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Multirelative $K$-theory and axioms for the $K$-theory of rings

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Abstract

$K$-groups are defined for a special type of $m$-tuples of ideals in a ring. It is shown that some of the properties of this multirelative $K$-theory characterize the $K$-theory of rings.

Introduction

Multirelative $K$-groups $K_n(R, a_1, \ldots, a_m)$ of an $m$-tuple $(a_1, \ldots, a_m)$ of ideals of a ring $R$ are recently used to derive properties of the absolute $K$-groups, e.g. by Levine [4] and by Bloch and Lichtenbaum [1]. Here it is shown how $K$-theory as defined in [3] can easily be extended to the multirelative case and that some of its properties can be taken as axioms for the $K$-theory of rings. Special types of $m$-tuples of ideals—the ‘normal’ $m$-tuples—play a crucial role. In fact we will only define multirelative $K$-groups for such $m$-tuples. The notion of normal $m$-tuple of ideals is introduced in Section 2. It already appeared in 1981 in a paper by Dayton and Weibel [2] on the $K$-theory of affine glued schemes under the name of ‘condition (CRT)’ (= Chinese Remainder Theorem).

In Section 4 we review briefly higher $K$-theory as defined in [3]. In Section 6 multirelative $K$-groups are defined, and in Section 7 it is shown that from some of their properties one can reconstruct the $K$-theory of rings.

1 Notations

In this paper ‘ring’ stands for a non-unital ring. Non-unital rings form a category which is denoted by $\mathcal{R}$.

Since the functors $GL$, $E$ and $K_1$ are product preserving functors from unital rings to groups, they can be extended to functors defined on $\mathcal{R}$ in the usual way: if $T$ is one of these functors, then put

$$T(R) := \text{Ker}(T(R^+) \rightarrow T(\mathbb{Z})),$$

where $R^+ = R \times \mathbb{Z}$ with multiplication given by

$$(r, k)(s, l) = (rs + ks + lr, kl)$$
is a ring with \((0,1)\) as unity element.

Here ‘ideal’ will always stand for ‘twosided ideal’.

By \(A\) we will denote the category of Abelian groups, by \(G\) the category of all groups, and by \(S\) the category of sets. The category of simplicial objects in a category \(C\) is denoted by \(sC\).

## 2 \(m\)-cubes and normal \(m\)-tuples

In this section the notion of normality of an \(m\)-tuple of ideals is considered. Only the group structure is involved in its definition, and since we can use later a similar notion for groups instead of rings we give a more general definition. By \(m\) we will denote the set \(\{1, \ldots, m\}\).

**Definition 1.** An \(m\)-tuple \((B_1, \ldots, B_m)\) of normal subgroups of a group \(A\)—also denoted as \((A, B_1, \ldots, B_m)\)—is called normal if for all subsets \(I\) and \(J\) of \(m\)

\[
\bigcap_{i \in I} B_i \cdot \prod_{j \in J} B_j = \bigcap_{i \in I} \left( B_i \cdot \prod_{j \in J} B_j \right).
\]

The condition is trivially fulfilled when \(I \cap J \neq \emptyset\). In the case of Abelian groups it reads in the additive notation as

\[
\bigcap_{i \in I} B_i + \sum_{j \in J} B_j = \bigcap_{i \in I} \left( B_i + \sum_{j \in J} B_j \right).
\]

Note that in the special case of an \(m\)-tuple of ideals in a commutative ring the condition is a local one since it involves only intersections and sums of ideals.

The subsets of \(m\) are ordered by inclusion. This ordered set determines in the usual way a category \(C_m\). For every pair \((I, J)\) of subsets with \(I \subseteq J\) there is the unique morphism \(\rho^I_J\) from \(I\) to \(J\) in \(C_m\).

**Definition 2.** Let \(\mathcal{D}\) be a category. An \(m\)-cube in \(\mathcal{D}\) is a functor

\[
D: C_m \to \mathcal{D}, \quad I \mapsto D_I, \quad \rho^I_J \mapsto r^I_J.
\]

The morphisms in \(C_m\) are generated by the \(\rho^I_J\), where \(#J = #I + 1\). An \(m\)-cube in a category \(\mathcal{D}\) is a commutative diagram in \(\mathcal{D}\) having the shape of an \(m\)-dimensional cube. The edges of the cube correspond to the images of these generating morphisms.

**Definition 3.** Let \(D: C_m \to \mathcal{D}\) be an \(m\)-cube in \(\mathcal{D}\). It is said to be a split \(m\)-cube if for every pair of subsets \((I, J)\) of \(m\) satisfying \(I \subseteq J\) there is a morphism \(s^I_J: D_J \to D_I\) in \(\mathcal{D}\) such that

1. \((S1)\) \(s^I_I s^K_J = s^K_I\) for all \(I \subseteq J \subseteq K\),
2. \((S2)\) \(r^I_J s^I_J = 1_{D_J}\) for all \(I \subseteq J\),

2
(S3) \( r_J^I \cap J = s^{I \cup J}_J r_J^I \) for all \( I \) and \( J \).

(Of course such a split \( m \)-cube can also be seen as a functor defined on a category which is obtained from \( C_m \) by adjoining extra morphisms \( \sigma_I^J : J \rightarrow I \).)

In condition (S3) one only needs the case where \( \#(I \setminus J) = \#(J \setminus I) = 1 \).

