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§0. Introduction. The theory of Uniformly Reflexive Structures (URS) studied by Wagner and Strong ([8],[6],[1]), is an elegant axiomatization of parts of recursion theory. The theory abstracts some properties of the function \( n(m) \) (i.e. the \( n \)th partial recursive function applied to \( m \)) by considering arbitrary domains with a binary operation application. The standard URS is \( \mathcal{X} \) with domain \( \omega \cup \{\infty\} \) and application \( n.m = n(m) \) if defined, else.

However the URS are not complete for the description of recursion theory. Real computations do have a length, a feature which is missing in the URS. In fact there are sentences in the language of URS undecided by the axioms. E.g. let \( e = \lambda x.xx \), i.e. \( ex = xx \) for all \( x \), then \( ee = \infty \) is such a sentence. But this sentence holds in the intended interpretation \( \mathcal{X} \) as follows from an argument using length of computation.

Moreover in a URS it is not always possible to represent the partial recursive functions.

To overcome these defects we introduce a concept of a norm.

A Normed Uniformly Reflexive Structure (NURS) is a URS which a norm \( |. . . . | \) can be defined satisfying:
1. \( |x;y| \in \omega \cup \{\infty\} \)
2. \( |x;y| = \infty \iff x.y = \infty \)
3. \( |s.x.y;z| > |x.z;y.z| + |x;z| + |y;z| \), if \( |s.x.y;z| \neq \infty \)

The intended interpretation of \( |x;y| \) is "the length of computation of \( x.y \)".

The following facts motivate the introduction of NURS. As was intended \( \mathcal{X} \) is a NURS. Wagners (highly) constructible URS are NURS. In every NURS \( ee = \infty \) holds. More generally, for a NURS \( \mathcal{X} \) and a term \( M \) of the theory, \( M \) has no normal form \( \iff \mathcal{X} \vdash M = \infty \). In a NURS all splinters are semi-computable, and hence can be used to represent the partial recursive functions.
The use of length of computation in recursion theory has also been stressed by Y. Moschovakis [3]. In fact the axioms of the norm in a URS imply Moschovakis' condition on the length of computation.

Familiarity with URS is assumed. See e.g. Wagner [8] and Strong [6].

In §1 the defects of URS mentioned above are shown. A formal theory WS, convenient for the study of URS, is introduced in §2. The term model of an extension of WS provides some counter examples for the relation between semi-computable and recursively enumerable. The results about the NURS are proved in §3.

§1. The definition of a URS given below is not exactly the same as those of Wagner and Strong. The axioms are written down in a way showing the correspondence with combinatory logic. Axiom 7 is added; it implies that we may assume that terms with different normal forms are unequal in a URS (2.10).

1.1. Def. A URS is a structure \( \mathcal{U} = \langle U, \ast, i, k, s, \delta, \cdot \rangle \) such that the following holds where \( a, b, c \) are variables ranging over \( U - \{ \ast \} \):

1. \( \ast \cdot a = a \cdot \ast = \ast \cdot \ast = \ast \)
2. \( i \cdot a = a \)
3. \( k \cdot a \cdot b = a \)
4. \( s \cdot a \cdot b \cdot c = (a \cdot c) \cdot (b \cdot c) \); \( s \cdot a \cdot b \neq \ast \)
5. \( a = b \rightarrow \delta \cdot a \cdot b = k \); \( a \neq b \rightarrow \delta \cdot a \cdot b = k \cdot i \)
6. \( i \neq k \)
7. \( s \cdot a \cdot b = s \cdot a' \cdot b' \rightarrow a = a' \land b = b' \).

1.2. Def. Kleenes URS, \( \mathcal{K} \), is the structure \( \langle \omega^*, \ast, i, k, s, \delta, \cdot \rangle \) such that \( \omega^* = \omega \cup \{ \ast \} \) with, \( \ast \notin \omega \), \( n \cdot m = \{n\}(m) \) if defined

\( \ast \cdot n = n \cdot \ast = \ast \cdot \ast = \ast \), and \( i, k, s, \delta \) are to be found by the s-m-n theorem such that axioms 2, ..., 7 hold. As an example we construct \( k \).

