $\# T \leq \# \mathcal{A}$. In fact, we will show that $\# T = \# \mathcal{A}$.

First note that $\mathcal{A}$ is infinite, and $\mathcal{A} \subseteq T$, so $T$ is infinite as well. To prove that $\# T = \# \mathcal{A}$ write the elements of $T$ as words (with the use of brackets). Indeed, with $Q := \mathcal{A} \cup \{\cdot, +, \cdot, \cdot\}$, there is an obvious injection from $T$ into the set $Q^*$ of words over $Q$. Since $\mathcal{A}$ is infinite, and $Q \setminus \mathcal{A}$ is finite we have $\# Q = \# \mathcal{A}$ by Hilbert’s hotel. Recall that $Q^* = \bigcup_{n=0}^{\infty} Q^n$. Since $Q$ is infinite, we also have $\# (\mathbb{N} \times Q) = \# Q$ and even $\# (Q \times Q) = \# Q$ (see Theorem 3.7.7 of [2]), so $\# Q = \# (Q^n)$ for all $n > 0$. It follows that

$$
\# (Q^*) = \# (\bigcup_{n=0}^{\infty} Q^n) = \# (1 + \bigcup_{n=1}^{\infty} Q) = \# (1 + \mathbb{N} \times Q) = \# Q.
$$

Since there is an injection from $T$ to $Q^*$ we have $\# \mathcal{A} \leq \# T \leq \# (Q^*) = \# Q = \# \mathcal{A}$ and so $\# T = \# \mathcal{A}$. Hence $\# \text{Ran} f_B \leq \# \mathcal{A}$.

Since $\text{Ran} f_B$ is a $*$-algebra that contains $\text{Ran} f$, the closure $\overline{\text{Ran} f_B}$ of $\text{Ran} f_B$ with respect to the norm on $\mathcal{B}'$ is a $C^*$-algebra that contains $\text{Ran} f$. As $\mathcal{B}'$ is the smallest $C^*$-subalgebra that contains $\text{Ran} f$, we see that $\mathcal{B}' = \overline{\text{Ran} f_B}$.

Let $S$ be the set of all Cauchy sequences in $\text{Ran} f_B$. As every point in $\mathcal{B}'$ is the limit of a Cauchy sequence in $\text{Ran} f_B$, we get $\# \mathcal{B}' \leq \# S$. Thus:

$$
\# \mathcal{B}' \leq \# S
\leq \# (\text{Ran} f_B)^N \quad \text{as } S \subseteq (\text{Ran} f_B)^N
\leq \# (\mathcal{A}^N) \quad \text{as } \# \text{Ran} f_B \leq \# \mathcal{A}.
$$

Thus we have proven our claim.

Hence Conditions (i) hold and $U : \text{C}_{\text{MIU}} \rightarrow \text{C}_{\text{PU}}$ has a left adjoint. 

We have seen that $U : \text{C}_{\text{MIU}} \rightarrow \text{C}_{\text{PU}}$ has a left adjoint $F : \text{C}_{\text{PU}} \rightarrow \text{C}_{\text{MIU}}$. This adjunction gives a comonad $FU$ on $\text{C}_{\text{MIU}}$, which in turns gives us two categories: the Eilenberg–Moore category $\mathcal{E.M}(FU)$ of $FU$-coalgebras and the Kleisli category $\mathcal{K}l(FU)$. We claim that $\text{C}_{\text{PU}}$ is isomorphic to $\mathcal{K}l(FU)$ since $\text{C}_{\text{MIU}}$ is a subcategory of $\text{C}_{\text{PU}}$ with the same objects.

This is a special case of a more general phenomenon which we discuss in the next section (in terms of monads instead of comonads), see Theorem 9.

## 2 Kleislian Adjunctions

Beck’s Theorem (see [6], VI.7) gives a criterion for when an adjunction $F \dashv U$ “is” an adjunction between $\text{C}$ and $\mathcal{E.M}(FU)$. We give a similar (but easier) criterion for when an adjunction “is” an adjunction between $\text{C}$ and $\mathcal{K}l(FU)$. The criterion is not new; e.g., it is mentioned in [5] (paragraph 8.6) without proof or reference, and it can be seen as a consequence of Exercise VI.5.2 of [6] (if one realises that an equivalence which is bijective on objects is an isomorphism). Proofs can be found in the appendix.
**Notation 6.** Let $F: C \to D$ be a functor with right adjoint $U$. Denote the unit of the adjunction by $\eta: \text{id}_D \to UF$, and the counit by $\varepsilon: FU \to \text{id}_C$.

