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A Symmetrization for Finite Two-Person Games

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Abstract: The symmetrization method of Gale, Kuhn and Tucker for matrix games is extended for bimatrix games. It is shown that the equilibria of a bimatrix game and its symmetrization correspond two by two. A similar result is found with respect to quasi-strong, regular and perfect equilibria.

Key Words: Bimatrix Game, Equilibria, Symmetrization

1 Introduction

This paper is devoted to a symmetrization method for finite two-person games (bimatrix games), which originates from a method for matrix games by Gale et al. (1950).

Already in the fundamental paper of von Neumann (1928) attention is paid to symmetric matrix (two-person, zero-sum) games and it is observed that these games have value zero. An alternative proof of this result is given by Brown and von Neumann (1950), where it is extended to a proof of existence of a value for general matrix games by referring to a symmetrization method of Gale, Kuhn and Tucker (1950). Also an alternative to this method is provided.

In a symmetric game both players have the same strategic possibilities and there is no discrimination in the payoffs. Therefore a symmetrization of a game is an extension of this game to fair play. In fact, where in real-life situations people play games, they tend to symmetrize by means of tossing, exchanging roles in a second play, etc. This motivates the study of symmetrizations for finite two-person games that are not zero-sum. Furthermore symmetric games play an important role in the new field of sociobiology, founded by Maynard Smith (1982), where evolutionary stable strategies correspond to special symmetric equilibria.

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Griesmer, Hoffman and Robinson (1963) propose a symmetrization method for bimatrix games. For such games this method and the method of von Neumann (Brown and von Neumann, 1950) are extensively studied in a paper of Jansen, Potters and Tijs (1986). Correspondences between equilibria and some refinements of equilibria for a game and for its symmetrization are given with respect to either type. In this paper we deal with a similar correspondence, but now concerning the symmetrization method of Gale, Kuhn and Tucker. Therefore, in Section 3, we extend this method to the case of bimatrix games. In Section 4, we give a correspondence with respect to equilibria and in Section 5 we test three types of refinements, i.e. quasi-strong, regular and perfect equilibria, for a similar property. Section 2 is preliminary.

The method described in this paper yields a one-to-one correspondence between pairs of equilibria for a game and pairs of equilibria for its symmetrization. A similar statement holds for each of the three refinements discussed here. Jansen, Potters and Tijs showed that such a nice correspondence does not exist for the method of Griesmer, Hoffman and Robinson. On the contrary the method of von Neumann implies a similar one-to-one correspondence. However, in this case, the 'size' of the symmetrization of a game with m and n pure strategies for the players, respectively, is large: In this symmetrization both players have $m \cdot n$ pure strategies. For the symmetrization of Gale, Kuhn and Tucker this number is $m+n+1$ (and $m+n$ for the symmetrization of Griesmer, Hoffman and Robinson). This makes the latter symmetrization more interesting for computational purposes.

Notation: We denote by $e_1, \ldots, e_n$ the standard basis vectors of $\boldsymbol{R}^n$. $1_n \in \boldsymbol{R}^n$ corresponds to $(1,1,\ldots,1)$. For $x,y \in \boldsymbol{R}^n$ we define $(x,y) := \sum_{i=1}^{n} x_i y_i$ and we write $x \geq y$ $(x > y)$ if $x_i \geq y_i$ $(x_i > y_i)$ for all $i \in \{1,\ldots,n\}$. A matrix A is called strictly positive (strictly negative), and we write $A > 0$ $(A < 0)$ if all the entries of A are positive (negative). By A^t we denote the transpose of a matrix A.

2 Preliminaries

Let $A = [a_{ij}]_{i=1}^{m}{}_{j=1}^{n}$ and $B = [b_{ij}]_{i=1}^{m}{}_{j=1}^{n}$ be two real $m \times n$ matrices. We consider the finite two-person game $(\Delta_m, \Delta_n, K, L)$, where for $t \in N$

$$\Delta_t := \left\{ x \in \boldsymbol{R}^t \,|\, x \geq 0 \quad \text{and} \quad \sum_{i=1}^{t} x_i = 1 \right\},$$

and where the payoff-functions $K,L:\Delta_m\times\Delta_n\rightarrow\mathbf{R}$ are defined by $K(p,q)=pAq:=\sum_{i=1}^{m}\sum_{j=1}^{n}p_i a_{ij}q_j$ and $L(p,q)=pBq:=\sum_{i=1}^{m}\sum_{j=1}^{n}p_i b_{ij}q_j$. We call this game the $m\times n$ *bimatrix game* (A,B).

For the game (A,B) we define an *equilibrium* as a strategy pair $(\bar{p},\bar{q})\in\Delta_m\times\Delta_n$ with $\bar{p}A\bar{q}\geq pA\bar{q}$ and $\bar{p}B\bar{q}\geq\bar{p}Bq$ for all $(p,q)\in\Delta_m\times\Delta_n$. Nash (1951) showed that the set $E(A,B)$ of all equilibria for (A,B) is non-empty.

For a strategy $p\in\Delta_m$ we define its *carrier* by $C(p):=\{i\in\{1,\ldots,m\}|p_i>0\}$ and the set of *pure best replies* of the second player against p by $PB(B,p):=\{j\in\{1,\ldots,n\}|pBe_j=\max_{l\in\{1,\ldots,n\}}pBe_l\}$. For a strategy $q\in\Delta_n$, its carrier $C(q)$ and the set $PB(A,q)$ of pure best replies of the first player against q are defined similarly.

