COMPLETENESS OF THE PROPOSITIONS-AS-TYPES
INTERPRETATION OF INTUITIONISTIC LOGIC
INTO ILLATIVE COMBINATORY LOGIC

WIL DEKKERS, MARTIN BUNDER, AND HENK BARENDREGT

Abstract. Illative combinatory logic consists of the theory of combinators or lambda calculus extended
by extra constants (and corresponding axioms and rules) intended to capture inference. In a preceding
paper, [2], we considered 4 systems of illative combinatory logic that are sound for first order intuitionistic
propositional and predicate logic. The interpretation from ordinary logic into the illative systems can be
done in two ways: following the propositions-as-types paradigm, in which derivations become combinators,
or in a more direct way, in which derivations are not translated. Both translations are closely related in a
canonical way. In the cited paper we proved completeness of the two direct translations. In the present
paper we prove that also the two indirect translations are complete. These proofs are direct whereas in
another version, [3], we proved completeness by showing that the two corresponding illative systems are
conservative over the two systems for the direct translations. Moreover we shall prove that one of the
systems is also complete for predicate calculus with higher type functions.

§1. Introduction. In [2] we introduced 4 systems of illative combinatory logic
(ICL). We derived roughly the following soundness result
\[ \Delta \vdash_L A \Rightarrow [\Delta] \vdash_C [A], \]
where \( L \) represents propositional or predicate logic and \([\cdot]\) one of two possible
translations of each system into an ICL system \( C \). Of the interpretations one is the
propositions-as-types interpretation due to Curry, Howard and de Bruijn, the other
is a more direct interpretation.

We derived completeness results for the direct translations of propositional and
predicate calculus into 2 of the 4 systems of ICL. These, again roughly, took the
following form
\[ [\Delta] \vdash_C [A] \Rightarrow \Delta \vdash_L A. \]
In the present paper we shall prove that also the two indirect translations are
complete. These completeness results imply the consistency of the ICL's involved.
Moreover we shall prove that one of the systems is also complete for predicate
calculus with higher type functions.

§2. Some definitions of preceding paper. This paper is a continuation of a preced-
ing paper, [2], by the same authors. We will refer to that paper most of the time as B;

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so for example Definition B2.1 means Definition 2.1 in [2]. In this section we repeat some definitions of [2] that are needed in the present paper. For an introduction into ICL-systems, motivations, examples, propositions with proofs, and references we refer to [2].

**Definition 2.1** (Definition B2.1; the logic system PROP). PROP is the \( \ominus \) fragment of intuitionistic propositional logic determined as follows.

(i) The set of formulas of PROP, notation \( \mathbb{F}_{\text{PROP}} \), is defined by the following abstract syntax

\[
\mathbb{F}_{\text{PROP}} = \mathbb{V} | \mathbb{F}_{\text{PROP}} \ominus \mathbb{F}_{\text{PROP}}.
\]

Here \( \mathbb{V} \) is a set of propositional variables.

(ii) Let \( \Gamma \subseteq \mathbb{F}_{\text{PROP}} \) and \( \varphi \in \mathbb{F}_{\text{PROP}} \). Then \( \Gamma \vdash_{\text{PROP}} \varphi \) is defined by the following system of natural deduction.

\[
\begin{array}{c}
\varphi \in \Gamma \Rightarrow \Gamma \vdash \varphi; \\
\Gamma \vdash \varphi \ominus \psi, \; \Gamma \vdash \varphi \Rightarrow \Gamma \vdash \psi; \\
\Gamma, \varphi \vdash \psi \Rightarrow \Gamma \vdash \varphi \ominus \psi.
\end{array}
\]

**Definition 2.2** (Definition B2.2; the logic system PRED). PRED is the \( \{\ominus, \forall\} \) fragment of first order many-sorted intuitionistic predicate calculus of a given signature \( s \).

Below as an example, we will treat a version of PRED with \( s \) the signature of the structure

\[ \langle A_1, A_2, f, g, P, a \rangle \]

with

- \( A_1, A_2 \) nonempty sets;
- \( f : A_1 \to A_1 \) a unary function;
- \( g : A_1 \to A_2 \to A_1 \) a binary function;
- \( P \subseteq A_1 \) a unary relation;
- \( a \in A_1 \) a constant.

(All results also hold for arbitrary signatures.)

(i) The set of terms of PRED, notation \( \mathbb{T}_{\text{PRED}} \), is defined by the following abstract syntax.

\[
\begin{align*}
\mathbb{T}_{\text{PRED}} &= \mathbb{T}_{A_1} | \mathbb{T}_{A_2}; \\
\mathbb{T}_{A_1} &= \mathbb{V}^{A_1} | a | fT_{A_1} | gT_{A_1}T_{A_2}; \\
\mathbb{T}_{A_2} &= \mathbb{V}^{A_2}.
\end{align*}
\]

(ii) The set of formulas of PRED, notation \( \mathbb{F}_{\text{PRED}} \), is defined by the following abstract syntax.

\[
\mathbb{F}_{\text{PRED}} = \mathbb{P}T_{A_1} | \mathbb{F}_{\text{PRED}} \ominus \mathbb{F}_{\text{PRED}} | \forall \mathbb{V}^{A_1} \mathbb{F}_{\text{PRED}}.
\]

(iii) \( \Gamma \vdash_{\text{PRED}} \varphi \) is axiomatised by the following system of natural deduction.
**PRED**

<table>
<thead>
<tr>
<th>[ \varphi \in \Gamma ]</th>
<th>[ \Gamma \vdash \varphi ]</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \Gamma \vdash \varphi \supset \psi ]</td>
<td>[ \Gamma \vdash \varphi ] [ \vdash \varphi \psi ]</td>
</tr>
<tr>
<td>[ \Gamma, \varphi \vdash \psi ]</td>
<td>[ \vdash \varphi \supset \psi ]</td>
</tr>
<tr>
<td>[ \Gamma \vdash \forall x^A \varphi ] , [ t \in T_{A_i} ]</td>
<td>[ \vdash \varphi[x^A_i := t] ]</td>
</tr>
<tr>
<td>[ \Gamma \vdash \varphi ] , [ x^A \in \text{FV}(\Gamma) ]</td>
<td>[ \vdash \forall x^A \varphi ]</td>
</tr>
</tbody>
</table>

The systems PROP and PRED are being interpreted in ICL's by two kinds of embeddings. The first kind is 'direct' and the second kind is according to the 'propositions-as-types' and 'proofs-as-terms' paradigm. As there are two logical systems, PROP and PRED, there are four systems of ICL. These systems are called \( \mathcal{JP}, \mathcal{JE}, \mathcal{JF}, \) and \( \mathcal{JG} \) respectively. Their use for the two kinds of interpretation is as follows. Let \([ \ ]^1 \) be the direct, and \([ \ ]^2 \) the propositions-as-types translation. Then the following table shows the systems of ICL that are used for the two translations of PROP and PRED.

| PROP | \[ \mathcal{JP} \] | \[ \mathcal{JF} \] |
| PRED | \[ \mathcal{JE} \] | \[ \mathcal{JG} \] |

For example

\([ \ ]^2 : \text{PRED} \to \mathcal{JG} \)

Now we will describe the four systems of ICL.

**Definition 2.3** (Definition B2.3; the four systems of ICL). Let \( T = \Lambda(\Xi, L) \), be the set of type-free lambda terms extended by the extra constants \( \Xi \) and \( L \).

(i) Define the following terms in \( T \).

\[
\begin{align*}
P & \equiv \lambda xy.\Xi(Kx)(Ky), \\
F & \equiv \lambda xyz.\Xi(xyz), \\
G & \equiv \lambda xyz.\Xi(xyz), \\
H & \equiv L o K,
\end{align*}
\]

where \( K \equiv \lambda pq.p \), \( M o N \equiv \lambda x.M(Nx) \) and \( S \equiv \lambda pqr.pr(qr) \).

Write \( X \supset Y \) for \( P XY \).

(ii) Define the following four systems of illative combinatory logic \( \mathcal{JP}, \mathcal{JE}, \mathcal{JF} \) and \( \mathcal{JG} \). \( T \) is the set of terms of all 4 systems. A basis \( \Gamma \) is a set of terms. All four systems have as rules the following:

All systems

\[
\begin{align*}
X \in \Gamma & \Rightarrow \Gamma \vdash X; \\
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash X, X =_{\beta\eta} Y & \Rightarrow \Gamma \vdash Y.
\end{align*}
\]
The four systems have the following specific rules:

\[ \mathcal{FP} \]

- \( P_e \) \( \Gamma \vdash X \supset Y, \ \Gamma \vdash X \Rightarrow \Gamma \vdash Y \);
- \( P_i \) \( \Gamma, X \vdash Y, \ \Gamma \vdash XY \Rightarrow \Gamma \vdash X \supset Y \);
- \( P_H \) \( \Gamma, X \vdash HY, \ \Gamma \vdash HY \Rightarrow \Gamma \vdash H(X \supset Y) \).

\[ \mathcal{E} \]

- \( \Xi_e \) \( \Gamma \vdash \Xi XY, \ \Gamma \vdash XV \Rightarrow \Gamma \vdash YV \);
- \( \Xi_i \) \( \Gamma, X \vdash Yx, \ \Gamma \vdash LX, \ x \notin \text{FV}(\Gamma, X, Y) \Rightarrow \Gamma \vdash \Xi XY \);
- \( \Xi_H \) \( \Gamma, X \vdash H(Yx), \ \Gamma \vdash LX, \ x \notin \text{FV}(\Gamma, X, Y) \Rightarrow \Gamma \vdash H(\Xi XY) \).

\[ \mathcal{F} \]

- \( F_e \) \( \Gamma \vdash FXYZ, \ \Gamma \vdash XV \Rightarrow \Gamma \vdash Y(ZV) \);
- \( F_i \) \( \Gamma, X \vdash Y(Zx), \ \Gamma \vdash LX, \ x \notin \text{FV}(\Gamma, X, Y, Z) \Rightarrow \Gamma \vdash FXYZ \);
- \( F_L \) \( \Gamma, X \vdash L(Yx), \ \Gamma \vdash LX, \ x \notin \text{FV}(\Gamma, X, Y) \Rightarrow \Gamma \vdash L(FXY) \).

\[ \mathcal{G} \]

- \( G_e \) \( \Gamma \vdash GXYZ, \ \Gamma \vdash XV \Rightarrow \Gamma \vdash YV(ZV) \);
- \( G_i \) \( \Gamma, X \vdash Yx(Zx), \ \Gamma \vdash LX, \ x \notin \text{FV}(\Gamma, X, Y, Z) \Rightarrow \Gamma \vdash GXYZ \);
- \( G_L \) \( \Gamma, X \vdash L(Yx), \ \Gamma \vdash LX, \ x \notin \text{FV}(\Gamma, X, Y) \Rightarrow \Gamma \vdash L(GXY) \).

**DEFINITION 2.4** (Definition B2.8; interpretation of PROP). Let \( r \) be a closed term in \( \Lambda(\Xi, L) \). Two maps (for \( i = 1, 2 \))

\[ [-]_i^r : \mathbb{F}_{\text{PROP}} \to \Lambda(\Xi, L) \]

and two maps

\[ \Gamma_i^r : \mathbb{F}_{\text{PROP}} \to \text{illative contexts} \]

are defined by the following table. (Note that these illative contexts are effectively grammatical conditions on the variables (propositional, individual) that appear in a proposition.)