Recall that $UF$ is a monad with unit $\eta$ and as multiplication, for $C$ from $C$,

$$\mu_C := U\varepsilon_C: UFUFC \to UFC.$$ 

Let $\mathcal{K}(UF)$ be the Kleisli category of the monad $UF$. So $\mathcal{K}(UF)$ has the same objects as $C$, and the morphisms in $\mathcal{K}(UF)$ from $C_1$ to $C_2$ are the morphisms in $C$ from $C_1$ to $UFC_2$. Given $C$ from $C$ the identity in $\mathcal{K}(UF)$ on $C$ is $\eta_C$. If $C_1, C_2, C_3, f: C_1 \to C_2, g: C_2 \to C_3$ from $C$ are given, $g$ after $f$ in $\mathcal{K}(UF)$ is

$$g \circ f := \mu_{C_3} \circ UF g \circ f.$$ 

Let $V: C \to \mathcal{K}(UF)$ be given by, for $f: C_1 \to C_2$ from $C$,

$$Vf := \eta_{C_2} \circ f: C_1 \to UFC_2.$$ 

Let $G: \mathcal{K}(UF) \to C$ be given by, for $f: C_1 \to UFC_2$ from $C$,

$$Gf := \mu_{C_2} \circ UF f: UFC_1 \to UFC_2.$$ 

The following is Exercise VI.5.1 of [6].

**Lemma 7.** Let $F: C \to D$ be a functor with a right adjoint $U$. Then there is a unique functor $L: \mathcal{K}(UF) \to D$ (called the comparison functor) such that $U \circ L = G$ and $L \circ V = F$ (see Notation 6).

$\mathcal{K}(UF)$

$\downarrow$

$\downarrow$ $U$

$V$

$\downarrow$

$C$

$L$

$G$

$D$

**Definition 8.** Let $C$ and $D$ be categories.

(i) A functor $F: C \to D$ is called **Kleislian** when it has a right adjoint $U: D \to C$, and the functor $L: \mathcal{K}(UF) \to D$ from Lemma 7 is an isomorphism.

(ii) We say that $D$ is **Kleislian over $C$** when there is a Kleislian functor $F: C \to D$.

**Theorem 9.** Let $F: C \to D$ be a functor with a right adjoint $U$.

The following are equivalent.

(i) $F$ is Kleislian (see Definition 8).

(ii) $F$ is bijective on objects (i.e. for every object $D$ from $D$ there is a unique object $C$ from $C$ such that $FC = D$).

**Corollary 10.** The embedding $U^{op}: (C^{\ast}_{\text{MIU}})^{op} \to (C^{\ast}_{\text{PU}})^{op}$ is Kleislian (see Def. 8).

**Proof.** By Theorem 9 we must show that $U^{op}$ has a left adjoint and is bijective on objects. Since the embedding $U: C^{\ast}_{\text{MIU}} \to C^{\ast}_{\text{PU}}$ has a left adjoint $F: C^{\ast}_{\text{PU}} \to C^{\ast}_{\text{MIU}}$; it follows that $F^{op}: (C^{\ast}_{\text{PU}})^{op} \to (C^{\ast}_{\text{MIU}})^{op}$ is the right adjoint of $U^{op}$. Thus $U^{op}$ has a left adjoint. Further, as $C^{\ast}_{\text{MIU}}$ and $C^{\ast}_{\text{PU}}$ have the same objects, $U$ is bijective on objects, and so is $U^{op}$. Hence $U^{op}$ is Kleislian. $\Box$
In summary, the embedding \( U : C^*_{\text{MIU}} \rightarrow C^*_{\text{PU}} \) has a left adjoint \( F \) (and so \( F^{\text{op}} : (C^*_{\text{MIU}})^{\text{op}} \rightarrow (C^*_{\text{PU}})^{\text{op}} \) is right adjoint to \( U^{\text{op}} \)), and the unique functor from the Kleisli category \( \mathcal{K}(FU) \) of the monad \( FU \) on \( (C^*_{\text{MIU}})^{\text{op}} \) to \( (C^*_{\text{PU}})^{\text{op}} \) that makes the two triangles in the diagram below on the left commute is an isomorphism.