It then reads

(S3') \( r_I^J s_I^{I \cup J} = s_J^{I \cup J} r_I^J \) for all \( j, k \notin I \) with \( j \neq k \).

This can easily be seen as follows. Put \( K = I \cap J \), \( I \setminus K = \{i_1, \ldots, i_p\} \) and \( J \setminus K = \{j_1, \ldots, j_q\} \). Then the result follows from the diagram

\[
\begin{array}{cccccc}
D_K & \rightarrow & D_{K \cup \{i_1\}} & \rightarrow & \cdots & \rightarrow & D_I \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
D_{K \cup \{j_1\}} & \rightarrow & D_{K \cup \{i_1, j_1\}} & \rightarrow & \cdots & \rightarrow & D_{I \cup \{j_1\}} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\vdots & & \vdots & & \vdots & & \vdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
D_J & \rightarrow & D_{J \cup \{i_1\}} & \rightarrow & \cdots & \rightarrow & D_{I \cup J}
\end{array}
\]

where the horizontal maps are \( r \)-maps and the vertical maps are \( s \)-maps.

**Definition 4.** An \( m \)-tuple \( T = (A, B_1, \ldots, B_m) \) of normal subgroups determines an \( m \)-cube in \( G \):

\[
I \mapsto T_I = A \big/ \prod_{i \in I} B_i.
\]

When \( I \subseteq J \), then \( \prod_{i \in J} B_i \subseteq J \) and \( 1_A \) induces a group homomorphism \( r_I^J : T_I \rightarrow T_J \). This \( m \)-cube is said to be *induced* by the \( m \)-tuple \( T \). Similarly for an \( m \)-tuple of ideals in a ring.

**Proposition 2.1.** Let \( D : C_m \rightarrow D \) be an \( m \)-cube in \( G \), which is split as an \( m \)-cube in \( S \). Then \( D \) is induced by a normal \( m \)-tuple of normal subgroups of \( D_\emptyset \).

**Proof.** For \( i \in m \) put

\[
B_i = \text{Ker}\left( r_I^i : D_\emptyset \rightarrow D_{\{i\}} \right).
\]

We will first show that the cube is induced by the \( m \)-tuple \( (D_\emptyset, B_1, \ldots, B_m) \). Since the cube splits in \( S \), the homomorphisms \( D_\emptyset \rightarrow D_I \) are surjective. To show that for each \( I \subseteq m \)

\[
\text{Ker}(D_\emptyset \rightarrow D_I) = \prod_{i \in I} B_i.
\]
This can be done by induction on $\#(I)$. For $\#(I) = 0$ it is trivial. Let $\#(I) > 0$. Choose $k \in I$. By induction hypothesis
\[ \ker(D_\emptyset \rightarrow D_{I \setminus \{k\}}) = \prod_{i \in I \setminus \{k\}} B_i. \]

Since the cube splits in $\mathcal{S}$ we have a commutative diagram with exact rows and columns:

\begin{align*}
1 & \rightarrow B_k \cap \prod_{i \in I \setminus \{k\}} B_i & \rightarrow & \rightarrow B_k & \rightarrow & \ker\left(r_{I \setminus \{k\}}^I\right) & \rightarrow & 1 \\
& \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & \rightarrow \prod_{i \in I \setminus \{k\}} B_i & \rightarrow & D_\emptyset & \rightarrow & D_{I \setminus \{k\}} & \rightarrow & 1 \\
& \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & \rightarrow \ker\left(r_I^I\right) & \rightarrow & D_{\{k\}} & \rightarrow & D_I & \rightarrow & 1 \\
& \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\
1 & & & 1 & & & 1 & \\
\end{align*}

Hence
\[ \ker(r_I^I)/B_k \cong \prod_{i \in I \setminus \{k\}} B_i \big/ \left( B_k \cap \prod_{i \in I \setminus \{k\}} B_i \right) \cong \prod_{i \in I} (B_i/B_k), \]
and therefore,
\[ \ker(r_I^I) = \prod_{i \in I} B_i. \]

For the normality of the $m$-tuple let $I, J \subseteq \underline{m}$ and consider the commutative square
\[
\begin{array}{ccc}
D_\emptyset & \xrightarrow{(r_I^I)_{(i)}} & D_\emptyset/B_i \\
\downarrow r_{(i)}^I & & \downarrow (r_{J \cup \{i\}}^I)_{(i)} \\
\prod_{j \in J} B_j & \xrightarrow{(r_{J \cup \{i\}}^I)_{(i)}} & \prod_{j \in J \cup \{i\}} B_j
\end{array}
\]

Since the $m$-cube is split in $\mathcal{S}$ the vertical homomorphisms have compatible sections in $\mathcal{S}$. So $r_I^I$ induces a surjective homomorphism on the kernels of the horizontal homomorphisms. This holds for all $I, J \subseteq \underline{m}$. Therefore, the $m$-tuple $(D_\emptyset, B_1, \ldots, B_m)$ is normal. \(\square\)

For the Abelian case we also prove the converse.

**Proposition 2.2.** Let $T = (A, B_1, \ldots, B_m)$ be a normal $m$-tuple of subgroups of an Abelian group $A$. Then the induced $m$-cube is split in the category $\mathcal{S}$.  

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Proof. By taking kernels of the surjective homomorphisms in the induced $m$-cube it can be extended to a diagram of $3^n$ Abelian groups. We will give a detailed description of this diagram and show how a splitting of the cube can be obtained from it.