Let \( \psi(x, y) = x \). Then \( \psi \) is partial recursive. Hence

\[
\begin{align*}
x &= \psi(x, y) \\
&= \{e\}(x, y) \quad \text{for some index } e \text{ of } \psi. \\
&= \{s_1^b\}(e, x)(y) \\
&= \{\{k\}(x)\}(y) \quad k \text{ index of } \lambda x. s_1^b(e, x). \\
&= k \cdot x \cdot y.
\end{align*}
\]
By pumping up the indices, cf. Rogers [4], p.83, we can assure that axiom 7 holds.

1.3. **Theorem.** Let $e = s.i.i$. Then $e.e = *$ is independent in the theory of the URS. 1)

**Proof.** It will be shown that $e.e = *$ is true in $\mathcal{X}$ but false in a modification $\mathcal{X}^s$.

We have $\mathcal{X} \not\models e.a = (1.a)(1.a) = a.a$, i.e. $(e)(a) = (a)(a)$.

The computation of $(e)(a)$ runs as follows:
Read $a$; compute $(a)(a)$. Hence the computation of $(e)(e)$ is:
Read $e$; compute $(e)(e)$; Read $e$; compute $(e)(e)$; ...
Therefore $(e)(e)$ is undefined. Hence $\mathcal{X} \not\models e.e = *$.

Let $\mathcal{X}^* = (\omega^*, *, i, k, s^*, \delta, *)$ be the following modification of $\mathcal{X}$.

\[ a \cdot b = a.b \quad \text{if} \ a \neq e \text{ or } b \neq e \]
\[ = 0 \quad \text{else} \]

Then $*$ is partial recursive. Let $s^*.a.b.c = (a \cdot c) \cdot (b \cdot c)$.

Again by pumping up the indices we may assume that $s^* \neq e$, $s^* a \neq e$ for all $a$ and $s^*.a.b = e$ iff $a = b = i$. Hence
$s^*.a.b.c = s^*.a.b.c = (a \cdot c) \cdot (b \cdot c)$, unless perhaps
$s^*.a.b.c = c = e$. But then $a = b = i$ and $(i \cdot e) \cdot (i \cdot e) = e \cdot e$.

It is clear that $i, k, \delta \neq e$ and the axioms 2, 3 and 5 follow.
Axiom 7 can be assured as in 1.2. Clearly $\mathcal{X}^* \not\models e.e = *$. \[ \Box \]

Another defect of the URS is the following. The partial recursive functions can be represented in a URS provided one has an infinite semi-computable (SC) splinter, Strong [6], 3.2. However, H. Friedman has shown that there is a URS without infinite SC splinter.

1.4. **Def.** Let $\alpha$ be a non-standard model of Peano arithmetic with universe $A$. Let $\mathcal{X}_\alpha$ be the structure $(A^*, *, i, k, s, \delta, *)$ where $* \notin A$, $i, k, s, \delta$ are as in 1.2 and $*$ is defined by
\[ a \cdot b = c \quad \text{if} \ \alpha \not\models \{a\}(b) = c \quad \text{i.e.} \ \alpha \not\models \exists z \left( T(a, b, z) \wedge U(z) = c \right) \]
\[ = * \quad \text{else} \]

$U$ and $T$ are the components of Kleene's normal form theorem. Then $\mathcal{X}_\alpha$ is a URS; e.g. $\mathcal{X}_\alpha \not\models k.a.b = a$ holds since $(\{k\}(a))(b) = a$ is provable in Peano arithmetic, hence $\alpha \not\models \{k\}(a)(b) = a$.

1) Compare this with the following: Let $E = \{x | x \in x\}$. Then $E \not\in E$ is independent in ZF without foundation, but refusible in ZF itself.
1.5. **Theorem** (H. Friedman). $\mathcal{A}$ is a URS without infinite SC splinter.

Proof. If $\mathcal{A}$ would contain an infinite SC splinter, each splinter would be SC, Strong [6] 3.11. Therefore the set of standard numbers would be SC. But this is absurd since SC sets are definable ($x \in A \iff f(x) \neq *$), and the set of standard numbers is not. $\blacklozenge$

1.6. Cor. There exists a URS with an infinite non SC splinter on which the partial recursive functions can be represented.

