MATHEMATICS Proceedings A 89 (3), September 29, 1986

Are there countable topological combinatory algebras?

by Henk Barendregt[1] and Jan van Mill[2]

[1] *Mathematisch Instituut, Rijksuniversiteit Utrecht, Budapestlaan 6, 3508 TA Utrecht, the Netherlands*
[2] *Wiskundig Seminarium, Vrije Universiteit, De Boelelaan 1081, 1081 HV Amsterdam, the Netherlands*

Communicated by Prof. H. Freudenthal at the meeting of March 24, 1986

A *topological combinatory algebra* is an applicative structure $(D, .)$ provided with a topology on D such that on D the notions of continuous, algebraic and representable function all three coincide (see [1] § 5.1 for terminology). Such structures are always infinite (except the trivial one point algebra; we will not consider it).

One way of finding combinatory algebras is to construct a cpo (complete partial ordering) D, provided with its Scott topology, such that the set of continuous maps $[D \to D]$ with the proper topology is a retract of D. In this way one obtains a topological combinatory algebra.

In § 1 it will be shown that for an infinite cpo D the set $[D \to D]$ is always uncountable. The question arises whether there are countable topological combinatory algebras. This would be impossible if there are always uncountably many continuous functions on an infinite topological space. This is not so: in § 2 an example is given of a countable space with only the identity and the constant functions continuous. The question in the title remains open.

ACKNOWLEDGEMENT

A sketch of the proof of theorem 1 is due to Furio Honsell. Kees Doets and Dick de Jongh made useful comments.

§ 1. LEMMA 1. (i) Let $(D, \leq)$ be an infinite poset. Then D contains an infinite chain or an infinite antichain.

(ii) Let $(D, \leq)$ be an infinite chain. Then D contains an infinite increasing or an infinite decreasing chain.

PROOF. (i) Partition $[D]^2$, the two element subsets of D, in $X_1 = \{\{x, y\} | x < y$ or $y < x\}$ and $X_2 = \{\{x, y\} | x, y$ incomparable$\}$. By Ramsey's theorem [4] IX 3.1 there is an infinite homogeneous set $D_0 \subseteq D$. If $[D_0]^2 \subseteq X_1$ (X_2 respectively) then D_0 is a chain (antichain).

(ii) Let $D_1 = \{x_0, x_1, \ldots\} \subseteq D$. Partition $[D_1]^2$ in $X_1 = \{\{x_i, x_j\} | i < j \Rightarrow x_i \leq x_j\}$ and $X_2 = [D_1]^2 - X_1$. Then again by Ramsey's theorem we find an infinite increasing or an infinite decreasing chain. □

LEMMA 2. Every element in a cpo is below a maximal element.

PROOF. By the definition of cpo the lemma of Zorn applies to the sets $\{y | x \leq y\}$. □

DEFINITION. Let $(D, \leq)$ be a cpo with its Scott topology. $X \subseteq D$ is called a *subspace* if $(X, \leq)$ is a cpo and on this structure the Scott topology coincides with the subspace topology inherited from D.

EXAMPLE. In $(\omega+2, \leq) = (\{0, 1, 2, \ldots, \omega, \omega+1\}, \leq)$ the subset $\{x | x \neq \omega\}$ is not a subspace, but the subset $\omega = \{0, 1, 2, \ldots\}$ is

LEMMA 3. Let $(D, \leq)$ be a cpo.

(i) If $x_0 \leq x_1 \leq \ldots$ is an increasing chain in D then $D_0 = \{x_0, x_1, \ldots, \bigcup x_i\}$ is a subspace.

(ii) If $x_0 \geq x_1 \geq \ldots$ is a decreasing chain in D, then $\{\bot, \ldots x_2, x_1, x_0\}$ is a subspace. ($\bot$ denotes the least element of D).

(iii) If $D_1 \subseteq D$ is not a subspace, then D contains an infinite increasing chain.

PROOF. (i) The opens of D_0 are ϕ, D_0 and the sets $O_n = \{y \in D_0 | x_n \leq y\}$. These belong to the subspace topology: $O_n = D_0 \cap \{z \in D | z \not\leq x_{n-1}\}$. Moreover, if O is open in D, then $O \cap D_0$ is open in D_0 since $\bigcup$ has the same meaning in D_0 as in D.

(ii) Similar.

(iii) Suppose $O_1 \subseteq D_1$ is Scott open in D_1 but not of the form $O \cap D_1$ with O open in D. Define $O^* = \{y \in D | \exists x \in O_1 x \leq y\}$. Then $O^* \cap D_1 = O_1$, hence O^* is not Scott open in D. But O^* is closed upward in D, so for some directed $X \subseteq D$ one has $\bigcup X \in O^*$ and $X \cap O^* = \phi$. Then no element of X is the supremum of X. Hence for every $x \in X$ there is a $y \in X$ incomparable with x. Since X is directed it follows that for each $x \in X$ there is an $x' \in X$ with $x < x'$, i.e. $X \subseteq D$ contains an infinite increasing chain. If O is Scott open in D but $O \cap D_1$ not in D_1, then the reasoning is easier. □

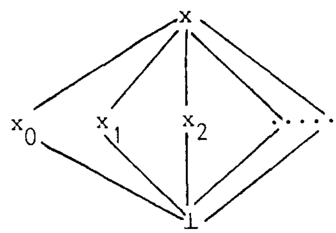
THEOREM 1. If D is an infinite cpo, then $[D \rightarrow D]$ is uncountable.