For each pair $(I, J)$ of disjoint subsets of $m$ define

$$C^I_J = \bigcap_{i \in I} B_i + \sum_{j \in J} B_j / \sum_{j \in J} B_j.$$  

Then for each such pair $(I, J)$ and each $k \notin I \cup J$ we have a surjective homomorphism $C^I_J \to C^I_{J \cup \{k\}}$ induced by $r^I_J \circ r^I_{J \cup \{k\}}: A_J \to A_{J \cup \{k\}}$, where we use the notation

$$A_J = A \bigcap_{j \in J} B_j.$$  

Thus $A_J = C^0_J$. The kernel of the surjective homomorphism $C^I_J \to C^I_{J \cup \{k\}}$ is

$$\left(\bigcap_{i \in I} B_i + \sum_{j \in J} B_j\right) \cap \left(\bigcap_{i \in I} B_i + \sum_{j \in J} B_j + B_k + \sum_{j \in J} B_j\right) / B_k + \sum_{j \in J} B_j.$$  

We have the inclusions

$$\bigcap_{i \in I \cup \{k\}} B_i + \sum_{j \in J} B_j \subseteq \left(\bigcap_{i \in I} B_i + \sum_{j \in J} B_j\right) \cap \left(\bigcap_{i \in I} B_i + \sum_{j \in J} B_j + B_k + \sum_{j \in J} B_j\right) \subseteq \bigcap_{i \in I \cup \{k\}} \left(\bigcap_{i \in I} B_i + \sum_{j \in J} B_j\right).$$  

By normality these groups are equal, so we have a short exact sequence

$$0 \to C^I_{J \cup \{k\}} \to C^I_J \to C^I_{J \cup \{k\}} \to 0.$$  

For each pair $(I, J)$ of disjoint subsets of $m$ satisfying $I \cup J = m$ choose a section

$$t^I_J: C^I_J \to C^I_{\emptyset}(\subseteq C^0_{\emptyset} = A)$$  

of the map $C^I_{\emptyset} \to C^I_J$ induced by $r^I_{\emptyset}: A \to A_J$ and satisfying $t^I_J(0) = 0$. Next define maps $t^I_J: C^I_J \to C^I_{\emptyset}$ for every disjoint pair $(I, J)$ using induction to the number of elements of the complement of $I \cup J$. So, let $(I, J)$ be a disjoint pair of subsets of $m$ with $\#(I \cup J) = n < m$ and assume that sections $t^K_L: C^K_{\emptyset} \to C^K_{\emptyset}$ have already been defined for pairs $(K, L)$ with $K \cup L$ having more than $n$ elements.

Choose $k \in m \setminus (I \cup J)$. Let $x \in C^I_J$, then for $y = r^I_{\emptyset} t^I_{J \cup \{k\}} r^I_{J \cup \{k\}}(x)$ we have

$$r^I_{J \cup \{k\}}(y) = r^I_{J \cup \{k\}} t^I_{J \cup \{k\}} r^I_{J \cup \{k\}}(x) = r^I_{J \cup \{k\}}(x),$$  

so, $x - y \in C^I_{J \cup \{k\}}$. Now define $t^I_J$ by

$$t^I_J(x) = t^I_{J \cup \{k\}}(x - y) + t^I_{J \cup \{k\}} r^I_{J \cup \{k\}}(y).$$  

It easily verified that this map is a section of $r: C^I_{\emptyset} \to C^I_J$. Furthermore it is independent of the choice of $k$: if also $l \notin I \cup J$, then in both cases the image of
an $x \in C_I^J$ under $t_I^J$ is determined in the same way by the images of the same elements in the following four groups

$$
\begin{array}{cccc}
C_I^{(l,k)} & C_I^{(l)} & C_{J \cup \{k\}}^{(l)} & C_J^{(k)} \\
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & C_I^{(l,k)} & C_I^{(l)} & C_{J \cup \{k\}}^{(l)} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & C_I^{(k)} & C_I^{(k)} & C_{J \cup \{k\}}^{(k)} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & C_J^{(l,k)} & C_J^{(l)} & C_J^{(k)} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0
\end{array}
$$

Thus we obtain a splitting of the cube, where the sections $s_I^J$ of the homomorphisms $r_I^J$, where $I \subseteq J$, are the maps $r_I^J t_I^J$. In particular, condition (S3') follows from the above diagram for $I = \emptyset$.

\[ \square \]

3 Operations on normal $m$-tuples of ideals

By $\mathcal{R}_m$ we will denote the category of all normal $m$-tuples of ideals. Such an $m$-tuple is denoted as $(R, a_1, \ldots, a_m)$, where $R$ is a ring and $a_1, \ldots, a_m$ are ideals of $R$. A morphism $\phi: (R, a_1, \ldots, a_m) \to (S, b_1, \ldots, b_m)$ is just a ringhomomorphism $\phi: R \to S$ satisfying $\phi(a_i) \subseteq b_i$ for all $i \in m$.

The following notations will simplify notations for long exact sequences of multirelative $K$-theory. Another advantage will be that they are useful to indicate functoriality properties.

For each $m \geq 1$ the functor $D: \mathcal{R}_m \to \mathcal{R}_{m-1}$ is the functor that deletes the last ideal:

$$D(R, a_1, \ldots, a_m) = (R, a_1, \ldots, a_{m-1})$$

and which has no effect on morphisms.