\[
\begin{array}{ccc}
\mathcal{K}(FU) & \cong & (C^*_{\text{MIU}})^{\text{op}} \\
\downarrow_{F^{\text{op}}} & & \downarrow_{(C^*_{\text{PU}})^{\text{op}}} \\
\text{Set} & \cong & \text{Set}_{\text{multi}}
\end{array}
\]

For the category \( \text{Set}_{\text{multi}} \) of multimaps between sets used in the introduction to describe the semantics of non-deterministic programs the situation is the same, see the diagram above to the right.

(The functor \( V \) is the obvious embedding. The right adjoint \( G \) of \( V \) sends a multimap \( f \) from \( X \) to \( Y \) to the function \( Gf : \mathcal{P}(X) \rightarrow \mathcal{P}(Y) \) that assigns to a subset \( A \in \mathcal{P}(X) \) the image of \( A \) under \( f \). Note that \( GV = \mathcal{P} \).

3 Discussion

3.1 Variations

**Example 11** (Subunital maps). Let \( C^*_{\text{PSU}} \) be the category of \( C^* \)-algebras and the positive linear maps \( f \) between them that are subunital, i.e. \( f(1) \leq 1 \). The morphisms of \( C^*_{\text{PSU}} \) are called \( \text{PsU-maps} \).

It is not hard to see that the products in \( C^*_{\text{PSU}} \) are the same as in \( C^*_{\text{MIU}} \), and that the equaliser in \( C^*_{\text{MIU}} \) of a pair \( f, g \) of MIU-maps is the equaliser of \( f, g \) in \( C^*_{\text{PSU}} \) as well. Thus the embedding \( U : C^*_{\text{MIU}} \rightarrow C^*_{\text{PSU}} \) preserves limits. Using the same argument as in Theorem 5 but with “PU-map” replaced by “PsU-map” one can show that \( U \) satisfies the Solution Set Condition. Hence \( U \) has a left adjoint by Freyd’s Adjoint Function Theorem, say \( F : C^*_{\text{PSU}} \rightarrow C^*_{\text{MIU}} \).

Since \( C^*_{\text{PSU}} \) has the same objects as \( C^*_{\text{MIU}} \) (namely the \( C^* \)-algebras) the functor \( U^{\text{op}} : (C^*_{\text{MIU}})^{\text{op}} \rightarrow (C^*_{\text{PSU}})^{\text{op}} \) is bijective on objects and thus Kleislian (by Th. 9).

Hence \( (C^*_{\text{PSU}})^{\text{op}} \) is Kleislian over \( (C^*_{\text{MIU}})^{\text{op}} \).

**Example 12** (Bounded linear maps). Let \( C_p \) be the category of positive bounded linear maps between \( C^* \)-algebras. We will show that \( (C_p)^{\text{op}} \) is not Kleislian over \( (C^*_{\text{MIU}})^{\text{op}} \). Indeed, if it were then \( (C_p)^{\text{op}} \) would be cocomplete, but it is not: there is no \( \omega \)-fold product of \( C \) in \( C_p \). To see this, suppose that there is a \( \omega \)-fold product \( \mathcal{P} \) in \( C_p \) with projections \( \pi_i : \mathcal{P} \rightarrow C \) for \( i \in \omega \). Since \( \pi_i \) is a bounded linear map for \( i \in \omega \), it has finite operator norm, say \( \| \pi_i \| \). By symmetry, \( \| \pi_i \| = \| \pi_j \| \) for all \( i, j \in \omega \). Write \( K := \| \pi_0 \| = \| \pi_1 \| = \| \pi_2 \| = \cdots \). Define \( f_i : C \rightarrow C \) by \( f_i(z) = iz \) for all \( z \in C \) and \( i \in \omega \). Then \( f_i \) is a positive bounded linear map for each \( i \in \omega \). Since \( \mathcal{P} \) is the \( \omega \)-fold product of \( C \), there is a (unique positive) bounded linear map \( f : C \rightarrow \mathcal{P} \) such that \( \pi_i \circ f = f_i \) for all \( i \in \omega \). For each \( N \in \omega \) we have

\[
N = \| f_N(1) \| \leq \| f_N \| = \| \pi_N \circ f \| \leq \| \pi_N \| \| f \| = K \| f \|.
\]

Thus \( K \| f \| \) is greater than any number, which is absurd.

