THEORETICAL PEARLS

Applications of Plotkin-terms: partitions and morphisms for closed terms

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Abstract

This theoretical pearl is about the closed term model of pure untyped lambda-terms modulo \( \beta \)-convertibility. A consequence of one of the results is that for arbitrary distinct combinators (closed lambda terms) \( M, M', N, N' \) there is a combinator \( H \) such that

\[ HM = HM' \neq HN = HN'. \]

The general result, which comes from Statman [1998], is that uniformly r.e. partitions of the combinators, such that each “block” is closed under \( \beta \)-conversion, are of the form \( \{H^{-1}\{M\}\}_{M \in A^0} \). This is proved by making use of the idea behind the so-called Plotkin-terms, originally devised to exhibit some global but non-uniform applicative behavior. For expository reasons we present the proof below. The following consequences are derived: a characterization of morphisms and a counter-example to the perpendicular lines lemma for \( \beta \)-conversion.

1. Introduction

We use notations from recursion theory and lambda calculus, see Rogers [1987] and Barendregt [1984].

NOTATION. (i) \( \varphi_x \) is the \( e \)-th partial recursive function of one argument.

(ii) \( W_e = \text{dom}(\varphi_e) \subseteq \mathbb{N} \) is the r.e. set with index \( e \).

(iii) \( A \) is the set of lambda-terms and \( A^0 \) is the set of closed-lambda terms (combinators).

(iv) \( W_e = \{M \in A^0 \mid \#M \in W_e\} \subseteq A^0 \); here \( \#M \) is the code of the term \( M \).

1.1. DEFINITION. (i) Inspired by Visser [1980] we define a Visser-partition (V-partition) of \( A^0 \) to be a family \( \{W_e\}_{e \in S} \) such that
(1) $S \subseteq \mathbb{N}$ is an r.e. set.
(2) $\forall e \in S \forall M, N (M \in W_e \& N = M) \Rightarrow N \in W_e$.
(3) $W_e \cap W_{e'} \neq \emptyset \Rightarrow W_e = W_{e'}$.

(ii) A family $\{W_e\}_{e \in S}$ is a pseudo-V-partition if it satisfies just 1 and 2.

1.2. DEFINITION. Let $\{W_e\}_{e \in S}$ be a V-partition.

1. The partition is said to be covering if $\bigcup_{e \in S} W_e = \Lambda^\emptyset$.
2. The partition is said to be inhabited if $\forall e \in S \ W_e \neq \emptyset$.
3. A V-partition $\{W_e\}_{e \in S'}$ is said to be (extensionally) equivalent with $\{W_e\}$ if these families define the same collection of non-empty sets, i.e. if

$$\{W_e \mid e \in S \& W_e \neq \emptyset\} = \{W_e' \mid e \in S' \& W_e \neq \emptyset\}.$$

1.3. EXAMPLE. Let $H$ be some given combinator. Define

$$W_e(M, H) = \{N \in \Lambda^\emptyset \mid HN = HM\},$$

Then $\{W_e\}_{e \in S_H}$, with $S_H = \{e(M, H) \mid M \in \Lambda^\emptyset\}$, is an example of a covering and inhabited V-partition. We denote this V-partition by $\{W_e(M, H)\}_{M \in \Lambda^\emptyset}$.

1.4. PROPOSITION. (i) Every V-partition is effectively equivalent to an inhabited one.
(ii) Every V-partition can effectively be extended to a covering one.

PROOF. (i) Given $\{W_e\}_{e \in S}$ define $S' = \{e \in S \mid W_e \neq \emptyset\}$. Then $\{W_e\}_{e \in S'}$ is the required modified partition.

(ii) Given $\{W_e\}_{e \in S}$ define

$$W_e(M) = \{N \mid N = M \lor \exists e \in S \ M, N \in W_e\}.$$

Then $\{W_e(M)\}_{M \in \Lambda^\emptyset}$ is the required V-partition. 