Proof. Let $\mathbb{N}$ be the standard model of Peano arithmetic. Let $\mathfrak{A} \equiv \mathbb{N}$ be a non-standard model. For each partial recursive function $\psi$ with index $e$ we have

$$\mathbb{N} \models \{e\}(n) = m \iff \psi(n) = m$$

$$\mathfrak{A} \not\models \exists z \ T(e, n, z) \iff \psi(n) \text{ is undefined.}$$

Therefore, since $\mathfrak{A} \equiv \mathbb{N}$, $\mathfrak{A} \not\models \{e\}(n) = m \iff \psi(n) = m$

Hence, there exists a URS such that only partial recursive functions with recursive domain can be represented on any of the infinite splinters.

1.7. **Theorem.** There exists a URS such that for no infinite splinter $X$ the partial recursive functions can be represented on $X$.

Proof. Let $\mathfrak{A}$ be a non-standard model of Peano arithmetic in which only the recursive r.e. sets are definable on $\omega$, see [2], Exc.7,p123. Let $\psi$ be a partial recursive function with non recursive domain $A$. Then $\psi$ is not representable on the splinter of standard integers for otherwise $A$ would be definable on $\omega$. But then $\psi$ is not representable on any infinite splinter $X$, since all infinite splinters are in bijective computable correspondence, [6], 3-7. $\blacklozenge$

§2. The following theory WS is convenient for the study of URS.

2.1. **Def.** WS has the following language.

**Alphabet:** $x_0, x_1, \ldots$ variables

$I, K, S, A, -$ constants

$>, =$ reduction, equality

$(,)$ brackets
Terms are inductively defined by
1. A variable or constant is a term
2. If $M, N$ are terms, so is $(MN)$.

Formulas are $M \supset N$ and $M = N$ where $M, N$ are terms.

Notation: $x, y, z, \ldots$ denote arbitrary variables

$M, N, L$ denote arbitrary terms

$M_1 \ldots M_n$ stands for $\ldots (M_1M_2) \ldots M_n$

$M \subseteq M'$ if $M$ is a subterm of $M'$

$x \in M$ if $x$ occurs in $M$

$M$ is closed if for no $x$ $x \in M$

$\equiv$ denotes syntactic equality.

If $M$ is a closed WS term and $U=(U, \ast, i, k, s, \delta, \lambda)$ is a URS, then $M^U$ is the obvious interpretation of $M$ in $U$: $\ast^U = \ast$, $1^U = i$, etc, $(MN)^U = M^UN^U$; $U \vdash M = N$ iff $M^U = N^U$.

A term $M$ is in normal form (nf) if it has no subterms of the form $\ast$, $IA$, $KAB$, $SABC$ or $\Delta AB$.

WS is defined by the following axioms and rules:

**I**
0. $\ast M \supset \ast$, $M \ast \ast$
1. $IM \supset M$
2. $KMN \supset M$ if $N$ is in nf
3. $SMNL \supset ML(NL)$
4. a $\Delta MM \supset K$ if $M$ is in nf
   b $\Delta MN \supset KI$ if $M, N$ are in nf and $M \neq N$

**II**
1. $M \supset M$
2. $M \supset M' = ZM \supset ZM'$, $MZ \supset M'Z$
3. $M \supset N$, $N \supset L \Rightarrow M \supset L$

**III**
1. $M \supset N \Rightarrow M = N$
2. $M = N \Rightarrow N = M$
3. $M = N$, $N = L \Rightarrow M = L$

2.2. (Church-Rosser theorem) If $WS \vdash M = N$, then for some term $Z$

$WS \vdash M \supset Z$ and $WS \vdash N \supset Z$.

Proof. Well-known. See e.g. [5, T. 11, p. 144].

2.3. **Def.** A WS-term $M$ has a nf if $WS \vdash M = M'$ and $M'$ is in nf.

By 2.2 the normal form of a term is unique if it exists. If $M$ has a nf, all its reduction sequences terminate, by the restriction in axioms I2, 4.
2.4. Def. Let \( \mathcal{U} \) be a URS with domain \( U \). WS(\( \mathcal{U} \)) is the theory WS modified as follows. For each \( a \in U, a \) is an additional constant. A term of WS(\( \mathcal{U} \)) is in nf, if it does not contain a subterm \(*, IA, \text{ etc. or } aM\). WS(\( \mathcal{U} \)) has the additional axioms 
\[ aM \Rightarrow a.M \text{.} \] 
Axiom I4.b should be replaced by 
\[ \Delta MN \Rightarrow KI \quad \text{if } M, N \text{ are nf's and } \forall \not\vDash M \neq N. \]

Clearly \( \not\vDash WS(\mathcal{U}) \).
2.2 and 2.3 apply also to WS(\( \mathcal{U} \)).