**Example 13** (Completely positive maps). For clarity’s sake we recall what it means for a linear map \( f \) between \( C^* \)-algebras to be completely positive (see \cite{8}). For this we need some notation. Given a \( C^* \)-algebra \( \mathcal{A} \), and \( n \in \mathbb{N} \) let \( M_n(\mathcal{A}) \) denote the set of \( n \times n \)-matrices with entries from \( \mathcal{A} \). We leave it to the
Quantum Programs as Kleisli Maps

M where A_i and \parallel_i for all

We must show that c in C

CPU-maps

The completely positive linear maps that preserve the unit are called

It is easy to see that \parallel - \parallel_\pi on M_n(\mathcal{A}) given by for A \in M_n(\mathcal{A}),

where \xi((M_\pi(A))(A)) : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n} is the bounded linear map represented by the matrix (M_\pi(A))(A), and \|\xi((M_\pi(A))\| is the operator norm of \xi((M_\pi(A)) in \mathcal{B}(\mathcal{H}^{\otimes n}).

It is easy to see that \parallel - \parallel_\pi satisfies the C*-identity, \|A^*A\|_\pi = \|A\|_\pi^2 for all A \in M_n(\mathcal{A}). It is less obvious that M_n(\mathcal{A}) is complete with respect to \parallel - \parallel_\pi. To see this, first note that \|A_{ij}\| \leq \|A\|_\pi for all i, j. So given a Cauchy sequence A_1, A_2, \ldots in M_n(\mathcal{A}) we can form the entrywise limit A, that is, A_{ij} = \lim_{m \to \infty} A_{ij}. We leave it to the reader to check that A_{ij} is the limit of A_1, A_2, \ldots, and thus M_n(\mathcal{A}) is complete with respect to \parallel - \parallel_\pi. Hence M_n(\mathcal{A}) is a C*-algebra with norm \parallel - \parallel_\pi.

The C*-norm \parallel - \parallel_\pi does not depend on \pi. Indeed, let \mathcal{H}_1 and \mathcal{H}_2 be Hilbert spaces and let \pi_1 : \mathcal{A} \to \mathcal{B}(\mathcal{H}_1) and \pi_2 : \mathcal{A} \to \mathcal{B}(\mathcal{H}_2) be isometric MIU-maps; we will show that \parallel - \parallel_{\pi_1} = \parallel - \parallel_{\pi_2}. Recall that the norm \parallel - \parallel_\pi induces an order \leq_\pi on M_n(\mathcal{A}) given by 0 \leq_\pi A if \|A - |A|\|_\pi \leq \|A\|_\pi where A \in M_n(\mathcal{A}). Since \|A\|_\pi^2 = \inf{ \lambda \in [0, \infty] : A^*A \leq_\lambda } for all A \in M_n(\mathcal{A}), to prove \parallel - \parallel_{\pi_1} = \parallel - \parallel_{\pi_2} it suffices to show that the orders \leq_{\pi_1} and \leq_{\pi_2} coincide. But this is easy when one recalls that A \in M_n(\mathcal{A}) is positive iff A is of the form B^*B for some B \in M_n(\mathcal{A}).

The completely positive linear maps that preserve the unit are called CPU-maps. Let C_{CPU} be the category of CPU-maps between C*-algebras. Since M_n(f) is a MIU-map when f is a MIU-map and a MIU-map is positive, we see that any MIU-map is completely positive. Thus C_{MIU} is a subcategory of C_{CPU}. We claim that (C_{CPU})^{op} is Kleislian over (C_{MIU})^{op}.