For each $m \geq 1$ the functor $M: \mathcal{R}_m \to \mathcal{R}_{m-1}$ is the functor that deletes the last ideal and that takes the ring and the other ideals modulo this ideal:

$$M(R, a_1, \ldots, a_m) = (R/\overline{a_m}, \overline{a_1}, \ldots, \overline{a_{m-1}}),$$

where $\overline{a}_j = a_j + a_i/a_i$, and which maps a morphism to the induced morphism.

A functor morphism $\phi: D \to M$ of the functors $D, M: \mathcal{R}_m \to \mathcal{R}_{m-1}$ is defined as follows: let $A = (R, a_1, \ldots, a_m)$, then $\phi_A: D(A) \to M(A)$ is the canonical ringhomomorphism $R \to R/\overline{a_m}$.
Every $A \in \mathcal{R}_m$ has an *underlying ideal* $I(A)$, which is defined as the intersection of the $m$ ideals in $A$: when $A = (R, a_1, \ldots, a_m)$, then

$$I(A) = a_1 \cap \cdots \cap a_m.$$ 

Thus defined, $I(A)$ is functorial in $A$.

### 4 Higher K-theory of rings

In [3] the definition of higher $K$-groups is as follows. Let $R \in \mathcal{R}$. Choose a simplicial ring $\mathbf{R}$ with an augmentation $\varepsilon : \mathbf{R} \to R$ such that

- $\mathbf{R}$ is aspherical, i.e. $\pi_n(\mathbf{R}) = 0$ for all $n \geq 1$,
- $\mathbf{R}_m$ is free for all $m \geq 0$, say $\mathbf{R}_m$ is free on a set $X_m$ of generators,
- the sets $X_m$ of free generators are stable under degeneracies: $s_j(X_m) \subseteq X_{m+1}$ for all $m \geq 0$,
- the augmentation $\varepsilon$ induces an isomorphism $\pi_0(\mathbf{R}) \cong R$.

Then for $n \geq 3$ the group $K_n(R)$ is defined as the $(n-2)$nd homotopy group of the simplicial group $GL(\mathbf{R})$, and the groups $K_1(R)$ and $K_2(R)$ are given by the exactness of

$$0 \to K_2(R) \to \pi_0(GL\mathbf{R}) \to GL(R) \to K_1(R) \to 0.$$ 

The groups $K_n(R)$ for $n \geq 3$ are Abelian because $GL(\mathbf{R})$ is a simplicial group.

The group $K_1(R)$ is Abelian since it is the cokernel of $GL(\mathbf{R}_0) \to GL(R)$, and $K_2(R)$ is Abelian because it is the cokernel of $GL(\mathbf{R}_1) \to GL(\mathbf{Z}_0)$, where $\mathbf{Z}_0 = \{ (x_0, x_1) \mid \varepsilon(x_0) = \varepsilon(x_1) \}$. In [3] it is shown using a comparison theorem that the higher $K$-groups are thus well-defined and that they are actually functors.

For the purpose of this paper we will confine to a functorial resolution $\mathbf{Fr}(R)$ of a ring $R$, which we now describe. Let $F : S \to \mathcal{R}$ the free ring functor and let $U : \mathcal{R} \to S$ be the underlying set functor, then the functor $FU : \mathcal{R} \to \mathcal{R}$ together with the obvious functor morphisms $\nu : FU \to (FU)^2$ and $\eta : FU \to I$ is a cotriple. Put

$$\mathbf{Fr}_n = (FU)^{n+1}.$$ 

Face and degeneracy morphisms are given by

$$d_i = (FU)^i \eta(FU)^{n-1-i} \quad \text{and} \quad s_j = (FU)^i \nu(FU)^{n-1-i}.$$ 

The augmentation is then given by $\eta$.

A property of this functorial resolution is that, when applied to a surjective ringhomomorphism $R \to S$, it gives a dimensionwise surjective homomorphism $\mathbf{Fr}R \to \mathbf{Fr}S$ of simplicial rings, and since the ringhomomorphisms are dimensionwise split it also gives a surjective simplicial grouphomomorphism $GL(\mathbf{Fr}R) \to GL(\mathbf{Fr}S)$. This is often convenient when considering homotopy fibres, because surjective simplicial grouphomomorphisms are fibrations themselves. So instead of taking a homotopy fibre one just takes a fibre, i.e. the kernel of the simplicial group homomorphism.
5 Cubes in a simplicial group

Let $A$ be a simplicial group with augmentation $d_0: A_0 \to A$. It is a contravariant functor $A: \Omega_+^{op} \to \mathcal{G}$ from the category $\Omega_+$ of finite ordered sets

$$[n] = \{0, \ldots, n\} \quad (n \geq -1)$$

(where $[-1] = \emptyset$) and monotone (= order preserving) maps to the category of groups. (Here we use the notation $A_{-1} = A$.) We will show that $A$ determines an $m$-cube of groups for every nonnegative integer $m$. Instead of the ordered set of subsets of $\mathbb{N}$ for the description of an $m$-cube the ordered set of subsets of $[m]$ will be used for this purpose.