2.5. (Abstraction) Let M be a WS(\( \mathcal{U} \)) term not containing \(*.\)
Then there exists a WS(\( \mathcal{U} \)) term \( \lambda x. M \) such that
1. \( \lambda x. M \) is in nf; \( x \notin \lambda x. M \)
2. WS(\( \mathcal{U} \)) \( \vdash (\lambda x. M)N = [x/N]M \) for \( N \) in nf.

Proof. As in combinatory logic.

Note, however, that also there exists a WS term \( \lambda x. * \) in nf such that \( \not\vDash (\lambda x.*)a = * \) for all \( \mathcal{U} \).
Take e.g. \( \lambda x.* = S(K\omega)(K\omega) \) with \( \omega = \lambda x.\Delta(KI)(xx) \).

2.6. Def. Let \( M \sim M' \) denote \( Mx = M'x \) for \( x \notin MM' \).

2.7. (Fixed Point Theorem) There exists a WS term FP such that
1. WS \( \vdash FP f \sim f(FP f) \)
2. FP f is in nf.

Proof. Let \( \omega_f = \lambda xz.f(xx)z \) and FP f = \( \omega_f \omega_f \).

2.8. Lemma. Let M be a WS(\( \mathcal{U} \)) term. Then M is a nf \( \Rightarrow \)
\( \not\vDash M \neq * \).

Proof. The set of normal forms NF can be defined inductively 
by 1. \( a, I, K, S, \Delta \in NF \). 2. \( AB \in NF \Rightarrow KA, SA, \Delta A \text{ and } SAB \in NF \). Then 
the result follows inductively realizing that in a URS 
\( k.a, s.a, \delta.a, a.s.a.b \neq * \).

The pumping up of indices used in 1.2 and 1.3 can be done in 
each URS due to axiom 7.

2.9. Lemma.
Then there exists a term P such that for all \( \mathcal{U} \)
1. \( \not\vDash \mathcal{U} \vDash Pab \neq * \)
2. \( \not\vDash \mathcal{U} \vDash Pab \sim a \)
3. \( \not\vDash \mathcal{U} \vDash Pab = Pa'b' \Rightarrow a = a' \wedge b = b' \).
Proof. Let \( P = \lambda abx. K(ax)b \). Clearly \( P \) satisfies 1 and 2.

By writing out \( P \) in terms of \( I, K \) and \( S \), one sees that \( P \) satisfies 3 due to axiom 7.

2.10. Cor. Let \( M \neq M' \) be WS terms in nf. Then we may assume \( \forall \mathcal{U} \models M \neq M' \) for all \( \mathcal{U} \).

Proof. By changing if necessary the basic constants \( i, k, s, \) and \( \delta \), using \( P \). See e.g. [7], p. 133 bottom.

What we may we will.

2.11. Cor. \( WS(\mathfrak{U}) \) is a conservative extension of WS.

Proof. The only axiom of WS not in \( WS(\mathfrak{U}) \) is \( I4b \). However, this follows from the modified axiom by 2.10. Hence \( WS(\mathfrak{U}) \) is an extension of WS. If \( M, N \) are WS terms and \( WS(\mathfrak{U}) \vdash M = N \) (or \( \vdash M \geq N \)), then the proof involves only WS terms (unless \( WS \vdash M = N = \ast \)). The \( WS(\mathfrak{U}) \) axioms only can hold for \( A \neq B \), by 2.10.

Hence \( WS \vdash M = N \) (or \( M \geq N \)).

2.12. Theorem 1. \( WS(\mathfrak{U}) \vdash M = N \Rightarrow \forall \mathcal{U} \not\models M = N \)

Proof. 1. Induction on the length of proof of \( M = N \) using 2.10.

2. By 1. and 2.6.

The converse of 2.12. 1,2 are false. E.g. in \( \mathfrak{U} \not\models EE \neq \ast \)

where \( E = SII \). But \( EE \) has no nf. However, if \( \mathcal{U} \) is a NURS the converse of 2.12.2 is true. See 3.3.

2.13. Def. Let \( WS^* \) be WS augmented by the axioms:

\[
M \geq \ast \quad \text{if} \quad M \text{ has no nf.}
\]

For each NURS \( \mathcal{U} \) we will have the completeness result:

\[
WS^* \vdash M = N \iff \forall \mathcal{U} \not\models M = N, \quad \text{for closed} \ M, N; \]

see 3.5.

