§0. Introduction. The theory of Uniformly Reflexive Structures (URS) studied by Wagner and Strong ([8],[6],[1]), is an elegant axiomatization of parts of recursion theory. The theory abstracts some properties of the function \( \{n\}(m) \) (i.e. the \( n \)th partial recursive function applied to \( m \)) by considering arbitrary domains with a binary operation \( \cdot \). The standard URS is \( \mathcal{K} \) with domain \( \omega \cup \{\ast\} \) and application \( n \cdot m = \{n\}(m) \) if defined, else.

However the URS are not completely adequate for the description of recursion theory. Real computations do have a length, a feature which is missing in the URS. In fact there are sentences in the language of URS undecided by the axioms. E.g. let \( e = \lambda x.xx \), i.e. \( ex = xx \) for all \( x \), then \( ee = \ast \) is such a sentence. But this sentence holds in the intended interpretation \( \mathcal{K} \) as follows from an argument using length of computation.

Moreover in a URS it is not always possible to represent the partial recursive functions.

To overcome these defects we introduce a concept of a norm. A Normed Uniformly Reflexive Structure (NURS) is a URS which

1. \( |x \cdot y| \in \omega \cup \{\ast\} \) can be defined satisfying:
   \[ s \cdot x \cdot y \cdot z \geq |x \cdot z \cdot y \cdot z| + |x \cdot z| + |y \cdot z| \], if \( s \cdot x \cdot y \cdot z \neq \ast \)

The intended interpretation of \( |x \cdot y| \) is "the length of computation of \( x \cdot y \)".

The following facts motivate the introduction of NURS. As was intended \( \mathcal{K} \) is a NURS. Wagners (highly) constructible URS are NURS. In every NURS \( ee = \ast \) holds. More generally, for a NURS \( \mathcal{N} \) and a term \( M \) of the theory, \( M \) has no normal form \( \iff \mathcal{N} \vdash M = \ast \).

In a NURS all splinters are semi-computable, and hence can be used to represent the partial recursive functions.
The use of length of computation in recursion theory has also been stressed by Y. Moschovakis [3]. In fact the axioms of the norm in a URS imply Moschovakis' condition on the length of computation.

Familiarity with URS is assumed. See e.g. Wagner [8] and Strong [6].

In §1 the defects of URS mentioned above are shown. A formal theory WS, convenient for the study of URS, is introduced in §2. The term model of an extension of WS provides some counter examples for the relation between semi-computable and recursively enumerable. The results about the NURS are proved in §3.

§1. The definition of a URS given below is not exactly the same as those of Wagner and Strong. The axioms are written down in a way showing the correspondence with combinatory logic. Axiom 7 is added; it implies that we may assume that terms with different normal forms are unequal in a URS (2.10).

1.1. Def. A URS is a structure $\mathcal{U} = (U, *, i, k, s, \delta, \cdot)$ such that the following holds where $a, b, c$ are variables ranging over $U - \{\ast\}$:

1. $* a = a * = * * = *$
2. $i a = a$
3. $k a b = a$
4. $s a b c = (a c)(b c) ; s a b \neq *$
5. $a = b \rightarrow \delta a b = k ; a \neq b \rightarrow \delta a b = k i$
6. $i \neq k$
7. $s a b = s a' b' \rightarrow a = a' \land b = b'$.

1.2. Def. Kleenes URS, $\mathcal{K}$, is the structure $(\omega^*, *, i, k, s, \delta, \cdot)$ such that $\omega^* = \omega \cup \{\ast\}$ with, $* \notin \omega$, $n.m = \{n\}(m)$ if defined $\ast$ else $*$.

Let $\Psi(x, y) = x$. Then $\Psi$ is partial recursive. Hence

$x = \Psi(x, y)$

$= \{e\}(x, y)$ for some index $e$ of $\Psi$.

$= \{s^1_1(e, x)\}(y)$

$= \{k\}(x)\}(y)$ k index of $\lambda x. s^1_1(e, x)$.

$= k x y$. 


By pumping up the indices, cf. Rogers [4], p. 83, we can assure that axiom 7 holds.