It is well-known that a strategy pair (p,q) is an equilibrium for (A,B) if and only if $C(p)\subset PB(A,q)$ and $C(q)\subset PB(B,p)$.

For the game (A,B) a subset T of the equilibrium set $E(A,B)$ is called a *Nash subset* if $(p,q),(r,s)\in T$ implies $(p,s),(r,q)\in T$. Jansen (1981) showed that $E(A,B)$ is the finite union of maximal Nash subsets, where a Nash subset is called *maximal* if it is not properly contained in another Nash subset.

An $n\times n$ bimatrix game (A,B) is called symmetric if $B=A^t$. It is easily verified that for a symmetric game $(A,A^t),(p,q)\in E(A,A^t)$ if and only if $(q,p)\in E(A,A^t)$.

Finally in this section we note that for a bimatrix game (A,B), the equilibrium set $E(A,B)$ is invariant under the operation where to every entry of one of the matrices the same real number is added. So without loss of generality we may always assume $A>0$ and $B<0$.

3 Gale, Kuhn and Tucker Symmetrization Method for Bimatrix Games

In this section we extend the method of Gale, Kuhn and Tucker for symmetrizing a matrix game to the case of a bimatrix game. We consider an $m\times n$ bimatrix game (A,B) such that $A>0$ and $B<0$. We will call the symmetric bimatrix game (C,C^t), where C is the $(m+n+1)\times(m+n+1)$ matrix

$$C=\begin{bmatrix}0 & A & -1_m\\ B^t & 0 & 1_n\\ 1_m & -1_n & 0\end{bmatrix},$$

the *Gale, Kuhn and Tucker symmetrization* of (A,B), in short *GKT-symmetrization*.

In order to show that there is a one-to-one correspondence between pairs of equilibria for the game (A,B) and pairs of equilibria for the game (C,C^t), we first describe how an equilibrium for (C,C^t) yields two equilibria for (A,B). Therefore we need a lemma and some notation.

For a strategy $\tau \in \Delta_{m+n+1}$ we let $\tau_x := (\tau_1,\ldots,\tau_m)$, $\tau_y := (\tau_{m+1},\ldots,\tau_{m+n})$ and $\tau_z := \tau_{m+n+1}$. Then $\tau = (\tau_x, \tau_y, \tau_z)$.

Lemma 1: Let (C,C^t) be the GKT-symmetrization of an $m\times n$ bimatrix game (A,B) with $A>0$ and $B<0$. Let $(\varrho,\sigma)\in E(C,C^t)$. Then $\varrho_x \neq 0$, $\varrho_y \neq 0$, $\varrho_z \neq 0$, $\sigma_x \neq 0$, $\sigma_y \neq 0$ and $\sigma_z \neq 0$.

Proof: We prove the lemma by following the scheme below.

$$\begin{array}{ccccc}
\varrho_x \neq 0 & \overset{(a)}{\Rightarrow} & \sigma_y \neq 0 & \overset{(b)}{\Rightarrow} & \varrho_z \neq 0 \\
(f)\Uparrow & & & & \Downarrow(c) \\
\sigma_z \neq 0 & \overset{(e)}{\Leftarrow} & \varrho_y \neq 0 & \overset{(d)}{\Leftarrow} & \sigma_x \neq 0
\end{array}$$

Note that, since $\varrho,\sigma\in\Delta_{m+n+1}$, we already have $\varrho \geq 0$, $\sigma\geq 0$ and $\varrho\neq 0$, $\sigma\neq 0$.

We start at the upper left corner of the scheme:

(a) Assume $\varrho_x \neq 0$. Then there is an $i_0 \in \{1,\ldots,m\}$ such that

$$e_{i_0} C\sigma = e_{i_0} A\sigma_y - \sigma_z \geq \begin{cases} \max\limits_{i\in\{1,\ldots,m\}} e_i A\sigma_y - \sigma_z \\ \max\limits_{j\in\{1,\ldots,n\}} \sigma_x B e_j + \sigma_z \\ (\sigma_z, 1_m) - (\sigma_y, 1_n)\ . \end{cases}$$

Suppose $\sigma_y = 0$. Then $e_{i_0} C\sigma = e_{i_0} A\sigma_y - \sigma_z = -\sigma_z \leq 0$. This yields $0 \geq (\sigma_x, 1_m) - (\sigma_y, 1_n) = (\sigma_x, 1_m)$. So $\sigma_x = 0$. This again implies $0 \geq \max\limits_{j\in\{1,\ldots,n\}} \sigma_x B e_j + \sigma_z = \sigma_z$. So $\sigma_z = 0$. Consequently $\sigma = 0$, which is a contradiction. Hence $\sigma_y \neq 0$.

(b) Assume $\sigma_y \neq 0$. Then there is a $j_0 \in \{m+1,\ldots,m+n\}$ such that

$$\varrho C^t e_{j_0} = \varrho_x B e_{j_0 - m} + \varrho_z \geq \begin{cases} \max\limits_{i\in\{1,\ldots,m\}} e_i A\varrho_y - \varrho_z \\ \max\limits_{j\in\{1,\ldots,n\}} \varrho_x B e_j + \varrho_z \\ (\varrho_x, 1_m) - (\varrho_y, 1_n)\ . \end{cases}$$

Suppose $\varrho_z = \varrho_x = 0$. Then $0 \geq \max\limits_{i\in\{1,\ldots,m\}} e_i A\varrho_y$. Since $A>0$ this implies $\varrho_y = 0$. Consequently $\varrho = 0$, which is a contradiction. Now suppose $\varrho_z = 0$,

$\varrho_x \neq 0$. Then, since $B<0$, we have $0>\varrho_x B e_{j_0-m} \geq \max_{i\in\{1,\ldots,m\}} e_i A \varrho_y - \varrho_z = \max_{i\in\{1,\ldots,m\}} e_i A \varrho_y$. This contradicts $A>0$. Consequently $\varrho_z \neq 0$.