The main theorem comes in two version. The second more sharp version is needed for the construction of so called inevitably consistent equations, see Statman [1999].

1.5. THEOREM (Main theorem). (i) Let $\{W_e\}_{e \in S}$ be a V-partition. Then one can construct effectively a combinator $H$ such that for all $M, N \in \Lambda^\emptyset$

$$HM = HN \iff M = N \lor \exists e \in S \ M, N \in W_e.$$

The construction of $H$ is effective in the code of the underlying r.e. set $S$.

(ii) Let $\{W_e\}_{e \in S}$ be a pseudo-V-partition. Then one can construct effectively a combinator $H$ such that if $\{W_e\}_{e \in S}$ is an actual V-partition, then (*) holds.

The theorem will be proved in §2. It has several consequences. In order to state these we have to formulate the notion of morphism on $\Lambda^\emptyset$ and the so-called perpendicular lines lemma.

1.6. DEFINITION. Let $\varphi : \Lambda^\emptyset \to \Lambda^\emptyset$ be a map. Then $\varphi$ is a morphism if

1. $\varphi(M) = E_{f(\#M)}$, for some recursive function $f$.
2. $M = N \Rightarrow \varphi(M) = \varphi(N)$. 

1.7. **Lemma.** (i) Let $F$ be a combinator and define $\varphi_H(M) \equiv HM$. Then $\varphi_H$ is a morphism.

(ii) Let $F, G$ be combinators such that for all $M \in \Lambda^\emptyset$ there exists a unique $N \in \Lambda^\emptyset$ with $FM = GN$. Then there is a map $\varphi_{F,G}$ such that $FM = G\varphi_{F,G}(M)$, for all $M$, which is a morphism.

**Proof.** (i) For the coding $#$ let $\text{app}$ be the recursive function such that $\#(PQ) = \text{app}(\#P, \#Q)$. Define $f(m) = \text{app}(\#H, m)$. Then $\varphi_H(M) = \text{Ec}_{f(\#M)}$. It is obvious that $\varphi_H$ preserves $\beta$-equality.

(ii) Let $R(m, n)$ be an r.e. relation. Then we have $R(m, n) \iff \exists z T(m, n, z)$, for some recursive $T$. Let $< n, z >$ be a recursive pairing with recursive inverses $< n, z >.0 = n, < n, z >.1 = z$. Define ($\mu$ is the least number operator)

$$\iota_n. R(m, n) = (\mu p. T(m, p.0, p.1)).0.$$ 

Then $\exists n \in NR(m, n) \Rightarrow R(m, \iota_n. R(m, n))$. In order to construct the morphism $\varphi_{F,G}$, define

$$f(m) = \iota_n. F(\text{Ec}_n) = G(\text{Ec}_n).$$

By the assumption (existence) $f$ is total. Define $\varphi_{F,G}(M) = \text{Ec}_{f(\#M)}$. Now $f(\#M) = n \Rightarrow F(\text{Ec}_n) = G(\text{Ec}_n)$. Therefore $FM = G\varphi_{F,G}(M)$, for all $M$. The condition

$$M = M' \Rightarrow \varphi_{F,G}(M) = \varphi_{F,G}(M')$$

holds by the assumption (unicity). \hfill \blacksquare

One may wonder whether dropping the unicity condition in lemma 1.7 (ii) one may obtain a morphism by making a right uniformization. This is not the case.

1.8. **Proposition.** There exists combinators $F, G$ such that $\forall M \exists N FM = GN$ but without any morphism satisfying $\forall M FM = G\varphi(M)$.