Let $\Omega(m)$ be the category of injective monotone maps

$$\alpha: [k] \to [m - 1].$$

A morphism from $\alpha: [k] \to [m - 1]$ to $\beta: [l] \to [m - 1]$ is a monotone map $\gamma: [k] \to [l]$ such that $\beta \gamma = \alpha$. It exists if and only if $\text{Im}(\alpha) \subseteq \text{Im}(\beta)$, and it is unique if it exists.

For each $I \subseteq [m - 1]$ there is a unique injective monotone map

$$\alpha_I: [k] \to [m - 1],$$

where $k = m - 1 - \#(I)$ and $\text{Im}(\alpha_I) = [m - 1] \setminus I$. If $I \subseteq J \subseteq [m - 1]$, then $\text{Im} (\alpha_I) \supseteq \text{Im} (\alpha_J)$, so then there is a unique $\gamma^j_I: \alpha_J \to \alpha_I$,

i.e. a monotone $\gamma^j_I: [m - 1 - \#(J)] \to [m - 1 - \#(I)]$ such that $\alpha_I \gamma^j_I = \alpha_J$.

**Definition 5.** Let $A$ be an augmented simplicial group and let $m$ be a nonnegative integer. Then the $m$-cube of $A$ is the $m$-cube $A(m): C_m \to \mathcal{G}$ with

$$\begin{cases} A(m)_I = A_{[m - 1 - \#(I)]} & \text{for all } I \subseteq [m - 1], \\ r^I_J = A(\gamma^j_I): A(m)_I \to A(m)_J & \text{for all } I \subseteq J \subseteq [m - 1]. \end{cases}$$

**Lemma 5.1.** Let the augmentation $d_0: A_0 \to A_{-1}$ induce a surjective homomorphism $\pi_0(A) \to A_{-1}$. Then for all integers $i, j, m$ such that $0 \leq j < i \leq m$

$$d_i^m(\text{Ker}(d_j^m)) = \text{Ker}(d_j^{m-1}).$$

**Proof.** Let $x \in \text{Ker}(d_j^m)$. Then, since $i > j$, $d_j d_i(x) = d_{i-1}d_j(x) = 1$. So $d_i(\text{Ker}(d_j)) \subseteq \text{Ker}(d_j)$. Now, let $y \in \text{Ker}(d_j^{m-1})$. There is an $x \in A_m$ such that $d_j(x) = 1$ and $d_i(x) = y$. For $m > 1$ this is the case because a simplicial group is a Kan-complex, while for $m = 1$ it follows from the condition on the augmentation. \qed
Proposition 5.1. Let $A$ be a simplicial group with an augmentation $d_0 : A \to A$ that induces an isomorphism $\pi_0(A) \to A$. Then for all $m \geq 1$ the $m$-cube $A(m)$ is induced by the $m$-tuple

$$(A_{m-1}, \text{Ker}(d_0), \ldots, \text{Ker}(d_{m-1})).$$

Proof. All face maps are surjective, so it remains to show that for all $J \subseteq [m-1]$

$$\text{Ker}(r^0_J) = \prod_{j \in J} \text{Ker}(d_j^{(m-1)}).$$

For $J = \emptyset$ this is trivially true. Let $J$ be nonempty and proceed by induction. Let $x \in \text{Ker}(r^0_J)$. Let $k \in J$ be maximal. Then $r^0_{\{k\}}(x) = d_k(x) \in \text{Ker}(r^0_{\{k\}})$.

By induction this group is equal to $\prod_{j \in J'} \text{Ker}(d_j^{(m-2)})$, where $J' = J \setminus \{k\}$. (Here we used the maximality of $k$ in $J$ and the same result for the $(m-1)$-cube $A(m-1)$.) By the lemma we have

$$d_k\left(\prod_{j \in J'} \text{Ker}(d_j^{(m-1)})\right) = \prod_{j \in J'} \text{Ker}(d_j^{(m-2)}).$$

Choose $y \in \prod_{j \in J'} \text{Ker}(d_j^{(m-1)})$ such that $d_k(y) = d_k(x)$. Then $xy^{-1} \in \text{Ker}(d_k)$. It follows that

$$\text{Ker}(r^0_J) \subseteq \prod_{j \in J} \text{Ker}(d_j^{(m-1)}).$$

For the other inclusion note that $d_j = r^0_{\{j\}}$ and

$$r^0_J r^0_{\{j\}} = r^0_J.$$

□

Proposition 5.2. Let $A$ be as in Proposition 5.1 and assume moreover that $A$ is aspherical. Then the $m$-tuple

$$(A_{m-1}, \text{Ker}(d_0), \ldots, \text{Ker}(d_{m-1}))$$

is normal.

Proof. The edges of the $m$-cube are the face maps. Normality means that these maps preserve intersections of (the images of) the normal subgroups $\text{Ker}(d_0), \ldots, \text{Ker}(d_{m-1})$. By induction it suffices to show this for the face maps $d_i^{(m-1)}$. Let $J \subseteq [m-1]$. Then to show that

$$d_i\left(\bigcap_{j \in J} \text{Ker}(d_j)\right) = \bigcap_{j \in J} d_i(\text{Ker}(d_j)).$$

for $i \notin J$. The inclusion of the left hand side in the right hand side is trivial. So let $x \in \bigcap_{j \in J} d_i(\text{Ker}(d_j))$. Then for $j \in J$ there is an $y_j \in \text{Ker}(d_j)$ such that $x = d_i(y_j)$. For $j < i$ it follows that $d_j(x) = d_j d_i(y_j) = d_{i-1} d_j(x_j) = 1$. Similarly for $j > i$ we have $d_{j-1}(x) = 1$. So, since a simplicial group is a Kan-complex and for $J = [m-1]$ since $A$ is aspherical, there is a $y \in A_{m-1}$ such that $d_j(y) = 1$ for all $j \in J$ and $d_i(y) = x$. This shows that $x \in d_i\left(\bigcap_{j \in J} \text{Ker}(d_j)\right)$. □
6 Multirelative $K$-theory