2.14. Def. \( \mathcal{U}(WS^*_o) \) (respectively \( \mathcal{U}(WS^*_c) \)) is the term model consisting of arbitrary (respectively closed) WS terms modulo provable equality in \( WS^* \). Clearly they are URS.

Similarly we define \( \mathcal{U}(WS^*_o, c(\mathcal{U})) \).

These term models can be used for some counter-examples.
2.15. **Def.** A subset $X$ of a URS $\mathcal{N}$ is RE if $X = \emptyset$ or $X = \text{Ra } f = \{a \mid \exists x (fx = a)\}$ for some total $f$ in $\mathcal{N}$ (i.e. $\forall a$ $fa \neq *$).

In $\mathcal{N}$, $X$ is RE $\iff$ $X$ is SC.

2.16. **Theorem 1**. For $\mathcal{N}(WS^*)$ we have

1. $X$ is SC $\iff$ $X$ is RE
2. $X$ is RE $\iff$ $X$ is SC
3. $X$ is computable $\iff$ $X$ is finite or cofinite.

**Proof.**

2.16.1 **Def.** The family of $F$, $\mathcal{J}(F)$, is the set

$$\{N \mid \exists F' \preceq F' \land N \in F'\}.$$ If $F$ has a nf, $\mathcal{J}(F)$ is finite.

Each reduction of $FA$ to a nf can be written in the form

$$FA \succeq \delta M_0[A] \succeq \delta M_1[A] \succeq \delta M_1'[A] \succeq \ldots \succeq M[A]$$

(\*)

where $\succeq_\delta$ is axiomatized leaving out the $\Delta$ reduction axioms and $\succeq_\beta$ is axiomatized leaving out the $\ast, I, K, S$ axioms. A may not actually occur in $M[A]$. Referring to the sequence (\*) we define:

2.16.2 **Def.** $\text{Diag}_n(F,A) = \{\Delta C_1[A]C_2[A] \mid \Delta C_1[B]C_2[B] \subseteq M_n\}$.

$B$ satisfies $\text{Diag}_n(F,A) \iff \Delta C_1[A]C_2[A] = \Delta C_1[B]C_2[B]$, for all members of $\text{Diag}_n(FA)$.

2.16.3 **Lemma.** Let $FA$ have a nf for all $A$. Let $xa \notin F$. Consider the sequence (\*) for $F(xa)$. Then

0. $B$ satisfies $\text{Diag}_n(F,xa) = M_n[B] \succeq M_n'[B]$.

1. $xa$ is never "active" (i.e. in a subterm of the form $((xa)P) \in M_n[xa], M_n'[xa]$).

2. For almost all, i.e. all except finitely many, $B$ satisfies $\text{Diag}_n(F,xa)$.

**Proof.** 0 is obvious.

1. follows by substituting for $xa$ a nf $\omega$ such that $\omega P$ has no nf for all $P$.

2. by realizing that the only possible exceptions are in $\mathcal{J}(F)$.

2. $\Rightarrow$ 1. follows as 0 with $\omega$ satisfying $\cup \text{Diag}_n(F,xa)$ and using 0.

\[\Box\]

1) A different example of 1. was given in Wagner [8], 6.13.

3. was proved by Strong [7] for the URS $\mathcal{N}(WS^*)$. 

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2.16.4 Cor. Let $F_A$ have a nf for all $A$. Let $x_a \notin F$ and $x_a \notin M$, the nf of $F(x_a)$. Then for almost all $B$

$$F(B) = F(x_a).$$

Proof. Let $\text{Diag}(F,x_a) = \cup \text{Diag}_n(F,x_a)$ which is finite. This is satisfied by almost all $B$ (2.16.3.2). Thus (2.16.3.0)

$$FB \supseteq M[B].$$

Also $F(x_a) \supseteq M[x_a]$. But then, since $x_a \notin M[x_a]$, $FB = F(x_a)$.

More easily one can prove the following.

2.16.5 Cor. Let $F(x_a)$ have a nf, where $x_a \notin F$, $x_a \notin$ the nf of $F(x_a)$. Then for $x' \notin F$ $F(x'a) = F(x_a)$.

Proof. Since $x'a$ is a non-active term, it does not matter if it occurs in an active place.