**Proof.** Let $\Delta = \Upsilon \emptyset$ and define $F = \lambda x. (x, \Delta, l)$ and $G = \lambda y. (Ey, y \Omega \Delta, yl)$. Then, see Statman [1986],

$$FM =_\beta GN \iff (N =_\beta c_n \lor N =_\beta l) \land EN =_\beta M. \quad (1)$$

Any morphism $\varphi$ such that $FM = G\varphi(M)$ would solve the convertibility problem recursively: one has by (1)

$$M = M' \iff \varphi(M) = \varphi(M'), \quad (2)$$

and since $\varphi(M), \varphi(M')$ have nf's by (1), the RHS of (2) is decidable. \hfill \blacksquare

1.9. **Proposition.** Not every morphism is of the form $\varphi_H$.

**Proof.** Let $F, G \in \Lambda^\emptyset$ be such that $F \circ G = I$. Then $F, G$ determine a so-called inner model $[ ] = [ ]^{F,G}$ as follows.

$$[x] = x;$$
$$[PQ] = F[P][Q];$$
$$[\lambda x.P] = G(\lambda x.[P]).$$
Using the condition on $F, G$ it can be proved that

$$M =_\beta N \Rightarrow [M] = [N].$$

Therefore defining $\varphi(M) = [M]$ we obtain a morphism.

Now take $F \equiv \lambda y. ul, \Gamma \equiv \lambda z. yz$. Then indeed $F \circ G = 1$ and for the resulting inner model one has $[I] = \lambda y. yI$ and $[[\Omega]] = (\lambda y. (\lambda z. zI))I(\lambda y. (\lambda z. zI))$.

Suppose towards a contradiction that the resulting $\varphi$ is of the form $\varphi_H$. Then $H I = \lambda y. yI$, so $H$ is solvable and hence has a hnf $\lambda x_1 \ldots x_n \ldots M_1 \ldots M_m$. But $H \Omega = (\lambda y. (\lambda z. zI))I(\lambda y. (\lambda z. zI))$, which is unsolvable. Therefore the head-variable $x_i$ is $x_1$. But then $H \Omega = \lambda x_2 \ldots x_n. \Omega M_1^* \ldots M_m^*$ which is not of the correct form.\[\square\]

The following is a corollary to the main theorem.

1.10. COROLLARY. Every morphism $\varphi$ is of the form $\varphi_{F,G}$.

PROOF. Let $\varphi$ be a given morphism. Define

$$W_e(N) = \{ Z \mid \exists M \in A^0 [\varphi(M) = N \land [Z = \langle c_0, M \rangle \lor Z = \langle c_1, N \rangle]]\}.$$  

Then $\{W_e(N)\}$ is a $V$-partition. By the main theorem there exists an $H$ such that

$$H(c_0, M) = H(c_1, N) \Leftrightarrow (c_0, M) = (c_1, N) \lor N = \varphi(M) \Leftrightarrow N = \varphi(M).$$

Define

$$F = \lambda n.H(c_0, n);$$

$$G = \lambda n.H(c_1, n).$$

Then $F M = G N \Leftrightarrow N = \varphi(M)$. Therefore $\varphi = \varphi_{F,G}$.\[\square\]

Note that for a given morphism $\varphi$ one can define by

$$W_{e(M, \varphi)} = \{N \in A^0 \mid \varphi(M) = \varphi(N)\}.$$  

This is an inhabited $V$-partition. It is not difficult to show that that each $V$-partition is equivalent to one of the form $\{W_{e(M, \varphi)}\}$. Note that $\{W_{e(M, H)}\} = \{W_{e(M, \varphi_H)}\}$, see lemma 1.7. The following result shows that covering $V$-partitions are always of this more restricted form.

1.11. COROLLARY. If $\{W_e\}$ is a covering $V$-partition, then $\{W_e\}$ is equivalent to $\{W_{e(M, H)}\}_{M \in A^0}$ for some $H$, effectively found from $\{W_e\}$.