A normal $m$-tuple of ideals $A = (R, a_1, \ldots, a_m)$ induces an $m$-cube in $\mathcal{R}$

$$A: I \to R / \sum_{i \in I} a_i,$$

which by Proposition 2.2 is split in $\mathcal{S}$. Application of $\text{Fr}$ to this $m$-cube gives an $m$-cube of simplicial rings which is dimensionwise split in $\mathcal{R}$. Put

$$\text{Fr}(R, a_i) := \text{Ker}(\text{Fr}(R) \to \text{Fr}(R/a_i)).$$

This is a simplicial ideal. The $m$-cube is then induced by the $m$-tuple

$$(\text{Fr}(R), \text{Fr}(R, a_1), \ldots, \text{Fr}(R, a_m)),$$

of simplicial ideals, an object of the category $s\mathcal{R}_m$ of normal $m$-tuples of simplicial ideals. We also define the simplicial ideal

$$\text{Fr}(R, a_1, \ldots, a_m) := \bigcap_{i=1}^m \text{Fr}(R, a_i).$$

Application of $GL$ gives an $m$-cube of simplicial groups, which is dimensionwise split in $\mathcal{G}$. This $m$-cube is induced by the $m$-tuple

$$(GL\text{Fr}(R), GL\text{Fr}(R, a_1), \ldots, GL\text{Fr}(R, a_m))$$

of simplicial normal subgroups. For $n \geq 3$ we define multirelative $K_n$ by

$$K_n(R, a_1, \ldots, a_m) := \pi_{n-1}(GL\text{Fr}(R, a_1, \ldots, a_m)).$$

Multirelative $K_2$ and $K_1$ are then given by the exactness of

$$0 \to K_2(R, a_1, \ldots, a_m) \to \pi_0(GL\text{Fr}(R, a_1, \ldots, a_m)) \to GL(a_1 \cap \cdots \cap a_m) \to K_1(R, a_1, \ldots, a_m) \to 0.$$

These multirelative $K_1$ and $K_2$ are Abelian groups for the same reason as in the absolute case.

Now let $A \in \mathcal{R}_m$ with $m \geq 1$. Then $\phi_*: GL\text{Fr}(DA) \to GL\text{Fr}(MA)$ is a fibration with fibre $GL\text{Fr}(A)$. The long exact sequence of homotopy groups is a long exact sequence of multirelative $K$-groups which can easily be extended to include multirelative $K_2$ and $K_1$.

**Proposition 6.1.** Let $A \in \mathcal{R}_m$ with $m \geq 1$. Then we have a functorial exact sequence

$$\cdots \to K_n(A) \to K_n(DA) \to K_n(MA) \to K_{n-1}(A) \to \cdots \to K_1(MA).$$

□
The connecting map $K_n(MA) \to K_{n-1}(A)$ will be denoted by $\delta$ and the map $K_n(A) \to K_n(DA)$ by $\iota$. To put it in an even more functorial way, we have an exact sequence of functors and functor morphisms

$$\cdots \to K_n \overset{\iota}{\to} K_n D \overset{K_n(\phi)}{\to} K_n M \overset{\delta}{\to} K_{n-1} \to \cdots \to K_1 M.$$ 

In the remaining part of this section multirelative $K_0$ is defined and the long exact sequence for multirelative $K$-theory is extended with multirelative $K_0$-groups.

**Definition 6.** For a normal $m$-tuple $A$ of ideals we define

$$K_0(A) = K_0(I A).$$

Thus defined, $K_0$ is a functor from $\mathcal{R}_m$ to $\mathcal{A}$.

For $m = 1$ we take the long exact sequence to be the long exact sequence of an ideal in a ring. Now assume that $m \geq 1$ and that we have an extended long exact sequence

$$\cdots \to K_1 D \to K_1 M \to K_0 \to K_0 D \to K_0 M$$

of functors $\mathcal{R}_m \to \mathcal{A}$. We will show that there is also such a sequence of functors $\mathcal{R}_{m+1} \to \mathcal{A}$.

Let $A = (R, a_1, \ldots, a_{m+1}) \in \mathcal{R}_{m+1}$. Put $b = IA = \bigcap_{i=1}^{m+1} a_i$. We have exact sequences for the following $m$-tuples of ideals