(c) Assume $\varrho_z \neq 0$. Then

$$e_{m+n+1} C\sigma = (\sigma_x, 1_m) - (\sigma_y, 1_n) \geq \begin{cases} \max_{i\in\{1,\ldots,m\}} e_i A \sigma_y - \sigma_z \\ \max_{j\in\{1,\ldots,n\}} \sigma_x B e_j + \sigma_z \end{cases} .$$

Suppose $\sigma_x = 0$. Then $0 \geq -(\sigma_y, 1_n) \geq \max_{j\in\{1,\ldots,n\}} \sigma_x B e_j + \sigma_z = \sigma_z$. Hence $\sigma_z = 0$. Consequently $0 \geq (-\sigma_y, 1_n) \geq \max_{i\in\{1,\ldots,m\}} e_i A \sigma_y$. Since $A>0$, this implies $\sigma_y = 0$. Consequently $\sigma = 0$, which is a contradiction. Hence $\sigma_x \neq 0$.

The implications (d), (e) and (f) can be proved in a similar way. Since $\varrho \neq 0$ ($\sigma \neq 0$), at least one of the vectors ϱ_x, ϱ_y and ϱ_z (σ_x, σ_y and σ_z) has a positive coordinate. So we are in the situation of the scheme. ■

In view of Lemma 1, for an equilibrium (ϱ, σ) for (C, C^t), ϱ_x and ϱ_y can be normalized so that they become strategies in Δ_m and Δ_n respectively. These strategies are important in

Theorem 1: Let (C, C^t) be the GKT-symmetrization of an $m \times n$ bimatrix game (A,B) with $A>0$ and $B<0$. Let $(\varrho,\sigma) \in E(C,C^t)$. Then $(\varrho_x(\varrho_x, 1_m)^{-1}, \sigma_y(\sigma_y, 1_n)^{-1}) \in E(A,B)$ and $(\sigma_x(\sigma_x, 1_m)^{-1}, \varrho_y(\varrho_y, 1_n)^{-1}) \in E(A,B)$.

Proof: Since $(\varrho,\sigma) \in E(C,C^t)$ we have

$$e_i C\sigma = \begin{cases} e_i A \sigma_y - \sigma_z & \text{for } i \in \{1,\ldots,m\} \\ \sigma_x B e_{i-m} + \sigma_z & \text{for } i \in \{m+1,\ldots,m+n\} \\ (\sigma_x, 1_m) - (\sigma_y, 1_n) & \text{for } i = m+n+1 \end{cases} \tag{1}$$

and

$$\varrho C^t e_j = \begin{cases} e_j A \varrho_y - \varrho_z & \text{for } j \in \{1,\ldots,m\} \\ \varrho_x B e_{j-m} + \varrho_z & \text{for } j \in \{m+1,\ldots,m+n\} \\ (\varrho_x, 1_m) - (\varrho_y, 1_n) & \text{for } j = m+n+1 \end{cases} . \tag{2}$$

By Lemma 1, $\varrho_x \neq 0$ and $\sigma_y \neq 0$.

Suppose $(\varrho_x)_i > 0$. Then, since $C(\varrho) \subset PB(C,\sigma)$, we obtain $e_i A\sigma_y - \sigma_z = \max_{k\in\{1,\ldots,m\}} \{e_k A\sigma_y - \sigma_z\}$. Hence $e_i A\sigma_y = \max_{k\in\{1,\ldots,m\}} e_k A\sigma_y$. Similarly, if $(\sigma_y)_j > 0$, then $\varrho_x B e_j = \max_{l\in\{1,\ldots,n\}} \varrho_x B e_l$.

This implies $C(\varrho_x(\varrho_x, 1_m)^{-1}) \subset PB(A, \sigma_y(\sigma_y, 1_n)^{-1})$, and $C(\sigma_y(\sigma_y, 1_n)^{-1}) \subset PB(B, \varrho_x(\varrho_x, 1_m)^{-1})$. Hence $(\varrho_x(\varrho_x, 1_m)^{-1}, \sigma_y(\sigma_y, 1_n)^{-1}) \in E(A,B)$.

Similarly one shows $(\sigma_x(\sigma_x, 1_m)^{-1}, \varrho_y(\varrho_y, 1_n)^{-1}) E(A,B)$. ■

If $(\varrho,\sigma) \in E(C,C^t)$, then also $(\sigma,\varrho) \in E(C,C^t)$. According to Theorem 1 these two equilibria of (C,C^t) yield the same two equilibria for (A,B). We now concern ourselves with a converse statement. Therefore we need the following definition.

For (p,q), $(\tilde{p},\tilde{q}) \in E(A,B)$, the *GKT-product* $(p,q)*(\tilde{p},\tilde{q})$ is defined as

$$\left(\left(\frac{p}{2-pBq}, \frac{\tilde{q}}{2+\tilde{p}A\tilde{q}}, 1-\frac{1}{2-pBq}-\frac{1}{2+\tilde{p}A\tilde{q}}\right),\right.$$
$$\left.\times\left(\frac{\tilde{p}}{2-\tilde{p}B\tilde{q}}, \frac{q}{2+pAq}, 1-\frac{1}{2-\tilde{p}B\tilde{q}}-\frac{1}{2+pAq}\right)\right).$$

Note that the GKT-product $*$ is well-defined since $A>0$ and $B<0$.