PROOF. Let $H$ be the combinator constructed effectively from $\{W_e\}$. We will show that $W_{e(M, H)} = \{N \mid H N = H M\}$ is equivalent to $\{W_e\}$. Claim. For $N \in W_e$ one has $W_e = W_{e(M, H)}$. Indeed,

$$N \in W_e \Leftrightarrow M = N \lor M, N \in W_e \Leftrightarrow H N = H M \Leftrightarrow N \in W_{e(M, H)}.$$
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Therefore, noting that \( M \in \mathcal{W}_e(M, H) \),

\[
\{ W_e \mid M \in \mathcal{W}_e(M, H), W_e \neq \emptyset \} \subseteq \{ \mathcal{W}_e(M, H) \mid \mathcal{W}_e(M, H) \neq \emptyset, M \in \mathcal{W}_e(M, H) \}.
\]

The converse inclusion holds also, since every \( M \) belongs to some \( \mathcal{W}_e \) and hence \( \mathcal{W}_e(M, H) = \mathcal{W}_e \) for this \( e \). □

The following theorem states that if a combinator, seen as function of \( n \) arguments, is constant—modulo Böhm-tree equality—on \( n \) perpendicular lines, then it is constant everywhere.

1.12. THEOREM (Perpendicular lines lemma). Let \( F \) be a combinator. Suppose that for \( n \in \mathbb{N} \) there are combinators \( M_{ij} \), \( 1 \leq i \neq j \leq n \), and \( N_1, \ldots, N_n \) such that for all combinators \( Z \) one has (\( \equiv \) denotes Böhm-tree equality, i.e. \( M \equiv N \Leftrightarrow BT(M) = BT(N)\))

\[
\begin{align*}
F \quad Z \quad M_{12} \quad \ldots \quad M_{1n-1} \quad M_{1n} & \equiv N_1; \\
F \quad M_{21} \quad Z \quad \ldots \quad M_{2n-1} \quad M_{2n} & \equiv N_2; \\
\vdots \\
F \quad M_{n1} \quad M_{n2} \quad \ldots \quad M_{nn-1} \quad Z & \equiv N_n.
\end{align*}
\]

Then for all \( P_1, \ldots, P_n \in \mathcal{W}_e \) one has

\[ FP_1 \ldots P_n \equiv N_1 \equiv N_2 \equiv \ldots \equiv N_n. \]

PROOF. This is the restriction to closed terms of a theorem in Barendregt [1984], theorem 14.4.12, having the same proof. □

1.13. COROLLARY. The perpendicular lines lemma is false for any \( n > 1 \), if \( \equiv \) is replaced by \( =_\beta \).

PROOF (For \( n = 1 \) the perpendicular lines lemma is trivially true for \( =_\beta \)). Let \( n > 1 \). For notational simplicity we assume \( n = 2 \) and give a counter example. Define

\[
\begin{align*}
\mathcal{W}_{e_1} &= \{ N \in \mathcal{W}_e \mid N = (S, S) \} \\
\mathcal{W}_{e_2} &= \{ N \in \mathcal{W}_e \mid \exists Z \in \mathcal{W}_e [N = (1, Z) \lor N = (Z, 1)] \}
\end{align*}
\]

Then \( \{ \mathcal{W}_e \} \subset \mathcal{W}_{e_1} \cap \mathcal{W}_{e_2} \) is a V-partition. Let \( H \) be the combinator obtained from this partition by the main theorem. Then for all \( Z \in \mathcal{W}_e \)

\[ H(S, S) \neq H(1, Z) = H(Z, 1). \]

Now define \( F \equiv \lambda xy. H(x, y) \). Then for all \( Z \in \mathcal{W}_e \)

\[ FSS \neq F1Z = FZ1. \]

This is indeed a counterexample. □

We do believe the conjecture in Barendregt [1984], stating that the perpendicular line lemma with \( \equiv \) replaced by \( =_\beta \) is correct for open terms.
2. Proof of the main theorem

In order to prove the main theorem 1.5, let a V-partition determined by \( S \) be fixed in this section. By proposition 1.4 it may be assumed that the partition is inhabited.

