SOLVABILITY IN LAMBDA CALCULI

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ABSTRACT. This paper discusses for various versions of the type-free λ-calculus the concept of solvability, introduced in [1]. For the λK-calculus an equivalent notion of head normal form was introduced by Wadsworth [17].

Arguments are given for the utility of this concept and the proposal that the unsolvable terms should be considered as terms without a meaning.

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INTRODUCTION.

We assume familiarity with the λ K- and λI-calculus as presented in [6], [7], and Scott's models D_∞, P_ω (see [15] [16]).

The discussion starts with a comparison between the λI- and λK-calculus. It turns out that in the I-case terms without a normal form (nf) can be equated consistently, but not so in the K-case. The general theorem is that the theory identifying all unsolvable terms is consistent. Moreover it will be shown that this theory has a unique Hilbert-Post complete extension(1).

For the I-case this HP complete extension is the extensional λ I-calculus in which all terms without a nf are equated : for the K-case it is the set of equations true in any of Scott's lattice theoretic models for the λ-calculus.

The representation of the partial recursive functions gives other arguments for the usefulness of the notion of solvability. For the λI-calculus partial recursive functions were represented in such a way that if f(n) is undefined, then, the representing term has no nf. Taking this representation literally for the λK-calculus causes several difficulties, which can be avoided by interpreting undefinedness as unsolvability. In this way it is possible to give a representation faithful w.r.t. the definitions of the μ-recursive functions.

The following versions of the type-free λ-calculus will be considered :

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1) Let $\mathfrak{T}$ be some set of equation between λ-terms. $\mathfrak{T}$ is *consistent* iff the theory $\lambda+\mathfrak{T}$ does not prove every equation. $\mathfrak{T}$ is *Hilbert-Post (HP) complete* iff (i) $\mathfrak{T}$ is consistent. (ii) $\lambda+\mathfrak{T} \nvdash M = N \Rightarrow \lambda+\mathfrak{T} + M = N$ is inconsistent. HP complete theories are maximal consistent collections of equations.

the λI- and the λK-calculus, with and without extensionality and their combinatory versions. For an introduction see [3], [6] or [8].

NOTATIONS. $\lambda I(\eta) \vdash$ denotes provability in the λ I-calculus (+ extensionality). Similarly for $\lambda K(\eta)$. Λ_I, Λ_K denote the set of λI-, resp. λK-terms, Λ_I°, Λ_K° denote the subsets of terms without free variables.

If $\lambda(\eta)$ Λ, Λ° are used in a certain context, then these symbols can be read throughout as $\lambda I(\eta)$, Λ_I, Λ_I° or $\lambda K(\eta)$, Λ_K, Λ_K°.
The same applies to derived notions (e.g. I- vs. K-solvable).
Also the notion of consistency depends on the system in which one works.

Ω denotes the term $(\lambda x.xx)(\lambda x.xx) \in \Lambda_I^\circ$.

1. λ I-vs. λ K-solvability.

The λI-calculus was introduced by Church. The λK-calculus is the λ-variant of the theory of combinators introduced by Schönfinkel and Curry.

In [7], 3S3 Church is cited as giving the following arguments for his preference of the λI-calculus over the λK-calculus.
1) What Church wanted to do with the λ-calculus could be done with the λ I-version, e.g. the representation of the partial recursive functions.
2) If one - like Church - considers as significant only terms having a normal form(2), then significant λ K-terms may have non-significant parts (e.g. $KI\Omega$). However, this is not so for λ I-terms.
3) The λ K-calculus might lead to inconsistencies.

The reason behind argument 1) can be explained by the existence of several approximations for K definable in the λI-calculus.
In the first place there exists a $K_1 \in \Lambda_I^\circ$, such that $K_1 x n = x$ for all numerals n (for Church numerals take $K_1 = \lambda xy.yIIx$). Secondly, more recently it is shown, that for a finite set of normal forms there exists a local K :

1.1. Theorem. (Barendregt, Klop) *Let $\mathfrak{N} \subset \Lambda_I^\circ$ be a finite set of terms having a normal form. Then there exists a $K_{\mathfrak{N}} \in \Lambda_I^\circ$ such that*

$$\lambda \vdash K_{\mathfrak{N}} MN = M \text{ for all M and all } N \in \mathfrak{N} \cup \{K_{\mathfrak{N}}\}.$$