<table>
<thead>
<tr>
<th>( \varphi )</th>
<th>( [\varphi]_1^r )</th>
<th>( \Gamma_1^r(\varphi) )</th>
<th>( [\varphi]_2^r )</th>
<th>( \Gamma_2^r(\varphi) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>( rp )</td>
<td>( H(rp) )</td>
<td>( rp )</td>
<td>( L(rp) )</td>
</tr>
<tr>
<td>( \psi \supset \chi )</td>
<td>( [\psi]_1^r \supset [\chi]_1^r )</td>
<td>( \Gamma_1^r(\psi), \Gamma_1^r(\chi) )</td>
<td>( [\psi]_2^r \supset [\chi]_2^r )</td>
<td>( \Gamma_2^r(\psi), \Gamma_2^r(\chi) )</td>
</tr>
</tbody>
</table>

**REMARK 2.5.** The \( r \) in the above can be replaced by \( I \) (i.e., omitted). In other versions of this paper however we specialised to \( r = K \). (cf., Remark 7.1.) Moreover in Proposition B2.15 we used it to derive a relation between the two interpretations.
Definition 2.6 (Definition B2.12; interpretation of PRED).

(i) $\Lambda^t(\Xi, L)$ is $\Lambda(\Xi, L)$ extended by the extra constants $A_1, A_2, P, f, g, a$ associated with the signature $s$ of the many-sorted structure of our example. Because we are going to interpret many sorted predicate logic with sorts $A_1, A_2$, it is useful to have among the free variables of the $\lambda$-calculus infinite sets $\mathcal{Y}_1, \mathcal{Y}_2$, with $\mathcal{Y}_i = \{x_i, y_i, z_i \ldots\}$. $x, y, z$ denote arbitrary variables.

(ii) Let $r$ be a closed term in $\Lambda^t(\Xi, L)$. Two maps (for $i = 1, 2$)

\[
[-]^i_r : \mathcal{F}_{PRED} \rightarrow \Lambda^t(\Xi, L)
\]

and a map

\[
\Gamma : \mathcal{F}_{PRED} \rightarrow \text{illative contexts}
\]

are defined by the following tables:

<table>
<thead>
<tr>
<th>$t$</th>
<th>$[t]^i_r$</th>
<th>$\Gamma(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^A_i$</td>
<td>$x_j$</td>
<td>$A_j x_j$</td>
</tr>
<tr>
<td>$a$</td>
<td>$a$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$f s$</td>
<td>$f[s]^i_r$</td>
<td>$\Gamma(s)$</td>
</tr>
<tr>
<td>$g s t$</td>
<td>$g[s]^i_r[t]^i_r$</td>
<td>$\Gamma(s), \Gamma(t)$</td>
</tr>
</tbody>
</table>

(iii)

\[
\Gamma^1_{r, s} = \langle A_1, A_2, F A_1 A_1 f, F A_1 (F A_2 A_1) g, F A_1 H (r o P), A_1 a \rangle,
\]

\[
\Gamma^2_{r, s} = \langle A_1, A_2, F A_1 A_1 f, F A_1 (F A_2 A_1) g, F A_1 L (r o P), A_1 a \rangle
\]

and

\[
\Gamma^{i,+}_{r, s} = \Gamma^i_{r, s} \cup \{A_2 x_2\} \text{ where } x_2 \in \mathcal{Y}_2 \text{ is some variable.}
\]

The definitions $\Gamma^i_{r, s}$ and $\Gamma^{i,+}_{r, s}$ of course refer to our example of a many-sorted predicate calculus with signature $s$.

§3. Summary of results in preceding and present paper. We had as aim in the preceding paper to prove

(i) $\Delta \vdash_{PROP} \varphi \iff \forall \text{ closed } r[\ldots, [\Delta]^1_r \vdash_{\mathcal{F}P} [\varphi]^i_r]$;

(ii) $\Delta \vdash_{PRED} \varphi \iff \forall \text{ closed } r[\ldots, [\Delta]^1_r \vdash_{\mathcal{F}\Xi} [\varphi]^1_r]$;

(iii) $\Delta \vdash_{PROP} \varphi \iff \forall \text{ closed } r \exists M[\ldots, [\Delta]^1_r \vdash_{\mathcal{F}P} [\varphi]^2_r M]$;

(iv) $\Delta \vdash_{PRED} \varphi \iff \forall \text{ closed } r \exists M[\ldots, [\Delta]^1_r \vdash_{\mathcal{F}G} [\varphi]^1_r M]$;

where $\ldots$ stands for $\Gamma^i_r(\Delta, \varphi)$ in the cases of PROP and for $\Gamma^{i,+}_r(\Delta, \varphi)$ in the cases of PRED.

The proofs of soundness ($\Rightarrow$) were easy. (cf., Propositions B2.11 and B2.14). In Propositions B3.14 and B3.11 we proved completeness for the 2 direct translations

\[
[\ ]^1 : \text{PROP} \rightarrow \mathcal{F}P \quad \text{and} \quad [\ ]^1 : \text{PRED} \rightarrow \mathcal{F}\Xi,
\]

(i) $\Leftarrow$ and (ii) $\Leftarrow$, by specialising
to \( r = 1 \). In the present paper we prove completeness for the 2 indirect translations \([2] : \text{PROP} \to \mathcal{F} \) and \([2] : \text{PRED} \to \mathcal{G} \), \((iii) \equiv \) and \((iv) \equiv \), also by specialising to \( r = 1 \). In Section 7 we point out that completeness for all 4 cases can be proved also by specialising to \( r = K \).

The completeness proof for the direct translation \([1] : \text{PRED} \to \mathcal{E} \) was given in the following way. We had to show

\[
\Gamma_{1,1}^\Delta, [\Delta]^1, \Gamma(\Delta, \varphi) \vdash_{\mathcal{E}} [\varphi]^1 \Rightarrow \Delta \vdash_{\text{PRED}} \varphi.
\]

(cf., page B781.) Write \( \Gamma' = \Gamma_{1,1}^\Delta, [\Delta]^1, \Gamma(\Delta, \varphi) \). First we defined in B3.2 a set \( \Theta \) in order to analyze the terms \( M \) such that \( \Gamma' \vdash_{\mathcal{E}} M \). We defined \( \Theta = \{ M \mid \exists N \in \Theta \mid N \equiv_{\beta \eta} M \} \) and it was clear that \( \Gamma' \subseteq \Theta \). We showed in Proposition B3.10 that \( \Theta \) is an invariant (modulo \( \beta \eta \)-equality) for derivations in \( \mathcal{E} \):

\[
\Gamma \vdash_{\mathcal{E}} M, \Gamma \subseteq \Theta \Rightarrow M \in \Theta.
\]

Using this result we then proved in Proposition B3.11

\[
\Gamma' \vdash_{\mathcal{E}} [\varphi]^1 \Rightarrow \Delta \vdash_{\text{PRED}} \varphi.
\]

The completeness proof for \([1] : \text{PROP} \to \mathcal{P} \) \((i) \equiv \) was done in a similar but easier way.

In the present paper we will prove completeness of \([2] : \text{PROP} \to \mathcal{F} \) and \([2] : \text{PRED} \to \mathcal{G} \)

\((iii) \equiv \) and \((iv) \equiv \).

We start with the most difficult case, the proof of completeness of \([2] : \text{PRED} \to \mathcal{G} \). We define in Definition 4.2 a set \( \Theta_G \) and we will show in Proposition 4.10 that \( \Theta_G \) is an invariant (modulo \( \beta \eta \)-equality) for derivations in \( \mathcal{G} \):

\[
\Gamma \vdash_{\mathcal{G}} M, \Gamma \subseteq \Theta_G \Rightarrow M \in \Theta_G.
\]

The invariant \( \Theta_G \) is more complicated than \( \Theta \) of B3.2 and we cannot use it to prove completeness directly in a way similar to the proof of completeness for \( \mathcal{E} \) relative to \( \text{PRED} \) by means of \( \Theta \) in B3.11. In fact we shall use \( \Theta_G \) in order to show in Lemma 4.18 that in a proof of \( \Gamma \vdash_{\mathcal{G}} M \), where \( \Gamma \subseteq \Theta_G \), all cut formulas can be eliminated. Then we define in Definition 4.21 a second invariant \( \Theta_1 \). This \( \Theta_1 \) plays a role similar to that of \( \Theta \) in B3.2. Using this invariant we prove in 4.22–4.25 the completeness in a way similar to the proof of completeness of \([1] : \text{PRED} \to \mathcal{E} \) in B3.2–B3.11.

Completeness of \( \mathcal{F} \) for \( \text{PROP} \) is easier. There is no need for cut elimination and we give the proof directly, using an invariant \( \Theta_2 \) similar to \( \Theta \) and \( \Theta_1 \).

In the last part of the paper we shall introduce another logic system: Predicate Calculus with Higher Type Functions, notation \( \text{PRED} \lambda \). We shall prove in Proposition 6.12 that \( \mathcal{G} \) is also complete for \( \text{PRED} \lambda \to \) under the second translation \([2] \).

\section{Completeness of \( \mathcal{G} \) for \( \text{PRED} \).}

We will show for \( \varphi \in \mathbb{F}_{\text{PRED}}, \Delta \subseteq \mathbb{F}_{\text{PRED}} : \)

\[
\forall \text{closed } r \exists M \left[ \Gamma_{r,s}^{2+, \Delta} \vdash_{\mathcal{G}} \Delta \vdash_{\text{PRED}} [\varphi]^2 \right] \Rightarrow \Delta \vdash_{\text{PRED}} \varphi.
\]

It is sufficient to show

\[
\Gamma_{1,s}^{2+, \Delta}, \Gamma(\Delta, \varphi) \vdash_{\mathcal{G}} [\varphi]^2 \Rightarrow \Delta \vdash_{\text{PRED}} \varphi.
\]
We omit the index \( I \) in the notation and we prove
\[
\Gamma^{2,+}, [\Delta]^2, \Gamma(\Delta, \varphi) \vdash_{\mathcal{F}G} [\varphi]^2 M \Rightarrow \Delta \vdash_{\text{PRED}} \varphi.
\]

The proof is again in several steps. First we define in Definition 4.2 an invariant \( \mathcal{O}_G \) (modulo \( \beta\eta \)-equality) for derivations in \( \mathcal{F}G \). Using this invariant we show in Lemma 4.18 that in a proof of \( \Gamma^{2,+}, [\Delta]^2, \Gamma(\Delta, \varphi) \vdash_{\mathcal{F}G} N \) all cuts can be eliminated. Then we define in Definition 4.21 a second invariant \( \mathcal{O}_1 \), similar to \( \mathcal{O} \) in B3.2 and using this invariant we prove in 4.22 - 4.25 the completeness in a way similar to the proof of completeness of \( [\ ]^1 : \text{PRED} \rightarrow \mathcal{F}Z \) in B3.2 - B3.11. Instead of \( \vdash_{\mathcal{F}G} \) we shall mostly write \( \vdash \).

Now we repeat Definition B3.2 of the invariant \( \mathcal{O} \) for derivations in \( \mathcal{F}Z \) for the manysorted structure of our example, in order to contrast it with the one for derivations in \( \mathcal{F}G \) of Definition 4.2 and also because we shall refer to it often in the sequel. Instead of \( x_i, \mathcal{F}_i \) and \( \mathcal{T}_i \) we shall write \( x^{A_i}, \mathcal{F}^{A_i} \) and \( \mathcal{T}^{A_i} \).