2.1. LEMMA. Let \( \{ W_e \}_{e \in S} \) be an inhabited V-partition.

(i) There exists a total recursive function \( f = f_S \) such that
\[
\forall e \in S \ W_e = \{ f((2e + 1)2^n) | n \in \mathbb{N} \}.
\]

(ii) There exists a combinator \( E_S \) such that
\[
\forall e \in S \ W_e = \{ E (\varepsilon_{(2e+1)2^n}) | n \in \mathbb{N} \}.
\]

PROOF. (i) By elementary recursion theory there exists a recursive function \( h \) such that \( W_e = \text{Range}(\psi_{h(e)}) \) and \( \psi_{h(e)} \) is total, for all \( e \in S \). Observing that \( e,n \) are uniquely determined by \( k = (2e + 1)2^n \), define \( f \) by \( f(0) = 0, f((2e + 1)2^n) = \psi_{h(e)}(n) \).

(ii) Take \( E_S = E \circ F_S \), where \( F_S \) lambda defines \( f_S \) and \( \varepsilon_{e \# M} = M \) for all \( M \in \mathcal{A}^\theta \).

2.2. DEFINITION. (i) Define
\[
\text{odd}(0) = 0; \quad \text{odd}((2e + 1)2^n) = 2e + 1.
\]

(ii) Define \( M \sim N \) iff \( M = N \lor M = E_m, N = E_n \) and \( \text{odd}(m) = \text{odd}(n) \), for some \( m, n \).

Notice that \( M \sim N \) iff \( M = N \) or \( \exists e \in S M, N \in W_e \). Therefore we have to prove that there exists a combinator \( H \) such that
\[
HM = HN \iff M \sim N.
\]

The proof consists in constructing a combinator \( H = H_S \) such that

1. \( M \sim N \Rightarrow HM = HN \), proposition 2.4;
2. \( HM = HN \Rightarrow M \sim N \), proposition 2.9.

The second part of the main theorem easily follows by inspecting the proof.

2.3. DEFINITION. (i) Define
\[
T \equiv \lambda xyz. xy(zxy);
A \equiv \lambda f gx y z. f(a(Ex))[f(S^+x)y(g(S^+x))z];
B \equiv \lambda f g x. f(Sx)(a(E(Tx))(g(S^+x))(gx)).
\]

(ii) By the double fixed-point theorem there exists terms \( F, G \) such that
\[
F \rightarrow AFG;
G \rightarrow BFG.
\]
To be explicit, write

\[ D \equiv (\lambda xy.y(xxy)); \]
\[ Y \equiv DD; \]
\[ G \equiv Y(\lambda u.B(Y(\lambda v.Avw))u); \]
\[ F \equiv Y(\lambda u.AuG). \]

(iii) Finally define

\[ H \equiv \lambda xa.Fc_1(ax)(Ge_1). \]

**Notation.** Write

\[ F_k \equiv Fc_k; \]
\[ G_k \equiv Ge_k; \]
\[ E_k \equiv Ec_k; \]
\[ a_k \equiv aE_k; \]
\[ H_k[a] \equiv F_k[a](F_{k+1}MG_{k+1}N); \]
\[ G_k[a] \equiv F_k[a](F_{k+1}aG_{k+1}G_k). \]

Note that by construction

\[ F_kMN \rightarrow F_k a_k(F_{k+1}MG_{k+1}N); \]
\[ G_k \rightarrow F_{k+1}aG_{k+1}G_k. \]

By reducing \( F \), respectively \( G \), it follows that

\[ H_k[a] \equiv F_k a_k G_k \rightarrow C_k[H_{k+1}[a]] \]
\[ H_k[a] \equiv F_k a_k G_k \rightarrow C_k[H_{k+1}[a]]. \]