Proof. See [12], 9. ∎

Since Church believed in his thesis, the fact that all partial recursive functions can be represented in the λI-calculus was satisfactory for him : all computational processes can be represented in the λ I-calculus. However this representation is not the most efficient one. For the representation of programming languages the λI-calculus has the disadvantage that a call by name mechanism

(2) *On several occasions, Scott strongly disagreed with this positions e.g.* [14], p. 159, 165. Below it will be shown that the views of Church and Scott are nevertheless compatible.

cannot be implemented : a λI-computation of 0.2^{100} has to first evaluate 2^{100} and then map the result on 0. In the λK-calculus 0.2^{100} can reduce immediately to 0. This feature of the λK-calculus is due to the fact mentioned in argument 2, which therefore is rather an *advantage* of this theory.

As to argument 3, the λK-calculus itself is consistent, as follows from the extension of the Church-Rosser theorem to the λK-calculus, see e.g. [8], [3]. However it would seem that Church is right that the λK-calculus may lead to inconsistencies. For if one interprets the «non-significance» of terms having no nf in such a way that they are to be identified, then the λ K-calculus becomes inconsistent. (This is not so for the λI-calculus, see 1.16).

1.2. Theorem. It is inconsistent to identify in the K-calculus all terms without a nf.

Proof. Let $M_i \equiv \lambda x.xi\Omega$ where Ω is a term without nf. Then

$$\lambda K^{+} M_0 = M_1 \vdash 0 = M_0 K = M_1 K = 1 \quad \blacksquare$$

However it will become clear that in the λ K-calculus not all terms without a nf should be considered as meaningless.

The inconsistency shown in 1.2 is due to the different «solving» behaviour of M_0, M_1.

1.3. Definition. A closed term M is *solvable* iff $\lambda \vdash M N_1 .. N_n = I$ for some n and some $N_1 .... N_n \in \Lambda$. An arbitrary term M is solvable iff its closure $\lambda\vec{x}.M$ is solvable.

To see the particular role if I (= λ x.x) in this definition, note that M (closed) is solvable iff $\forall P \; \exists \vec{N} \; M\vec{N} = P$.

The definition of solvability can be given for the various versions of the λ -calculus. Most of these concepts are equivalent.

1.4. Theorem. (i) M is solvable in one of the versions of the λ-calculus iff M is solvable in this version extended with extensionality.
ii) Solvability is invariant under the $\lambda \rightleftarrows$ CL translations.

Proof. See [9], 5 or [4] $\blacksquare$

Hence there are essentially only two concepts of solvability :

Definition 1.5. Let $M \in \Lambda^{\circ}$

(i) M is *I-solvable* iff $\exists \vec{N} \in \Lambda_I \; \lambda \vdash M\vec{N} = I$

(ii) M is *K-solvable* iff $\exists \vec{N} \in \Lambda_K \; \lambda K \vdash M\vec{N} = I$.

An arbitrary $M \in \Lambda$ is I- or K-solvable iff its closure is.

In both cases M may be a λK-term. Since the λK-calculus is conservative over the λI-calculus (see [2] 1.3), it does not matter if for $M \in \Lambda_I$ I-solvability is defined by provability in λI or λ K.

In both cases a syntactic characterization can be given.

1.6. Definition (Wadsworth) (i) A λ-term M *is in head normal form* (hnf) iff M is of the form $\lambda x_1 \ldots x_n . y N_1 .. N_n$, for some n, k $\geq$ 0.

(ii) A λK-term M *has a hnf* iff $\lambda K \vdash M = M'$ and M' is in hnf.

1.7. Theorem. (i) (Barendregt) For the λI-calculus : M is solvable $\Leftrightarrow$ M has a normal form.

(ii) (Wadsworth) For the λK-calculus : M is solvable $\Leftrightarrow$ M has a head normal form.

Proof. See [2] 2.1 or [12] 7 and [18] 5.3. ■

Head normal forms are a proper extension of the normal forms. Hence both for the I- and K-case terms with a normal form are solvable.

Passing from the λI- to the λK-calculus there is a choice for the generalization of the class of λI-terms $\{M \mid M \text{ has a nf}\} = \{M \mid M \text{ is solvable}\}$.

