Multirelative $K$-theory and axioms for the $K$-theory of rings

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Abstract

$K$-groups are defined for a special type of $m$-tuples of ideals in a ring.
It is shown that some of the properties of this multirelative $K$-theory characterize the $K$-theory of rings.

Introduction

Multirelative $K$-groups $K_n(R, a_1, \ldots, a_m)$ of an $m$-tuple $(a_1, \ldots, a_m)$ of ideals of a ring $R$ are recently used to derive properties of the absolute $K$-groups, e.g. by Levine [4] and by Bloch and Lichtenbaum [1]. Here it is shown how $K$-theory as defined in [3] can easily be extended to the multirelative case and that some of its properties can be taken as axioms for the $K$-theory of rings. Special types of $m$-tuples of ideals—the ‘normal’ $m$-tuples—play a crucial role. In fact we will only define multirelative $K$-groups for such $m$-tuples. The notion of normal $m$-tuple of ideals is introduced in Section 2. It already appeared in 1981 in a paper by Dayton and Weibel [2] on the $K$-theory of affine glued schemes under the name of ‘condition (CRT)’ (= Chinese Remainder Theorem).

In Section 4 we review briefly higher $K$-theory as defined in [3]. In Section 6 multirelative $K$-groups are defined, and in Section 7 it is shown that from some of their properties one can reconstruct the $K$-theory of rings.

1 Notations

In this paper ‘ring’ stands for a non-unital ring. Non-unital rings form a category which is denoted by $\mathcal{R}$.

Since the functors $GL$, $E$ and $K_1$ are product preserving functors from unital rings to groups, they can be extended to functors defined on $\mathcal{R}$ in the usual way: if $T$ is one of these functors, then put

$$T(R) := \text{Ker}(T(R^+) \to T(\mathbb{Z})),$$

where $R^+ = R \times \mathbb{Z}$ with multiplication given by

$$(r,k)(s,l) = (rs + ks + lr, kl)$$
is a ring with \((0,1)\) as unity element.

Here ‘ideal’ will always stand for ‘twosided ideal’.

By \(\mathcal{A}\) we will denote the category of Abelian groups, by \(\mathcal{G}\) the category of all groups, and by \(\mathcal{S}\) the category of sets. The category of simplicial objects in a category \(\mathcal{C}\) is denoted by \(s\mathcal{C}\).

## 2 \(m\)-cubes and normal \(m\)-tuples

In this section the notion of normality of an \(m\)-tuple of ideals is considered. Only the group structure is involved in its definition, and since we can use later a similar notion for groups instead of rings we give a more general definition. By \(m\) we will denote the set \(\{1, \ldots, m\}\).

**Definition 1.** An \(m\)-tuple \((B_1, \ldots, B_m)\) of normal subgroups of a group \(A\)—also denoted as \((A, B_1, \ldots, B_m)\)—is called normal if for all subsets \(I\) and \(J\) of \(m\)

\[
\bigcap_{i \in I} B_i \cdot \prod_{j \in J} B_j = \bigcap_{i \in I} \left( B_i \cdot \prod_{j \in J} B_j \right).
\]

The condition is trivially fulfilled when \(I \cap J = \emptyset\). In the case of Abelian groups it reads in the additive notation as

\[
\bigcap_{i \in I} B_i + \sum_{j \in J} B_j = \bigcap_{i \in I} \left( B_i + \sum_{j \in J} B_j \right).
\]

Note that in the special case of an \(m\)-tuple of ideals in a commutative ring the condition is a local one since it involves only intersections and sums of ideals.

The subsets of \(m\) are ordered by inclusion. This ordered set determines in the usual way a category \(C_m\). For every pair \((I, J)\) of subsets with \(I \subseteq J\) there is the unique morphism \(\rho^I_J\) from \(I\) to \(J\) in \(C_m\).

**Definition 2.** Let \(\mathcal{D}\) be a category. An \(m\)-cube in \(\mathcal{D}\) is a functor

\[
D: C_m \to \mathcal{D}, \quad I \mapsto D_I, \quad \rho^I_J \mapsto r^I_J.
\]

The morphisms in \(C_m\) are generated by the \(\rho^I_J\), where \(#J = #I + 1\). An \(m\)-cube in a category \(\mathcal{D}\) is a commutative diagram in \(\mathcal{D}\) having the shape of an \(m\)-dimensional cube. The edges of the cube correspond to the images of these generating morphisms.

**Definition 3.** Let \(D: C_m \to \mathcal{D}\) be an \(m\)-cube in \(\mathcal{D}\). It is said to be a split \(m\)-cube if for every pair of subsets \((I, J)\) of \(m\) satisfying \(I \subseteq J\) there is a morphism \(s^I_J: D_J \to D_I\) in \(\mathcal{D}\) such that

(S1) \(s^I_I s^K_J = s^K_K\) for all \(I \subseteq J \subseteq K\),

(S2) \(r^I_J s^I_J = 1_{D_J}\) for all \(I \subseteq J\),
(S3) \( r_{I \cup J}^I s_{I \cap J}^I = s_{I \cup J}^I r_{I \cup J}^I \) for all \( I \) and \( J \).

(Of course such a split \( m \)-cube can also be seen as a functor defined on a category which is obtained from \( C_m \) by adjoining extra morphisms \( \sigma^I_J : J \to I \).)