**Definition 4.1 (Invariant for derivations in \( \mathcal{F}Z \)).**
\[
\mathcal{F} \equiv \mathcal{F}^{A_1} \cup \mathcal{F}^{A_2},
\]
\[
\mathcal{F}^{A_1} \equiv \mathcal{V}^{A_1} \cup A \cup \mathcal{F}^{A_1}, A \mathcal{F}^{A_2},
\]
\[
\mathcal{F}^{A_2} \equiv \mathcal{V}^{A_2},
\]
\[
\mathcal{P} \equiv \mathcal{P}^{A_1} \cup \Xi A_1(\lambda x^{A_1} \mathcal{P}) \cup \Xi(K\mathcal{P})(K\mathcal{P});
\]
\[
\mathcal{G} \equiv \mathcal{L} A_1 \cup A_1 \mathcal{F}^{A_1} \cup \Xi A_1(\lambda x^{A_1} \mathcal{P}) \cup \Xi(K\mathcal{P})(K\mathcal{P}) \cup L(K\mathcal{P});
\]
\[
\mathcal{O} \equiv \mathcal{G} \cup \mathcal{P}.
\]

Now we will define the invariant \( \mathcal{O}_G \) for derivations in \( \mathcal{F}G \). \( \mathcal{O}_G \) is more complicated than the invariant \( \mathcal{O} \) for \( \mathcal{F}Z \) for several reasons. If \( \Gamma^{1,+}, [\Delta]^1, \Gamma(\Delta, \varphi) \vdash \mathcal{L} X \) in system \( \mathcal{F}Z \) then \( X = A_i \) or \( X = \mathcal{K}p \) with \( p \in \mathcal{P} \). But if \( \Gamma^{2,+}, [\Delta]^2, \Gamma(\Delta, \varphi) \vdash \mathcal{L} X \) in system \( \mathcal{F}G \) then we can have for example \( X = \mathcal{F}A_1 A_2 \) or even \( X = \mathcal{F}p A_2 \). Therefore we define an extension \( \mathcal{L}_G \) of the set of sorts \( \{A_1, A_2\} \) containing these \( X \)'s. We need for each \( l \in \mathcal{L}_G \) a set of variables \( \mathcal{V}_G^l \), similar to the sets \( \mathcal{V}^{A_1} \) and \( \mathcal{V}^{A_2} \). If we took a different set of variables for each \( l \in \mathcal{L}_G \), then the Substitution Lemma 4.6, that is needed in the Proof of Proposition 4.10, would not hold. Therefore we define for each \( l \in \mathcal{L}_G \) the skeleton \( sk(l) \), such that \( sk(p) \) is the same for all \( p \in \mathcal{P}_G \) and we define \( \mathcal{V}^l_G = \mathcal{V}^{l'}_G \) iff \( sk(l_1) = sk(l_2) \).

The sets \( \mathcal{T}^l_G \) will contain \( \lambda \)-abstractions: if \( l \equiv \mathcal{F}l_1 l_2 \) and \( l_2 \in \mathcal{T}_G \), then \( \lambda x^{l_1} t^{l_2} \in \mathcal{T}^l_G \). This may seem a little bit strange, but in fact it is very natural. For example we have \( \vdash \mathcal{F}A_1 A_1(\lambda x^{A_1} x^{A_1}) \) by rule \( F_1 \). Hence \( \lambda x^{A_1} x^{A_1} \) should be in \( \mathcal{T}^{A_1 A_1} \). Moreover in the Proof of Lemma 4.9 we really need that \( \lambda x^{l_1} t^{l_2} \in \mathcal{T}^{F_{l_1} l_2} \).

Finally we had \( \Xi(K\mathcal{P})(K\mathcal{O}) \in \mathcal{G} \) whereas now we have \( \Xi p(\lambda x^p \mathcal{O}) \in \mathcal{G} \), where \( x^p \) may occur free in \( \mathcal{O} \).

**Definition 4.2 (Invariant \( \mathcal{O}_G \) for derivations in \( \mathcal{F}G \)).**
(i) \( \Lambda^s(\Xi, \mathcal{L}) \) is, as in Definition B2.12, \( \Lambda(\Xi, \mathcal{L}) \) extended by the extra constants \( A_1, A_2, P, f, g, a \) associated with the signature \( s \) of the manysorted structure of our example.

(ii) Now we will define simultaneously a set \( \mathcal{L}_G \) of sorts (an extension of the set \( \{A_1, A_2\} \)), for each \( l \in \mathcal{L}_G \) the skeleton \( sk(l) \), a set of variables \( \mathcal{V}_G^l \) and a set of terms \( \mathcal{T}^l_G \), sets \( \mathcal{P}_G, \mathcal{H}_G, \mathcal{G}_G \) and \( \mathcal{O}_G \).
Definition of $L_G$.

$$L_G \equiv A_1 \mid A_2 \mid Pt^{A_1} \mid G_l(\lambda x^l_1 . l_2) \quad (t^{A_1} \in T_G^{A_1}, \ l_1, l_2 \in L_G, \ x^l_1 \in \nu^{l_1}_G).$$

Definition of $sk(l)$.

$$sk(A_i) \equiv A_i;$$

$$sk(Pt^{A_1}) \equiv P x^{A_1} \quad \text{(i.e., } sk(Pt^{A_1}) \text{ is independent of } t^{A_1});$$

$$sk(G_l(\lambda x^l_1 . l_2)) \equiv G(sk(l_1))(\lambda x^l_1 . sk(l_2)).$$

Definition of $\nu^G_l$.

We assume that for each skeleton $sk(l)$ there is a set of variables

$$\nu^G_{sk(l)} \equiv \{ x_{sk(l)}, y_{sk(l)}, x^{sk(l)}_1, y^{sk(l)}_1, x^{sk(l)}_2, y^{sk(l)}_2, \ldots \}$$

such that $\nu^G_{sk(l_1)} \cap \nu^G_{sk(l_2)} = \emptyset$ if $sk(l_1) \neq sk(l_2)$, and for each $l \in L_G$ we define

$$\nu^G_l \equiv \nu^G_{sk(l)}, \quad x^l \equiv x^{sk(l)}, \quad x^l_1 \equiv x^{sk(l)}_1, \quad y^l \equiv y^{sk(l)}, \quad y^l_1 \equiv y^{sk(l)}_1.$$

Definition of $T_G^l$.

$$T_G^l \equiv a \mid \nu^G_{sk(l)} \mid T_G^l G_l(\lambda x^l_1 . l) T_G^l, \quad \text{if } l \equiv A_1;$$

$$T_G^l \equiv \nu^G_{sk(l)} \mid T_G^l G_l(\lambda x^l_1 . l) T_G^l, \quad \text{if } l \equiv A_2 \text{ or } l \equiv Pt^{A_1};$$

$$T_G^l \equiv f \mid \nu^G_{sk(l)} \mid T_G^l G_l(\lambda x^l_1 . l) T_G^l \mid \lambda x^{A_1} T_G^{A_1}, \quad \text{if } l \equiv FA_1 A_1;$$

$$T_G^l \equiv g \mid \nu^G_{sk(l)} \mid T_G^l G_l(\lambda x^l_1 . l) T_G^l \mid \lambda x^{A_1} T_G^{FA_2 A_1}, \quad \text{if } l \equiv FA_1(FA_2 A_1);$$

$$T_G^l \equiv \nu^G_{sk(l)} \mid T_G^l G_l(\lambda x^l_1 . l) T_G^l \mid \lambda x^{l_2} T_G^{l_3}, \quad \text{if } l \equiv GL(\lambda x^{l_2} l_3);$$

$$l \notin \{ FA_1 A_1, FA_1(FA_2 A_1) \}.$$


$$P_G \equiv P T_G^l \mid G_l(\lambda x^l . P_G);$$

$$H_G \equiv (Pt^{A_1}) T^{Pt^{A_1}} \mid \Xi l(\lambda x^l . H_G);$$

$$E_G \equiv L L_G \mid A_i T_G^l \mid \Xi l(\lambda x^l . E_G);$$

$$G_G \equiv H_G \mid G_G.$$

(iii) Notation.

$$l, l_1, l_2, l', \ldots \quad \text{denote elements of } L_G;$$

$$p, p_1, p_2, p', \ldots \quad \text{denote elements of } P_G.$$

Remarks 4.3. (i) If $FV(l_2) = \emptyset$ then $G_l(\lambda x^{l_1} . l_2) = \beta F_l l_2$. In that case we will denote $G_l(\lambda x^{l_1} . l_2)$ usually by $F_l l_2$. In fact we did that already in the above definition.

(ii) The grammar in the definition of $G_G$ is not context-free and this causes some inaccuracies in the notation. In the definition of $P_G$ (and similarly at some other places) instead of $G_l(\lambda x^l . P_G)$ we should write $G L_G(\lambda x L_G . P_G)$, but then we do not know that these two $L_G's$ represent the same $l \in L_G$. (We may not have $G_l(\lambda x^{l_2} . P_G)$.)
(iii) In the definition of $\mathcal{F}^l$ instead of the clause $\mathcal{F}^l_{G}(λx^l,l')t^l$ where $l = l'[x^l := t^l]$, but that does not make a difference because we have for $l ∈ \mathcal{L}_G$

$$\mathcal{F}^l_{G}[x^l := t^l] = \mathcal{F}^l,$$

as follows easily from

$$\forall^l_{G}[x^l := t^l] = \forall^l$$

in Lemma 4.6.

**Lemma 4.4.** (i) $\exists l(λx^l, pt) = G(l(λx^l, pt))(λx^l, t)$;
(ii) $G(l(λx^l, t)) = \exists l(λx^l, pt) + F(t)$;
(iii) $\exists l(λx^l, pt^p) = G(l(λx^l, p) + G(l(λx^l, p)))$, for $l G(l(λx^l, p) + G(l(λx^l, p))) = λx^l, t^p$;
(iv) $G(l(λx^l, t)p) + G(l(λx^l, t)p) = \exists l(λx^l, pt)$, for $t^p = t G(l(λx^l, p) + G(l(λx^l, p)))$;
(v) $G = P G = P G = G$ modulo βη-equality.

**Proof.** (i), (ii) immediately from the definition of $G$;
(iii), (iv) from (i), (ii);
(v) from (iii) and (iv). 

**Remark 4.5.** By (v) in the previous lemma we have $G = P G = P G$. In B3.2 we defined $\mathcal{L} = P | \mathcal{G}$ whereas now we have $G = G$. This corresponds with the fact that now we treat derivations $\Gamma \vdash [λ]^2 M$ instead of $\Gamma \vdash [λ]^2 M$.

**Lemma 4.6.** Let $W = \mathcal{L}_G, \mathcal{F}^l, P G, H G, G G, P G$, or $G G$ and let $w ∈ W$, $x^l ∈ V^l_{G}$, $t^l ∈ \mathcal{F}^l_{G}$, then

$$w[x^l := t^l] ∈ W,$$

moreover if $W = \mathcal{L}_G$ then

$$sk(w[x^l := t^l]) \equiv sk(w),$$

and hence

$$\forall^w_{G}[x^l := t^l] \equiv \forall^w.$$  

**Proof.** By induction on the structure of $w$. Instead of $x^l$ and $t^l$ we shall write $x$ and $t$.

**Case** $w ∈ \mathcal{L}_G$.

$w \equiv A_l$. This is trivially ok.

$w \equiv P [t^l]$. This follows from the case $w \equiv t^l$.

$w \equiv G(l_2(λy^l, l_3)$. Then $w[x := t] \equiv G(l_2[x := t])(λy^l, l_3[x := t]) \equiv G(l_2[x := t])(λy^l, l_3[x := t])$
(by the second part of the IH for $l_2$) and this is in $\mathcal{L}_G$ by
the first part of the IH for $l_2$ and $l_3$. The second part of the
IH for $l_2$ and $l_3$ yields $sk(w[x := t]) \equiv sk(w).$
Case \( w = t' \in \mathcal{T}_G \).

\( l \equiv A_1 \). Now \( w = x_l^{A_1} a \) or \( t_1^{G_l(\lambda x_l^{A_1} a)} t_2^{l_1} \). All these cases are easy.

\( l \equiv A_2 \) or \( Pr_{A_1}^l \). Easy.

\( l \equiv G_l(\lambda y^{l_2} b). \) Now \( w = x_l^{l_2} t_1^{G_l(\lambda x_l^{l_2} b, l_1)}, \) \( \lambda y^{l_2} t_2^{l_1} f \) or \( g \). Easy again.

Case \( w \in \mathcal{H}_G \).

\( w \equiv Pr_{A_1}^l. \) This follows from the case \( w \equiv t_1^{A_1} \).

\( w \equiv G_l(\lambda y^{l_2} p). \) Then \( w[x := t] = G(l_2[x := t])(\lambda y^{l_2} p[x := t]) = G(l_2[x := t])(\lambda y^{l_2}[x := t])p[x := t]) \).