We claim that solvability is the more natural and fruitful concept.

The unsolvable (i.e. not solvable) terms not only have no nf, but even hereditarily so.

1.8. Theorem. Let M $\in \Lambda^{\circ}$.

Then M is solvable $\Leftrightarrow$ $\exists \vec{N}$ $M\vec{N}$ has a nf.

Proof. $\Rightarrow$ by definition . $\Leftarrow$ since normal forms are solvable. ■

This property of hereditary undefinedness motivates the following :

1.9. Proposal. The unsolvable terms should be considered as being meaningless (undefined).

Adopting proposal 1.9 reconciles the positions of Church and Scott concerning the significance of terms. Church had in mind the λ I-calculus, hence his standpoint is by 1.7 (i) equivalent to 1.9. Scott on the other hand was speaking about the λK-calculus, where the solvable terms form a proper extension of the terms having a normal form. That Scott will surely accept 1.9 follows from 1.18 below.

The definition of «M is in hf», viz. «M has no redex subterms», is syntactical. It was pointed out by Wadsworth that the concept of normalizability is essential syntactical for by his 2.13 it follows that there is no set $\mathfrak{N}\mathfrak{F} \subset D_\infty$ such that

$$M \text{ has a nf} \Leftrightarrow [M]^{D_\infty} \in \mathfrak{N}\mathfrak{F}$$

On the other hand, by its very definition, the property of solvability is semantical, i.e. makes sense in models, although not necessarily first order. (In D_∞ and P_ω solvability is a first order property as follows from 1.18). This is another argument for the preferability of taking the unsolvable terms rather than the terms without a nf, as the class of terms without a meaning.

Another argument for 1.9 is that the unsolvable terms are from a computational point of view not very informative, they are *generic* (in the sense of algebraic geometry : if a generic point has a certain property, the whole space has this property).

1.10. Theorem (genericity lemma). Let Ω be unsolvable. If $F\Omega = I$, then $FM = I$ for all M.

Proof. See [4] or [1] 3.2.3. ■

1.11. Corollary. Let Ω be unsolvable. If $F\Omega$ is solvable, then FM is solvable for all M.

Proof. Let $F\Omega$ be solvable. Then $F\Omega\vec{N} = I$.

Hence by 1.10 (applied to $F' \equiv \lambda x.Fx\vec{N}$)

$FM\vec{N} = I$, i.e. FM is solvable, for all M. ■

In [14] 1.8 Scott introduced the following notion :

1.12. Definition. $M \subseteq N \Leftrightarrow \forall F$ closed [FM defined $\Rightarrow$ FN defined] .

Interpreting «defined» as «solvable», as is proposed in 1.9, we derive the following axiom of Scott [14] 1.10.

1.13. Corollary. If Ω is undefined, then $\Omega \subseteq M$ for all M.

Proof. By 1.11. ■

The heriditary undefinedness of the unsolvable terms makes it plausible that they can be identified.

1.14. Definition. $\mathscr{H} = \{\Omega = \Omega' \mid \Omega, \Omega'$ are unsolvable$\}$. If it is necessary to distinguish between the I- and K-case we use the notations $\mathscr{H}_I$ and $\mathscr{H}_K$.

1.15. Theorem. The theory extended with $\mathscr{H}$ (+ extensionality) is consistent (both for the I- and K-case).

Proof. The first proof in [1] went via the genericity lemma.
See 2.4 for a more elegant version of this proof. A model theoretic proof for the K-case follows from 1.19. ■

Relativizing 1.15 to the I-case establishes :

1.16. Corollary. In the λI-calculus (+ extensionality) it is consistent to equate all terms without a nf.

1.17. Definition. A model $\mathfrak{M}$ of the λ-calculus is *sensible* iff all unsolvable terms are equal in $\mathfrak{M}$.

The term model of $\mathscr{H}$, which exists by 1.15, is a sensible model. Also Scotts models are sensible.

1.18. Theorem. (Hyland ;Wadsworth). Let D_∞ and P_ω be Scott's well-known models for the λ-calculus, with least elements $\perp$ and $\emptyset$ respectively.