$$B = DA = (R, a_1, \ldots, a_m),$$

$$\overline{B} = (R/b, a_1/b, \ldots, a_m/b)$$

and

$$(R, a_1, \ldots, a_m, b).$$

These $m$-tuples are normal and their $K$-groups fit into a commutative diagram.
Let the dashed arrow be the composition \( K_1(B) \to K_1(DB) \to K_0(b) \). By an easy diagram chase we see that the sequence with the dashed arrow is exact as well. The identity on \( R \) is a morphism
\[
(R, a_1, \ldots, a_m, b) \to A
\]
in \( R_{m+1} \). So we have a commutative diagram with exact rows:
\[
\begin{array}{ccc}
K_1(R, a_1, \ldots, a_m, b) & \longrightarrow & K_1(B) \\
\downarrow & & \downarrow 1 \\
K_1(A) & \longrightarrow & K_1(DA) \longrightarrow K_1(MA).
\end{array}
\]
It now suffices to show that the morphism \( \alpha \) in this diagram is an isomorphism. The \((m+1)\)-tuple \((R/b, a_1/b, \ldots, a_{m+1}/b)\) induces an exact sequence
\[
K_1(R/b, a_1/b, \ldots, a_{m+1}/b) \to K_1(B) \to K_1(MA).
\]
The group \( K_1(R/b, a_1/b, \ldots, a_{m+1}/b) \) is a quotient of \( GL((a_1/b) \cap \cdots \cap (a_{m+1}/b)) = \{1\} \), so \( \alpha \) is injective. On the other hand, since the \((m+1)\)-tuple \( A \) of ideals is normal, the identity on \( R \) induces an isomorphism \( I(B) \to I(MA) \) and hence also an isomorphism
\[
GL(I(B)) \simto GL(I(MA)).
\]
Since the multirelative \( K_1 \) is a quotient of the general linear group of the underlying ideal, the map \( \alpha \) is surjective. This proves:

**Theorem 1.** Let \( A \in R_m \) for \( m \geq 1 \). Then we have a functorial exact sequence
\[
\cdots \to K_n(A) \to K_n(DA) \to K_n(MA) \to K_{n-1}(A) \to \cdots \to K_0(MA).
\]

\[\square\]

### 7 Axioms for multirelative \( K \)-theory

It will be shown in this section that an axiomatic approach to multirelative \( K \)-theory is possible. We take some of the properties of multirelative \( K \)-groups as axioms and show that they determine all of multirelative \( K \)-theory.

**Axioms**

**Multirelative \( K \)-theory** consists of functors
\[
K_n : R_m \to A \quad \text{for} \ m \text{ and } n \text{ integers } \geq 0,
\]
morphisms
\[
\delta : K_{n+1}M \to K_n \quad \text{(for } m \text{ and } n \text{ integers } \geq 0)\]
of functors \( R_{m+1} \to A \) and morphisms
\[
t : K_n \to K_n D \quad \text{(for } m \text{ and } n \text{ integers } \geq 0)\]
of functors \( R_{m+1} \to A \), such that
the following sequence is an exact sequence of functors $\mathcal{R}_{m+1} \to \mathcal{A}$ for all non-negative integers $m$ and $n$

$$K_{n+1} D \xrightarrow{K_{n+1} \phi} K_{n+1} M \xrightarrow{\delta} K_n \xrightarrow{i} K_n D \xrightarrow{K_n \phi} K_n M.$$ 

(MK2) $K_n(R) = 0$ for all $n \geq 0$ and all free associative non-unital rings $R$,

(MK3) $K_0(A) = K_0(IA)$ for all $A \in \mathcal{R}_m$ for all $m$.

Loosely speaking, the multirelative $K$-groups are only defined for normal $m$-tuples of ideals and they fit into exact sequences the way one can expect, the (absolute) $K$-groups of free non-unital rings are trivial and the multirelative $K_0$ is just the Grothendieck group of the intersection of the ideals.

Let $(R, a_1, \ldots, a_m)$ be a normal $m$-tuple of ideals. It induces an $m$-cube

$$I \to R_I = R \bigg/ \sum_{i \in I} a_i,$$

which is split in $\mathcal{S}$. Application of $\Fr$ gives an $m$-cube

$$I \to \Fr(R_I)$$

of aspherical simplicial rings, which is dimensionwise split in $\mathcal{R}$.

**Proposition 7.1.** Let $m$ and $n$ be positive integers. Then the $(m + n)$-tuple

$$\left( \Fr(R)_{n-1}, \Fr(R, a_1)_{n-1}, \ldots, \Fr(R, a_m), \Ker(d_0^{(n-1)}), \ldots, \Ker(d_{n-1}^{(n-1)}) \right)$$

is normal.

**Proof.** First we show that the induced $(m + n)$-cube is

$$(I_1, I_2) \to \Fr(R_{I_1})_{n-1} \#(I_2),$$

where the cube is indexed by pairs of subsets of $m$ and $[n-1]$. This set of pairs is ordered by componentwise inclusion:

$$(I_1, I_2) \leq (J_1, J_2) \iff I_1 \subseteq J_1 \quad \text{and} \quad I_2 \subseteq J_2.$$

The homomorphism

$$\Fr(R)_{n-1} \to \Fr(R_{I_1})_{n-1} \#(I_2)$$

is the composition

$$\Fr(R)_{n-1} \to \Fr(R_{I_1})_{n-1} \to \Fr(R_{I_1})_{n-1} \#(I_2),$$

the first map being induced by $\emptyset \subseteq I_1$ and the second by $[n - 1] \setminus I_2 \subseteq [n - 1]$. Both homomorphisms are surjective. The first one has kernel $\bigcap_{i \in I_1} \Fr(R, a_i)_{(n-1)}$ and the second one $\bigcap_{i \notin I_2} \Ker(d_i)$, where the $d_i$ are face maps of $\Fr(R_{I_1})$. Since
\( \text{Fr}(R) \) and \( \text{Fr}(R_{I_j}) \) are both aspherical, elements of the second kernel can be lifted to elements of \( \bigcap_{i \notin I_2} \ker(d_i) \), where the \( d_i \) are face maps of \( \text{Fr}(R) \).