1.3. Theorem. Let e = s.i.i. Then e.e = * is independent in the theory of the URS.

Proof. It will be shown that e.e = * is true in \( \mathcal{K}^0 \) but false in a modification \( \mathcal{K}^* \).
We have \( \mathcal{K} \not\models e.a = (i.a)(i.a) = a.a, \) i.e. \( \{e\}(a) = \{a\}(a) \).

The computation of \( \{e\}(a) \) runs as follows:
Read a; compute \( \{a\}(a) \). Hence the computation of \( \{e\}(e) \) is:
Read e; compute \( \{e\}(e) \); Read e; compute \( \{e\}(e) \); ...
Therefore \( \{e\}(e) \) is undefined. Hence \( \mathcal{K} \not\models e.e = * \).
Let \( \mathcal{K}^* = (\omega^*, *, i, k, s^*, \delta, *) \) be the following modification of \( \mathcal{K} \).

Axiom 7 can be assured as in 1.2. Clearly \( \mathcal{K}^* \not\models e.e = * \).

Another defect of the URS is the following. The partial recursive functions can be represented in a URS provided one has an infinite semi-computable (SC) splinter, Strong [6], 3.2. However, H. Friedman has shown that there is a URS without infinite SC splinter.

1.4. Def. Let \( \mathcal{M} \) be a non-standard model of Peano arithmetic with universe A. Let \( \mathcal{K}(\mathcal{M}) \) be the structure \( (A^*, *, i, k, s, \delta, \alpha) \) where
\( \alpha \notin A \), \( i, k, s, \delta \) are as in 1.2 and \( \alpha \) is defined by
\( \alpha \cdot a = a \) \( \alpha \cdot * = * \).

Let \( a \cdot b = c \) if \( \mathcal{M} \not\models \{a\}(b) = c \) i.e. \( \mathcal{M} \not\models \exists z (T(a, b, z) \land U(z) = c) \).

U and T are the components of Kleene's normal form theorem. Then \( \mathcal{K}(\mathcal{M}) \) is a URS; e.g. \( \mathcal{K}(\mathcal{M}) \models k.a.b = a \) holds since \( \{k\}(a) \) holds in Peano arithmetic, hence \( \mathcal{M} \not\models \{k\}(a) \).

1) Compare this with the following: Let \( E = \{x \mid x \in x \} \). Then \( E \in E \) is independent in ZF without foundation, but refusable in ZF itself.
1.5. **Theorem** (H. Friedman). \( \mathcal{U} \) is a URS without infinite SC splinter.

Proof. If \( \mathcal{U} \) would contain an infinite SC splinter, each splinter would be SC, Strong [6] 3.11. Therefore the set of standard numbers would be SC. But this is absurd since SC sets are definable \( (x \in A \iff f(x) \neq *) \), and the set of standard numbers is not. \( \Box \)

1.6. **Cor.** There exists a URS with an infinite non SC splinter on which the partial recursive functions can be represented.

Proof. Let \( \mathbb{N} \) be the standard model of Peano arithmetic. Let \( \mathfrak{A} \equiv \mathbb{N} \) be a non-standard model. For each partial recursive function \( \psi \) with index \( e \) we have

\[
\mathbb{N} \models \{ e \} (n) = m \iff \psi(n) = m
\]

\[
\mathfrak{A} \models \exists z \, T(e,n,z) \iff \psi(n) \text{ is undefined.}
\]

Therefore, since \( \mathfrak{A} \equiv \mathbb{N} \), \( \mathfrak{A} \models \{ e \} (n) = m \iff \psi(n) = m \)

\[
\mathfrak{A} \models \exists z \, T(e,n,z) \iff \psi(n) \text{ is undefined.}
\]

However, there exists a URS such that only partial recursive functions with recursive domain can be represented on any of the infinite splinters.

1.7. **Theorem.** There exists a URS such that for no infinite splinter \( X \) the partial recursive functions can be represented on \( X \).

Proof. Let \( \mathfrak{A} \) be a non-standard model of Peano arithmetic in which only the **recursive** r.e. sets are definable on \( \omega \), see [2], Exc.7,p123.

Let \( \psi \) be a partial recursive function with non recursive domain \( A \). Then \( \psi \) is not representable on the splinter of standard integers for otherwise \( A \) would be definable on \( \omega \). But then \( \psi \) is not representable on any infinite splinter \( X \), since all infinite splinters are in bijective computable correspondence, [6],3-7. \( \Box \)

§2. The following theory WS is convenient for the study of URS.