Then M is unsolvable $\Leftrightarrow D_\infty \models M = I \Leftrightarrow P_\omega \models M = \emptyset$.

Proof. See [10] and [18] . ■

1.19. Corollary. D_∞ and P_ω are sensible models.

2. The Hilbert Post completion of $\mathscr{H}$.

Let $\mathrm{Th}(D_\infty)$ be the set of equations true in Scott's D_∞ models for the λ-calculus. Hyland and Wadsworth gave a syntactic characterization of $\mathrm{Th}(D_\infty)$

independent of the initial lattic D. Moreover they showed that Th(D_∞) is in the λ K-calculus the *unique* HP-complete extension $\mathcal{H}^*$ of $\mathcal{H}$. The existence of $\mathcal{H}^*$ can be proved without going into the details of D_∞ .

2.1. Definition

(i) Let M, N $\in \Lambda^\circ$. M and N are *solvably equivalent*, notation M $\approx$ N, iff for all F [FM is solvable $\Leftrightarrow$ FN is solvable].

(ii) Let M, N $\in \Lambda$ be arbitrary. M $\approx$ N iff $\lambda\vec{x}M \approx \lambda\vec{x}N$, where x is a string of variables such that FV(M) $\cup$ FV(N) $\subset \{x\}$.

Note that the definition 2.1 (ii) is independent of the choice of $\vec{x}$.

2.2. Definition. $\mathcal{H}^* = \{M = N \mid M \approx N\}$

2.3. Theorem.

(i) $\lambda + \mathcal{H}^* \vdash M = N \Rightarrow M = N \in \mathcal{H}^*$

(ii) $\mathcal{H}^*$ is consistent.

Proof.

(i) Induction on the length of proof of $\lambda + \mathcal{H}^* \vdash M = N$.

(ii) Note that I $\not\approx \Omega$ and use (i). ■

2.4. Theorem

(i) $\mathcal{H} \subset \mathcal{H}^*$; (ii) $\mathcal{H}$ is consistent.

Proof. (i) Suppose $\Omega = \Omega' \in \mathcal{H}$. Then Ω, Ω' are unsolvable. We may assume Ω, Ω' are closed. Claim $\Omega \approx \Omega'$. Indeed if FΩ were solvable, then by 1.11 F Ω' is solvable and conversely.

(ii) By (i), since $\mathcal{H}^*$ is consistent. ■

2.5. Theorem (Böhm). Let M,N have different $\beta\eta$ nf's. Then M = N is inconsistent.

Proof. For the K-case this is proved in [5]. For the I-case the proof is the same using the proof of 1.1, see [4]. ■

2.6. Lemma. (Jacopini) Let $\omega_3 = \lambda x.xxx$ and $\Omega_3 = \omega_3\omega_3$. Then $I = \Omega_3$ is inconsistent.

Proof. $I = \Omega_3 \vdash I = \Omega_3 = \Omega_3\omega_3 = I\omega_3 = \omega_3$.

Now 2.5 applies to I, ω_3. ■

2.7. Lemma. $\mathcal{H}$ + M = N is consistent $\Rightarrow$ M = N $\in \mathcal{H}^*$

Proof. Suppose M = N $\notin \mathcal{H}$, i.e. M $\not\approx$ N. Let, say, FM be solvable and FN unsolvable. Then $\exists\vec{N}$ $\lambda \vdash FM\vec{N} = I$ and $\lambda + \mathcal{H} \vdash FN\vec{N} = \Omega_3$, since $FN\vec{H}$ and Ω_3 are unsolvable. Therefore $\lambda + \mathcal{H} + M = N \vdash I$ $FM\vec{N} = FN\vec{N} = \Omega_3$, hence by 2.6 $\mathscr{H}+M = N$ is inconsistent. ∎

2.8. Theorem. $\mathscr{H}^*$ is the unique HP-complete extension of $\mathscr{H}$.

Proof. By 2.3 (ii), 2.4 (i) $\mathscr{H}^*$ is a consistent extension of $\mathscr{H}$.
Let $\mathfrak{T} \supset \mathscr{H}$ be consistent. Then by 2.7 $\mathfrak{T} \subset \mathscr{H}^*$. Hence $\mathscr{H}^*$ contains all consistent extensions of $\mathscr{H}$ and is therefore the unique maximal consistent, i.e. HP-complete, theory extending $\mathscr{H}$. ∎

Relativizing definition 2.2 to the K- and I-cases one obtains the HP-complete theories $\mathscr{H}^*_K$, $\mathscr{H}^*_I$.
Both sets have an interesting characterization, 2.9 and 2.12.