Let us show that U preserves limits. To show that U preserves equalisers, let f, g : \mathcal{A} \to \mathcal{B} be MIU-maps. Then \mathcal{C} := \{ x \in \mathcal{A} : f(x) = g(x) \} is a C*-subalgebra of \mathcal{A} and the embedding e : \mathcal{C} \to \mathcal{A} is an isometric MIU-map. Then e is the equaliser of f, g in C_{MIU}; we will show that e is the equaliser of f, g in C_{CPU}. Let \mathcal{E} be a C*-algebra, and let c : \mathcal{E} \to \mathcal{A} be a CPU-map such that f \circ c = g \circ c. Let d : \mathcal{E} \to \mathcal{C} be the restriction of c. It turns out we must prove that d is completely positive. Let n \in \mathbb{N} be given. We must show that M_n d : M_n \mathcal{C} \to M_n \mathcal{C} is positive. Note that M_n e is an injective MIU-map and thus an isometry. So in order to prove that M_n d is positive it suffices to show that M_n e \circ M_n d = M_n (e \circ d) = M_n c is positive, which it is since c is completely positive. Thus e is the equaliser of f, g in C_{CPU}. Hence U preserves equalisers.

To show that U preserves products, let I be a set and for each i \in I let \mathcal{A}_i be a C*-algebra. We will show that \bigoplus_{i \in I} \mathcal{A}_i is the product of the \mathcal{A}_i in C_{CPU}. Let \mathcal{E} be a C*-algebra, and for each i \in I, let f_i : \mathcal{E} \to \mathcal{A}_i be a CPU-map. As before, let f : \mathcal{E} \to \bigoplus_{i \in I} \mathcal{A}_i be the map given by f(x)(i) = f_i(x) for all i \in I and x \in \mathcal{E}. Leaving the details to the reader it turns out that it suffices to show that f is completely positive. Let n \in \mathbb{N} be given. We must prove that M_n f : M_n(\mathcal{E}) \to M_n(\bigoplus_{i \in I} \mathcal{A}_i) is positive. Let \varphi : M_n(\bigoplus_{i \in I} \mathcal{A}_i) \to \bigoplus_{i \in I} M_n(\mathcal{A}_i) be the unique MIU-map such that \pi_i \circ \varphi = M_n \pi_i for all i \in I. Then \varphi is a MIU-isomorphism and thus to prove that M_n f is positive, it suffices to show that \varphi \circ M_n f is positive. Let i \in I be given. We must prove that \pi_i \circ \varphi \circ M_n f is positive. But we have \pi_i \circ \varphi \circ M_n f = M_n \pi_i \circ M_n f = M_n(\pi_i \circ f) = M_n f_i, which is positive since f is completely positive. Thus \bigoplus_{i \in I} \mathcal{A}_i is the product of the \mathcal{A}_i in C_{CPU} and hence U preserves limits.
With the same argument as in Theorem 9 the functor \( U \) satisfies the Solution Set Condition and thus \( U \) has a left adjoint. It follows that \( U^{\text{op}}: (C^*_{\text{MIU}})^{\text{op}} \longrightarrow (C^*_{\text{CPU}})^{\text{op}} \) is Kleislian.

**Example 14 (W*-algebras).** Let \( W^*_{\text{NMIU}} \) be the category of von Neumann algebras (also called \( W^* \)-algebras) and the MIU-maps between them that are normal, i.e., preserve suprema of upwards directed sets of self-adjoint elements. Let \( W^*_{\text{NPU}} \) be the category of von Neumann and normal PU-maps. Note that \( W^*_{\text{NMIU}} \) is a subcategory of \( W^*_{\text{NPU}} \). We will prove that \( (W^*_{\text{NPU}})^{\text{op}} \) is Kleislian over \( (W^*_{\text{NMIU}})^{\text{op}} \).