In fact we have $(p,q)*(\tilde{p},\tilde{q}) \in \Delta_{m+n+1} \times \Delta_{m+n+1}$. Furthermore, if $(\varrho,\sigma) := (p,q)*(\tilde{p},\tilde{q})$, then $(\sigma,\varrho) = (\tilde{p},\tilde{q})*(p,q)$. The next theorem shows the relevance of the GKT-product.

Theorem 2: Let (C,C^t) be the GKT-symmetrization of an $m\times n$ bimatrix game (A,B) with $A>0$ and $B<0$. Let $(p,q),(\tilde{p},\tilde{q}) \in E(A,B)$. Then $(p,q)*(\tilde{p},\tilde{q}) \in E(C,C^t)$ and $(\tilde{p},\tilde{q})*(p,q) \in E(C,C^t)$.

Proof: Let, for $(p,q),(\tilde{p},\tilde{q}) \in E(A,B)$, $(\varrho,\sigma) := (p,q)*(\tilde{p},\tilde{q})$. Then $(\varrho,\sigma) \in \Delta_{m+n+1} \times \Delta_{m+n+1}$ and $\varrho_x \neq 0$, $\varrho_y \neq 0$, $\varrho_z \neq 0$, and $\sigma_x \neq 0$, $\sigma_y \neq 0$, $\sigma_z \neq 0$ by construction.

(a) For $i \in \{1,\ldots,m\}$ such that $\varrho_i > 0$ we originally had $p_i > 0$. From $C(p) \subset PB(A,q)$ it then follows that $e_i A q = \max_{k\in\{1,\ldots,m\}} e_k A q$. Then we obtain from the GKT-product

$$e_i C\sigma = e_i A q(2+pAq)^{-1} - (1-(2-\tilde{p}B\tilde{q})^{-1} - (2+pAq)^{-1})$$
$$= \max_{k\in\{1,\ldots,m\}} e_k C\sigma .$$

Since $\max_{k\in\{1,\ldots,m\}} e_k A q = pAq$, the last expression also equals $(2-\tilde{p}B\tilde{q})^{-1} - (2+pAq)^{-1}$.

(b) Similarly, for a $j\in\{m+1,\ldots,m+n\}$ with $\varrho_j>0$

$$e_j C\sigma = \max_{l\in\{m+1,\ldots,m+n\}} e_l C\sigma = (2-\tilde{p}B\tilde{q})^{-1} - (2+pAq)^{-1} .$$

(c) Since $e_{m+n+1}C\sigma = (2-\tilde{p}B\tilde{q})^{-1} - (2+pAq)^{-1}$, it follows from (a) and (b) that $\max_{k\in\{1,\ldots,m+n+1\}} e_k C\sigma = (2-\tilde{p}B\tilde{q})^{-1} - (2+pAq)^{-1}$.

(d) Combining (a), (b) and (c) we obtain $C(\varrho) \subset PB(C,\sigma)$. Similarly one shows $C(\sigma) \subset PB(C^t,\varrho)$. Hence $(\varrho,\sigma)\in E(C,C^t)$, or equivalently $(p,q)*(\tilde{p},\tilde{q})\in E(C,C^t)$. Similarly one proves $(\tilde{p},\tilde{q})*(p,q)\in E(C,C^t)$. ■

The next theorem shows that each equilibrium for the GKT-symmetrization of a bimatrix game is the GKT-product of two equilibria for this bimatrix game.

Theorem 3: Let (C,C^t) be the GKT-symmetrization of an $m\times n$ bimatrix game (A,B) with $A>0$ and $B<0$. Let $(\varrho,\sigma)\in E(C,C^t)$. Then there are equilibria $(p,q),(\tilde{p},\tilde{q})\in E(A,B)$ such that $(\varrho,\sigma) = (p,q)*(\tilde{p},\tilde{q})$.

Proof: From Lemma 1 we obtain $\varrho_z>0$. So $\varrho C\sigma = (\sigma_x,1_m)-(\sigma_y,1_n)$ and $\varrho C^t\sigma = (\varrho_x,1_m)-(\varrho_y,1_n)$. Let $p=(\varrho_x,1_m)^{-1}\varrho_x$, $q=(\sigma_y,1_n)^{-1}\sigma_y$, $\tilde{p}=(\sigma_x,1_m)^{-1}\sigma_x$ and $\tilde{q}=(\varrho_y,1_n)^{-1}\varrho_y$. From Lemma 1 we obtain $p,\tilde{p}\in\Delta_m$ and $q,\tilde{q}\in\Delta_n$. By Theorem 1, $(p,q),(\tilde{p},\tilde{q})\in E(A,B)$. Furthermore $\max_{j\in\{1,\ldots,n\}} \varrho_x B e_j + \varrho_z = (\varrho_x,1_n)-(\varrho_y,1_n)$ (cf. (2)). Consequently

$$\max_{j\in\{1,\ldots,n\}} pBe_j = (\varrho_x,1_m)^{-1}[(\varrho_x,1_m)-(\varrho_y,1_n)-\varrho_z] .$$