2.1. **Def.** WS has the following language.

**Alphabet:** \( x_0, x_1, \ldots \) variables

I, K, S, A, \( \cdot \) constants

\( \rightarrow, = \) reduction, equality

(,) brackets
Terms are inductively defined by
1. A variable or constant is a term
2. If M, N are terms, so is (MN).

Formulas are M \geq N and M = N where M, N are terms.

Notation: x, y, z, ... denote arbitrary variables
M, N, L denote arbitrary terms
M_1 M_2 ... M_n stands for \((... (M_1 M_2) ... M_n)\)
M \subset M' if M is a subterm of M'
x \in M if x occurs in M
M is closed if for no x \(x \notin M\)
\(\equiv\) denotes syntactic equality.

If M is a closed WS term and \(\forall x (U, \cdot, i, k, s, \delta, \cdot)\) is a URS, then \(M^\forall\) is the obvious interpretation of M in \(\forall x: \_^\forall U = \_^\forall, I^\forall = i,\) etc, \((MN)^\forall = M^\forall N^\forall; \forall \# M = N \iff M^\forall = N^\forall\).

A term M is in normal form (nf) if it has no subterms of the form \(\_ U, KAB, SABC\) or \(\Delta AB\).

WS is defined by the following axioms and rules:

I 0. \(\_ \geq \_\)
1. \(\_ M \geq M\)
2. \(KMN \geq M\) if N is in nf
3. \(SMNL \geq ML(NL)\)
4.a \(\Delta MM \geq K\) if M is in nf
  b \(\Delta MN \geq KI\) if M, N are in nf and M \(\not\equiv N\)

II 1. \(M \geq M\)
2. \(M \geq M' \Rightarrow ZM \geq ZM', MZ \geq M'Z\)
3. \(M \geq N, N \geq L \Rightarrow M \geq L\)

III 1. \(M \geq N \Rightarrow M = N\)
2. \(M = N \Rightarrow N = M\)
3. \(M = N, N = L \Rightarrow M = L\)

2.2. (Church-Rosser theorem) If WS \(\vdash M = N\), then for some term Z WS \(\vdash M \geq Z\) and WS \(\vdash N \geq Z\).
Proof. Well-known. See e.g. [5], T. 14, p. 144.

2.3. Def. A WS-term M has a nf if WS \(\vdash M = M'\) and M' is in nf.

By 2.2 the normal form of a term is unique if it exists. If M has a nf, all its reduction sequences terminate, by the restriction in axioms I2, 4.
2.4. Def. Let $\mathcal{V}$ be a URS with domain $U$. WS($\mathcal{V}$) is the theory WS modified as follows. For each $a \in U$, $a$ is an additional constant. A term of WS($\mathcal{V}$) is in nf, if it does not contain a subterm $\omega$, IA, etc. or $aM$. WS($\mathcal{V}$) has the additional axioms $aM \geq a\cdot a$. Axiom $\text{I}_{4.\cdot}$ should be replaced by

$$\Delta MN \geq KI$$

if $M, N$ are nf's and $\forall \mathcal{V} \not\vdash M \neq N$.

Clearly $\forall \mathcal{V} \not\vdash WS(\mathcal{V})$.

2.2 and 2.3 apply also to WS($\mathcal{V}$).

2.5. (Abstraction) Let $M$ be a WS($\mathcal{V}$) term not containing $\omega$. Then there exists a WS($\mathcal{V}$) term $\lambda x.M$ such that

1. $\lambda x. M$ is in nf; $x \not\in \lambda x. M$
2. WS($\mathcal{V}$) $\vdash (\lambda x. M)N = [x/N]M$ for $N$ in nf.

Proof. As in combinatory logic.

Note, however, that also there exists a WS term $\lambda x. \cdot$ in nf such that $\forall \mathcal{V} \not\vdash (\lambda x. \cdot)a = \cdot$ for all $\mathcal{V}$.

Take e.g. $\lambda x. \cdot = S(K\omega)(K\omega)$ with $\omega = \lambda x. A(KI)(xx)$.