**2.4. Proposition.** \( M \sim N \Rightarrow HM = HN. \)

**Proof.** By lemma 2.1 it suffices to show \( HE_k = HE_{2k} \) for all \( k \).

\[ HE_k = \lambda a.H_1[a] \]
\[ = \lambda a.C_1[C_2[\ldots C_{k-1}[H_k[a]]\ldots]]; \]
\[ = \lambda a.C_1[C_2[\ldots C_{k-1}[C_k[H_k[a]]]\ldots]]; \]
\[ = \lambda a.H_1[a_2k]; \]
\[ = \lambda a.C_1[C_2[\ldots C_{k-1}[C_k[H_k[a_2k]]]\ldots]]; \]
\[ = \lambda a.C_1[C_2[\ldots C_{k-1}[C_k[H_k[a]]]\ldots]]; \]
\[ = \lambda a.C_1[C_2[\ldots C_{k-1}[C_k[H_k[a]]]\ldots]]; \]

by (1),

by (2),

by (1). \( \blacksquare \)
As a piece of art we exhibit in more detail the reduction flow (contracted redxes are underlined).

\[
\begin{align*}
&HE_k \\
&\lambda a. F_1 a_k G_1 \\
&\lambda a. F_1 a_1 (F_2 a_2 G_2 G_1) \\
&\lambda a. F_1 a_1 (F_2 a_2 (F_3 a_2 G_2 G_1) G_1) \\
&\ldots \\
&\lambda a. F_1 a_1 (F_2 a_2 (F_3 a_3 (\ldots (F_k a_k G_k G_{k-1}\ldots) G_2 G_1) \\
&\lambda a. F_1 a_1 (F_2 a_2 (F_3 a_3 (\ldots (F_k a_k (F_{k+1} a_{2k} G_{k+1} G_k) G_{k-1}\ldots) G_2 G_1)
\end{align*}
\]

And also

\[
\begin{align*}
&HE_{2k} \Rightarrow \ldots \Rightarrow \\
&\lambda a. F_1 a_1 (F_2 a_2 (F_3 a_3 (\ldots (F_k a_k (F_{k+1} a_{2k} G_{k+1} G_k) G_{k-1}\ldots) G_2 G_1)
\end{align*}
\]

For the converse implication we need the fine structure of the reduction.

2.5. DEFINITION. Define

\[
\begin{align*}
&D_k^0[M] \equiv F_k(aM) \equiv Y(\lambda u.AuG)c_k(aM) \\
&D_k^1[M] \equiv (\lambda y.y(Dy))((\lambda u.AuG)c_k(aM) \\
&D_k^2[M] \equiv (\lambda u.AuG)F_k(aM) \\
&D_k^3[M] \equiv AFGc_k(aM) \\
&D_k^4[M] \equiv (\lambda x.y.F_k(aEx)((F_{S+x} y(1+x))z))c_k(aM) \\
&D_k^5[M] \equiv (\lambda x.y.F_k(aEx)((F_{S+x} y(1+x))z))(aM) \\
&D_k^6[M] \equiv (\lambda y.F_k(aEy)((F_{S+y} y(1+x))z))(aM) \\
&D_k^7[M] \equiv (\lambda y.F_k(aEy)((F_{S+y} y(1+x))z))(aM) \\
\end{align*}
\]

2.6. LEMMA. Let \( F_k(aM)N \) head-reduce in \( 8p + q \) steps to \( W \). Then

\[
\begin{align*}
&W \equiv D_k^0[M]N, \quad \text{if } p = 0; \\
&\equiv D_k^p[Ex]((H_{k+1}[E_k])^{p-1}(H_{k+1}[M]N)), \quad \text{else.}
\end{align*}
\]

PROOF. Note that \( F_k(aM)N \equiv D_k^0[M]N \). Moreover,

\[
\begin{align*}
&D_k^0[M]N \rightarrow_h D_k^{q+1}[M]N, \quad \text{for } q < 7; \\
&D_k^1[M]N \rightarrow_h D_k^0[Ex]((H_{k+1}[M]N).
\end{align*}
\]