In condition (S3) one only needs the case where \( \#(I \setminus J) = \#(J \setminus I) = 1 \).

It then reads

(S3') \( r_{I \cup \{k\}}^I s_{I \cup \{j\}}^I = s_{I \cup \{j,k\}}^I r_{I \cup \{j,k\}}^I \) for all \( j, k \not\in I \) with \( j \neq k \).

This can easily be seen as follows. Put \( K = I \cap J \), \( I \setminus K = \{i_1, \ldots, i_p\} \) and \( J \setminus K = \{j_1, \ldots, j_q\} \). Then the result follows from the diagram

\[
\begin{array}{cccccc}
D_K & \to & D_{K \cup \{i_1\}} & \to & \cdots & \to & D_I \\
\uparrow & & \uparrow & & \uparrow & & \\
D_{K \cup \{j_1\}} & \to & D_{K \cup \{i_1,j_1\}} & \to & \cdots & \to & D_{J \cup \{j_1\}} \\
\uparrow & & \uparrow & & \uparrow & & \\
\vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \\
D_J & \to & D_{J \cup \{i_1\}} & \to & \cdots & \to & D_{J \cup J}
\end{array}
\]

where the horizontal maps are \( r \)-maps and the vertical maps are \( s \)-maps.

**Definition 4.** An \( m \)-tuple \( T = (A, B_1, \ldots, B_m) \) of normal subgroups determines an \( m \)-cube in \( G \):

\[
I \mapsto T_I = A \big/ \prod_{i \in I} B_i.
\]

When \( I \subseteq J \), then \( \prod_{i \in I} B_i \subseteq J \) and \( 1_A \) induces a group homomorphism \( r_I^J : T_I \to T_J \). This \( m \)-cube is said to be induced by the \( m \)-tuple \( T \). Similarly for an \( m \)-tuple of ideals in a ring.

**Proposition 2.1.** Let \( D : C_m \to D \) be an \( m \)-cube in \( G \), which is split as an \( m \)-cube in \( S \). Then \( D \) is induced by a normal \( m \)-tuple of normal subgroups of \( D_\emptyset \).

**Proof.** For \( i \in m \) put

\[
B_i = \text{Ker} \left( r_{\emptyset}^{\{i\}} : D_\emptyset \to D_{\{i\}} \right).
\]

We will first show that the cube is induced by the \( m \)-tuple \((D_\emptyset, B_1, \ldots, B_m)\). Since the cube splits in \( S \), the homomorphisms \( D_\emptyset \to D_I \) are surjective. To show that for each \( I \subseteq m \)

\[
\text{Ker}(D_\emptyset \to D_I) = \prod_{i \in I} B_i.
\]
This can be done by induction on \(#(I)\). For \(#(I) = 0\) it is trivial. Let \(#(I) > 0\).
Choose \(k \in I\). By induction hypothesis

\[
\text{Ker}(D_\emptyset \to D_{I \setminus \{k\}}) = \prod_{i \in I \setminus \{k\}} B_i.
\]

Since the cube splits in \(S\) we have a commutative diagram with exact rows and columns:

\[
\begin{array}{cccc}
1 & 1 & 1 \\
\downarrow & \downarrow & \downarrow \\
1 & B_k \cap \prod_{i \in I \setminus \{k\}} B_i & B_k & \text{Ker}(r^I_{I \setminus \{k\}}) & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
1 & \prod_{i \in I \setminus \{k\}} B_i & D_\emptyset & D_{I \setminus \{k\}} & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
1 & \text{Ker}(r^I_I) & D_k & D_I & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
1 & 1 & 1 & & \\
\end{array}
\]

Hence

\[
\text{Ker}(r^0_I)/B_k \cong \prod_{i \in I \setminus \{k\}} B_i \bigg/ \left( B_k \cap \prod_{i \in I \setminus \{k\}} B_i \right) \cong \prod_{i \in I}(B_i/B_k),
\]

and therefore,

\[
\text{Ker}(r^0_I) = \prod_{i \in I} B_i.
\]

For the normality of the \(m\)-tuple let \(I, J \subseteq \underline{m}\) and consider the commutative square

\[
\begin{array}{ccc}
D_\emptyset & \xrightarrow{(r^0_I)_{(i)}} & \prod_{i \in I} D_\emptyset / B_i \\
\downarrow r^0_j & & \downarrow (r^0_{\cup I})_{(i)} \\
\prod_{J \subseteq \underline{m}} B_j & \xrightarrow{r^0_J}_{(i)} & \prod_{J \subseteq \underline{m}} B_j
\end{array}
\]

Since the \(m\)-cube is split in \(S\) the vertical homomorphisms have compatible sections in \(S\). So \(r^0_I\) induces a surjective homomorphism on the kernels of the horizontal homomorphisms. This holds for all \(I, J \subseteq \underline{m}\). Therefore, the \(m\)-tuple \((D_\emptyset, B_1, \ldots, B_m)\) is normal. 

For the Abelian case we also prove the converse.

**Proposition 2.2.** Let \(T = (A, B_1, \ldots, B_m)\) be a normal \(m\)-tuple of subgroups of an Abelian group \(A\). Then the induced \(m\)-cube is split in the category \(S\).
Proof. By taking kernels of the surjective homomorphisms in the induced $m$-cube it can be extended to a diagram of $3^n$ Abelian groups. We will give a detailed description of this diagram and show how a splitting of the cube can be obtained from it.