2.6. Def. Let $M \sim M'$ denote $Mx = M'x$ for $x \not\in MM'$.

2.7. (Fixed Point Theorem) There exists a WS term FP such that

1. WS $\vdash FP f \sim f(FP f)$
2. FP $f$ is in nf.

Proof. Let $\omega_f = \lambda xz. f(xx)z$ and $FP f = \omega_f \omega_f$.

2.8. Lemma. Let $M$ be a WS($\mathcal{V}$) term. Then $M$ is a nf $\Rightarrow$

$$\forall \mathcal{V} \not\vdash M \neq \cdot$$

Proof. The set of normal forms NF can be defined inductively by 1. $a, I, K, S, \Delta \in NF$. 2. $AB \in NF \Rightarrow KA, SA, \Delta A$ and $SAB \in NF$. Then the result follows inductively realizing that in a URS $k.a, s.a, \delta.a, s.a.b \neq \cdot$.

The pumping up of indices used in 1.2 and 1.3 can be done in each URS due to axiom 7.

2.9. Lemma.

Then there exists a term $P$ such that for all $\mathcal{V}$

1. $\forall \mathcal{V} \not\vdash Pab \neq \cdot$
2. $\forall \mathcal{V} \not\vdash Pab \sim a$
3. $\forall \mathcal{V} \not\vdash Pab = Pa'b' \rightarrow a = a' \land b = b'$. 
Proof. Let $P = \lambda a x. k(a x)b$. Clearly $P$ satisfies 1 and 2. By writing out $P$ in terms of $I$, $K$ and $S$, one sees that $P$ satisfies 3 due to axiom 7.

2.10. Cor. Let $M \neq M'$ be WS terms in nf. Then we may assume $\forall N \models M \neq M'$ for all $N$.
Proof. By changing if necessary the basic constants $i, k, s$, and $\delta$, using $P$. See e.g. [3], p. 133 bottom.

What we may we will.

2.11. Cor. $WS(\mathcal{W})$ is a conservative extension of $WS$.
Proof. The only axiom of $WS$ not in $WS(\mathcal{W})$ is $I4b$. However, this follows from the modified axiom by 2.10. Hence $WS(\mathcal{W})$ is an extension of $WS$. If $M, N$ are WS terms and $WS(\mathcal{W}) \vdash M = N$ (or $\vdash M \gg N$), then the proof involves only WS terms (unless $WS \vdash M = N = *$). The $WS(\mathcal{W})$ axioms only can hold for $A \neq B$, by 2.10. Hence $WS \vdash M = N \ (\vdash M \gg N)$.

2.12. Theorem 1. $WS(\mathcal{W}) \vdash M = N \Rightarrow \forall N \models M = N$
Proof. 1. Induction on the length of proof of $M = N$ using 2.10.
2. By 1. and 2.6.

The converse of 2.12. 1, 2 are false. E.g. in $\mathcal{W} \models EE \neq *$ where $E = SII$. But $EE$ has no nf. However, if $\mathcal{U}$ is a NURS the converse of 2.12. 2 is true. See 3.3.

2.13. Def. Let $WS^*$ be WS augmented by the axioms:

For each NURS $\mathcal{U}$ we will have the completeness result:

$WS^* \vdash M = N \iff \mathcal{U} \models M = N$, for closed $M, N$;
see 3.5.

2.14. Def. $\mathcal{U}(WS^*)$ (respectively $\mathcal{U}(WS^*_{oc})$) is the term model consisting of arbitrary (respectively closed) WS terms modulo provable equality in $WS^*$. Clearly they are URS.
Similarly we define $\mathcal{U}(WS^*_{oc}, (\mathcal{U}))$.

These term models can be used for some counter-examples.
2.15. Def. A subset $X$ of a URS $W$ is RE if $X = \emptyset$ or $X = \text{Ra}\ f = \{a \mid \exists x \ (fx = a)\}$ for some total $f$ in $M$ (i.e. $\forall a \ fa \neq \ast$).
In $\mathfrak{W}$, $X$ is RE $\iff$ $X$ is SC.

2.16. Theorem 1). For $\mathfrak{W}(\text{WS}^c)$ we have
1. $X$ is SC $\iff$ $X$ is RE
2. $X$ is RE $\iff$ $X$ is SC
3. $X$ is computable $\iff$ $X$ is finite or cofinite.