2.9. Theorem. (Hyland ; Wadsworth) $M = N \in \mathscr{H}^*_K \Leftrightarrow D_\omega \models M = N$.

Proof. See [10], [18] or [4]. ∎

2.10. Corollary. $D^\circ_\omega = \{ [M]^{D_\infty} | M \text{ closed term}\}$ is algebraically simple.

Proof. If D°_∞ would have a proper homomorphic image D', then Th(D') would be a consistent extension of $\mathscr{H}^*$. ∎

2.11. Lemma.(Curry) $M \in \Lambda$ has a β-nf $\Leftrightarrow$ M has a $\beta\eta$-nf. ∎

Proof. See [2], 1.4. ∎

2.12. Theorem. The set of equations provable in $\lambda I\eta + \{M=N \mid M, N\ \lambda I\text{-terms without nf}\}$ (i.e. in $\lambda I\eta + \mathscr{H}_I$) is $\mathscr{H}^*_I$ and hence HP-complete.

Proof. The theory $\lambda I\eta + \mathscr{H}_I$ is consistent by 1.16. Hence by 2.8 relativized to the I-case $\lambda I\eta + \mathscr{H}_I \vdash M = N \Rightarrow M = N \in \mathscr{H}^*_I$.

Now suppose $M \approx_I N$. Claim $\lambda I\eta + \mathscr{H}_I \vdash M = N$. Case 1. M,N are I-unsolvable.
Then $\mathscr{H}_I \vdash M = N$.

Case 2. Exactly one of M,N is I-solvable. Then not $M \approx_I N$.

Case 3. M,N are I-solvable. By 1.7 (i) and 2.11 M,N have $\beta\eta$-nf's.

Subcase 3.1 M,N have identical $\beta\eta$-nf's.
Then $\lambda I\eta \vdash M = N$.

Subcase 3.2 M,N have different $\beta\eta$-nf's.
Then by 2.5 not $M \approx_I N$, since $\mathscr{H}^*_I$ is consistent. ∎

It should be noted that $\mathscr{H}^*_I$ and $\mathscr{H}^*_K$ are overlapping sets of equations, and hence by their HP-completeness incompatible, i.e. $\mathscr{H}^*_I \cup \mathscr{H}^*_K$ is inconsistent.

2.13. Theorem. (Wadsworth) Let $H \equiv \lambda jxy.x(jy)$ and $J \equiv FPH$, i.e. the fixed point of H. Then $D_\infty \models I = J$.

Proof. See [18], 3.2. ∎

2.14. Remark. $\mathcal{H}_I^*$ and $\mathcal{H}_K^*$ are overlapping.

Proof. $I = J \in \mathcal{H}_K^*$ by 2.13 and 2.9, but $I = J \notin \mathcal{H}_I^*$.
since J has no nf.

$\Omega = \lambda x.x\Omega \in \mathcal{H}_I^*$ since both terms have no nf, hence are I-unsolvable,

but $\Omega = \lambda x.x\Omega \notin \mathcal{H}_K^*$ since $\lambda x.x\Omega$ is K-solvable and Ω not. ∎

3. The representation of the partial recursive functions.

In § 1 it was noted that the notions of solvability and having a normal form are the same for the λ I-calculus, but not so for the λ K-calculus. Definitions involving the concept of normal form which worked well for the I-case, have some disadvantages for the K-case.

3.1. Definition (Church) A partial function $f : \omega^k \to \omega$ is λ*I-definable* iff for some $F \in \Lambda_I^\circ$

$$\lambda I \vdash F\vec{n} = m \qquad \text{if } f(\vec{n}) = m$$

$$F\vec{n} \text{ has no nf} \qquad \text{if } f(\vec{n}) \text{ is undefined.}$$

In this situation F is said to *represent* f.

Here $0, 1, ..$ is some sequence $\in \Lambda_I^\circ$ representing the integers.