It suffices to show that \( U \) has a left adjoint. Again we follow the lines of the proof of Theorem 5. Products and equalisers in \( W^*_{\text{NMIU}} \) are the same as in \( C^*_{\text{MIU}} \). It is not hard to see that the embedding \( U: W^*_{\text{NMIU}} \longrightarrow W^*_{\text{NPU}} \) preserves limits. To see that \( U \) satisfies the Solution Set Condition we use the same method as before: given a von Neumann algebra \( \mathcal{A} \), find a suitable cardinal \( \kappa \) such that the following is a solution set.

\[
I := \{ (c, \psi) : \mathcal{C} \text{ is a von Neumann algebra on a subset of } \kappa \\
\text{and } c: \mathcal{A} \longrightarrow \mathcal{C} \text{ is a normal PU-map} \},
\]

Only this time we take \( \kappa = \#(\mathcal{A}(\mathcal{A})) \) instead of \( \kappa = \#(\mathcal{A}^\mathbb{N}) \). We leave the details to the reader, but it follows from the fact that given a subset \( X \) of a von Neumann algebra \( \mathcal{B} \) the smallest von Neumann subalgebra \( \mathcal{B}' \) that contains \( X \) has cardinality at most \( \#(\mathcal{A}(\mathcal{A})) \). Indeed, if \( \mathcal{H} \) is a Hilbert space such that \( \mathcal{B} \subseteq \mathcal{B}(\mathcal{H}) \) (perhaps after renaming the elements of \( \mathcal{B} \)), then \( \mathcal{B}' \) is the closure (in the weak operator topology on \( \mathcal{B}(\mathcal{H}) \) of the smallest \( * \)-subalgebra containing \( X \). Thus any element of \( \mathcal{B}' \) is the limit of a filter — a special type of net, see paragraph 12 of [9] — of \( * \)-algebra terms over \( X \), of which there are no more than \( \#(\mathcal{A}(\mathcal{A})) \).

By a similar reasoning one sees that the opposite \( (W^*_{\text{NCPA}})^{\text{op}} \) of the category of normal completely positive subunital linear maps between von Neumann algebras is Kleislian over \( (W^*_{\text{NMIU}})^{\text{op}} \). The existence of the adjoint to the inclusion \( W^*_{\text{NMIU}} \rightarrow W^*_{\text{NCPA}} \) is key in our construction of a model of Selinger and Valiron’s quantum lambda calculus by von Neumann algebras, see [11].

### 3.2 Concrete description

In this note we have shown that the embedding \( U: C^*_{\text{MIU}} \longrightarrow C^*_{\text{CPU}} \) has a left adjoint \( F \), but we miss a concrete description of \( F \mathcal{A} \) for all but the simplest \( C^* \)-algebras \( \mathcal{A} \). What constitutes a “concrete description” is perhaps a matter of taste or occasion, but let us pose that it should at least enable us to describe the Eilenberg–Moore category \( \mathcal{E} \cdot \mathcal{M} \text{(FU)} \) of the comonad \( \text{FU} \). More concretely, it should settle the following problem.

**Problem 15.** Writing \( \text{BOUS} \) for the category of positive linear maps that preserve the unit between Banach order unit spaces, determine whether \( \mathcal{E} \cdot \mathcal{M} \text{(FU)} \cong \text{BOUS} \).

(An order unit space is an ordered vector space \( V \) over \( \mathbb{R} \) with an element 1, the order unit, such that for all \( v \in V \) there is \( \lambda \in [0, \infty) \) such that \(-\lambda \cdot 1 \leq v \leq \lambda \cdot 1 \). The smallest such \( \lambda \) is denoted by \( \|v\| \). See [4] for more details. If \( v \mapsto \|v\| \) gives a complete norm, \( V \) is called a Banach order unit space.)

### 3.3 MIU versus PU

A second “problem” is to give a physical description (if there is any) of what it means for a quantum program’s semantics to be a MIU-map (and not just a PU-map). A step in this direction might be to define for a \( C^* \)-algebra \( \mathcal{A} \), a PU-map \( \varphi: \mathcal{A} \rightarrow \mathbb{C} \), and \( a, b \in \mathcal{A} \) the quantity

\[
\text{Cov}_\varphi(a, b) := \varphi(a^* b) - \varphi(a)^* \varphi(b)
\]
and interpret it as the covariance between the observables $a$ and $b$ in state $\varphi$ of the quantum system $\mathcal{A}$. Let $T : \mathcal{A} \rightarrow \mathcal{B}$ be a PU-map between $C^\ast$-algebras (so perhaps $T$ is the semantics of a quantum program). Then it is not hard to verify that $T$ is a MIU-map if and only if $T$ preserves covariance, that is,

$$\text{Cov}_{\varphi}(Ta, Tb) = \text{Cov}_{\varphi \circ T}(a, b) \quad \text{for all } a, b \in \mathcal{A}.$$ 

### 4 Acknowledgements

Example[2] and Example[3] were suggested by Robert Furber. I’m grateful that Jianchao Wu and Sander Uijlen spotted several errors in a previous version of this text. Kenta Cho realised that the results of this paper might be used to construct a model of the quantum lambda calculus. I thank them, and Bart Jacobs, Sam Staton, Wim Veldman, and Bas Westerbaan for their help.