For each pair $(I, J)$ of disjoint subsets of $m$ define

$$C'_I = \bigcap_{i \in I} B_i + \sum_{j \in J} B_j \bigg/ \sum_{j \in J} B_j.$$ 

Then for each such pair $(I, J)$ and each $k \notin I \cup J$ we have a surjective homomorphism $C'_I \to C'_{I \cup \{k\}}$, induced by $r^I_{J \cup \{k\}} : A_I \to A_{I \cup \{k\}}$, where we use the notation

$$A_J = A \bigg/ \sum_{j \in J} B_j.$$ 

Thus $A_J = C^0_J$. The kernel of the surjective homomorphism $C'_I \to C'_{I \cup \{k\}}$ is

$$\left( \bigcap_{i \in I} B_i + \sum_{j \in J} B_j \right) \cap \left( B_k + \sum_{j \in J} B_j \right) \bigg/ \sum_{j \in J} B_j.$$ 

We have the inclusions

$$\bigcap_{i \in I \cup \{k\}} B_i + \sum_{j \in J} B_j \subseteq \left( \bigcap_{i \in I} B_i + \sum_{j \in J} B_j \right) \cap \left( B_k + \sum_{j \in J} B_j \right) \subseteq \bigcap_{i \in I \cup \{k\}} \left( B_i + \sum_{j \in J} B_j \right).$$

By normality these groups are equal, so we have a short exact sequence

$$0 \to C'_{I \cup \{k\}} \to C'_I \to C'_{I \cup \{k\}} \to 0.$$ 

For each pair $(I, J)$ of disjoint subsets of $m$ satisfying $I \cup J = m$ choose a section

$$t'_I : C'_I \to C'_0 (\subseteq C^0_0 = A)$$

of the map $C'_0 \to C'_I$ induced by $r^0_I : A \to A_I$ and satisfying $t'_I(0) = 0$. Next define maps $t'_I : C'_I \to C'_0$ for every disjoint pair $(I, J)$ using induction to the number of elements of the complement of $I \cup J$. So, let $(I, J)$ be a disjoint pair of subsets of $m$ with $\#(I \cup J) = n < m$ and assume that sections $t'_L : C'_L \to C'_0$ have already been defined for pairs $(K, L)$ with $K \cup L$ having more than $n$ elements.

Choose $k \in m \setminus (I \cup J)$. Let $x \in C'_I$, then for $y = r^0_I t'_I \cap r^J_{I \cup \{k\}}(x)$ we have

$$r^J_{I \cup \{k\}}(y) = r^0_I t'_I \cap r^J_{I \cup \{k\}}(x) = r^J_{I \cup \{k\}}(x),$$

so, $x - y \in C'_{I \cup \{k\}}$. Now define $t'_I$ by

$$t'_I(x) = t'_I(x - y) + t'_I y r^J_{I \cup \{k\}}(x).$$

It easily verified that this map is a section of $r : C'_0 \to C'_I$. Furthermore it is independent of the choice of $k$: if also $l \notin I \cup J$, then in both cases the image of
an \( x \in C_I^J \) under \( t_I^J \) is determined in the same way by the images of the same elements in the following four groups
\[
\begin{align*}
C_I^J \cup \{l,k\}, & \quad C_I^J \cup \{l\}, & \quad C_I^J \cup \{k\}, & \quad \text{and } C_I^J \cup \{k,l\} : \\
0 & \quad 0 & \quad 0 & \quad 0 \\
\downarrow & \quad \downarrow & \quad \downarrow & \quad \downarrow \\
0 & \quad C_I^J \cup \{l,k\} & \quad C_I^J \cup \{l\} & \quad C_I^J \cup \{k\} & \quad 0 \\
\downarrow & \quad \downarrow & \quad \downarrow & \quad \downarrow \\
0 & \quad C_I^J \cup \{k\} & \quad C_I^J & \quad C_I^J \cup \{k\} & \quad 0 \\
\downarrow & \quad \downarrow & \quad \downarrow & \quad \downarrow \\
0 & \quad C_I^J \cup \{k\} & \quad C_I^J \cup \{l\} & \quad C_I^J \cup \{k,l\} & \quad 0 \\
\downarrow & \quad \downarrow & \quad \downarrow & \quad \downarrow \\
0 & \quad 0 & \quad 0 & \quad 0
\end{align*}
\]

Thus we obtain a splitting of the cube, where the sections \( s_I^J \) of the homomorphisms \( r_I^J \), where \( I \subseteq J \), are the maps \( r_I^J \uparrow \Phi \). In particular, condition \((S3')\) follows from the above diagram for \( I = \emptyset \).

3 Operations on normal \( m \)-tuples of ideals

By \( \mathcal{R}_m \) we will denote the category of all normal \( m \)-tuples of ideals. Such an \( m \)-tuple is denoted as \( (R, a_1, \ldots, a_m) \), where \( R \) is a ring and \( a_1, \ldots, a_m \) are ideals of \( R \). A morphism \( \phi: (R, a_1, \ldots, a_m) \to (S, b_1, \ldots, b_m) \) is just a ring homomorphism \( \phi: R \to S \) satisfying \( \phi(a_i) \subseteq b_i \) for all \( i \in [m] \).

The following notations will simplify notations for long exact sequences of multirelative \( K \)-theory. Another advantage will be that they are useful to indicate functoriality properties.