3.2. Theorem (Kleene). Let $f : \omega^k \to \omega$ be a partial function. Then
f is λI-definable $\Leftrightarrow$ f is partial recursive.

Proof. See [11]. ∎

In [7] 13A definition 3.1 was copied literally for the λK-calculus and the corresponding version of 3.2 was proved.

This had the following disadvantages :

1. Because of the call by name character of the λK-calculus, the representation of a composition is not necessarily the composition of the representations.
E.g. Let f(n) = 0 and g(n) be undefined for all n. Then

$$F = \lambda x.0 \text{ and } G = \lambda x\Omega \quad \lambda\text{ K-represent f and g.}$$

$f \circ g$ is the totally undefined function (= g). However

$F \circ G = \lambda x.F(Gx) = \lambda x.\,0$, which does not represent g.

The proof in [7] of the K-version of 3.2 avoids this difficulty by *writing the partial recursive functions in Kleene's normal form*

$$f(\vec{x}) = U(\mu z\, T(e,\vec{x},z)).$$

Then a representation of f can be found from its index e and the representations of U and T. However this representation is not intensional, i.e. does not preserve the definition of f as a μ-recursive function.

2. Having a normal form is not invariant under the standard translations between the various versions of the λ-calculus. E.g. S(K)(SII)) (K(SII)) is in combinatory

logic a normal form, but not its λ-translation, which is convertible to $\lambda a.\Omega$. Therefore the representation had to be proved separately for the different systems. See [7], 13 A4. Disadvantage 2 applies as well to the I-case.

Definition 3.1 can be modified in the spirit of proposal 1.9.

3.3. Definition. A partial function f : $\omega^k \to \omega$ is *strongly* λ-*definable* iff for some term F

$$\lambda \vdash F\vec{n} = m \qquad \text{if } f(\vec{n}) = m$$
$$F\vec{n} \text{ is unsolvable} \qquad \text{if } f(n) \text{ is undefined.}$$

3.4. Theorem. The strongly λ-definable functions are exactly the partial recursive functions.

Proof. If f is strongly λ-definable its graph is r.e. (since the λ-calculus is recursively axiomatized) hence f is partial recursive. Conversely we show as a prime example how the representations of two functions can be used to represent their composition. Let f,g be functions of one argument and be represented by F,G. Let h(x) = f(g(x)). Let $\square$ be a sequence of terms such that for all n, $\lambda \vdash n\square = I$. For Church's numerals we can take $\square = II$, for the numerals in [3] we can take $\square = FII$.

Now h can be represented by $H = \lambda x.(Gx\square)(F(Gx))$: if $h(n) = m$, then g(n) is defined and the «jamming» factor $Gn\square$ vanishes, i.e. becomes I. Therefore H n = f(g(x)). Let $\square$ be a sequence of terms such that for all n, or f(g(n)) is undefined. In the first case Gn is unsolvable, hence also Gn(F(Gn)), i.e. Hn is unsolvable. Also in the second case Hn is unsolvable.

Similarly one can give a definition-preserving representation of functions defined by primitive recursion or minimalization. ∎

It has been stressed by Kreisel [13], p. 177, 178 that in connection with the so called «superthesis», Church's thesis expresses less then we know. When we say that all mechanically computable numbertheoretic functions are λ-definable or recursive, we merely speak of the results of computations, of their graphs. But we have in mind that λ-terms correspond to our procedures for defining these functions. As far as the μ-recursive and the λ-definable functions are concerned, strong definability proves the equivalence not only in the sense of Church's thesis, but also of the super thesis : definitions are preserved.

Apart from the interest of representing the partial recursive intensionality, the following use has been made of strong definability.

3.5. Theorem (Wadsworth). Let φ_x be the partial recursive function with index x. There is a strong representation of the partial recursive functions, φ_x being represented by F_x, such that

$$\varphi_x \cong \varphi_y \Leftrightarrow D_\infty \models F_x = F_y.$$

Proof. See [19] or [4]. ∎

3.6. Corollary. (Wadsworth) $\mathcal{H}^* = \mathrm{Th}(D_\infty)$ is Π^0_2 complete.

Proof. By its very definition $\mathcal{H}^*$ is Π^0_2. By 3.5 it is complete, since $\varphi_x \cong \varphi_y$ is. ■

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