Funding was received from the European Research Council under grant agreement No 320571.

### References


### A Additional Proofs

**Proof of Lemma**[7] Define $LC := FC$ for all objects $C$ of $\mathcal{K} \ell(UF)$ and

$$Lf := \varepsilon_{FC_2} \circ Ff$$

for $f : C_1 \rightarrow UFC_2$ from $C$. We claim this gives a functor $L : \mathcal{K} \ell(UF) \rightarrow D$.

**($L$ preserves the identity)** Let $C$ be an object of $\mathcal{K} \ell(UF)$, that is, an object of C. Then the identity on $C$ in $\mathcal{K} \ell(UF)$ is $\eta_C$. We have $L(\eta_C) = \varepsilon_{FC} \circ F\eta_C = id_{FC}$. 


To prove that $L(g \circ f) = Lg \circ LF$. We have:

$$L(g \circ f) = L(\mu_{C_3} \circ UF \circ f)$$

by def. of $g \circ f$

$$= \varepsilon_{FC_3} \circ F \mu_{C_3} \circ FUF \circ f$$

by def. of $L$

$$= \varepsilon_{FC_3} \circ FU \varepsilon_{FC_3} \circ FUF \circ f$$

by def. of $\mu_{C_3}$

$$= \varepsilon_{FC_3} \circ Fg \circ \varepsilon_{FC_2} \circ Ff$$

by nat. of $\eta$

$$= Lg \circ LF$$

by def. of $L$

Hence $L$ is a functor from $\mathcal{K}(UF)$ to $D$.

Let us prove that $U \circ L = G$. For $f: C_1 \rightarrow UF$ from $C$ we have

$$ULf = U(\varepsilon_{FC_2} \circ Ff)$$

by def. of $L$

$$= U\varepsilon_{FC_2} \circ UFf$$

$$= \mu_{C_2} \circ UFf$$

by def. of $\mu_{C_2}$

$$= Gf$$

by def. of $Gf$.

Let us prove that $L \circ V = F$. For $f: C_1 \rightarrow C_2$ from $C$ be given, we have

$$LV = L(\eta_{C_2} \circ f)$$

by def. of $V$

$$= \varepsilon_{FC_2} \circ F \eta_{C_2} \circ Ff$$

by def. of $L$

$$= Ff$$

by counit–unit eq.

We have proven that there is a functor $L: \mathcal{K}(UF) \rightarrow D$ such that $U \circ L = G$ and $L \circ V = F$. We must still prove that it is as such unique.

Let $L': \mathcal{K}(UF) \rightarrow D$ be a functor such that $U \circ L' = G$ and $L' \circ V = F$. We must show that $L = L'$. Let us first prove that $L'$ and $L$ agree on objects. Let $C$ be an object of $\mathcal{K}(UF)$, i.e., $C$ is an object of $C$. Since $L' \circ V = F$ and $VC = C$ we have $L'C = L'VC = FC = LC$. Now, let $f: C_1 \rightarrow UF$ from $C$ be given (so $f$ is a morphism in $\mathcal{K}(UF)$ from $C_1$ to $C_2$). We must show that $L'f = UL \equiv \varepsilon_{FC_2} \circ Ff$. Note that since $F$ is the left adjoint of $U$ there is a unique morphism $\overline{f}: FC_1 \rightarrow FC_2$ in $D$ such that $UF \circ \eta_{C_1} = f$.