For each \( m \geq 1 \) the functor \( D: \mathcal{R}_m \to \mathcal{R}_{m-1} \) is the functor that deletes the last ideal:
\[
D(R, a_1, \ldots, a_m) = (R, a_1, \ldots, a_{m-1})
\]
and which has no effect on morphisms.

For each \( m \geq 1 \) the functor \( M: \mathcal{R}_m \to \mathcal{R}_{m-1} \) is the functor that deletes the last ideal and that takes the ring and the other ideals modulo this ideal:
\[
M(R, a_1, \ldots, a_m) = (R/\overline{a_m}, \overline{a_1}, \ldots, \overline{a_{m-1}}),
\]
where \( \overline{a_j} = a_j + a_i/a_i \), and which maps a morphism to the induced morphism.

A functor morphism \( \phi: D \to M \) of the functors \( D, M: \mathcal{R}_m \to \mathcal{R}_{m-1} \) is defined as follows: let \( A = (R, a_1, \ldots, a_m) \), then \( \phi_A: D(A) \to M(A) \) is the canonical ring homomorphism \( R \to R/a_m \).
Every $A \in \mathcal{R}_m$ has an underlying ideal $I(A)$, which is defined as the intersection of the $m$ ideals in $A$: when $A = (R, a_1, \ldots, a_m)$, then

$$I(A) = a_1 \cap \cdots \cap a_m.$$  

Thus defined, $I(A)$ is functorial in $A$.

4 Higher $K$-theory of rings

In [3] the definition of higher $K$-groups is as follows. Let $R \in \mathcal{R}$. Choose a simplicial ring $R$ with an augmentation $\varepsilon: R \to R$ such that

- $R$ is aspherical, i.e. $\pi_n(R) = 0$ for all $n \geq 1$,
- $R_m$ is free for all $m \geq 0$, say $R_m$ is free on a set $X_m$ of generators,
- the sets $X_m$ of free generators are stable under degeneracies: $s_j(X_m) \subseteq X_{m+1}$ for all $m \geq 0$,
- the augmentation $\varepsilon$ induces an isomorphism $\pi_0(R) \to R$.

Then for $n \geq 3$ the group $K_n(R)$ is defined as the $(n-2)$nd homotopy group of the simplicial group $GL(R)$, and the groups $K_1(R) = K_2(R)$ are given by the exactness of

$$0 \to K_2(R) \to \pi_0(GLR) \to GL(R) \to K_1(R) \to 0.$$  

The groups $K_n(R)$ for $n \geq 3$ are Abelian because $GL(R)$ is a simplicial group. The group $K_1(R)$ is Abelian since it is the cokernel of $GL(R_0) \to GL(R)$, and $K_2(R)$ is Abelian because it is the cokernel of $GL(R_1) \to GL(Z_0)$, where $Z_0 = \{(x_0, x_1) \mid \varepsilon(x_0) = \varepsilon(x_1)\}$. In [3] it is shown using a comparison theorem that the higher $K$-groups are thus well-defined and that they are actually functors.

For the purpose of this paper we will confine to a functorial resolution $Fr(R)$ of a ring $R$, which we now describe. Let $F: \mathcal{S} \to \mathcal{R}$ the free ring functor and let $U: \mathcal{R} \to \mathcal{S}$ be the underlying set functor, then the functor $FU: \mathcal{R} \to \mathcal{R}$ together with the obvious functor morphisms $\nu: FU \to (FU)^2$ and $\eta: FU \to I$ is a cotriple. Put

$$Fr_n = (FU)^{n+1}.$$  

Face and degeneracy morphisms are given by

$$d_i = (FU)^i \eta(FU)^{n-1-i} \quad \text{and} \quad s_j = (FU)^i \nu(FU)^{n-1-i}.$$  

The augmentation is then given by $\eta$.

A property of this functorial resolution is that, when applied to a surjective ringhomomorphism $R \to S$, it gives a dimensionwise surjective homomorphism $FrR \to FrS$ of simplicial rings, and since the ringhomomorphisms are dimensionwise split it also gives a surjective simplicial groupomorphism $GL(FrR) \to GL(FrS)$. This is often convenient when considering homotopy fibres, because surjective simplicial groupomorphisms are fibrations themselves. So instead of taking a homotopy fibre one just takes a fibre, i.e. the kernel of the simplicial group homomorphism.
5 Cubes in a simplicial group

Let $A$ be a simplicial group with augmentation $d_0: A_0 \to A$. It is a contravariant functor $A: \Omega_+^{op} \to \mathcal{G}$ from the category $\Omega_+$ of finite ordered sets

$$[n] = \{0, \ldots, n\} \quad (n \geq -1)$$

(where $[-1] = \emptyset$) and monotone (= order preserving) maps to the category of groups. (Here we use the notation $A_{-1} = A$.) We will show that $A$ determines an $m$-cube of groups for every nonnegative integer $m$. In stead of the ordered set of subsets of $[m]$ for the description of an $m$-cube the ordered set of subsets of $[m-1]$ will be used for this purpose.

Let $\Omega(m)$ be the category of injective monotone maps

$$\alpha: [k] \to [m-1].$$

A morphism from $\alpha: [k] \to [m-1]$ to $\beta: [l] \to [m-1]$ is a monotone map $\gamma: [k] \to [l]$ such that $\beta \gamma = \alpha$. It exists if and only if $\text{Im}(\alpha) \subseteq \text{Im}(\beta)$, and it is unique if it exists.