PROOF. *Case* 1. For some $x \in D$ the set $x\downarrow = \{x \in D \mid y \leq x\}$ is infinite. Then by lemma 1 there is below x either (i) an infinite increasing chain or (ii) an infinite decreasing chain or (iii) an infinite antichain.

Case 1(i). Then D contains the subspace $D' = \{x_0 \leq x_1 \leq \ldots \leq \bigcup x_i\}$ which is an algebraic lattice. Therefore the uncountably many monotonic maps $D' \to D'$ are all continuous. Since algebraic lattices are continuous, D' is an injective topological space, i.e. the continuous maps can be extended continuously to $D \to D'$, see [2] II, 3.5. It follows that $[D \to D]$ is uncountable.

Case 1(ii). Then D contains the subspace $D' = \{\bot \leq \ldots \leq x_1 \leq x_0\}$ and we proceed as in case 1(i).

Case 1(iii). Then D contains the infinite Chinese lantern



This is an algebraic lattice with uncountably many monotonic maps. If the lantern is a subspace, then we are done. If not, then by lemma 3(iii) we are back to case 1(i).

Case 2. For every $x \in D$ the set $x\downarrow$ is finite. Then by lemma 2 the set M of maximal elements is infinite. For $A \subseteq M$ define

$$f(\chi) = \bot \quad \text{if } x \notin M$$

$$= m_0 \text{ if } x \in A$$

$$= m_1 \text{ if } x \in M - A$$

where $m_0, m_1 \in M$ are distinct. Then f is continuous. Again $[D \to D]$ is uncountable. □

§ 2. A COUNTABLE SPACE WITH ONLY COUNTABLY MANY CONTINUOUS SELF-MAPS

In this section we shall produce an example of a countable space with only the identity and the constant functions continuous. Let us call two topological spaces X and Y *orthogonal* provided that for every open subset U of X, every continuous function $f: U \to Y$ is constant, and vica versa. The main ingredient in the construction of our example is that spaces such as described in the following theorem exist: we shall postpone the proof until the end of this section.

THEOREM 2. There is a sequence of countable spaces Σ_n, $n \in \mathbb{N}$, with the following properties:

(i) each Σ_n is compact, connected, locally connected, T_1 and anti-Hausdorff *,

(ii) the sequence $\langle \Sigma_n : n \in \mathbb{N} \rangle$ is pairwise orthogonal.

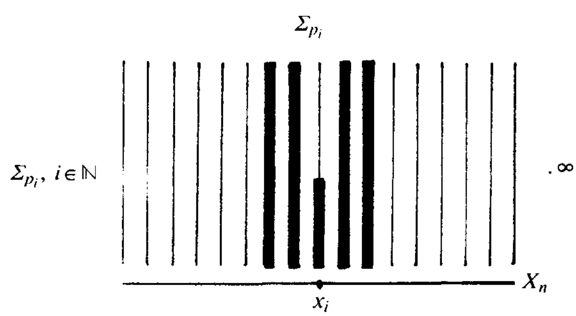
We shall now construct our example from the spaces described in Theorem 2. Let $\{A_n : n \in \mathbb{N}\}$ be a partition of $\mathbb{N}$ into countably many infinite sets. By induction on $n \in \mathbb{N}$, we shall construct a compact, connected, locally connected, countable T_1-space X_n such that:

(1) X_{n-1} is a closed subspace of X_n,
(2) if $i<n$, $f: X_i \to X_n$ is continuous such that $f(x) \notin X_i$ for certain $x \in X_i$, then f is constant,
(3) if $i<n$, $f: X_n \to X_n$ is continuous and $f(X_i) \subseteq X_i$ then either $f \restriction X_i$ is constant or $f \restriction X_i$ is the identity on X_i,
(4) if $k>n$, $p \in A_k$ and $U \subseteq \Sigma_p$ is open then every continuous function $f: U \to X_n$ is constant,
(5) if $k>n$, $p \in A_k$ and $U \subseteq X_n$ is *connected* and open then every continuous function $f: U \to \Sigma_p$ is constant.

Let $X_1 = \Sigma_p$ for certain $p \in A_1$. By Theorem 2 we find that $(4)_1$ and $(5)_1$ are satisfied.

Suppose that we constructed the spaces X_i for $1 \leq i \leq n$ satisfying $(1)_n$ through $(5)_n$. We shall construct X_{n+1}.