Since $\varrho\in\Delta_{m+n+1}$, we have $(\varrho_x,1_m)+(\varrho_y,1_n)+\varrho_z=1$. Hence

$$\max_{j\in\{1,\ldots,n\}} pBe_j = (\varrho_x,1_m)^{-1}[2(\varrho_x,1_m)-1] .$$

Since $(p,q)\in E(A,B)$, $pBq = \max_{j\in\{1,\ldots,n\}} pBe_j$. Hence $pBq = (\varrho_x,1_m)^{-1} \times [2(\varrho_x,1_m)-1]$, or

$$(\varrho_x,1_m) = (2-pBq)^{-1} . \quad (3)$$

Similarly one obtains

$$(\sigma_y, 1_n) = (2 + pAq)^{-1} \tag{4}$$

$$(\sigma_x, 1_m) = (2 - \tilde{p}B\tilde{q})^{-1} \tag{5}$$

$$(\varrho_y, 1_n) = (2 + \tilde{p}A\tilde{q})^{-1} \ . \tag{6}$$

Now, if we use (3) – (6) in the definition of the GKT-product of the strategy pairs (p,q) and $(\tilde{p},\tilde{q})$, we find $(p,q)*(\tilde{p},\tilde{q}) = (\varrho,\sigma)$. ■

Theorems 1 – 3 yield a one-to-one correspondence between pairs of equilibria for a bimatrix game, and (symmetric) pairs of equilibria for the GKT-symmetrization of this bimatrix game. In the last theorem of this section we show that there is a similar correspondence with respect to maximal Nash subsets for the two games.

In order to describe this correspondence, we define the GKT-product also for maximal Nash subsets. Let (A,B) be an $m \times n$ bimatrix game and let (C,C^t) be the GKT-symmetrization of (A,B). Let S and T be maximal Nash subsets for (A,B). Then the GKT-product of S and T is defined as

$$S*T := \{(p,q)*(r,s) \,|\, (p,q) \in S \quad \text{and} \quad (r,s) \in T\} \ .$$

Theorem 4: Let (C,C^t) be the GKT-symmetrization of an $m \times n$ bimatrix game (A,B) with $A>0$ and $B<0$.

(i) If S and T are maximal Nash subsets for (A,B), then $S*T$ and $T*S$ are maximal Nash subsets for (C,C^t).
(ii) If U is a maximal Nash subset for (C,C^t), then $U = S*T$ for some maximal Nash subsets S and T for (A,B).

Proof: (i). Let S and T be maximal Nash subsets for (A,B). We only show that $S*T$ is a maximal Nash subset for (C,C^t).

(ia) First we show that $S*T$ is a Nash subset. Take $(\varrho_1,\sigma_1),(\varrho_2,\sigma_2) \in S*T$. By definition $(\varrho_1,\sigma_1) = (p_1,q_1)*(\tilde{p}_1,\tilde{q}_1)$ for some $(p_1,q_1) \in S$ and $(\tilde{p}_1,\tilde{q}_1) \in T$ and $(\varrho_2,\sigma_2) = (p_2,q_2)*(\tilde{p}_2,\tilde{q}_2)$ for some $(p_2,q_2) \in S$ and $(\tilde{p}_2,\tilde{q}_2) \in T$.
By Theorem 2, $(\varrho_1,\sigma_1),(\varrho_2,\sigma_2) \in E(C,C^t)$.
If we show that $(\varrho_1,\sigma_2) \in S*T$ and $(\varrho_2,\sigma_1) \in S*T$, it follows that $S*T$ is a Nash subset. We only show $(\varrho_1,\sigma_2) \in S*T$.

By construction

$$\varrho_1 = \left(\frac{p_1}{2-p_1 B q_1}, \frac{\tilde{q}_1}{2+\tilde{p}_1 A \tilde{q}_1}, 1-(2-p_1 B q_1)^{-1}-(2+\tilde{p}_1 A \tilde{q}_1)^{-1}\right),$$

$$\sigma_2 = \left(\frac{\tilde{p}_2}{2-\tilde{p}_2 B \tilde{q}_2}, \frac{q_2}{2+p_2 A q_2}, 1-(2-\tilde{p}_2 B \tilde{q}_2)^{-1}-(2+p_2 A q_2)^{-1}\right).$$

Since (p_1, q_1) and (p_2, q_2) are elements of S, we find $p_1 B q_1 = p_1 B q_2 = \max_{q \in \Delta_n} p_1 B q$ and $p_2 A q_2 = p_1 A q_2$.
Similarly $\tilde{p}_1 A \tilde{q}_1 = \tilde{p}_2 A \tilde{q}_1$ and $\tilde{p}_2 B \tilde{q}_2 = \tilde{p}_2 B \tilde{q}_1$.
Consequently we find $(\varrho_1, \sigma_2) = (p_1, q_2) * (\tilde{p}_2, \tilde{q}_1) \in S * T$. Hence $S * T$ is a Nash subset.

(ib) Now suppose U is a maximal Nash subset for (C, C^t) containing $S * T$. Let $(\tau_1, \omega_1), (\tau_2, \omega_2) \in U$. From Theorem 3 we obtain for $i = 1,2$ that $(\tau_i, \omega_i) = (r_i, s_i) * (\tilde{r}_i, \tilde{s}_i)$, where $(r_i, s_i), (\tilde{r}_i, \tilde{s}_i) \in E(A, B)$.
Evidently also (τ_1, ω_2) and (τ_2, ω_1) are elements of U. From the expression for the GKT-products $(r_i, s_i) * (\tilde{r}_i, \tilde{s}_i)$ for $i = 1, 2$, we find

$$\tau_1 = \left(\frac{r_1}{2-r_1 B s_1}, \frac{\tilde{s}_1}{2+\tilde{r}_1 A \tilde{s}_1}, 1-(2-r_1 B s_1)^{-1}-(2+\tilde{r}_1 A \tilde{s}_1)^{-1}\right),$$

$$\omega_2 = \left(\frac{\tilde{r}_2}{2-\tilde{r}_2 B \tilde{s}_2}, \frac{s_2}{2+r_2 A s_2}, 1-(2-\tilde{r}_2 B \tilde{s}_2)^{-1}-(2+r_2 A s_2)^{-1}\right).$$

Since $(\tau_1, \omega_2) \in E(C, C^t)$, we obtain from Theorem 3 that there are $(p, q), (\tilde{p}, \tilde{q}) \in E(A, B)$ such that $(\tau_1, \omega_2) = (p, q) * (\tilde{p}, \tilde{q})$.