For each $I \subseteq [m-1]$ there is a unique injective monotone map

$$\alpha_I: [k] \to [m-1],$$

where $k = m - 1 - \#(I)$ and $\text{Im}(\alpha_I) = [m-1] \setminus I$. If $I \subseteq J \subseteq [m-1]$, then $\text{Im}(\alpha_I) \supseteq \text{Im}(\alpha_J)$, so then there is a unique

$$\gamma_I^J: \alpha_J \to \alpha_I,$$

i.e. a monotone $\gamma_I^J: [m-1 - \#(J)] \to [m-1 - \#(I)]$ such that $\alpha_I \gamma_I^J = \alpha_J$.

**Definition 5.** Let $A$ be an augmented simplicial group and let $m$ be a nonnegative integer. Then the $m$-cube of $A$ is the $m$-cube $A(m): C_m \to \mathcal{G}$ with

$$\begin{cases}
A(m)_I = A_{[m-1-\#(I)]} & \text{for all } I \subseteq [m-1], \\
r_I^J = A(\gamma_I^J): A(m)_I \to A(m)_J & \text{for all } I \subseteq J \subseteq [m-1].
\end{cases}$$

**Lemma 5.1.** Let the augmentation $d_0: A_0 \to A_{-1}$ induce a surjective homomorphism $\pi_0(A) \to A_{-1}$. Then for all integers $i, j, m$ such that $0 \leq j < i \leq m$

$$d_i^{(m)}(\text{Ker}(d_j^{(m)})) = \text{Ker}(d_j^{(m-1)}).$$

**Proof.** Let $x \in \text{Ker}(d_j^{(m)})$. Then, since $i > j$, $d_j d_i(x) = d_{i-1} d_j(x) = 1$. So $d_i(\text{Ker}(d_j)) \subseteq \text{Ker}(d_j)$. Now, let $y \in \text{Ker}(d_j^{(m-1)})$. There is an $x \in A_m$ such that $d_j(x) = 1$ and $d_i(x) = y$. For $m > 1$ this is the case because a simplicial group is a Kan-complex, while for $m = 1$ it follows from the condition on the augmentation. \qed
Proposition 5.1. Let $A$ be a simplicial group with an augmentation $d_0: A \to A$ that induces an isomorphism $\pi_0(A) \to A$. Then for all $m \geq 1$ the $m$-cube $A(m)$ is induced by the $m$-tuple $(A_{m-1}, \text{Ker}(d_0), \ldots, \text{Ker}(d_{m-1}))$.

Proof. All face maps are surjective, so it remains to show that for all $J \subseteq [m-1]$

$$\text{Ker}(r^0_J) = \prod_{j \in J} \text{Ker}(d^{(m-1)}_j).$$

For $J = \emptyset$ this is trivially true. Let $J$ be nonempty and proceed by induction. Let $x \in \text{Ker}(r^0_J)$. Let $k \in J$ be maximal. Then $r^0_{\{k\}}(x) = d_k(x) \in \text{Ker}(r^0_{\{k\}})$. By induction this group is equal to $\prod_{j \in J'} \text{Ker}(d^{(m-2)}_j)$, where $J' = J \setminus \{k\}$. (Here we used the maximality of $k$ in $J$ and the same result for the $(m-1)$-cube $A(m-1)$.) By the lemma we have

$$d_k \left( \prod_{j \in J'} \text{Ker}(d^{(m-1)}_j) \right) = \prod_{j \in J'} \text{Ker}(d^{(m-2)}_j).$$

Choose $y \in \prod_{j \in J'} \text{Ker}(d^{(m-1)}_j)$ such that $d_k(y) = d_k(x)$. Then $xy^{-1} \in \text{Ker}(d_k)$. It follows that

$$\text{Ker}(r^0_J) \subseteq \prod_{j \in J} \text{Ker}(d^{(m-1)}_j).$$

For the other inclusion note that $d_j = r^0_{\{j\}}$ and

$$r^0_{\{j\}} r^0_J = r^0_J.$$

$\square$

Proposition 5.2. Let $A$ be as in Proposition 5.1 and assume moreover that $A$ is aspherical. Then the $m$-tuple

$$(A_{m-1}, \text{Ker}(d_0), \ldots, \text{Ker}(d_{m-1}))$$

is normal.

Proof. The edges of the $m$-cube are the face maps. Normality means that these maps preserve intersections of (the images of) the normal subgroups $\text{Ker}(d_0), \ldots, \text{Ker}(d_{m-1})$. By induction it suffices to show this for the face maps $d^{(m-1)}_i$. Let $J \subseteq [m-1]$. Then to show that

$$d_i \left( \bigcap_{j \in J} \text{Ker}(d_j) \right) = \bigcap_{j \in J} d_i(\text{Ker}(d_j)).$$

for $i \not\in J$. The inclusion of the left hand side in the right hand side is trivial. So let $x \in \bigcap_{j \in J} d_i(\text{Ker}(d_j))$. Then for $j \in J$ there is an $y_j \in \text{Ker}(d_j)$ such that $x = d_i(y_j)$. For $j < i$ it follows that $d_j(x) = d_j d_i(y_j) = d_{i-1} d_j(x_j) = 1$. Similarly for $j > i$ we have $d_{j-1}(x) = 1$. So, since a simplicial group is a Kan-complex and for $J = [m-1]$ since $A$ is aspherical, there is a $y \in A_{m-1}$ such that $d_j(y) = 1$ for all $j \in J$ and $d_i(y) = x$. This shows that $x \in d_i \left( \bigcap_{j \in J} \text{Ker}(d_j) \right).$ $\square$
6 Multirelative $K$-theory