2.16.6 Cor. Suppose $RA F \subseteq$ closed normal forms. Then $Ra F$ is finite.

Proof. Take $x_a \notin F$. By the assumption, never $x_a \subseteq M$, the nf of $FA$. Hence for almost all $B$, $FB = F(x_a)$.

Now we can prove 2.16.

1. Take $X = \{K^n I \mid n \in \omega\}$. Then $X$ is an infinite splinter hence SC (since $\mathcal{U}^o(WS^*)$ is a NURS, see §3). Suppose $X$ were RE, say $X = Ra F$. Then $F$ satisfies the assumption of 2.16.6, but $Ra F = X$ is not finite. Contradiction.

2. Take $X = Ra F$, with $F_a = x_a$. Suppose $X$ were SC, i.e.

$GM = I \quad \text{if } M \in X$

* else

for some $G$. Take $a \in G$. Then $x_a \notin G$. Also $x_a \notin I$ which is the nf of $G(x_a)$. Hence for $x' \notin G$ it follows by 2.16.5 that $G(x'a) = G(x_a) = I$, i.e. $x'a \in X$, a contradiction.

3. Let $X = \emptyset$ be computable. Define $GM = M \quad \text{if } M \in X$

* $M_o$ else

for some $M_o \in X$.

Then $X = Ra G$. Suppose the complement of $X$ is not finite. Then there is a variable $x \notin Ra G \cup \mathcal{F}(G)$. Then $x_a \notin G$, $x_a \notin$ the nf of $G(x_a)$. Hence by 2.16.4 $GB = G(x_a)$ for almost all $B$, i.e. $X = Ra G$ is finite.
§3. For NURS it is convenient to define for elements of 
\( \omega \cup \{\infty\} \): 
\( p \geq q \) \iff \( p = \infty \vee p > q \). Then \( \geq \) is transitive and 
axiom 3 for a norm can be stated as 
\[ |s \cdot a \cdot b ; c| \geq |a \cdot c ; b ; c| + |a ; c| + |b ; c|. \]

3.1. Examples of NURS.
1. \( \mathcal{X} \) becomes a NURS by defining 
\[ |e ; x| = \mu z \ T(e, x, z) \] 
if defined 
\[ \infty \] 
else

Then an examination of the properties of the \( T \) predicate shows 
that this defines a norm on \( \mathcal{X} \).
2. \( \mathcal{U} \langle WS^* \rangle \) are NURS by defining 
\[ |F; X| = \text{the length of the inside out reduction of } FX \text{ to nf} \] 
\[ \infty \] 
if \( FX \) has no nf.

The inside out reduction only reduces redeces SABC, etc. when 
A, B and C are normal forms.

3. Let \( \mathcal{U} \) be a (highly) constructible URS in the sense of [8]. Then \( \mathcal{U} \) 
is a NURS:
Let 
\[ f(e; x) = \min \{e, x \in \Lambda_n\} \] 
if defined 
\[ \infty \] 
else.

Take 
\[ |e; x| = \mu f(e; n) \]. This is a norm on \( \mathcal{U} \), for let 
\( f(sxy, z) = n \), 
then 
\( sxy = \phi_s(x, y), n > 0 \) 
and 
\( (x, z), (y, z), (xz, yz) \in \Lambda_{n-1} \) (see 
[8], p. 20–21 for the notation). Then 
\( f(x, z), f(y, z), f(xz, yz) \leq n-1 \), 
and 
\[ |sxy; z| = \mu n > 3 \mu n^{-1} > |x; z| + |y; z| + |xz; yz|. \]

4. Let \( \mathcal{U} \) be a non-standard model of Peano arithmetic. Then \( \mathcal{U}_\alpha \) 
is not a NURS. This follows from 1.5 and 3.4. Similarly it follows 
from 1.3 and 3.2 that \( \mathcal{U}_* \) is not a NURS.

The sentence \( EE = \ast \), with \( E = SII \), which was independent in the 
theory of URS becomes true in all NURS.

3.2. Let \( E = SII \) and \( \mathcal{U} \) be a NURS. Then 
\( \mathcal{U} \models EE = \ast \).

Proof. Suppose \( EE \neq \ast \). Then \( |E; E| \neq \infty \). But then 
\[ |E; E| = |SII; E| > |IE; IE| = |E; E| \] 
, a contradiction. \( \square \)

More general

3.3. Theorem. Let \( \mathcal{U} \) be a NURS and \( M \) a WS(\( \mathcal{U} \)) term. Then 
\( M \) has no nf \( \iff \) \( \mathcal{U} \models M = \ast \).
Proof. \* By 2.12.2.