Hence we have e.g. $\frac{p}{2-pBq} = \frac{r_1}{2-r_1 B s_1}$. Using $\sum_{i=1}^{m} p_i = \sum_{i=1}^{m} (r_1)_i = 1$, we find $\frac{1}{2-pBq} = \frac{1}{2-r_1 B s_1}$, and consequently $p = r_1$.
Similarly $q = s_2$, $\tilde{p} = \tilde{r}_2$ and $\tilde{q} = \tilde{s}_1$. So $(r_1, s_2) = (p, q) \in E(A, B)$ and $(\tilde{r}_2, \tilde{s}_1) = (\tilde{p}, \tilde{q}) \in E(A, B)$. By considering (τ_2, ω_1) in a similar way, we obtain (r_2, s_1), $(\tilde{r}_1, \tilde{s}_2) \in E(A, B)$. This implies that $\{(r_1, s_1), (r_2, s_2), (r_1, s_2), (r_2, s_1)\}$ and $\{(\tilde{r}_1, \tilde{s}_1), (\tilde{r}_2, \tilde{s}_2), (\tilde{r}_1, \tilde{s}_2), (\tilde{r}_2, \tilde{s}_1)\}$ are Nash subsets for (A, B).
Since the (τ_i, ω_i) are chosen arbitrary in U, we obtain that U is the GKT-product of two Nash subsets for (A, B). So in view of (ia) we obtain $U = S * T$ and hence $S * T$ is a maximal Nash subset.

(ii) Let U be a maximal Nash subset for (C, C^t). Similar to the proof above, we obtain that U is the GKT-product of two Nash subsets for (A, B). Each of these Nash subsets is contained in a maximal Nash subset for (A, B), and from (i) we obtain that the GKT-product of these maximal Nash subsets is a maximal Nash subset $\tilde{U}$ for (C, C^t). Evidently $U \subset \tilde{U}$. However, this implies $U = \tilde{U}$ and U is the GKT-product of two maximal Nash subsets for (C, C^t). ■

4 Behaviour of Refinements

In this section we investigate how three refinements of equilibria behave in the procedure of the GKT-symmetrization.

First we deal with quasi-strong equilibria, which were introduced by Harsanyi (1973).

An equilibrium (p,q) for a bimatrix game (A,B) is called *quasi-strong* if $C(p)=PB(A,q)$ and $C(q)=PB(B,p)$. The reader should note that not every equilibrium for a bimatrix game is quasi-strong. It is unknown whether every bimatrix game possesses a quasi-strong equilibrium.

Theorem 5: Let (C,C^t) be the GKT-symmetrization of an $m\times n$ bimatrix game (A,B) with $A>0$ and $B<0$.

(i) If $(p,q),(\tilde{p},\tilde{q})\in E(A,B)$ are quasi-strong, then both $(p,q)*(\tilde{p},\tilde{q})\in E(C,C^t)$ and $(\tilde{p},\tilde{q})*(p,q)\in E(C,C^t)$ are quasi-strong.

(ii) If $(\varrho,\sigma)\in E(C,C^t)$ is quasi-strong, then both $(\varrho_x(\varrho_x,1_m)^{-1},\ \sigma_y(\sigma_y,1_n)^{-1})\in E(A,B)$ and $(\sigma_x(\sigma_x,1_m)^{-1},\ \varrho_y(\varrho_y,1_n)^{-1})\in E(A,B)$ are quasi-strong.

Proof: (i) Let $(p,q),(\tilde{p},\tilde{q})\in E(A,B)$ both the quasi-strong and define $(\varrho,\sigma):=(p,q)*(\tilde{p},\tilde{q})$. Since, by Theorem 2, $(\varrho,\sigma)\in E(C,C^t)$, we obtain for $i\in\{1,\ldots,m+n+1\}$ that $\varrho_i>0$ implies $e_iC\sigma=\max_{k\in\{1,\ldots,m+n+1\}} e_kC\sigma$.

Suppose that for $i\in\{1,\ldots,m+n+1\}$

$$e_iC\sigma=\max_{k\in\{1,\ldots,m+n+1\}} e_kC\sigma\ . \tag{7}$$

Since $\varrho_x=\dfrac{p}{2-pBq}$, we find that $(\varrho_x)_{i_0}>0$ for at least one $i_0\in\{1,\ldots,m\}$.

Hence $\max_{k\in\{1,\ldots,m+n+1\}} e_kC\sigma=\max_{k\in\{1,\ldots,m\}} e_kC\sigma$.