A normal $m$-tuple of ideals $A = (R, a_1, \ldots, a_m)$ induces an $m$-cube in $R$

$$A: I \to R \square \sum_{i \in I} a_i,$$

which by Proposition 2.2 is split in $S$. Application of $Fr$ to this $m$-cube gives an $m$-cube of simplicial rings which is dimensionwise split in $R$. Put

$$Fr(R, a_i) := \text{Ker}(Fr(R) \to Fr(R/a_i)).$$

This is a simplicial ideal. The $m$-cube is then induced by the $m$-tuple

$$(Fr(R), Fr(R, a_1), \ldots, Fr(R, a_m)),$$

of simplicial ideals, an object of the category $\sR_m$ of normal $m$-tuples of simplicial ideals. We also define the simplicial ideal

$$Fr(R, a_1, \ldots, a_m) := \bigcap_{i=1}^m Fr(R, a_i).$$

Application of $GL$ gives an $m$-cube of simplicial groups, which is dimensionwise split in $G$. This $m$-cube is induced by the $m$-tuple

$$(GLFr(R), GLFr(R, a_1), \ldots, GLFr(R, a_m))$$

of simplicial normal subgroups. For $n \geq 3$ we define multirelative $K_n$ by

$$K_n(R, a_1, \ldots, a_m) := \pi_{n-2}(GLFr(R, a_1, \ldots, a_m)).$$

Multirelative $K_2$ and $K_1$ are then given by the exactness of

$$0 \to K_2(R, a_1, \ldots, a_m) \to \pi_0(GLFr(R, a_1, \ldots, a_m)) \to GL(a_1 \cap \cdots \cap a_m) \to K_1(R, a_1, \ldots, a_m) \to 0.$$ 

These multirelative $K_1$ and $K_2$ are Abelian groups for the same reason as in the absolute case.

Now let $A \in R_m$ with $m \geq 1$. Then $\phi_*: GLFr(DA) \to GLFr(MA)$ is a fibration with fibre $GLFr(A)$. The long exact sequence of homotopy groups is a long exact sequence of multirelative $K$-groups which can easily be extended to include multirelative $K_2$ and $K_1$.

**Proposition 6.1.** Let $A \in R_m$ with $m \geq 1$. Then we have a functorial exact sequence

$$\cdots \to K_n(A) \to K_n(DA) \to K_n(MA) \to K_{n-1}(A) \to \cdots \to K_1(MA).$$
The connecting map $K_n(MA) \to K_{n-1}(A)$ will be denoted by $\delta$ and the map $K_n(A) \to K_n(DA)$ by $\iota$. To put it in an even more functorial way, we have an exact sequence of functors and functor morphisms

$$\cdots \to K_n \xrightarrow{\iota} K_n D \xrightarrow{K_n(\phi)} K_n M \xrightarrow{\delta} K_{n-1} \to \cdots \to K_1 M.$$ 

In the remaining part of this section multirelative $K_0$ is defined and the long exact sequence for multirelative $K$-theory is extended with multirelative $K_0$-groups.

**Definition 6.** For a normal $m$-tuple $A$ of ideals we define

$$K_0(A) = K_0(I A).$$

Thus defined, $K_0$ is a functor from $\mathcal{R}_m$ to $\mathcal{A}$.

For $m = 1$ we take the long exact sequence to be the long exact sequence of an ideal in a ring. Now assume that $m \geq 1$ and that we have an extended long exact sequence

$$\cdots \to K_1 D \to K_1 M \to K_0 \to K_0 D \to K_0 M$$

of functors $\mathcal{R}_m \to \mathcal{A}$. We will show that there is also such a sequence of functors $\mathcal{R}_{m+1} \to \mathcal{A}$.

Let $A = (R, a_1, \ldots, a_{m+1}) \in \mathcal{R}_{m+1}$. Put $b = I A = \bigcap_{i=1}^{m+1} a_i$. We have exact sequences for the following $m$-tuples of ideals

$$B = DA = (R, a_1, \ldots, a_m),$$

$$\overline{B} = (R/b, a_1/b, \ldots, a_m/b)$$

and

$$(R, a_1, \ldots, a_{m-1}, b).$$

These $m$-tuples are normal and their $K$-groups fit into a commutative diagram.
Let the dashed arrow be the composition $K_1(B) \to K_1(DB) \to K_0(b)$. By an easy diagram chase we see that the sequence with the dashed arrow is exact as well. The identity on $R$ is a morphism

$$(R, a_1, \ldots, a_m, b) \to A$$

in $\mathcal{R}_{m+1}$. So we have a commutative diagram with exact rows:

$$
\begin{array}{ccccccc}
K_1(R, a_1, \ldots, a_m, b) & \longrightarrow & K_1(B) & \longrightarrow & K_1(B) & \longrightarrow & K_0(b) \\
\downarrow & & \downarrow 1 & & \downarrow \alpha & & \\
K_1(A) & \longrightarrow & K_1(DA) & \longrightarrow & K_1(MA). & & 
\end{array}
$$