Cases \( w \in \mathcal{H}_G, w \in \mathcal{G}_G. \) Similarly.

**Lemma 4.7.** Let \( cX_1 \ldots X_n = \beta_n M \) for some constant \( c \) and some \( M \in \mathcal{G}_G. \) Then \( n \in \{1, 2\} \) and \( M \equiv c Y_1 \ldots Y_n \) with \( Y_i = \beta_n X_i. \)

**Proof.** By Church-Rosser for \( \beta_n \)-reduction. 

In the definition of \( \mathcal{G}_G \) one of the clauses is \( A_1 \mathcal{T}_G^{A_1} \), where one might expect the more general clause \( l \mathcal{T}_G^l. \) But \( A_1 \mathcal{T}_G^{A_1} \) is sufficient as follows from the next lemma.

**Lemma 4.8.** \( l t^l \in \mathcal{G}_G \) for each \( l \in \mathcal{L}_G. \)

**Proof.** By induction on the structure of \( l. \) It is clear if \( l = A_1, A_2 \) or \( Pr_{A_1}^l. \) So let \( l = G_l(\lambda x^{l_1} l_2) \). Then \( t^l = G_l(\lambda x^{l_1} l_2) t_1^l = \Xi l_1(S(\lambda x^{l_1} l_2) t_1^l) = \Xi l_1(\lambda x^{l_1} l_2(t_1^l x^{l_1})). \)

Now \( t^l \in \mathcal{T}_G = \mathcal{F}_G^{G_l(\lambda x^{l_1} l_2)} \) and \( x^{l_1} \in \mathcal{F}_G^{l_1}, \) hence \( t_1^l x^{l_1} \in \mathcal{F}_G^{l_1}. \) So by the induction hypothesis one has \( l_2(t_1^l x^{l_1}) \in \mathcal{G}_G. \) Hence \( \Xi l_1(\lambda x^{l_1} l_2(t_1^l x^{l_1})) \in \mathcal{G}_G. \)

**Lemma 4.9.** \( IV \in \mathcal{G}_G \) for each \( I \in \mathcal{L}_G. \)

**Proof.** By induction on the structure of \( I. \) It is clear if \( I = A_1, A_2 \) or \( Pr_{A_1}^l. \) So let \( I = G_l(\lambda x^{l_1} l_2). \) Then \( IV = \Xi l_1(\lambda x^{l_1} l_2(V x^{l_1})). \) So \( l_2(V x^{l_1}) \in \mathcal{G}_G \) and hence by the induction hypothesis we have \( V x^{l_1} = t_2^l \in \mathcal{F}_G^{l_1}. \) So \( V = \lambda x^{l_1} t_2^l \in \mathcal{G}_G. \)

Now we will prove \( \Gamma \vdash M, \Gamma \subseteq \mathcal{G}_G \Rightarrow M \in \mathcal{G}_G. \) The proof is similar to the proof of Proposition B3.10, but without the induction loading. We used that induction loading to derive, in subcase \( \Xi_{(a)} \), that if \( \Gamma \vdash \Xi(Kp)(\lambda x. O) \) then \( x \in \text{FV}(O) \). But now we have \( \Gamma \vdash \Xi(p)(\lambda x. O) \) where \( x \) may occur free in \( O \).

**Proposition 4.10.** \( \mathcal{G}_G \) is an invariant (modulo \( \beta_n \)-equality) for derivations in \( \mathcal{F}_G \) : \( \Gamma \vdash M, \Gamma \subseteq \mathcal{G}_G \Rightarrow M \in \mathcal{G}_G. \)

**Proof.** By induction on the derivation of \( \Gamma \vdash M. \) We only consider the three specific rules \( \Xi_e, \Xi_1 \) and \( G_l. \) (The two general rules are easy and \( G_e, G_1 \) are equivalent to \( \Xi_e, \Xi_1 \).)

Case \( \Xi_e. \) \( \Gamma \vdash XY \) as direct consequence of \( \Gamma \vdash \Xi XY, \Gamma \vdash X V. \) By the IH one has \( \Xi XY \in \mathcal{G}_G, XV \in \mathcal{G}_G. \) From \( \Xi XY \in \mathcal{G}_G \) it follows that \( X = l, \) \( Y = \lambda x^{l_1} O \) where \( O \in \mathcal{G}_G. \) Now \( XV = IV \in \mathcal{G}_G, \) hence \( V = t_1^l \in \mathcal{F}_G^{l_1} \) by Lemma 4.9. So \( M = (\lambda x^{l_1} O) t_1^l = O[x' := t_1^l] \in \mathcal{G}_G \) by Lemma 4.6.
Case \( \Xi_l \). \( \Gamma \vdash M \) is \( \Gamma \vdash \Xi XY \) as direct consequence of
\( \Gamma \vdash LX, \Gamma, Xx \vdash Yx \) with \( x \notin \text{FV}(\Gamma, X, Y) \).

By the IH one has \( LX \in \mathcal{G}_G \), hence \( X = l \in \mathcal{L}_G \). Now \( x \) is any variable, so we may assume that \( x = x' \in \mathcal{F}_l \). Then \( lx \in \mathcal{G}_O \) by Lemma 4.8, hence by the IH we have \( Yx \in \mathcal{G}_O \). Let \( Yx = O \in \mathcal{G}_G \). Then \( M = \Xi l(\lambda x'.O) \in \mathcal{G}_G \).

Case \( \Xi_l \). \( \Gamma \vdash M \) is \( \Gamma \vdash L(GXY) \) as direct consequence of
\( \Gamma \vdash LX, \Gamma, Xx \vdash L(Yx) \) with \( x \notin \text{FV}(\Gamma, X, Y) \).

By the IH one has \( LX \in \mathcal{G}_G \), hence \( X = l \in \mathcal{L}_G \). Again we may assume that \( x = x' \in \mathcal{F}_l \). Then \( lx \in \mathcal{G}_G \), hence by the IH we have \( L(Yx) \in \mathcal{G}_O \). So \( Yx = l_1 \in \mathcal{L}_G \). We get \( GXY = Gl(\lambda x'.l_1) \in \mathcal{G}_G \). Hence \( L(GXY) \in \mathcal{G}_G \).

The invariant \( \mathcal{G}_G \) is more complicated than \( \mathcal{G} \) and we cannot prove completeness directly using this invariant, as in in B3.11 for \( \Xi \) relative to PRED. In fact we will use \( \mathcal{G}_G \) in order to show in 4.11-4.20 that in each proof \( \Gamma \vdash \Xi \mathcal{G} M \), where \( \Gamma \subseteq \mathcal{G}_G \), all cut formulas can be eliminated. Then using another invariant \( \mathcal{G}_1 \), similar to \( \mathcal{G} \) we prove in Proposition 4.25 completeness of \( \mathcal{G} \) for PRED in a way similar to the proof of completeness of \( \Xi \) for PRED.

**Cut elimination.** First we define the \( \Xi \)-length \( L_\Xi(\mathcal{O}) \) for \( \mathcal{O} \in \mathcal{G}_G \) (and \( L_\Xi(\mathcal{I}) \) for \( \mathcal{I} \in \mathcal{L}_G \) ) and state some lemma’s, without the (trivial) proofs.

**Definition** 4.11 (\( \Xi \)-length \( L_\Xi \)).
\[
L_\Xi((Pt^{A^l})\mathcal{I}^{t^{A^l}}) = L_\Xi(l\mathcal{I}) = L_\Xi(A_it^{A^l}) = 0,
L_\Xi(\Xi l(\lambda x'.O)) = 1 + L_\Xi(l\mathcal{I}) + L_\Xi(O),
L_\Xi(A_i) = L_\Xi(Pt^{A^l}) = 0,
L_\Xi(Gl(\lambda x'^l.l_1)) = 1 + L_\Xi(l_1) + L_\Xi(l_2).
\]

**Lemma** 4.12. (i) \( X, Y \in \mathcal{L}_G \cup \mathcal{G}_G, X \rightarrow_{\beta^n} Y \Rightarrow L_\Xi(X) = L_\Xi(Y) \).
(ii) \( X, Y \in \mathcal{L}_G \cup \mathcal{G}_G, X =_{\beta^n} Y \Rightarrow L_\Xi(X) = L_\Xi(Y) \).

**Definition** 4.13. Let \( M \in \mathcal{G}_G, M =_{\beta^n} \mathcal{O}. \) Then \( L_\Xi(M) = L_\Xi(O) \).

**Lemma** 4.14. \( L_\Xi(l^{t'}) = L_\Xi(l) \).

**Lemma** 4.15. \( X \in \mathcal{L}_G \cup \mathcal{G}_G, t^l \in \mathcal{T}_l \Rightarrow L_\Xi(X[x^l := t^l]) = L_\Xi(X) \).

**Lemma** 4.16.
\( \Xi XY \in \mathcal{G}_G, XV \in \mathcal{G}_G \Rightarrow L_\Xi(XV) < L_\Xi(\Xi XY) \) & \( L_\Xi(YV) < L_\Xi(\Xi XY) \).

**Proof.** \( \Xi XY = \Xi l(\lambda x'.O) \) and \( V \in \mathcal{T}_l \). Now use Lemma’s 4.14 and 4.15.

**Definition** 4.17. A cut elimination is a proof reduction of the following form
\[
\frac{D_1}{LX} \quad \frac{D_2}{YX} \quad \frac{D_3}{\Xi XY} \quad \frac{D_4}{XV} \quad \frac{[V/x]D_2}{YV}
\]
\( \Xi XY \) is called the cut formula.

**Lemma 4.18.** In a proof of \( \Gamma \vdash _F G M \) (or \( \Gamma \vdash _\Xi M \)), where \( \Gamma \subseteq \overline{G} \), all cut formulas can be eliminated by the above proof reduction.

**Proof.** By induction on the maximal \( \Xi \)-length of cut formulas in the proof of \( \Gamma \vdash _F G M \). Consider a cut in the proof with a cut formula \( \Xi XY \) of maximal \( \Xi \)-length. Eliminate this cut. Now new cut formulas may have been introduced at \( XV \) and \( YV \), however by Lemma 4.16 these have shorter \( \Xi \)-length. In the same way all other cut formulas of maximal \( \Xi \)-length can be eliminated and so by the induction hypothesis all cut formulas can be eliminated.

**Remark 4.19.** The above can easily be generalised to allow \( \beta \eta \)-equality steps (Eq-steps) as follows.

\[
\begin{array}{c}
D_1 & X_X \\
L X & D_2 \\
Eq \Xi XY & Y_X \\
\Xi X'Y' & Y'V \\
\Xi XV & X'V \\
Eq X'V & XV \\
\Xi XV & Y'V \\
\end{array}
\]

**Remark 4.20.** Similarly we can define cut elimination for \( P \) as a proof reduction of the following form.

\[
\begin{array}{c}
D_1 & X \\
H X & D_2 \\
\Xi XV & X \\
\Xi Y & Y \\
\Xi XY & Y \\
\end{array}
\]

This is in fact a special case of Definition 4.17. (Substitute \( KX \) for \( X \) and \( KY \) for \( Y \) in Definition 4.17.) Also in a proof of \( \Gamma \vdash _{_{P}} M \) where \( \Gamma \subseteq \overline{G} \) all cut formulas can be eliminated.

Now we define similar to the set \( \sigma \) in B3.2 a set \( \mathcal{L}_1 \) such that \( N \subseteq \overline{\mathcal{L}_1} \) for terms \( N \) of interest occurring in a cut free proof of \( \Gamma_{s^+}^2, \langle [\Delta] \rangle^2, \Gamma(\Delta, \varphi) \vdash _F \overline{[\varphi]^2 M} \).