For $k\in\{1,\ldots,m\}$ we have $e_kC\sigma=e_kAq(2+pAq)^{-1}-\sigma_z$. Consequently, if (7) is satisfied for an $i\in\{1,\ldots,m\}$, then

$$e_iAq(2+pAq)^{-1}-\sigma_z=e_iC\sigma=\max_{k\in\{1,\ldots,m+n+1\}} e_kC\sigma$$

$$=\max_{k\in\{1,\ldots,m\}} e_kC\sigma=\max_{k\in\{1,\ldots,m\}} e_kAq(2+pAq)^{-1}-\sigma_z.$$

So $e_i A q = \max_{k\in\{1,\ldots,m\}} e_k A q$. Since $C(p) = PB(A,q)$, we obtain that $p_i > 0$, and consequently $\varrho_i > 0$. Similarly one proves that if (7) is satisfied for an $i \in \{m+1,\ldots,m+n\}$, then also $\varrho_i > 0$.
Finally, by Lemma 1, $\varrho_{m+n+1} > 0$. Hence we have that for $i \in \{1,\ldots,m+n+1\}$, (7) implies $\varrho_i > 0$. This implies $C(\varrho) = PB(C,\sigma)$. Similarly one shows $C(\sigma) = PB(C^t,\varrho)$. Therefore (ϱ,σ) is a quasi-strong equilibrium for (C,C^t).

Similar arguments show that $(\sigma,\varrho) = (\tilde{p},\tilde{q}) * (p,q)$ is a quasi-strong equilibrium for (C,C^t).

(ii) The proof follows immediately from (1) and (2). ■

For a bimatrix game an isolated equilibrium is an equilibrium which is a maximal Nash subset itself. In view of Theorem 4 we obtain that Theorem 5 also holds if we replace the word quasi-strong by isolated. In Jansen (1987) a regularity concept for equilibria is introduced and it is proved that an equilibrium is *regular* if and only if it is isolated and quasi-strong. Hence we have

Corollary: Theorem 5 also holds if quasi-strong is replaced by regular.

Next we prove a result on perfect equilibria. This refinement was introduced by Selten (1975), who also showed that every bimatrix game possesses a perfect equilibrium. Instead of giving the original definition of perfectness we use the following equivalence proved by van Damme (1987): For a bimatrix game an equilibrium is *perfect* if and only if both equilibrium strategies are undominated, where for an $m \times n$ bimatrix game (A,B) a strategy $p \in \Delta_m$ $(q \in \Delta_n)$ is called *undominated* if for every $\tilde{p} \in \Delta_m$ $(\tilde{q} \in \Delta_n)$ such that $\tilde{p}A \geq pA$ $(B\tilde{q} \geq Bq)$ we have $\tilde{p}A = pA$ $(B\tilde{q} = Bq)$.

Theorem 6: Let (C,C^t) be the GKT-symmetrization of an $m \times n$ bimatrix game (A,B) with $A > 0$ and $B < 0$.

(i) If $(p,q),(\tilde{p},\tilde{q}) \in E(A,B)$ are perfect, then both $(p,q)*(\tilde{p},\tilde{q}) \in E(C,C^t)$ and $(\tilde{p},\tilde{q})*(p,q) \in E(C,C^t)$ are perfect.

(ii) If $(\varrho,\sigma) \in E(C,C^t)$ is perfect, then both $(\varrho_x(\varrho_x,1_m)^{-1},\ \sigma_y(\sigma_y,1_n)^{-1}) \in E(A,B)$ and $(\sigma_x(\sigma_x,1_m)^{-1},\ \varrho_y(\varrho_y,1_n)^{-1}) \in E(A,B)$ are perfect.

Proof: (i) Let $(p,q),(\tilde{p},\tilde{q}) \in E(A,B)$ both be perfect and define $(\varrho,\sigma) := (p,q)*(\tilde{p},\tilde{q})$. Since, by Theorem 2, $(\varrho,\sigma) \in E(C,C^t)$ we only have to show that ϱ and σ are undominated strategies for the game (C,C^t). Suppose a $\bar{\varrho} \in \Delta_{m+n+1}$ exists such that $\bar{\varrho}C \geq \varrho C$, or equivalently, using the GKT-product,

$$\begin{pmatrix} B\bar{\varrho}_y + \bar{\varrho}_z 1_m \\ \bar{\varrho}_x A - \bar{\varrho}_z 1_n \\ (\bar{\varrho}_y, 1_n) - (\bar{\varrho}_x, 1_m) \end{pmatrix}$$

$$\geq \begin{pmatrix} B\tilde{q}(2+\tilde{p}A\tilde{q})^{-1}+(1-(2+\tilde{p}A\tilde{q})^{-1}-(2-pBq)^{-1})1_m \\ pA(2-pBq)^{-1}-(1-(2+\tilde{p}A\tilde{q})^{-1}-(2-pBq)^{-1})1_n \\ (2+\tilde{p}A\tilde{q})^{-1}-(2-pBq)^{-1} \end{pmatrix} . \tag{8}$$

We consider two cases:

(a) $\bar{\varrho}_z \leq 1-(2+\tilde{p}A\tilde{q})^{-1}-(2-pBq)^{-1}$

(b) $\bar{\varrho}_z \geq 1-(2+\tilde{p}A\tilde{q})^{-1}-(2-pBq)^{-1}$.