It now suffices to show that the morphism $\alpha$ in this diagram is an isomorphism. The $(m+1)$-tuple $(R/b, a_1/b, \ldots, a_{m+1}/b)$ induces an exact sequence

$$K_1(R/b, a_1/b, \ldots, a_{m+1}/b) \to K_1(B) \to K_1(MA).$$

The group $K_1(R/b, a_1/b, \ldots, a_{m+1}/b)$ is a quotient of $GL((a_1/b) \cap \cdots (a_{m+1}/b)) = \{1\}$, so $\alpha$ is injective. On the other hand, since the $(m+1)$-tuple $A$ of ideals is normal, the identity on $R$ induces an isomorphism $I(B) \to I(MA)$ and hence also an isomorphism

$$GL(I(B)) \cong GL(I(MA)).$$

Since the multirelative $K_1$ is a quotient of the general linear group of the underlying ideal, the map $\alpha$ is surjective. This proves:

**Theorem 1.** Let $A \in \mathcal{R}_m$ for $m \geq 1$. Then we have a functorial exact sequence

$$\cdots \to K_1(A) \to K_1(DA) \to K_1(MA) \to K_{-1}(A) \to \cdots \to K_0(MA).$$

\[ \square \]

7 Axioms for multirelative $K$-theory

It will be shown in this section that an axiomatic approach to multirelative $K$-theory is possible. We take some of the properties of multirelative $K$-groups as axioms and show that they determine all of multirelative $K$-theory.

**Axioms**

**Multirelative $K$-theory** consists of functors

$$K_n: \mathcal{R}_m \to \mathcal{A} \quad \text{for } m \text{ and } n \text{ integers } \geq 0,$$

morphisms

$$\delta: K_{n+1}M \to K_n \quad \text{(for } m \text{ and } n \text{ integers } \geq 0)$$

of functors $\mathcal{R}_{m+1} \to \mathcal{A}$ and morphisms

$$\iota: K_n \to K_n D \quad \text{(for } m \text{ and } n \text{ integers } \geq 0)$$

of functors $\mathcal{R}_{m+1} \to \mathcal{A}$, such that
the following sequence is an exact sequence of functors $R_{m+1} \rightarrow A$ for all non-negative integers $m$ and $n$

$$K_{n+1}D \xrightarrow{K_{n+1}\phi} K_{n+1}M \xrightarrow{\delta} K_n \xrightarrow{\iota} K_nD \xrightarrow{K_n\phi} K_nM.$$  

(MK2) $K_n(R) = 0$ for all $n \geq 0$ and all free associative non-unital rings $R$,

(MK3) $K_0(A) = K_0(IA)$ for all $A \in \mathcal{R}_m$ for all $m$.

Loosely speaking, the multirelative $K$-groups are only defined for normal $m$-tuples of ideals and they fit into exact sequences the way one can expect, the (absolute) $K$-groups of free non-unital rings are trivial and the multirelative $K_0$ is just the Grothendieck group of the intersection of the ideals.

Let $(R, a_1, \ldots, a_m)$ be a normal $m$-tuple of ideals. It induces an $m$-cube

$$I \mapsto R_I = R \bigg/ \sum_{i \in I} a_i,$$

which is split in $\mathcal{S}$. Application of $\text{Fr}$ gives an $m$-cube

$$I \mapsto \text{Fr}(R_I)$$

of aspherical simplicial rings, which is dimensionwise split in $\mathcal{R}$.

**Proposition 7.1.** Let $m$ and $n$ be positive integers. Then the $(m+n)$-tuple

$$\left(\text{Fr}(R)_{n-1}, \text{Fr}(R, a_1)_{n-1}, \ldots, \text{Fr}(R, a_m), \text{Ker}(d_0^{(n-1)}), \ldots, \text{Ker}(d_{n-1}^{(n-1)})\right)$$

is normal.

**Proof.** First we show that the induced $(m+n)$-cube is

$$(I_1, I_2) \mapsto \text{Fr}(R_{I_1})_{n-1-\#(I_2)},$$

where the cube is indexed by pairs of subsets of $\underline{m}$ and $[n-1]$. This set of pairs is ordered by componentwise inclusion:

$$(I_1, I_2) \leq (J_1, J_2) \iff I_1 \subseteq J_1 \quad \text{and} \quad I_2 \subseteq J_2.$$  

The homomorphism

$$\text{Fr}(R)_{n-1} \rightarrow \text{Fr}(R_{I_1})_{n-1-\#(I_2)}$$

is the composition

$$\text{Fr}(R)_{n-1} \rightarrow \text{Fr}(R_{I_1})_{n-1} \rightarrow \text{Fr}(R_{I_1})_{n-1-\#(I_2)},$$

the first map being induced by $\emptyset \subseteq I_1$ and the second by $[n-1] \setminus I_2 \subseteq [n-1]$. Both homomorphisms are surjective. The first one has kernel $\bigcap_{i \in I_1} \text{Fr}(R, a_i)_{(n-1)}$ and the second one $\bigcap_{i \notin I_2} \text{Ker}(d_i)$, where the $d_i$ are face maps of $\text{Fr}(R_{I_1})$. Since
\textbf{Fr}(R) and \textbf{Fr}(R_{J_l}) are both aspherical, elements of the second kernel can be lifted to elements of }\bigcap_{i \notin I_2} \text{Ker}(d_i), \text{ where the } d_i \text{ are face maps of } \textbf{Fr}(R).