**Definition 4.21.** Definition of \( \mathcal{L}_1 \).

\[ \mathcal{L}_1 \equiv A_1 \mid A_2 \mid P_1. \]
Definition of \( \text{sk}(l) \) and \( \mathcal{V}_1^i \).

\[ \text{sk}(A_i) \equiv A_i; \]
\[ \text{sk}(P_t A_i) \equiv P x A_i \]  
(i.e., \( \text{sk}(P_t A_i) \) is independent of \( t^{A_1} \);)
\[ \text{sk}(G_l(\lambda x^{l_2}.p)) \equiv G(\text{sk}(l_2))(\lambda x^{l_2}.\text{sk}(p_1)). \]

\( \mathcal{V}_1^i \equiv \{ x_1^i, x_2^i, \ldots \} \), \( x_i^l = x_i^{\text{sk}(l)} \), \( x_i^{p_1} \neq x_i^{p_2} \) if \( \text{sk}(p_1) \neq \text{sk}(p_2) \);

Definition of \( \mathcal{T}_1 \).

\[ \mathcal{T}_1 \equiv \mathcal{V}_1^A \mid a \mid \mathcal{F}_1 \mathcal{A}_1 \mathcal{T}_1 \mathcal{A}_2 \]  
if \( l \equiv A_1 \);
\[ \mathcal{T}_1 \equiv \mathcal{V}_1^A \]  
if \( l \equiv A_2 \);
\[ \mathcal{T}_1 \equiv \mathcal{V}_1^I \mid \mathcal{G}_{\lambda x^{l_1}}(\mathcal{V}_1^I) \mathcal{T}_1 \]  
if \( l \equiv P_t A_1 \);
\[ \mathcal{T}_1 \equiv \mathcal{V}_1^I \mid \mathcal{G}_{\lambda x^{l_1}}(\mathcal{V}_1^I) \mathcal{T}_1 \mid \lambda x^{l_2}.\mathcal{T}_1 \mathcal{F}_1 \]  
if \( l \equiv G_l(\lambda x^{l_2}.p)) \).

Definition of \( \mathcal{P}_1, \mathcal{H}_1, \mathcal{G}_1 \) and \( \mathcal{E}_1 \).

\[ \mathcal{P}_1 \equiv P \mathcal{F}_1 \mathcal{A}_1 \mid G(\mathcal{V}_1^I) ; \]
\[ \mathcal{H}_1 \equiv (P_t A_1) ; \mathcal{G}_1 \mid \mathcal{E}(\mathcal{V}_1^I) ; \]
\[ \mathcal{G}_1 \equiv L \mathcal{L}_1 \mid A_i \mathcal{A}_1 \mid \mathcal{E}(\mathcal{V}_1^I) ; \]
\[ \mathcal{E}_1 \equiv \mathcal{H}_1 \mid \mathcal{G}_1. \]

Remarks 4.22. (i) Similar to Lemma 4.4(v) we now have \( \mathcal{H}_1 = \mathcal{P}_1 \mathcal{F}_1^\mathcal{P}_1 \).

(ii) If \( l \equiv A_i \) in \( G(\mathcal{V}_1^I) \) in the definition of \( \mathcal{P}_1 \), then \( x^l \) may occur free in \( p_1 \); but if \( l \equiv p \) then \( x^l \notin \text{FV}(p_1) \). So \( G(\mathcal{V}_1^I) \) may be split into \( G A_i(\lambda x^{A_i}.p_1) \) and \( G p(\mathcal{K}_p) \) and similarly for \( \mathcal{E}(\lambda x^l) \) in the definition of \( \mathcal{T}_1 \).

(iii) Note that \( \mathcal{E}_1 \neq \mathcal{E}_G \), hence Lemma 4.18 remains valid if \( \mathcal{E}_G \) is replaced by \( \mathcal{E}_1 \).

Lemma’s 4.6–4.9 mutatis mutandis also hold for \( \mathcal{E}_1 \). Moreover lemma’s corresponding to Lemma’s B3.7–B3.9 hold. Replace \([-] \) by \([-] \) and in the lemma corresponding to B3.7 take \( \mathcal{H}_1 \) instead of \( \mathcal{P}_1 \).

Lemma 4.23. Let \( p, p' \in \mathcal{P}_1 \). Then \( pM = p'M' \Rightarrow p = p' \).

Proof. Induction on \( p \).

The following lemma corresponds to Proposition B3.10.

Lemma 4.24. Let \( \Gamma \subseteq \mathcal{E}_1 \) and let \( \Gamma \vdash_{\mathcal{F} \mathcal{G}} N \) be a cut free proof. Then

(i) If the last applied rule in the proof is not the \( \Xi \)-rule, nor the \( G_\Xi \)-rule (modulo the \( \beta \)-rule), then \( N \in \mathcal{E}_1 \).

(ii) If \( N = l t \) with \( t \equiv \mathcal{L}_1, t \in \mathcal{F}_1 \) then \( N \in \mathcal{E}_1 \).

Proof. The proof of (i) and (ii) is by simultaneous induction on \( \Gamma \vdash N \). The two general cases are easy, so we only consider Cases \( \Xi_1 \), for (ii), and \( \Xi_\epsilon \).

Case \( \Xi_1 \). \( \Gamma \vdash N \) is \( \Gamma \vdash l t = \Xi XY \) as direct consequence of \( \Gamma \vdash LX, \Gamma, Xx \vdash Yx \).

Now \( lt = G l(\lambda x^{l_1}.p)t = \Xi l(\lambda x^{l_1}.p(tx^{l_1})) = \Xi XY \). The IH for \( \Gamma, l_1 x^{l_1} \vdash p(tx^{l_1}) \) yields \( p(tx^{l_1}) = O \in \mathcal{E}_1 \). So \( \Xi l(\lambda x^{l_1}.p(tx^{l_1})) = \Xi l(\lambda x^{l_1}.O) \in \mathcal{E}_1 \).
CASE \( \Xi_e \). \( \Gamma \vdash N \) is \( \Gamma \vdash XY \) as direct consequence of \( \Gamma \vdash X \). By the induction hypothesis for \( \Gamma \vdash \Xi XY \) we have \( \Xi XY \in \mathcal{F}_1 \). So \( \Xi XY = \Xi l_1(\lambda x^l, O) \). The IH for \( \Gamma \vdash l_1 V \) yields \( l_1 V \in \mathcal{F}_1 \), hence \( V \in \mathcal{F}_1 \). So \( XY = O[x^l := V] \in \mathcal{F}_1 \). 

PROPOSITION 4.25 (Completeness of \( \mathcal{G} \) for PRED). 

\[ \Gamma^2, [\Delta]^2, \Gamma(\Delta, \varphi) \vdash_{\mathcal{G}} \varphi^2 M \Rightarrow \Delta \vdash_{\text{PRED}} \varphi. \]

PROOF. The proof resembles very much the proof of Proposition B3.11. Note that \( \Gamma^2, [\Delta]^2 \subseteq \mathcal{F}_1 \) and \( \Gamma(\Delta, \varphi) \subseteq \mathcal{F}_1 \). Hence it is sufficient to prove

\[ (*) \quad \Gamma^g, [\Delta]^2 \vdash_{\mathcal{G}} N, \quad \Gamma \subseteq \mathcal{F}_1, \quad N = [\varphi]^2 M \Rightarrow \Delta \vdash_{\text{PRED}} \varphi. \]

Write \( \Gamma = \Gamma^g, [\Delta]^2 \). Then \( \Gamma \subseteq \mathcal{F}_1 \). The proof of \((*)\) is by induction on the derivation of \( \Gamma \vdash_{\mathcal{G}} N \). We may assume that the derivation is cut free because \( \mathcal{F}_1 \subseteq \mathcal{G} \).

CASE 1. \( \Gamma \vdash N \) because \( N \in \Gamma \).

\[ N = [\varphi]^2 M, \quad \text{so} \quad N \not\in \mathcal{F}_1, \quad \text{hence} \quad N \in [\Delta]^2. \]

We get from Lemma 4.23 that \( [\varphi]^2 = [\psi]^2 \) for some \( \psi \in \Delta \). So \( \varphi = \psi \) and hence \( \varphi \in \Delta \). Therefore \( \Delta \vdash_{\text{PRED}} \varphi \).

CASE 2. \( \Gamma \vdash N \) is a direct consequence of \( \Gamma \vdash N' \) and \( N = N' \).

Now \( N' = N = [\varphi]^2 M \) and by the IH for \( N' \) one has \( \Delta \vdash_{\text{PRED}} \varphi \).

CASE \( \Xi_e \). \( \Gamma \vdash N \) is \( \Gamma \vdash YV \) as direct consequence of \( \Gamma \vdash XY \). \( \Gamma \vdash XV \).

\( \Gamma \vdash N \) is cut free, hence from Lemma 4.24 we get that \( \Xi XY \in \mathcal{F}_1 \). We distinguish 3 subcases.

SUBCASE \( \Xi_e(a) \). \( X = A_l \), \( Y = \lambda x^A h \). Now \( h = pt^e = [\psi]^2 t^p \) for some \( p \in \mathcal{P} \), \( \psi \in \mathcal{F}_1 \). We have \( XY = A_l V \in \mathcal{F}_1 \) by Lemma 4.24. Therefore \( V = t^A = [t^A] \). Now \( N = [\varphi]^2 M = YV = (\lambda x^A \cdot \psi^2 t^p) \). Hence \( \Delta \vdash_{\text{PRED}} \varphi^2 \).

Also \( \Delta \vdash_{\text{PRED}} \varphi\).

Therefore it follows by modus ponens that \( \Delta \vdash_{\text{PRED}} \varphi \).

SUBCASE \( \Xi_e(b) \). \( X = p \), \( Y = \lambda x^p h \). Now \( Y = \lambda x^p t^p t^p \) where \( x^p \notin \text{FV} (p_2) \). Then \( N = [\varphi]^2 M = YV = p^t t^p \) for some \( t' \). So \( [\varphi]^2 = p^2 \). Let \( p = [\varphi_1]^2 \). Then \( \Xi XY = G_1(\lambda x^A h) t^t = F p^2 t^t = [\varphi_1] \varphi^2 t^t \) for some \( t'' \). So we have by the IH \( \Delta \vdash_{\text{PRED}} \varphi_1 \).

Also \( \Delta \vdash_{\text{PRED}} \varphi_1 \).

Therefore it follows by modus ponens that \( \Delta \vdash_{\text{PRED}} \varphi \).

SUBCASE \( \Xi_e(c) \). \( \Xi XY = \Xi l(\lambda x^l, g) \) This case is not applicable because \( YV \neq [\varphi]^2 M \).

CASE \( \Xi_i \). \( \Gamma \vdash N \) is \( \Gamma \vdash \Xi XY \) as direct consequence of \( \Gamma \vdash LX \). \( \Gamma, X x \vdash Y x \) with \( x \notin \text{FV}(\Gamma, X, Y) \).
\[\Xi XY = [\varphi]^2 M.\] So \([\varphi]^2 = G_{l_1}(\lambda x^{l_1}. p_2)\) for some \(l_1 \in \mathcal{L}_1\) and \(p_2 \in \mathcal{P}_1\), and \(\Xi XY = \Xi l_1(\lambda x^{l_1}. p_2(M x^{l_1}))\). We consider 2 subcases.

**Subcase 3(a).** \(X = A_1\). Let \(p_2 = [\psi]^2\). Then \(\varphi = \forall x A_1 \psi\). As \(\Gamma, X A_1 \vdash p_2(M x^{A_1})\) one has \(\Delta \vdash_{\text{PRED}} \psi\) by the IH. Now \(x^{A_1}\) does not occur in \(\Gamma\), so
\[\Delta \vdash_{\text{PRED}} \forall x A_1 \psi.\]

**Subcase 3(b).** \(X = p_1\). Let \(p_1 = [\varphi_1]^2, p_2 = [\varphi_2]^2\). Then \(\varphi = \varphi_1 \supset \varphi_2\). Now \(\Gamma, [\varphi_1]^2 x \vdash [\varphi_2]^2(M x)\). So one has by the IH \(\Delta, \varphi_1 \vdash_{\text{PRED}} \varphi_2\). Hence
\[\Delta \vdash_{\text{PRED}} \varphi_1 \supset \varphi_2.\]

**Case GL.** \(\Gamma \vdash N\) is \(\Gamma \vdash L(GXYZ)\) as direct consequence of
\[\Gamma \vdash LX, \, \Gamma, Xx \vdash L(Yx)\text{ with } x \notin \text{FV}(\Gamma, X, Y).\]
This case is not applicable because \(L(GXYZ) \neq [\varphi]^2 M\).