\* This will be proved in a number of steps.

3.3.1 Def. SC(M), the set of subcomputations of M, is defined inductively by:

If M is in normal form SC(M) = \emptyset; else M \in AB and SC(AB) = SC(A) \cup SC(B) \cup \{ [A^x;B^x] \}. Below we often omit the superscript \_M.

Clearly SC(M) is a finite set \( \mathbb{N} \cup \{ \infty \} \) and if M \supset M', then SC(M) \supset SC(M').

3.3.2 Def. \( IMI = \text{Max}(SC(M)) \). If SC(M) contains \( \infty \), \( IMI = \infty \).

3.3.3 Lemma. If M \supset M', then \( IMI > IM' \).

3.3.4 Lemma. \( IMI = \infty \iff \forall \in M = \ast \).

Proof. \( IMI = \infty \iff \in SC(M) \)

\[ \iff \text{for some } AB \subseteq M \quad |A;B| = \infty \]

\[ \iff \text{for some } AB \subseteq M \quad \forall \in AB = \ast \]

\[ \iff \forall \in M = \ast \] \( \square \)

3.3.5 Lemma. Let M \supset M' be an axiom of WSL, then \( IMI > IM' \).

Proof. Let M \equiv SABC and M' \equiv AC(BC).

Then SC(M) = \{ |S;A|, |SA;B|, |SAB;C| \} \cup SC(A) \cup SC(B) \cup SC(C).

SC(M') = \{ |A;C|, |B;C|, |AC;BC| \} \cup SC(A) \cup SC(B) \cup SC(C).

Since |SAB;C| > Max{|A;C|, |B;C|, |AC;BC|}

IMI > IM'. Equality may occur, e.g. if SC(C) contains the largest subcomputation.

If M \equiv KAB, M \equiv IA or M \equiv M', then M' \equiv A or M' \equiv M, hence M \supset M' and the result follows by 3.3.3.

If M \equiv \Delta AB, then M' \equiv K or \equiv KI, so SC(M) \supset SC(M') = \emptyset, hence IMI > IM'. Similarly if M \equiv aN. \( \square \)

3.3.6 Cor. If WS(L) M \supset M', then \( IMI > IM' \).

Proof. Induction on the length of proof of M \supset M'.

Let us consider only the case that M \supset M' is ZA \supset ZA' and is a direct consequence of A \supset A'. Then SC(ZA) = SC(Z) \cup SC(A) \cup \{ |Z;A| \} and similarly for SC(ZA'). Now \( \forall \in A = A' \), hence |Z;A| = |Z;A'|.

Hence \( IZA > IZA' \) by the induction hypothesis \( IMI > IM' \). \( \square \)

3.3.7 Def. A special redex is a WS\_L-term SABC, where A, B and C are in normal form.
3.3.8 **Lemma.** If $SABC$ is a special redex, then $\|SABC\| \geq \|AC(BC)\|$. 

**Proof.** Since $SC(A) = SC(B) = SC(C) = \emptyset$, $\|SABC\| = \max\{|S;A|, |SA;B|, |SAB;C|\} > |SAB;C| \geq \max\{|A;C|, |B;C|, |AC;BC|\} = \|AC(BC)\|$.

3.3.9 **Lemma.** Let $M$ be a WSO term without normal form. Then there exists a special redex $N$ without normal form in the family (see 2.16.1) of $M$, or else $\|M\| = \infty$.

**Proof.** Consider the finite set $T$ of subterms of $M$ partially ordered by $c$. Let $N$ be a minimal element of $T$ without a normal form. Then all subterms of $N$ have a normal form. Checking all possibilities it follows that $N$ is of the form $SABC$. Let $A^*, B^*$ and $C^*$ be the normal forms of $A$, $B$ and $C$. Now we have $M = \underbrace{\ldots(SABC)\ldots} \Rightarrow \underbrace{\ldots(SA^*B^*C^*)\ldots}$ and $SA^*B^*C^*$ is a special redex without normal form.

3.3.10 **Cor.** If $M$ has no normal form, then there exists a term $M'$ without normal form and $\|M\| \geq \|M'\|$.