Proof.
2.16.1 Def. The family of $F$, $J(F)$, is the set
$\{N \mid \exists F' \ F \supseteq F' \wedge N \subseteq F'\}$. If $F$ has a nf, $J(F)$ is finite.
Each reduction of $FA$ to a nf can be written in the form
$$FA \supseteq \delta M_1[A] \supseteq \delta M_1[A] \supseteq \delta M_1[A] \supseteq \ldots \supseteq M[A] \quad (*)$$
where $\supseteq$ is axiomatized leaving out the $\Delta$ reduction axioms and
$\supseteq$ is axiomatized leaving out the $\ast$, I, K, S axioms. A may not
actually occur in $M[A]$. Referring to the sequence $(*)$ we define:
2.16.2 Def. $\text{Diag}_n(F,A) = \{AC_1[A]C_2[A] \mid AC_1[A]C_2[A] \subseteq M_n\}$.
$B$ satisfies $\text{Diag}_n(F,A) \iff AC_1[A]C_2[A] = AC_1[B]C_2[B]$ , for
all members of $\text{Diag}_n(FA)$.

2.16.3 Lemma. Let $FA$ have a nf for all $A$. Let $xa \notin F$. Consider
the sequence $(*)$ for $F(xa)$. Then
0. $B$ satisfies $\text{Diag}_n(F,xa) = M_n[B] \supseteq \delta M_n[B]$.
1. $xa$ is never "active" (i.e. in a subterm of the form
   $(xa)P$) in $M_n[xa]$, $M_n'[xa]$.
2. For almost all, i.e. all except finitely many, $B$ satisfies
   $\text{Diag}_n(F,xa)$.

Proof. 0 is obvious.
1. follows by substituting for $xa$ a nf $\omega$ such that $\omega P$ has no nf
   for all $P$.
2. by realizing that the only possible exceptions are in $J(F)$.
$n \rightarrow n+1$ follows as $1$ with $\omega$ satisfying $\cup \text{Diag}_n(F,xa)$ and
   using 0.

1) A different example of 1. was given in Wagner [8], 6.13.
3. was proved by Strong [7] for the URS $\mathfrak{W}(\text{WS}^c)$.
2.16.4 Cor. Let $F_A$ have a nf for all $A$. Let $x_a \notin F$ and $x_a \notin M$, the nf of $F(x_a)$. Then for almost all $B$ $F(B) = F(x_a)$.

Proof. Let $\text{Diag}(F,x_a) = \cup \text{Diag}_n(F,x_a)$ which is finite. This is satisfied by almost all $B$ (2.16.3.2). Thus (2.16.3.0) $FB \supset M[B]$. Also $F(x_a) \supset M[x_a]$. But then, since $x_a \notin M[x_a]$, $FB = F(x_a)$.

More easily one can prove the following.

2.16.5 Cor. Let $F(x_a)$ have a nf, where $x_a \notin F$, $x_a \notin$ the nf of $F(x_a)$. Then for $x' \notin F$ $F(x'a) = F(x_a)$.

Proof. Since $x'a$ is a non-active term, it does not matter if it occurs in an active place.

2.16.6 Cor. Suppose $RA F \subset$ closed normal forms. Then $Ra F$ is finite.

Proof. Take $x_a \notin F$. By the assumption, never $x_a \subset M$, the nf of $FA$. Hence for almost all $B$, $FB = F(x_a)$.

Now we can prove 2.16.

1. Take $X = \{K^nI \mid n \in \omega\}$. Then $X$ is an infinite splinter hence SC (since $\forall (\omega \omega^*)$ is a NURS, see §3). Suppose $X$ were RE, say $X = Ra F$. Then $F$ satisfies the assumption of 2.16.6, but $Ra F = X$ is not finite. Contradiction.

2. Take $X = Ra F$, with $F_a = x_a$. Suppose $X$ were SC, i.e.

GM = I if $M \in X$

* else

for some $G$. Take $a \notin G$. Then $x_a \notin G$. Also $x_a \notin I$ which is the nf of $G(x_a)$. Hence for $x' \notin G$ it follows by 2.16.5 that $G(x'a) = G(x_a) = I$, i.e. $x'a \in X$, a contradiction.

3. Let $X = \emptyset$ be computable. Define

GM = M if $M \in X$

M_o else for some $M_o \in X$.