The rest is clear. At steps 16, 24 we obtain for example

\[
\begin{align*}
&D_k^7[Ex]((H_{k+1}[E_k])(H_{k+1}[E_k])(H_{k+1}[M]G_k)) \\
&D_k^0[Ex]((H_{k+1}[E_k])(H_{k+1}[M]G_k)) \rightarrow_h D_k^0[Ex]((H_{k+1}[E_k])(H_{k+1}[M]G_k)).
\end{align*}
\]

Remember that a standard reduction \( \sigma : M \rightarrow N \) always consists of a head-reduction followed by an internal reduction:

\[
\sigma : M \rightarrow_h W \rightarrow_i N.
\]
NOTATION. Write $M =_{s \leq n} N$ if there are standard reductions of length $\leq n$ from $M$ respectively $N$ to a common reduct $Z$. Similarly $M =_{i \leq n} N$ for internal standard reductions. Also the notations $=_{s < n}$ and $=_{i < n}$ will be used.

2.7. LEMMA. (i) $D^k_\psi [M] N =_{i \leq n} D^k_\psi [M'] N' \Rightarrow q = q' \& N =_{s \leq n} N'$.

(ii) $D^k_\psi [M] N =_{i \leq n} D^k_\psi [M'] N'$ & $q < 7 \Rightarrow M =_{s \leq n} M'$.

(iii) $D^k_\psi [M] N =_{i \leq n} D^k_\psi [M'] N' \Rightarrow H_{k+1} [M] =_{s \leq n} H_{k+1} [M']$.

PROOF. (i) Suppose $D^k_\psi [M] N =_{i \leq n} D^k_\psi [M'] N'$. Then by observing where the free variable $a$ occurs one can conclude that $q = q'$. Since the reductions to a common reduct are internal, the positions of $N, N'$ are not changed and hence $N =_{s \leq n} N'$.

(ii) Obvious from the definition of $D^k_\psi$.

(iii) In this case it follows that $D^k_\psi [E_k] (H_{k+1} [M] z) =_{i \leq n} D^k_\psi [E_k] (H_{k+1} [M'] z)$.

The conclusion $H_{k+1} [M] =_{s \leq n} H_{k+1} [M']$ depends on the fact that there are free variables $z$ to mark the residuals.

2.8. LEMMA. Suppose $G_k =_{s \leq n} (H_{k+1} [E_k])^d (H_{k+1} [M] G_k)$. Then $H_{k+1} [E (T c_k)] =_{s \leq n} H_{k+1} [M]$.

PROOF. By induction on $d$. If $d = 0$, then we have $G_k =_{s \leq n} H_{k+1} [M] G_k$. So there are standard reductions of these two terms to a common reduct. Observe that the head-reduction starting with $G_k$ begins as follows.

$$
G_k \equiv Y (\lambda u. B (Y (\lambda v. A v u)) u) c_k \\
\rightarrow_h (\lambda x. x (Y x)) (\lambda u. B (Y (\lambda v. A v u)) u) c_k \\
\rightarrow_h (\lambda u. B (Y (\lambda v. A v u)) u) G c_k \\
\rightarrow_h BFG c_k \\
\rightarrow_h (\lambda g x. F (S^+ k) (a (E S^+ (T x))) (g (S^+ k)) (g x)) G c_k \\
\rightarrow_h (\lambda x. F (S^+ k) (a (E S^+ (T x))) (G (S^+ k)) (G x)) c_k \\
\rightarrow_h F (S^+ k) (a (E S^+ (T c_k))) (G (S^+ k)) (G c_k).
$$

The heads of these terms are not of order 0 except the last one. But $H_{k+1} [X]$ is always of order 0. Therefore the mentioned standard reduction of $G_k$ goes at least to this last term $H_{k+1} [E S^+ (T c_k)] G_k$. But then $H_{k+1} [E S^+ (T c_k)] =_{s \leq n} H_{k+1} [M]$.