**Proof.** Let $N$ be as in 3.3.9, then $\|M\| > \|N\| \geq \|M'\|$ by 3.3.6 and 3.3.3. Let $N > M'$. Then $\|N\| > \|M'\|$ by 3.3.8. Since $N$ has no normal form, neither has $M'$.

Now the proof of 3.3.10 can be given.

Let $M$ be a term without normal form. Suppose $\not\exists M \neq \ast$. Then $\|M\| \neq \infty$ by 3.3.4. Hence by 3.3.10 there exists a sequence $M, M', M'', \ldots$ such that $\|M\| > \|M'\| > \|M''\| > \ldots$ is an infinite descending chain of integers.

3.4. **Theorem.** In a NURS $\forall$ all infinite splinters are SC.

**Proof.** Let $X = \{f^n, o\}$ be an infinite splinter. Define by the fixed point lemma a WSO($\forall$) term $H$ such that

$Hyx = I$ if $y = x$

$H(y) = \ast$ else.

Then $h = (H_0)^{\infty}$ is a semi-characteristic function of $X$.

If $a \in X$, clearly $H \circ a = I$, hence $h(a) = \ast$.

If $a \notin X$, then $H \circ a > H f(a) > \ldots$, i.e.

$H \circ a$ has no nf. Hence $h(a) = \ast$ by 3.3.
WS* is a complete axiomatization for the equations true in all NURS.

3.5. **Theorem.** Let \( \mathcal{U} \) be a NURS. Then for closed WS terms:

\[
WS^* \vdash M = N \iff \mathcal{U} \models M = N.
\]

**Proof.** \( \Rightarrow \) By 2.12.1, 3.3. \( \Leftarrow \) By 2.10, 3.3. \( \square \)

3.6. **Theorem.** Each URS can be embedded in a NURS (cf. Wagner [8], p. 31, 6.2), if the similarity type has no constants.

**Proof.** Clearly \( \mathcal{U} \subseteq \mathcal{V}(WS^*_\omega, \omega) \) which is a NURS by 3.1.2. \( \square \)

**Concluding remarks.**

A URS is almost a precomputation theory in the sense of Moschovakis [3]1). Restricting the attention to single-valued functions, his computation theories have an additional length of computation \( |e;\bar{x}| \) satisfying

\[
(+) \quad |S^m_n(e,\bar{x});\bar{y}| > |e;\bar{x},\bar{y}|, \text{ if defined.}
\]

Define in a NURS \( |e;\bar{x}| = |e;\bar{x}_1| + |e.\bar{x}_1;\bar{x}_2| + \ldots + |e.\bar{x}_1\ldots\bar{x}_{n-1};\bar{x}_n| \).

Then it follows readily from the definition of \( S^m_n \) in a URS ([8], 2.6) that this norm satisfies Moschovakis' axiom (+).

As suggested in [6], there is another way of extending a URS. A selection\(^2\) URS is an URS containing a "selection operator" \( c \) such that

\[
\exists a[f.a \neq \bot] = f.(c.f) \neq \bot.
\]

1) Not quite, because a URS does not need to contain a computable successor set.

2) In [6] such a URS is called "well-ordered". This name is a little absurd as can be argued as follows. Let \( \mathcal{A} \) be a model of Peano arithmetic of power continuum. Then \( \mathcal{V}(\omega^\omega) \) is a selection URS but cannot be well-ordered in ZF. On the other hand \( \mathcal{V}(WS^*_\omega) \) is countable and hence well-ordered, but has no selection operator.
In a selection URS a set is computable iff it is SC and co SC, [6], 3.4. This is not true in a general URS, [8], p. 39 bottom. Having a norm or a selection operator are independent of each other. \( X \) has a selection operator \( (c)(e) = (\mu x T(e, (x),_0, (x),_1))_0 \) Since this is provably in arithmetic a selection operator, \( X_p \) is a selection URS but not a NURS. Conversely, it is not difficult to show that \( U(WS^*) \) is not a selection URS, although it is a NURS.

In a NURS it would be natural to require for a selection operator \( c \)

\[ |c; a| \geq |a; c.a| \]

cf. [3], p. 225, (6-4).

Acknowledgement. The paper is an elaboration of part II of the author's dissertation. He wishes to thank his supervisor professor G. Kreisel for his stimulating personality.
References.