To prove that $L'f = ULf$, we show that both $Lf$ and $L'f$ have this property. We have

$$ULf \circ \eta_{C_1} = GF \circ \eta_{C_1}$$

as $U \circ L' = G$ by assump.

$$= \mu_{C_2} \circ UFf \circ \eta_{C_1}$$

by def. of $G$

$$= \mu_{C_2} \circ \eta_{UF\varepsilon_{C_2}} \circ f$$

by nat. of $\eta$

$$= f$$

as $UF$ is a monad.

By a similar argument we get $ULf \circ \eta_{C_1} = f$. Hence $Lf = L'f$.

**Proof of Theorem**

We use the symbols from Notation. Suppose that $L$ is an isomorphism. We must prove that $F$ is bijective on objects. Note that $F = L \circ V$, so it suffices to show that both $L$ and $V$ are bijective on objects. Clearly, $L$ is bijective on objects as $L$ is an isomorphism, and $V: C \rightarrow \mathcal{K}(UF)$ is bijective on objects since the objects of $\mathcal{K}(UF)$ are those of $C$ and $VC = C$ for all $C$ from $C$. 

\[\square\]
Suppose that \( (ii) \) holds. We prove that \( L \) is an isomorphism by giving its inverse. Let \( D \) be an object from \( D \). Note that since \( F \) is bijective on objects there is a unique object \( C \) from \( C \) such that \( FD = C \). Define \( KC := D \).

Let \( g : D_1 \to D_2 \) from \( D \) be given. Note that by definition of \( K \) we have:

\[
\begin{align*}
KD_1 &\xrightarrow{\eta_{KD_1}} UFKD_1 \\
&\xrightarrow{Ug} UD_1 \\
&\xrightarrow{UD_2} UFKD_2
\end{align*}
\]

Now, define \( Kg : KD_1 \to UFKD_2 \) in \( D \) by \( Kg := Ug \circ \eta_{KD_1} \).

We claim that this gives a functor \( K : D \to \mathcal{X}(UF) \).

**\( K \) preserves the identity** Let \( f : D_1 \to D_2 \) from \( D \) be given. We must prove that \( K(g \circ f) = K(g) \circ K(f) \). We have

\[
K(g) \circ K(f) = \mu_{KD_2} \circ UFKg \circ Kf
\]

by def. of \( \circ \)

\[
= \mu_{KD_2} \circ UFUg \circ UF \eta_{KD_2} \circ Uf \circ \eta_{KD_1}
\]

by def. of \( K \)

\[
= U \varepsilon_{D_2} \circ UFUg \circ UF \eta_{KD_2} \circ Uf \circ \eta_{KD_1}
\]

by def. of \( \mu \)

\[
= Ug \circ U \varepsilon_{D_2} \circ UF \eta_{KD_2} \circ Uf \circ \eta_{KD_1}
\]

by nat. of \( \varepsilon \)

\[
= Ug \circ Uf \circ \eta_{KD_1}
\]

by counit–unit eq.

\[
= K(g \circ f)
\]

by def of \( K \).

Hence \( K \) is a functor from \( D \) to \( \mathcal{X}(UF) \). We will show that \( K \) is the inverse of \( L \). For this we must prove that \( K \circ L = \text{id}_D \) and \( L \circ K = \text{id}_{\mathcal{X}(UF)} \).

For a morphism \( g : D_1 \to D_2 \) from \( D \), we have

\[
Lg = L(Ug \circ \eta_{KD_1})
\]

by def. of \( K \)

\[
= \varepsilon_{FKD_2} \circ FUg \circ F \eta_{KD_1}
\]

by def. of \( L \)

\[
= g \circ \varepsilon_{FKD_1} \circ F \eta_{KD_1}
\]

by nat. of \( \varepsilon \)

\[
= g
\]

by counit–unit eq.

For a morphism \( f : C_1 \to UFC_2 \) in \( C \) we have

\[
Kf = K(\varepsilon_{FC_2} \circ Ff)
\]

by def. of \( L \)

\[
KLf = U\varepsilon_{FC_2} \circ UFf \circ \eta_{KFC_1}
\]

by def. of \( K \)

\[
= U \varepsilon_{FC_2} \circ \eta_{UFC_2} \circ f
\]

by nat. of \( \eta \)

\[
= f
\]

by counit–unit eq.

Hence \( K \) is the inverse of \( L \), so \( L \) is an isomorphism.