If $d > 0$, then start the same argument as above, but at the intermediate conclusion

$$
H_{k+1} [E S^+ (T c_k)] G_k =_{s \leq n} (H_{k+1} [E_k])^d (H_{k+1} [M] G_k),
$$

one proceeds by concluding that

$$
G_k =_{s \leq n} H_{k+1} [E_k]^{d-1} (H_{k+1} [M] G_k)
$$

and uses the induction hypothesis.

2.9. PROPOSITION. $H_k [M] = H_k [N] \Rightarrow M \sim N$. 
PROOF. By the standardization theorem it suffices to show for all \( n \) that
\[
\forall k \in \mathbb{N} [H_k[M] =_{s \leq n} H_k[N] \Rightarrow M \sim N].
\]
This will be done by induction on \( n \). From \( H_k[M] =_{s \leq n} H_k[N] \) it follows that
\[
H_k[M] \rightarrow_h W_M \rightarrow_i Z
\]
\[
H_k[N] \rightarrow_h W_N \rightarrow_i Z.
\]
for some \( W_M, W_N, Z \).

Case 1. \( W_M, W_N \) are both reached after \( < 8 \) steps. Then by lemma 2.6 \( W_M \equiv D_k^d[M]G_k, W_N \equiv D_k^d[N]G_k \); By lemma 2.7(i) it follows that \( q = q' \). If \( q < 7 \), then by 2.7(ii) one has \( M = N \) so \( M \sim N \). If \( q = 7 \), then by 2.7(iii) one has \( H_{k+1}[M] =_{s \leq n} H_{k+1}[N] \) and by the induction hypothesis one has \( M \sim N \).

Case 2. \( W_M \) is reached after \( p \geq 8 \) steps and \( W_N \) after \( q < 8 \) steps. Then \( p = 8d + q \) and, keeping in mind lemma 2.7(i), it follows that \( W_M \equiv D_k^d[M]G_k, W_N \equiv D_k^d[N]G_k \), where \( R \equiv (H_{k+1}[E_k])^{d-1}(H_{k+1}[N]G_k) \). Then as in case 1 it follows that \( M \sim E_k \). Moreover, by lemma 2.8 \( H_{k+1}[E_{2k}] =_{s \leq n} H_{k+1}[N] \), so by the induction hypothesis \( E_{2k} \sim N \). So \( M \sim E_{k} \sim E_{2k} \sim N \).

Case 3. Both \( W_M, W_N \) are reached after \( \geq 8 \) steps. Then
\[
W_M \equiv D_k^d[M](H_{k+1}[E_k])^d(H_{k+1}[M]G_k); \quad W_N \equiv D_k^d[N](H_{k+1}[N]G_k).
\]
If \( d = d' \), then by lemma 2.7
\[
(H_{k+1}[E_k])^d(H_{k+1}[M]G_k) =_{s \leq n} (H_{k+1}[E_k])^d(H_{k+1}[N]G_k),
\]
so
\[
H_{k+1}[M] =_{s \leq n} H_{k+1}[N],
\]
since \( H_{k+1}[X] \) is always of order 0. Therefore by the induction hypothesis \( M \sim N \).

If on the other hand, say, \( d < d' \), then (writing \( d' = d + e \))
\[
W_M \equiv D_k^d[E_k](H_{k+1}[E_k])^d(H_{k+1}[M]G_k); \quad W_N \equiv D_k^d[E_k](H_{k+1}[E_k])^d(H_{k+1}[N]G_k).
\]
so
\[
H_{k+1}[M] =_{s \leq n} H_{k+1}[E_k] \quad G_k =_{s \leq n} (H_{k+1}[E_k])^{e-1}(H_{k+1}[N]G_k),
\]
since \( H_{k+1}[X] \) is always of order 0. Therefore by lemma 2.8
\[
H_{k+1}[E_{2k}] =_{s \leq n} H_{k+1}[N].
\]
Therefore by the induction hypothesis twice we obtain \( M \sim E_{k} \sim E_{2k} \sim N \).

References

Theoretical pearls