For the \((m + n)\)-tuple to be normal it suffices that the intersections of the images of the \(m + n\) ideals are preserved under the maps on the edges of the induced \((m + n)\)-cube. These are the homomorphisms

\[
\textbf{Fr}(R_{J_l}) \to \textbf{Fr}(R_{J \cup \{k\}}),
\]

where \(J \subseteq m, k \in m \setminus J\) and \(l \in [n - 1]\), and also the face maps

\[
d_i: \textbf{Fr}(R_{J_k}) \to \textbf{Fr}(R_{J_p}),
\]

where \(p \in [n - 1]\) and \(0 \leq i \leq p\). Without loss of generality we may assume that \(J = m, l = n - 1\) and \(p = n - 1\).

Because the \(m\)-cube \(J \to \textbf{Fr}(R_{J_k})\) is dimensionwise split we have short exact sequences

\[
0 \to \bigcap_{i \in I \cup \{k\}} \text{Fr}(R, a_i) \to \bigcap_{i \in I} \text{Fr}(R, a_i) \to \bigcap_{i \in I} \text{Fr}(R/a_k, \overline{a}_i) \to 0
\]

of aspherical simplicial rings. It follows that for all \(J \subseteq [n - 1]\) we have

\[
\bigcap_{i \in I} \text{Fr}(R, a_i)_{n-1} \cap \bigcap_{j \in J} \text{Ker}(d_j) = \bigcap_{j \in J} \text{Ker}(d'_j),
\]

where the \(d'_j\) are the face maps of \(\bigcap_{i \in I} \text{Fr}(R, a_i)\). Under \(\text{Fr}(R) \to \text{Fr}(R/a_k)\) this maps onto

\[
\bigcap_{j \in J} \text{Ker}(d''_j) = \bigcap_{i \in I} \text{Fr}(R/a_k, \overline{a}_i)_{n-1} \cap \bigcap_{j \in J} \text{Ker}(d''_j),
\]

where the \(d''_j\) are the face maps of \(\bigcap_{i \in I} \text{Fr}(R/a_k, \overline{a}_i)\) and \(d''_j\) those of \(\bigcap_{i \in I} \text{Fr}(R/a_k)\).

Because the simplicial rings \(\bigcap_{i \in I} \text{Fr}(R, a_i)\) are aspherical also the face maps \(d_i: \text{Fr}(R)_{n-1} \to \text{Fr}(R)_{n-2}\) preserve intersections

\[
\bigcap_{i \in I} \text{Fr}(R, a_i)_{n-1} \cap \bigcap_{j \in J} \text{Ker}(d_j).
\]

\[\square\]

\textbf{Theorem 2.} Let \(A = (R, a_1, \ldots, a_m) \in \mathcal{R}\). Then for all \(n \geq 0\) it follows from the axioms (KM1) and (KM2) that \(K_n(A)\) is naturally isomorphic to \(K_0\) of the following object of \(\mathcal{R}_{m+n}\):

\[(\text{Fr}(R)_{n-1}, \text{Fr}(R, a_1)_{n-1}, \ldots, \text{Fr}(R, a_m)_{n-1}, \text{Ker}(d_0), \ldots, \text{Ker}(d_{n-1})).\]

From axiom (KM3) it then follows that \(K_n(A)\) is determined. So (KM1), (KM2) and (KM3) can be taken as axioms for the (multirelative) \(K\)-theory of rings.

\textbf{Proof.} The proof follows from the following three lemmas. \[\square\]
Lemma 7.1. Let $m \geq -1$ and $q, n \geq 0$. Then

$$K_q(\text{Fr}(R)_n, \text{Fr}(R, a_1)_n, \ldots, \text{Fr}(R, a_m)_n) = 0.$$ 

Proof. Since for $m \geq 0$ the $(m - 1)$-tuples $D(A)$ and $M(A)$ are of the same type, the proof reduces by (MK1) to the case $m = -1$. For $m = -1$ the lemma follows from (MK2).

Put

$$A[n, p] = (\text{Fr}(R)_n, \text{Fr}(R, a_1)_n, \ldots, \text{Fr}(R, a_m)_n, \text{Ker}(d_0), \ldots, \text{Ker}(d_p)),$$

where $-1 \leq p \leq n$. It is an object of $\mathcal{R}_{m+p+1}$.

Lemma 7.2. For all $p < n$ and all $q > 0$ we have

$$K_q(A[n, p]) = 0.$$ 

Proof. For $p \geq 0$ we have

$$D(A[n, p]) = A[n, p - 1] \quad \text{and} \quad M(A[n, p]) = A[n - 1, p - 1].$$

By (MK1) the problem reduces to the case $p = -1$, which is covered by the previous lemma.

Lemma 7.3. For all $q, n \geq 0$ we have

$$K_q(A[n, n]) \cong K_{q+1}(A[n - 1, n - 1]).$$

Proof. This follows from (MK1) and the previous lemma.

From this lemma the theorem follows:

$$K_n(A) = K_n(A[-1, -1]) \cong K_{n-1}(A[0, 0]) \cong \cdots \cong K_0(A[n - 1, n - 1]).$$

References