§5. Completeness of \(\mathcal{F}\) for PROP. We must show for \(\varphi \in \mathcal{F}_{\text{PROP}}, \Delta \subseteq \mathcal{F}_{\text{PROP}}:\n\forall \text{closed } r \exists M \in \Lambda[[[\Delta]]^2, \Gamma_1^2(\Delta, \varphi) \vdash_{\mathcal{F}} [\varphi]^2 M] \implies \Delta \vdash_{\text{PROP}} \varphi.
\]
It is sufficient to show
\[\exists M \in \Lambda[[[\Delta]]^2, \Gamma_1^2(\Delta, \varphi) \vdash_{\mathcal{F}} [\varphi]^2 M] \implies \Delta \vdash_{\text{PROP}} \varphi.\]
We omit \(I\) in the notation and we prove
\[\Delta^2, \Gamma_1^2(\Delta, \varphi) \vdash_{\mathcal{F}} [\varphi]^2 M \implies \Delta \vdash_{\text{PROP}} \varphi.\]

The proof is much simpler than the proof of completeness of \(\mathcal{G}\) for PRED, because there is no need for cut elimination. We define only one invariant \(\mathcal{G}_2\) in 5.1, similar to \(\mathcal{G}\) and \(\mathcal{G}_1\) and using this invariant we prove the completeness in 5.2–5.5.

**Definition 5.1 (Invariant for derivations in \(\mathcal{F}\)).** \(\mathcal{G}_2\) is a set of proposition variables. For every \(p \in \mathcal{P}_2\) there is a different set \(\mathcal{V}_p^2\) of variables.

**Definition of \(\mathcal{V}_p^2\).**
\[\mathcal{V}_p^2 \equiv \mathcal{V}_2 \mid \mathcal{F}_2 p_1 \mathcal{V}_2, \text{ if } p \in \mathcal{V}_2^p;\]
\[\mathcal{F}^p_2 \equiv \mathcal{V}_2 \mid \mathcal{F}_2 p_1 \mathcal{F}_2, \mid \lambda x^{p_1}. \mathcal{F}_2^p, \text{ if } p \equiv \mathcal{F}_2 p_3.\]

**Definition of \(\mathcal{P}_2, \mathcal{H}_2, \mathcal{G}_2\) and \(\mathcal{G}_2\).**
\[\mathcal{P}_2 \equiv \mathcal{V}_2 \mid \mathcal{F}_2 \mathcal{P}_2;\]
\[\mathcal{H}_2 \equiv \forall ^{p}(v \in \mathcal{V}_2) \mid \Xi p_1(\lambda x^{p_1}. \mathcal{H}_2);\]
\[\mathcal{G}_2 \equiv \mathcal{L}_2 \mid \Xi \mathcal{P}_2(K \mathcal{G}_2);\]
\[\mathcal{G}_2 \equiv \mathcal{H}_2 \mid \mathcal{G}_2.\]

**Remark 5.2.** Similarly to Lemma 4.4(v) we have \(\mathcal{H}_2 = \mathcal{P}_2 \mathcal{P}_2^2\) modulo \(\beta\eta\)-equality. Note that in \(\Xi p_1(\lambda x^{p_1}. h)\) in the definition of \(\mathcal{H}_2\) we have \(h = p_2 t^{p_2}\) where \(x^{p_1} \notin \text{FV}(p_2)\).

We have lemma’s similar to Lemma’s 4.6–4.9, with similar, but easier, proofs. (Now \(\mathcal{L}_2 = \mathcal{P}_2.\)
Proposition 5.3. In \( \mathcal{S} \mathcal{F} \) we have

\[ \Gamma \vdash M, \Gamma \subseteq \mathcal{D}_2 \Rightarrow M \in \mathcal{D}_2. \]

Proof. Similar to the proof of B3.10. Instead of Case G\( \mathcal{L} \) we now have Case F\( \mathcal{L} \).

Lemma 5.4. (i) \( \mathcal{D}_2 \cap \mathcal{D}_2 = \emptyset \);
(ii) \( \mathcal{D}_2 \cap \overline{\mathcal{D}_2} = \emptyset \);
(iii) \([ \_ ]^2 : \mathbb{F}_{\text{PROP}} \to \mathcal{D}_2 \) is a bijection.
(iv) Let \( p, p' \in \mathcal{D}_2 \). Then \( pM = p'M' \Rightarrow p = p' \).

Proof. By easy inductions. For (ii) we use Church Rosser.

Proposition 5.5 (Completeness of \( \mathcal{S} \mathcal{F} \) for PROP).

\[ [\Delta]^2, \Gamma^2(\Delta, \varphi) \vdash_{\mathcal{S} \mathcal{F}} [\varphi]^2 M \Rightarrow \Delta \vdash_{\text{PROP}} \varphi. \]

Proof. Similar to the proof of Proposition 4.25 but easier. The subcases corresponding to subcases \( \Xi_{\setminus}(a) \) and \( \Xi_{\mid}(a) \) do not occur.

§6. Completeness of \( \mathcal{S} \mathcal{G} \) for \( \text{PREDA} \rightarrow \). In this section we will show completeness of \( \mathcal{S} \mathcal{G} \) over predicate logic extended with higher type functions. In this logic we may quantify over all functions based on \( A_1 \) and \( A_2 \) and also over functions ranging over functions etc. and we can state e.g.,

\[ \forall f_1 \in ((A_1 \to A_2) \to A_1) \forall f_2 \in (A_1 \to A_2)[P(f_1 f_2)]. \]

Definition 6.1. \( \text{PREDA} \rightarrow \), a preliminary version of Predicate Calculus with Higher Type Functions, is defined by

(i) Types of \( \text{PREDA} \rightarrow \)

\[ \mathbb{L}_{\text{PREDA} \rightarrow 0} = A_1 \mid A_2 \mid \mathbb{L}_{\text{PREDA} \rightarrow 0} \to \mathbb{L}_{\text{PREDA} \rightarrow 0}. \]

(ii) Variables of \( \text{PREDA} \rightarrow 0 \)

\[ \forall^1 = \{x^1_1, x^1_2, \ldots \} \text{ for } I \in \mathbb{L}_{\text{PREDA} \rightarrow 0}. \]

(iii) Terms of \( \text{PREDA} \rightarrow 0 \)

\[ T^1 = a \mid \forall^1 \mid T^1_1 \to \mathbb{T}^1_1, \quad \text{if } I = A_1; \]
\[ T^1 = \forall^1 \mid T^1_1 \to \mathbb{T}^1_1, \quad \text{if } I = A_2; \]
\[ T^1 = f \mid \forall^1 \mid T^1_1 \to \mathbb{T}^1_1, \quad \text{if } I = A_1 \to A_1; \]
\[ T^1 = g \mid \forall^1 \mid T^1_1 \to \mathbb{T}^1_1, \quad \text{if } I = A_1 \to A_2; \]
\[ T^1 = \forall^1 \mid T^1_1 \to \mathbb{T}^1_1, \quad \text{if } I \notin \{A_1, A_2, A_1 \to A_1, A_1 \to (A_2 \to A_1)\}. \]

(iv) Well-formed formulas of \( \text{PREDA} \rightarrow 0 \)

\[ \mathbb{F}_{\text{PREDA} \rightarrow 0} = \mathbb{P}^\mathbb{T} A_1 \mid \mathbb{F}_{\text{PREDA} \rightarrow 0} \supset \mathbb{F}_{\text{PREDA} \rightarrow 0} \mid (\forall x^1) \mathbb{F}_{\text{PREDA} \rightarrow 0}. \]

(v) Postulates of \( \text{PREDA} \rightarrow 0 \)
\[ \varphi \in \Gamma \Rightarrow \Gamma \vdash \varphi; \]
\[ \Gamma \vdash \varphi \supset \psi, \quad \Gamma \vdash \varphi \Rightarrow \Gamma \vdash \psi; \]
\[ \Gamma, \varphi \vdash \psi \Rightarrow \Gamma \vdash \varphi \supset \psi; \]
\[ \Gamma \vdash \forall x^1 \varphi, \quad t \in \mathbb{T}^1 \Rightarrow \Gamma \vdash \varphi[x^1 := t]; \]
\[ \Gamma \vdash \varphi, \quad x^1 \notin \text{FV}(\Gamma) \Rightarrow \Gamma \vdash \forall x^1 \varphi. \]

**Definition 6.2.** The mappings \([-]^i\) and \(\Gamma\) in B2.12 from \(\mathbb{F}_{\text{PREDA}}\) into \(\Lambda^i(\Xi, L)\) and illative contexts respectively are extended to mappings from \(\mathbb{F}_{\text{PREDA}}\) by altering some clauses and adding some clauses in the following way.

First we define \([A_j]^i = A_j\) \((j = 1, 2)\), \([l_1 \rightarrow l_2]^i = F[l_1]^i[l_2]^i\). We will denote \([l]^i\) by \(l\). Then we define

\[ [x^1]^i = x^i, \quad [t_1 t_2]^i = [t_1]^i [t_2]^i, \]
\[ \forall x^1 \varphi]^i = \Xi L(\lambda x^1. [\varphi]^j), \quad \forall x^1 \varphi]^2 = G L(\lambda x^1. [\varphi]^j), \]
\[ \Gamma(x^i) = 1 x^i, \quad \Gamma(\forall x^1 \varphi) = \Gamma(\varphi) - \{1 x^i\}, \quad \Gamma(t_1 t_2) = \Gamma(t_1) \cup \Gamma(t_2), \]
\[ \Gamma_{r,s}^1 = (LA_1, LA_2, FA_1 A_1 f, FA_1 (FA_2 A_1) g, FA_1 H(r o P), A_1 a), \]
\[ \Gamma_{r,s}^2 = (LA_1, LA_2, FA_1 A_1 f, FA_1 (FA_2 A_1) g, FA_1 L(r o P), A_1 a), \]
\[ \Gamma_{r,s}^{i+} = \Gamma_{r,s}^i \cup \{1 x^i | i \neq A_1\}, \]

[-]^i, \Gamma_s^i and \(\Gamma_{r,s}^{i+}\) are obtained from \([-]^i, \Gamma_r^i\) and \(\Gamma_{r,s}^{i+}\) by everywhere omitting \(r\) i.e., by replacing \(r o P\) by \(P\).