Then $X = Ra G$. Suppose the complement of $X$ is not finite. Then there is a variable $x \notin Ra G \cup \forall(G)$. Then $x_a \notin G$, $x_a \notin$ the nf of $G(x_a)$. Hence by 2.16.4 $GB = G(x_a)$ for almost all $B$, i.e. $X = Ra G$ is finite.
§3. For NURS it is convenient to define for elements of 
\( \omega \cup \{ \omega \} \): \( p \geq q \) iff \( p = \omega \lor p > q \). Then \( \geq \) is transitive and 
axiom 3 for a norm can be stated as 
\[ |s\cdot a\cdot b\cdot c| \geq |a\cdot c; b\cdot c| + |a; c| + |b; c|. \]

3.1. Examples of NURS.

1. \( \mathcal{X} \) becomes a NURS by defining 
\[ |e;x| = \mu z \ T(e,x,z) \quad \text{if defined} \]
\[ = \infty \quad \text{else} \]
Then an examination of the properties of the \( T \) predicate shows 
that this defines a norm on \( \mathcal{X} \).

2. \( \mathcal{V}(\text{WS}^*_0) \) are NURS by defining 
\[ |F;X| = \text{the length of the inside out reduction of } FX \text{ to } \text{nf} \]
\[ = \infty \quad \text{if } FX \text{ has no nf}. \]
The inside out reduction only reduces reduct \( \text{SABC, etc. when} \)
\( A, B \) and \( C \) are normal forms.

3. Let \( \mathcal{Y} \) be a (highly) constructible URS in the sense of [8]. Then \( \mathcal{Y} \)
is a NURS:
Let \( f(e;x) = \mu n \{ (e,x) \in \Lambda_n \} \) if defined 
\[ = \infty \quad \text{else}. \]
Take \( |e;x| = \mu f(e;n) \). This is a norm on \( \mathcal{Y} \), for let \( f(sxy,z) = n \),
then \( sxy = \phi_0(x,y), n > 0 \) and \( (x,z),(y,z),(xz,zy) \in \Lambda_{n-1} \) (see 
[8], p. 20-21 for the notation). Then \( f(x,z), f(y,z), f(xz,zy) \leq n-1 \),
and \( |sxy;z| = \mu n > 3.4^{n-1} > |x;z| + |y;z| + |xz;yz| \).

4. Let \( \mathcal{X} \) be a non-standard model of Peano arithmetic. Then \( \mathcal{X}_\mathcal{X} \)
is not a NURS. This follows from 1.5 and 3.4. Similarly it follows 
from 1.3 and 3.2 that \( \mathcal{X}^* \) is not a NURS.

The sentence \( EE = * \), with \( E = \text{SII} \), which was independent in the 
threeory of URS becomes true in all NURS.

3.2. Let \( E = \text{SII} \) and \( E^* \) be a NURS. Then 
\( \mathcal{V} \vdash EE = * \).
Proof. Suppose \( EE \neq * \). Then \( |E;E| \neq \infty \). But then 
\[ |E;E| = |\text{SII};E| > |IE;IE| = |E;E|, \quad \text{a contradiction}. \]

More general 

3.3. Theorem. Let \( \mathcal{V} \) be a NURS and \( M \) a WS(\( \mathcal{V} \) term. Then 
\( M \) has no nf \( \iff \mathcal{V} \vdash M = * \).
Proof. By 2.12.2.
This will be proved in a number of steps.

3.3.1 **Def.** \( \text{SC}(M) \), the set of subcomputations of \( M \), is defined inductively by:
If \( M \) is in normal form \( \text{SC}(M) = \emptyset \); else \( M \in AB \) and \( \text{SC}(AB) = \text{SC}(A) \cup \text{SC}(B) \cup \{|A; B^*|\} \). Below we often omit the superscript \( \mathcal{U} \).
Clearly \( \text{SC}(M) \) is a finite set \( c \cup \{0\} \) and if \( M \supset M' \), then
\( \text{SC}(M) \supset \text{SC}(M') \).

3.3.2 **Def.** \( \|M\| = \text{Max}\{\text{SC}(M)\} \). If \( \text{SC}(M) \) contains \( \infty \), \( \|M\| = \infty \).