(a) Suppose $\bar{\varrho}_y = 0$. Then the third line of (8) yields $-(\bar{\varrho}_x, 1_m) \geq (2+\tilde{p}A\tilde{q})^{-1} - (2-pBq)^{-1}$. Since $\bar{\varrho}_z = 1-(\bar{\varrho}_x, 1_m)$, we obtain

$$1+(2+\tilde{p}A\tilde{q})^{-1}-(2-pBq)^{-1} \leq \bar{\varrho}_z \leq 1-(2+\tilde{p}A\tilde{q})^{-1}-(2-pBq)^{-1} .$$

This implies $(2+\tilde{p}A\tilde{q})^{-1} \leq 0$, which contradicts $A>0$. So $\bar{\varrho}_y \neq 0$.
The first line of the inequality (8) yields $B\bar{\varrho}_y \geq B\tilde{q}(2+\tilde{p}A\tilde{q})^{-1}$, or equivalently, $(\bar{\varrho}_y, 1_n)B\bar{\varrho}_y(\bar{\varrho}_y, 1_n)^{-1} \geq B\tilde{q}(2+\tilde{p}A\tilde{q})^{-1}$.
Since $\bar{\varrho}_y(\bar{\varrho}_y, 1_n)^{-1} \in \Delta_n$, $\tilde{q}$ is undominated and $B<0$, we obtain from the last inequality that $(\bar{\varrho}_y, 1_n) \leq (2+\tilde{p}A\tilde{q})^{-1}$. Then the third line of (8) yields $(\bar{\varrho}_x, 1_m) \leq (2-pBq)^{-1}$. Since $\bar{\varrho}_z = 1-(\bar{\varrho}_x, 1_m)-(\bar{\varrho}_y, 1_n)$, the last two inequalities yield $\bar{\varrho}_z \geq 1-(2+\tilde{p}A\tilde{q})^{-1}-(2-pBq)^{-1}$. Thus $\bar{\varrho}_z = 1-(2+\tilde{p}A\tilde{q})^{-1}-(2-pBq)^{-1}$. This implies $(\bar{\varrho}_x, 1_m) = (2-pBq)^{-1}$ and $(\bar{\varrho}_y, 1_n) = (2+\tilde{p}A\tilde{q})^{-1}$. Then the undominatedness of p and $\tilde{q}$ for (A,B) implies $\bar{\varrho}C = \varrho C$. Consequently ϱ is undominated.

(b) A similar proof shows that also in this case ϱ is undominated.

Similarly one shows that also σ is undominated. Hence (ϱ, σ) is a perfect equilibrium for the game (C, C^t).

It is easily verified that then also $(\sigma, \varrho) = (\tilde{p}, \tilde{q}) * (p, q)$ is an undominated and hence perfect equilibrium for (C, C^t).

(ii) Let (ϱ, σ) be a perfect equilibrium for (C, C^t). We only show that $(\varrho_x(\varrho_x, 1_m)^{-1}, \sigma_y(\sigma_y, 1_n)^{-1})$ is a perfect equilibrium for (A,B). Suppose a $p \in \Delta_m$ exists such that $pA \geq \varrho_x(\varrho_x, 1_m)^{-1}A$. Define $\bar{\varrho} := (p(\varrho_x, 1_m), \varrho_y, \varrho_z) \in \Delta_{m+n+1}$. Then

$$\bar{\varrho}C = (B\varrho_y + \varrho_z 1_m, (\varrho_x, 1_m)pA - \varrho_z 1_n, (\varrho_y, 1_n) - (\varrho_x, 1_m)) \geq \varrho C .$$

Since ϱ is undominated, this implies $\bar{\varrho}C = \varrho C$, and in particular $pA = \varrho_x(\varrho_x, 1_m)^{-1}A$. Hence $\varrho_x(\varrho_x, 1_m)^{-1}$ is undominated. Similarly one shows that also $\sigma_y(\sigma_y, 1_n)^{-1}$ is undominated. Hence $(\varrho_x(\varrho_x, 1_m)^{-1}, \sigma_y(\sigma_y, 1_n)^{-1})$ is a perfect equilibrium for (A,B).

A similar proof shows that $(\sigma_x(\sigma_x, 1_m)^{-1}, \varrho_y(\varrho_y, 1_n)^{-1})$ is also a perfect equilibrium for (A,B). ■

Myerson (1978) introduced proper equilibria and showed that every bimatrix game possesses a proper equilibrium. For proper equilibria it is possible to prove an analogue to Theorem 6 (ii).

References

Brown GW, von Neumann J (1950) Solutions of Games by Differential Equations. Ann Math Stud 24:73 – 79

van Damme E (1987) Stability and Perfection of Nash Equilibria. Springer, Berlin

Gale D, Kuhn HW, Tucker AW (1950) On Symmetric Games. Ann Math Stud 24:81 – 87

Griesmer JH, Hoffman AJ, Robinson A (1963) On Symmetric Bimatrix Games. IBM Research Paper RC-959. IBM Corp, Thomas J Watson Research Center, Yorktown Heights, New York

Harsanyi JC (1973) Games with Randomly Disturbed Payoffs: A New Rationale for Mixed Strategy Equilibrium Points. Internat J Game Theory 2:1 – 23

Jansen MJM (1981) Maximal Nash Subsets for Bimatrix Games. Naval Res Logist Quart 28:147 – 152

Jansen MJM, Potters JAM, Tijs SH (1986) Symmetrizations of Two-Person Games. Methods Oper Res 54:385 – 402

Jansen MJM (1987) Regular Equilibrium Points of Bimatrix Games. OR Spektrum 9:87 – 92

Maynard Smith J (1982) Evolution and the Theory of Games. Cambridge University Press, Cambridge

Myerson RB (1978) Refinements of the Nash Equilibrium Concept. Internat J Game Theory 7:73 – 80

Nash JF (1951) Non-cooperative Games. Ann Math 54:286 – 295

von Neumann J (1928) Zur Theorie der Gesellschaftsspiele. Math Ann 100:295 – 320

Selten R (1975) Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games. Internat J Game Theory 4:25 – 55

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