For the \((m + n)\)-tuple to be normal it suffices that the intersections of the images of the \( m + n \) ideals are preserved under the maps on the edges of the induced \((m + n)\)-cube. These are the homomorphisms

\[
\text{Fr}(R_J)l \to \text{Fr}(R_{J \cup \{k\}})l,
\]

where \( J \subseteq m, k \in m \setminus J \) and \( l \in [n - 1] \), and also the face maps

\[
d_i: \text{Fr}(R_J)_p \to \text{Fr}(R_J)_{p-1},
\]

where \( p \in [n - 1] \) and \( 0 \leq i \leq p \). Without loss of generality we may assume that \( J = m, l = n - 1 \) and \( p = n - 1 \).

Because the \( m \)-cube \( J \hookrightarrow \text{Fr}(R_J) \) is dimensionwise split we have short exact sequences

\[
0 \to \bigcap_{i \in I \cup \{k\}} \text{Fr}(R, a_i) \to \bigcap_{i \in I} \text{Fr}(R, a_i) \to \bigcap_{i \in I} \text{Fr}(R/a_k, \bar{a}_i) \to 0
\]

of aspherical simplicial rings. It follows that for all \( J \subseteq [n - 1] \) we have

\[
\bigcap_{i \in I} \text{Fr}(R, a_i)_{n-1} \cap \bigcap_{j \in J} \ker(d_j) = \bigcap_{j \in J} \ker(d'_j),
\]

where the \( d'_j \) are the face maps of \( \bigcap_{i \in I} \text{Fr}(R, a_i) \). Under \( \text{Fr}(R) \to \text{Fr}(R/a_k) \) this maps onto

\[
\bigcap_{j \in J} \ker(d''_j) = \bigcap_{i \in I} \text{Fr}(R/a_k, \bar{a}_i)_{n-1} \cap \bigcap_{j \in J} \ker(d''_j),
\]

where the \( d''_j \) are the face maps of \( \bigcap_{i \in I} \text{Fr}(R/a_k, \bar{a}_i) \) and \( d''_j \) those of \( \bigcap_{i \in I} \text{Fr}(R/a_k) \).

Because the simplicial rings \( \bigcap_{i \in I} \text{Fr}(R, a_i) \) are aspherical also the face maps \( d_i: \text{Fr}(R)_{n-1} \to \text{Fr}(R)_{n-2} \) preserve intersections

\[
\bigcap_{i \in I} \text{Fr}(R, a_i)_{n-1} \cap \bigcap_{j \in J} \ker(d_j).
\]

\[ \square \]

**Theorem 2.** Let \( A = (R, a_1, \ldots, a_m) \in \mathcal{R} \). Then for all \( n \geq 0 \) it follows from the axioms (MK1) and (MK2) that \( K_n(A) \) is naturally isomorphic to \( K_0 \) of the following object of \( \mathcal{R}_{m+n} \):

\[
(\text{Fr}(R)_{n-1}, \text{Fr}(R, a_1)_{n-1}, \ldots, \text{Fr}(R, a_m)_{n-1}, \ker(d_0), \ldots, \ker(d_{n-1})).
\]

From axiom (MK3) it then follows that \( K_n(A) \) is determined. So (MK1), (MK2) and (MK3) can be taken as axioms for the (multirelative) \( K \)-theory of rings.

**Proof.** The proof follows from the following three lemmas.\[ \square \]
Lemma 7.1. Let \( m \geq -1 \) and \( q, n \geq 0 \). Then

\[ K_q(\text{Fr}(R)_n, \text{Fr}(R, a_1)_n, \ldots, \text{Fr}(R, a_m)_n) = 0. \]

**Proof.** Since for \( m \geq 0 \) the \((m - 1)\)-tuples \( D(A) \) and \( M(A) \) are of the same type, the proof reduces by (MK1) to the case \( m = -1 \). For \( m = -1 \) the lemma follows from (MK2).

Put

\[ A[n, p] = (\text{Fr}(R)_n, \text{Fr}(R, a_1)_n, \ldots, \text{Fr}(R, a_m)_n, \text{Ker}(d_0), \ldots, \text{Ker}(d_p)), \]

where \(-1 \leq p \leq n\). It is an object of \( R_{m+p+1} \).

Lemma 7.2. For all \( p < n \) and all \( q > 0 \) we have

\[ K_q(A[n, p]) = 0. \]

**Proof.** For \( p \geq 0 \) we have

\[ D(A[n, p]) = A[n, p - 1] \quad \text{and} \quad M(A[n, p]) = A[n - 1, p - 1]. \]

By (MK1) the problem reduces to the case \( p = -1 \), which is covered by the previous lemma.

Lemma 7.3. For all \( q, n \geq 0 \) we have

\[ K_q(A[n, n]) \cong K_{q+1}(A[n - 1, n - 1]). \]

**Proof.** This follows from (MK1) and the previous lemma.

From this lemma the theorem follows:

\[ K_n(A) = K_n(A[-1, -1]) \cong K_{n-1}(A[0, 0]) \cong \cdots \cong K_0(A[n - 1, n - 1]). \]

**References**