3.3.3 **Lemma.** If \( M \supset M' \), then \( \|M\| > \|M'\| \).

3.3.4 **Lemma.** \( \|M\| = \infty \iff \forall \not\vDash M = \ast \).
Proof. \( \|M\| = \infty \iff \exists \in \text{SC}(M) \)
\( \iff \text{for some } AB \subseteq M \quad |A; B| = \infty \)
\( \iff \text{for some } AB \subseteq M \quad \forall \not\vDash AB = \ast \)
\( \iff \forall \not\vDash M = \ast \).

3.3.5 **Lemma.** Let \( M \Rightarrow M' \) be an axiom of \( WS(\mathcal{A}) \). Then \( \|M\| > \|M'\| \).
Proof. Let \( M \equiv SABC \) and \( M' \equiv AC(BC) \).
Then \( \text{SC}(M) = \{|S; A|, |SA; B|, |SAB; C|\} \cup \text{SC}(A) \cup \text{SC}(B) \cup \text{SC}(C) \).
\( \text{SC}(M') = \{|A; C|, |B; C|, |AC; BC|\} \cup \text{SC}(A) \cup \text{SC}(B) \cup \text{SC}(C) \).
Since \( |SAB; C| > \text{Max}\{ |A; C|, |B; C|, |AC; BC| \} \)
\( \|M\| > \|M'\| \). Equality may occur, e.g. if \( \text{SC}(C) \) contains the largest subcomputation.
If \( M \equiv KAB, M \equiv IA \) or \( M \equiv M' \), then \( M' \equiv A \) or \( M' \equiv M \), hence \( M \supset M' \) and the result follows by 3.3.3.
If \( M \equiv \Delta AB, \) then \( M' \equiv K \) or \( \equiv KI \), so \( \text{SC}(M) \supset \text{SC}(M') = \emptyset \), hence \( \|M\| > \|M'\| \). Similarly if \( M \equiv aN \).

3.3.6 **Cor.** If \( WS(\mathcal{A}) \ M \Rightarrow M' \), then \( \|M\| > \|M'\| \).
Proof. Induction on the length of proof of \( M \Rightarrow M' \).
Let us consider only the case that \( M \Rightarrow M' \) is \( ZA \Rightarrow ZA' \) and is a direct consequence of \( A \Rightarrow A' \). Then \( \text{SC}(ZA) = \text{SC}(Z) \cup \text{SC}(A) \cup \{|Z; A|\} \)
and similarly for \( \text{SC}(ZA') \). Now \( \forall \not\vDash A = A' \), hence \( |Z; A| = |Z; A'| \).
Hence \( \|ZA\| > \|ZA'\| \) by the induction hypothesis \( \|A\| > \|A'\| \).

3.3.7 **Def.** A **special redex** is a \( WS(\mathcal{A}) \) term \( SABC \), where \( A, B \) and \( C \) are in normal form.
3.3.8 Lemma. If $SABC$ is a special redex, then

$$\|SABC\| \geq \|AC(BC)\|.$$  \(\Box\)

Proof. Since $SC(A) = SC(B) = SC(C) = \emptyset$

$$\|SABC\| = \max\{|S;A|,|SA;B|,|SAB;C|\} \geq |SAB;C| \geq \max\{|A;C|,|B;C|,|AC;BC|\} = \|AC(BC)\|.$$  \(\Box\)

3.3.9 Lemma. Let $M$ be a WSO\textsuperscript{W} term without normal form. Then there exists a special redex $N$ without normal form in the family (see 2.16.1) of $M$, or else $\forall x\in M \Rightarrow x\neq \ast$.

Proof. Consider the finite set $T$ of subterms of $M$ partially ordered by $c$. Let $N$ be a minimal element of $T$ without a normal form. Then all subterms of $N$ have a normal form. Checking all possibilities it follows that $N$ is of the form $SABC$. Let $A^*, B^*$ and $C^*$ be the normal forms of $A$, $B$ and $C$. Now we have

$$M = \text{(SABC)} \Rightarrow \text{(SA*B*C*)} \Rightarrow \text{(AC*BC*)}$$

and $SA*B*C^*$ is a special redex without normal form.  \(\Box\)

3.3.10 Cor. If $M$ has no normal form, then there exists a term $M'$ without normal form and $\|M\| \geq \|M'\|$.  \(\Box\)

Proof. Let $N$ be as in 3.3.9, then $\|M\| > \|N\|$ by 3.3.6 and 3.3.3. Let $N \geq M'$. Then $\|N\| > \|M'\|$ by 3.3.8. Since $N$ has no normal form, neither has $M'$.