**Proposition 6.3.** \(\mathcal{F} G\) is not complete for \(\text{PREDA} \rightarrow_0\).

**Proof.** Let \(\varphi = \forall x^1_{\sim} \rightarrow A_1 \forall x^2_{\sim} P(x_1 x_2) \supset \forall x^2_{\sim} P x x_2\). We will show

(*): \[ \forall \text{closed } r \exists M [\Gamma_{r,s}^{i+}, \Gamma(\varphi) \vdash_{\mathcal{G}} [\varphi]^2 M], \]

(**): \[ \vdash_{\text{PREDA} \rightarrow_0} \varphi. \]

**Proof of (**) Let \(X_1 = G(F A_1 A_1) (\lambda x_1. G A_1 (\lambda x_2. (r o P)(x_1 x_2))), \ Y_1 = G A_1 (\lambda x_2. (r o P)(x_1 x_2)), \ M = \lambda x_{\sim}, x_{\sim} I\) where \(I = \lambda x_{\sim}. x_{\sim} A_1\). We will prove (*) by showing \[ \forall \text{closed } r \[\Gamma_{r,s}^{i+}, \vdash_{\mathcal{G}} F X_1 Y_1 M]. \]

Note that the rules \(G_e\) and \(G_i\) imply the rules \(F_e\) and \(F_i\). Let \(r\) be a closed term in \(\Lambda^i(\Xi, L)\). We have by rule \(F_i\)

\[ LA_1 \vdash FA_1 A_1 I \]

and from this we get by rule \(G_e\) (taking \(X = FA_1 A_1, \ Y = \lambda x_1. G A_1 (\lambda x_2. (r o P)(x_1 x_2)), \ Z = x(= x_{X_1}), \ V = I) \]

\[ LA_1, X_1 x \vdash Y_1(xI). \]

So we have

(1) \[ LA_1, X_1 x \vdash Y_1(Mx). \]

Moreover rule \(F_e\) yields

\[ LA_1, FA_1 A_1 x_1, A_1 x_2 \vdash A_1(x_1 x_2). \]

Hence we get by applying rule \(F_e\) once more

\[ LA_1, FA_1 L(r o P), FA_1 A_1 x_1, A_1 x_2 \vdash L((r o P)(x_1 x_2)) \]
and by rule $G_{\lambda}$ twice

\[(2) \quad LA_1, FA_1 L(r \circ P) \vdash LX_1.\]

From (1) and (2) we get by rule $F_i$

\[LA_1, FA_1 L(r \circ P) \vdash FX_1 Y_1 M.\]

This proves (*). But $\not\forall_{\text{PRED} \lambda \to 0} \varphi$ because $\text{PRED} \lambda \to 0$ has no identity function in $F_{\text{FA}1} A_1$.

By similar examples one shows the need for other combinators or $\lambda$-terms. In fact all of these are needed in $\text{PRED} \lambda \to 0$ before completeness can be proved.

**Definition 6.4** (Predicate Calculus with Higher Type Functions, PRED$\lambda \to$). Types and variables are as for PRED$\rightarrow_{0}$.

**Terms of** PRED$\lambda \rightarrow$

- $T^l = a$ if $l = A_1$;
- $T^l = \forall^l | T^{l_1} \rightarrow_{\lambda} T^{l_1}$ if $l = A_2$;
- $T^l = f | \forall^l | T^{l_1} \rightarrow_{\lambda} T^{l_1} | \lambda^l \rightarrow_{\lambda} T^{l_1}$ if $l = A_1 \rightarrow A_1$;
- $T^l = g | \forall^l | T^{l_1} \rightarrow_{\lambda} T^{l_1} | \lambda^l \rightarrow_{\lambda} T^{l_1}$ if $l = A_1 \rightarrow (A_2 \rightarrow A_1)$;
- $T^l = \forall^l | T^{l_1} \rightarrow_{\lambda} T^{l_1} | \lambda^l \rightarrow_{\lambda} T^{l_1}$ if $l = l_2 \rightarrow l_3$,

where $l \notin \{A_1 \rightarrow A_1, A_1 \rightarrow (A_2 \rightarrow A_1)\}$.

The well-formed formulas of PRED$\lambda \rightarrow$ are defined as for PRED$\rightarrow_{0}$, but with the altered terms. The postulates of PRED$\lambda \rightarrow$ are as for PRED$\rightarrow_{0}$, plus

$\Gamma \vdash \varphi, \varphi = \exists_{\vec{y}} \psi \Rightarrow \Gamma \vdash \psi$.

**Remark 6.5.** Ordinary predicate logic PRED can be expressed by the pure type system (PTS), see [1].

\[
S = \ast^p, \ast^s, \ast^f, \square^p, \square^s \\
A = \ast^p : \square^p, \ast^s : \square^s \\
R = (\ast^p, \ast^p), (\ast^s, \ast^p), (\ast^s, \square^p) \\
(\ast^s, \ast^s, \ast^f), (\ast^s, \ast^f, \ast^f)
\]

The point is that $\ast^f$ is a sort for function spaces like $A_1 \rightarrow A_2$ or $A_1 \rightarrow (A_2 \rightarrow A_2)$, with $A_1, A_2 \in \ast^s$, but not for higher function spaces like $(A_1 \rightarrow A_2) \rightarrow A_2$. The system PRED$\lambda \rightarrow$ has higher function spaces and moreover it allows quantification over them. PRED$\lambda \rightarrow$ can be described more simply as the following PTS.

**PRED$\lambda \rightarrow$**

\[
S = \ast^p, \ast^s, \square^p, \square^s \\
A = \ast^p : \square^p, \ast^s : \square^s \\
R = (\ast^p, \ast^p), (\ast^s, \ast^p), (\ast^s, \square^p), (\ast^s, \ast^s)
\]

**Definition 6.6.** We extend the mappings $[-]_\lambda$ and $\Gamma$ of Definition 6.2 by

$[\lambda^l x^{l_1}. t^{l_2}]_\lambda' = \lambda^l x^{l_1}. [t^{l_2}]_\lambda'$, $\Gamma(\lambda^l x^{l_1}. t^{l_2}) = \Gamma(t^{l_2}) \setminus \{l_1 x^{l_1}\}$. 

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**Proposition 6.7** (Soundness of the interpretations for \( \text{PRED}_\lambda \).) Let \( \Delta \cup \{ \varphi \} \subseteq \mathcal{F}_{\text{PRED}_\lambda} \), then the following hold for all closed \( r \).

(i) \( \Delta \vdash_{\text{PRED}_\lambda} \varphi \Rightarrow \Gamma^{1}_{r^{+}}, [\Delta]^{1}_{r}, \Gamma(\Delta, \varphi) \vdash_{\mathcal{F}} \varphi^{1} \).

(ii) \( \Delta \vdash_{\text{PRED}_\lambda} \varphi \Rightarrow \Gamma^{2}_{r^{+}}, [\Delta]^{2}_{r}, \Gamma(\Delta, \varphi) \vdash_{\mathcal{G}} \varphi^{2} M \) for some \( M \).

**Proof.** We shall not give the proof here. It is similar to the proof of Proposition B2.14 but longer because many more trivial cases have to be considered.

Now in 6.8-6.12 we shall prove completeness of \( \mathcal{G} \) for \( \text{PRED}_\lambda \) under the indirect translation \([-]^{2} \). (Completeness of \( \mathcal{F} \Xi \) for \( \text{PRED}_\lambda \) under the direct translation \([-]^{1} \) also holds, but we shall not give the proof here.)

First we define in Definition 6.8 an invariant \( \sigma_{3} \), similar to \( \sigma \) in B3.2 and \( \sigma_{1} \) in Definition 4.21. We have \( \sigma_{3} \subseteq \sigma_{G} \), hence Lemma 4.18 yields that in a proof of \( \Gamma \vdash_{\mathcal{G}} M \) where \( \Gamma \subseteq \sigma_{3} \) all cut formulas can be eliminated. Then the proof of completeness of \( \mathcal{G} \) for \( \text{PRED}_\lambda \) is similar to the proof of completeness of \( \mathcal{G} \) for \( \text{PRED} \) in 4.22-4.25.

**Definition 6.8** (Invariant \( \sigma_{3} \) for \( \mathcal{G} \)).

\[
\sigma_{3} \equiv A_{1} | A_{2} | \mathcal{F}_{3} \mathcal{L}_{3};
\]

\[
\mathcal{L}_{3} \equiv \mathcal{L}_{3} | \mathcal{P}_{3};
\]

\[
\mathcal{V}^{l}_{3} \equiv \{ x_{1}^{j}, x_{2}^{j}, \ldots \}, \quad x_{1}^{j} \neq x_{2}^{j} \text{ if } l_{1} \neq l_{2};
\]

\[
\mathcal{F}^{l}_{3} \equiv a | \mathcal{V}^{l}_{3} | \mathcal{F}^{l}_{3};
\]

\[
\mathcal{T}^{l}_{3} \equiv \mathcal{V}^{l}_{3} | \mathcal{T}_{3}^{l} \mathcal{T}_{3}^{l}; \quad \text{if } l \equiv A_{1};
\]

\[
\mathcal{T}^{l}_{3} \equiv \mathcal{V}^{l}_{3} | \mathcal{T}_{3}^{l} \mathcal{T}_{3}^{l}; \quad \text{if } l \equiv A_{2};
\]

\[
\mathcal{T}^{l}_{3} \equiv f | \mathcal{V}^{l}_{3} | \mathcal{T}_{3}^{l} \mathcal{T}_{3}^{l} | \lambda x_{A_{1}} \mathcal{T}_{3}^{A_{1}}; \quad \text{if } l \equiv F_{A_{1}} A_{1};
\]

\[
\mathcal{T}^{l}_{3} \equiv g | \mathcal{V}^{l}_{3} | \mathcal{T}_{3}^{l} \mathcal{T}_{3}^{l} | \lambda x_{A_{1}} \mathcal{T}_{3}^{F_{A_{2}} A_{1}}; \quad \text{if } l \equiv F_{A_{1}} (F_{A_{2}} A_{1});
\]

\[
\mathcal{T}^{l}_{3} \equiv \mathcal{V}^{l}_{3} | \mathcal{T}_{3}^{l} \mathcal{T}_{3}^{l} | \lambda x_{A_{1}} \mathcal{T}_{3}^{l} \mathcal{T}_{3}^{l}; \quad \text{if } l \equiv L_{l_{2}} l_{3}, l \notin \{ F_{A_{1}} A_{1}, F_{A_{1}} (F_{A_{2}} A_{1}) \}.
\]

\[
\mathcal{B}_{3} \equiv P \mathcal{T}_{3}^{A_{1}} \equiv \mathcal{G}_{q}(\lambda x^{q}, \mathcal{B}_{3});
\]

\[
\mathcal{H}_{3} \equiv (P^{A_{1}}) t^{P^{A_{1}}} \equiv \mathcal{G}(\lambda x^{q}, \mathcal{H}_{3});
\]

\[
\mathcal{Q}_{3} \equiv \mathcal{L}_{3} \mathcal{L}_{3} | A_{i} \mathcal{T}_{3}^{A_{1}} \equiv \mathcal{G}(\lambda x^{q}, \mathcal{Q}_{3});
\]

\[
\mathcal{O}_{3} \equiv \mathcal{H}_{3} \mathcal{G}_{3}.
\]

\[
\mathcal{V}^{p}_{3} \equiv \{ x_{1}^{p}, x_{2}^{p}, \ldots \}, \quad x_{1}^{p} = x_{1}^{p_{1}} \neq x_{1}^{p_{2}} \text{ if } sk(p_{1}) \neq sk(p_{2});
\]

\[
sk(l) \equiv l;
\]

\[
sk(P^{A_{1}}) \equiv P x^{A_{1}} \quad \text{(i.e., } sk(P^{A_{1}}) \text{ is independent of } t^{A_{1}});\]

\[
sk(\mathcal{G}_{q}(\lambda x^{q}, p)) \equiv \mathcal{G}(sk(q))(\lambda x^{q}.sk(p));
\]

\[
\mathcal{T}^{p}_{3} \equiv \mathcal{V}^{p}_{3} | \mathcal{T}_{3}^{p} \mathcal{T}_{3}^{p}; \quad \text{if } p \equiv P^{A_{1}};
\]

\[
\mathcal{T}^{p}_{3} \equiv \mathcal{V}^{p}_{3} | \mathcal{T}_{3}^{p} \mathcal{T}_{3}^{p} | \lambda x^{q_{1}} \mathcal{T}_{3}^{p} \mathcal{T}_{3}^{p}; \quad \text{if } p \equiv \mathcal{G}_{q_{1}}(\lambda x^{q_{1}} p_{1}).
\]

**Remarks 6.9.** (i) Similar to Lemma 4.4(v) we now have \( \mathcal{H} = \mathcal{B}_{3} \mathcal{T}_{3}^{p} \mathcal{G}_{3} \) modulo \( \beta \eta \)-equality.

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(ii) If \( q \equiv l \) in \( Gq(\lambda x^q.p_1) \) in the definition of \( \mathcal{R}_3 \), then \( x^q \) may occur free in \( p_1 \); but if \( q \equiv p \) then \( x^q \notin \text{FV}(p_1) \). So \( Gq(\lambda x^q.p_1) \) may be split into \( G(\lambda x^p.p_1) \) and \( Gp(\lambda p_1) \) and similarly for \( \mathcal{E}q(\lambda x^q.g) \) in the definition of \( \mathcal{G}_3 \).

Lemma's 4.6–4.9 mutatis mutandis also hold for \( \mathcal{E}_3 \). (Replace \( l \) by \( q \).) Moreover lemma's corresponding to Lemma's B3.7–B3.9 hold. Replace \([-1] \) by \([-2] \) and in the lemma corresponding to B3.7 take \( \mathcal{H}_3 \) instead of \( \mathcal{R}_3 \).

**Lemma 6.10.** Let \( p, p' \in \mathcal{R}_3 \). Then \( pM = p'M' \Rightarrow p = p' \).

**Proof.** Induction on \( p \).

**Lemma 6.11.** Let \( \Gamma \subseteq \mathcal{G}_3 \) and let \( \Gamma \vdash \mathcal{F}_G N \) be a cut free proof. Then

(i) If the last applied rule in the proof is not the \( \Xi_l \)-rule, nor the \( \mathcal{G}_l \)-rule (modulo the \( \beta \eta \)-rule), then \( N \in \mathcal{R}_3 \).

(ii) If \( N = qt \) with \( q \in \mathcal{E}_3 \), \( t \in \mathcal{F}_G^q \), then \( N \in \mathcal{R}_3 \).

**Proof.** Similar to the Proof of Lemma 4.24.

**Proposition 6.12 (Completeness of \( \mathcal{F} \mathcal{G} \) for \( \mathcal{P} \mathcal{E} \mathcal{D} \rightarrow \) relative to \([-2] \)).**

\[ \Gamma^3 \vdash [\Delta]^2, \Gamma(\Delta, \varphi) \vdash \mathcal{F}_G [\varphi]^2 M \Rightarrow \Delta \vdash \mathcal{P} \mathcal{E} \mathcal{D} \rightarrow \varphi. \]

**Proof.** We have \( \mathcal{E}_3 \subseteq \mathcal{G}_3 \), hence by Lemma 4.18 we may assume that the proof of \( \Gamma^3 \vdash [\Delta]^2, \Gamma(\Delta, \varphi) \vdash \mathcal{F}_G [\varphi]^2 M \) is cut free. Then the proof is similar to the proof of completeness of \( \mathcal{F} \mathcal{G} \) for \( \mathcal{P} \mathcal{E} \mathcal{D} \) in Proposition 4.25.

**Remark 6.13 (Higher Order Predicate Calculus).** Let us define \( \mathcal{P} \mathcal{E} \mathcal{D} \lambda \omega \) as \( \mathcal{P} \mathcal{E} \mathcal{D} \lambda \) but with \( L_{\mathcal{P} \mathcal{E} \mathcal{D} \lambda \omega} = A_1 | A_2 | H | L_{\mathcal{P} \mathcal{E} \mathcal{D} \lambda \omega} \rightarrow L_{\mathcal{P} \mathcal{E} \mathcal{D} \lambda \omega} \).

Thus \( \mathcal{P} \mathcal{E} \mathcal{D} \lambda \omega \) allows quantification over all functions, but also over all predicates and propositions. As a PTS the system \( \mathcal{P} \mathcal{E} \mathcal{D} \lambda \omega \) is \( \mathcal{P} \mathcal{E} \mathcal{D} \lambda \) plus \( (\square p, \ast p), (\square p, \square p) \). Given an extension of \( \Gamma_{r,s}^l \) to include \( \vdash \mathcal{L} \mathcal{H} \), we would expect to prove the completeness of \( \mathcal{F} \mathcal{G} \) over \( \mathcal{P} \mathcal{E} \mathcal{D} \lambda \omega \), but we have not yet succeeded. In particular we have not been able to prove cut elimination for \( \mathcal{L} \mathcal{H} \in \Gamma_{r,s}^l \), so the methods employed here are not all available.

**§7. Conclusions and remarks.** In [2] and in this paper we proved soundness and completeness for the direct and indirect interpretations of propositional and predicate calculus into illative systems:

(i) \( \Delta \vdash \mathcal{P} \mathcal{E} \mathcal{D} \phi \quad \Leftrightarrow \quad \forall \text{closed } r[\ldots, [\Delta]_r \vdash \mathcal{F}_G [\phi]_r]; \)

(ii) \( \Delta \vdash \mathcal{P} \mathcal{E} \mathcal{D} \phi \quad \Leftrightarrow \quad \forall \text{closed } r[\ldots, [\Delta]_r \vdash \mathcal{F}_G [\phi]_r]; \)

(iii) \( \Delta \vdash \mathcal{P} \mathcal{E} \mathcal{D} \phi \quad \Leftrightarrow \quad \forall \text{closed } r \exists \mathcal{M}[\ldots, [\Delta]_r \vdash \mathcal{F}_G [\phi]_r \mathcal{M}]; \)

(iv) \( \Delta \vdash \mathcal{P} \mathcal{E} \mathcal{D} \phi \quad \Leftrightarrow \quad \forall \text{closed } r \exists \mathcal{M}[\ldots, [\Delta]_r \vdash \mathcal{F}_G [\phi]_r \mathcal{M}]; \)

where \( \ldots \) stands for \( \Gamma_{r,s}^l(\Delta, \varphi) \) in the cases of \( \mathcal{P} \mathcal{E} \mathcal{D} \phi \) and for \( \Gamma_{r,s}^{l+}, \Gamma(\Delta, \varphi) \) in the cases of \( \mathcal{P} \mathcal{E} \mathcal{D} \phi \).

**Remark 7.1.** The proof of soundness (\( \Rightarrow \)) was always simple. Completeness (\( \Leftarrow \)) was proved each time by specialising to \( r = 1 \). This is a very natural choice, but one may wonder if the proofs can be given for other \( r \), especially for \( r = K \). Inspecting the 4 proofs for \( r = 1 \) in this paper and in [2] one can verify that all these proofs need only minor changes to yield proofs for \( r = K \).
Let us consider as an example the proof of completeness of \( \mathcal{G} \) for \( \text{PRED} \) ((iv)\( \iff \)) taking \( r = \mathbf{K} \). Each \( Pt^{A_1} (t^{A_1} \in \mathcal{G}^{A_1}) \) in Section 4 must be replaced by \( K(Pt^{A_1}) \). (And a resulting \( K(Pt^{A_1})M \) we rewrite \( Pt^{A_1} \) again.) Let us denote for the moment the resulting set \( \mathcal{G} \) by \( \mathcal{G}_K \) and the set \( \mathcal{G} \) of Section 4 for \( r = \mathbf{I} \) by \( \mathcal{G}_I \). Then

\[
\mathcal{G}_K = K\mathcal{G}_I.
\]

In Definition 4.2 we define now

\[
\mathcal{G}_I^l \equiv \Lambda^l(\Xi, \mathbf{L}) \text{ if } l \equiv K(Pt^{A_1}).
\]

As a result then \( \mathcal{G}_I^p = \Lambda^l(\Xi, \mathbf{L}) \) for each \( p \in \mathcal{G}_K \). These are nearly all changes that are needed to get a proof for \( r = \mathbf{K} \) from the proof for \( r = \mathbf{I} \) in Section 4.

In another version of this paper, [3], we were not yet able to give completeness proofs of \( \mathcal{G} \) for \( \text{PRED} \) and \( \mathcal{F} \) for \( \text{PROP} \) ((iv)\( \iff \) and (iii)\( \iff \)) by specialising to \( r = \mathbf{I} \). We there gave completeness proofs by taking \( r = \mathbf{K} \), in a way different from the method sketched above. Let us consider now the proof of completeness of \( \mathcal{G} \) for \( \text{PRED} \) in [3] taking \( r = \mathbf{K} \). From

\[
[\varphi]^2_{K;\frac{2}{3}} = K[\varphi]^1_I
\]

in Lemma B2.15 we got

\[
(*) \quad [\varphi]^2_K \mathcal{M} = [\varphi]^1_I.
\]

Moreover in [3] we had a conservativity result

\[
(**) \quad \Gamma \vdash_{\mathcal{G}} M \Rightarrow \Gamma \vdash_{\mathcal{X}} M \text{ for 'suitable' } \Gamma \text{ and } M.
\]

Now the completeness proof in [3] was as follows

\[
\Gamma^+_K, [\Delta]^2_K, \Gamma(\Delta, \varphi) \vdash_{\mathcal{G}} [\varphi]^2_K \mathcal{M} \Rightarrow \quad \text{(by *)}
\]

\[
\Gamma^+_I, [\Delta]^1_I, \Gamma(\Delta, \varphi) \vdash_{\mathcal{G}} [\varphi]^1_I \Rightarrow \quad \text{(by ***)}
\]

\[
\Gamma^+_I, [\Delta]^1_I, \Gamma(\Delta, \varphi) \vdash_{\mathcal{X}} [\varphi]^1_I \Rightarrow \quad \text{(by Proposition B3.14)}
\]

\[
\Delta \vdash_{\text{PRED}} \varphi.
\]

(To prove this conservativity result we had to show, in a way similar to the proof in the present paper, that the proof of \( \Gamma \vdash_{\mathcal{G}} M \) could be made cut free.)

Also completeness of \( \mathcal{F} \) relative to \( \text{PROP} \) for \([\mathcal{F}^2_K \) followed from completeness of \( \mathcal{P} \) relative to \( \text{PROP} \) for \([\mathcal{F}^1_I \) by a conservativity result for \( \mathcal{F} \) relative to \( \mathcal{P} \).

We also proved elsewhere completeness of \( \mathcal{G} \) relative to \( \text{PRED}\) for \([\mathcal{G}^2_K \) followed because system \( \mathcal{G} \) is stronger then system \( \tilde{\mathcal{E}} \). That last completeness is not proved in the present paper.

**Remark 7.2.** We defined \( \mathcal{P}, \mathcal{F}, \mathcal{G} \) in B2.3 in terms of the constant \( \Xi \) and in B2.7 we obtained the following result.

The systems \( \mathcal{P}, \tilde{\mathcal{E}}, \mathcal{F} \) and \( \mathcal{G} \) are related as follows
where \( \rightarrow \) denotes nondecreasing strength, i.e., \( s_1 \rightarrow s_2 \) means that for all \( \Gamma, X \)

\[
\Gamma \vdash_{s_1} X \Rightarrow \Gamma \vdash_{s_2} X.
\]

Moreover we had conservativity results like

\[
\Gamma \vdash_{J_G} M \Rightarrow \Gamma \vdash_{J_E} M
\]

for 'suitable' \( \Gamma \) and \( M \) mentioned above. But in the proofs of soundness and completeness in the present paper we did not use these results. So one may wonder if soundness and completeness remain valid if we define \( P, F \) and \( G \) to be constants. This is indeed the case (only the bases \( \Gamma^1_{r,s} \) and \( \Gamma^2_{r,s} \) must be rewritten). In the soundness proofs we did not use the fact that \( P, F \) and \( G \) were defined in terms of \( \Xi \). Also completeness remains valid because the systems with \( P, F \) and \( G \) constant are less strong than the systems we considered. For example for system \( J_F \) then one does not have the relation

\[
F = \lambda y z. \Xi(\lambda y z. x (y \circ z)).
\]

**Remark 7.3.** In B4.1 we stated that it is possible to work with variants of the systems \( J_P, J_\Xi, J_F \) and \( J_G \) based on \( \beta \)-conversion only. This still holds. One may think that in the proof of Lemma 4.9 (and the proof of Lemma 6.11) the \( \eta \)-axiom is used. This however is not the case, because by Definition 4.2 not only \( f t A_1 \in J_{G_1} \), but also \( f \in J_{G_1}^A A_1 \) and similarly \( g \in J_{G_1}^{FA_2 A_1} \).

**References**