Now the proof of 3.3.\textsuperscript{=} can be given.

Let $M$ be a term without normal form. Suppose $\forall x\in M \Rightarrow x\neq \ast$. Then $\|M\| \neq \ast$ by 3.3.4. Hence by 3.3.10 there exists a sequence $M, M', M'', \ldots$ such that $\|M\| > \|M'\| > \|M''\| > \ldots$ is an infinite descending chain of integers.  \(\Box\)

3.4. Theorem. In a NURS $\forall x$ all infinite splinters are SC.

Proof. Let $X = \{f^n o\}$ be an infinite splinter. Define by the fixed point lemma a WSO($\forall x$) term $H$ such that

$$H(y)x = I \quad \text{if } y = x$$

$$H(fy)x \text{ else.}$$  

Then $h = (H_o)^{1/2}$ is a semi-characteristic function of $X$:

If $a \in X$, clearly $H \circ a = I$, hence $h_a \neq \ast$.

If $a \notin X$, then $H \circ a \gg H f(o)a \gg \ldots$, i.e.

$H \circ a$ has no nf. Hence $h_a = \ast$ by 3.3.  \(\Box\)
WS* is a complete axiomatization for the equations true in all NURS.

3.5. **Theorem.** Let \( \mathcal{U} \) be a NURS. Then for closed WS terms:

\[
\text{WS* } \vdash M = N \iff \forall x M = N.
\]

Proof. \( \Rightarrow \) By 2.12.1, 3.3. \( \Leftarrow \) By 2.10, 3.3. \( \square \)

3.6. **Theorem.** Each URS can be embedded in a NURS (cf. Wagner [8], p. 31, 6.2), if the similarity type has no constants.

Proof. Clearly \( \forall x \in \mathcal{N}(\text{WS}^*, \mathcal{O}(\mathbb{N})) \) which is a NURS by 3.1.2. \( \square \)

**Concluding remarks.**

A URS is almost a precomputation theory in the sense of Moschovakis [3]¹. Restricting the attention to single-valued functions, his computation theories have an additional length of computation \( |e;\vec{x}| \) satisfying

\[
(+) \quad |S^n_m(e,\vec{x});\vec{y}| > |e;\vec{x},\vec{y}|, \quad \text{if defined.}
\]

Define in a NURS \( |e;\vec{x}| = |e;x_1| + |e.x_1;x_2| + \ldots + |e.x_1 \ldots x_{n-1};x_n| \).

Then it follows readily from the definition of \( S^n_m \) in a URS ([8], 2.6) that this norm satisfies Moschovakis' axiom (+).

As suggested in [6], there is another way of extending a URS. A **selection**² URS is an URS containing a "selection operator" \( c \) such that

\[
\exists a[f.a \neq \ ] = f.(c.f) \neq .
\]

1) Not quite, because a URS does not need to contain a computable successor set.

2) In [6] such a URS is called "well-ordered". This name is a little absurd as can be argued as follows. Let \( \mathcal{M} \) be a model of Peano arithmetic of power continuum. Then \( \mathcal{M} \) is a selection URS but cannot be well-ordered in ZF. On the other hand \( \mathcal{N}(\text{WS}^*) \) is countable and hence well-ordered, but has no selection operator.
In a selection URS a set is computable iff it is SC and co SC, [6], 3.4. This is not true in a general URS, [8], p. 39 bottom.

Having a norm or a selection operator are independent of each other. \( \mathcal{X} \) has a selection operator \( \{c\}(e) = (\mu x T(e, (x)_0, (x)_1))_0 \). Since this is provably in arithmetic a selection operator, \( \mathcal{X}_0 \) is a selection URS but not a NURS. Conversely, it is not difficult to show that \( \mathcal{U}(WS^*_0) \) is not a selection URS, although it is a NURS,

In a NURS it would be natural to require for a selection operator \( c \)

\[ |c; a| \geq |a; c; a| \]

cf. [3], p. 225, (6-4).

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References.