Let $\{x_i : i \in \mathbb{N}\}$ enumerate X_n and let $\{p_i : i \in \mathbb{N}\}$ enumerate A_{n+1}. For convenience, assume that the spaces Σ_p, $p \in A_{n+1}$, are pairwise disjoint and that in addition they do not intersect X_n. The underlying set of X_{n+1} is $(X_n \cup \bigcup_{i \in \mathbb{N}} \Sigma_{p_i}) \cup \{\infty\}$, where ∞ is a point not in $X_n \cup \bigcup_{i \in \mathbb{N}} \Sigma_{p_i}$.



We shall define the topology on X_{n+1} in terms of basic neighborhoods. A basic neighborhood of a point $x \in \Sigma_{p_i}$, $i \in \mathbb{N}$, in X_{n+1} is a neighborhood of x in Σ_{p_i}. A basic neighborhood of the point $x_i \in X_n$ has the form

$$(U \cup \bigcup \{\Sigma_{p_j} : x_j \in U\}) \setminus F,$$

* This means that every two nonempty open sets intersect.

where $U \subseteq X_n$ is a neighborhood of x_i in X_n and $F \subseteq \Sigma_{p_i}$ is an arbitrary *finite* set. A basic neighborhood of ∞ has the form

$$\{\infty\} \cup (\bigcup_{i \in \mathbb{N}} \Sigma_{p_i} \setminus F),$$

where $F \subseteq \bigcup_{i \in \mathbb{N}} \Sigma_{p_i}$ is an arbitrary *finite* set.

It is clear that X_{n+1} is a countable, connected and locally connected, compact T_1-space and that X_n is a closed subspace of X_{n+1}. We shall prove that X_{n+1} satisfies $(2)_{n+1}$ through $(5)_{n+1}$.

LEMMA 4. $(2)_{n+1}$ holds.

PROOF. Take $i < n+1$ and a continuous function $f: X_i \to X_{n+1}$ such that $f(x) \notin X_i$ for certain $x \in X_i$. If $f(X_i) \subseteq X_n$ then f is constant by $(2)_n$. We therefore assume, without loss of generality, that $f(x) \in X_{n+1} \setminus X_n$.

Case 1. $\exists \hat{x} \in X_i$ such that $f(\hat{x}) \in \Sigma_{p_j}$ for certain $j \in \mathbb{N}$.

Let $a = f(\hat{x})$ and put $V = \{y \in X_i : f(y) = a\}$. Then V is closed (since X_{n+1} is T_1) and nonempty (since $\hat{x} \in V$). We will show that V is open. Take $y \in V$ arbitrarily. Since Σ_{p_j} is open in X_{n+1}, there is a connected open neighborhood U of y in X_i such that $f(U) \subseteq \Sigma_p$. By $(5)_i$ we find that $f(U)$ is a single point, so $f(U) = \{a\}$ since $f(y) = a$. Consequently, $U \subseteq V$ and we conclude that V is open. By connectivity of X_i we find that $V = X_i$, i.e. f is constant.

Case 2. $f(X_i) \cap \bigcup_{j \in \mathbb{N}} \Sigma_{p_j} = \phi$.

Then $f(x) = \infty$. If $f(X_i) \cap X_n \neq \phi$ then by connectivity of $f(X_i)$ we find $f(X_i) \cap \bigcup_{j \in \mathbb{N}} \Sigma_{p_j} \neq \phi$, which is not the case. Consequently, $f(X_i) \cap X_n = \phi$, whence $f(X_i) = \{\infty\}$. □

LEMMA 5. $(3)_{n+1}$ holds.

PROOF. Take $i \leq n$ and a continuous $f: X_{n+1} \to X_{n+1}$ such that $f(X_i) \subseteq X_i$. Suppose first that there is an $x \in X_n$ such that $f(x) \notin X_n$. By $(2)_{n+1}$ (lemma 4) we conclude that $f \restriction X_n$ is constant. Since $f(x) \notin X_n$, $X_i \subseteq X_n$ and $f(X_i) \subseteq X_i$, this is a contradiction. Therefore, $f(X_n) \subseteq X_n$. Let $g = f \restriction X_n$. If $i < n$ then by $(3)_n$, $g \restriction X_i = f \restriction X_i$ is constant or $g \restriction X_i = f \restriction X_i$ is the identity on X_i. We may therefore assume that $i = n$. Suppose that for certain $x_i \in X_n$ we have $f(x_i) \neq x_i$. We shall prove that $f \restriction X_n$ is constant. Since

$$E = (X_n \setminus \{x_i\}) \cup \bigcup \{\Sigma_{p_j} : j \neq i\}$$

is a neighborhood of $f(x_i)$ in X_{n+1}, there is a neighborhood F of x_i such that $f(F) \subseteq E$. The neighborhood F contains $\Sigma_{p_i} \setminus G$ for certain finite set $G \subseteq \Sigma_{p_i}$. Define $\pi: X_n \cup \bigcup_{i \in \mathbb{N}} \Sigma_{p_i} \to X_n$ by

$$\begin{cases} \pi \restriction X_n \text{ is the identity on } X_n, \\ \pi(\Sigma_{p_i}) = \{x_i\}. \end{cases}$$

Then π is clearly continuous. Then $\pi f(\Sigma_{p_i} \setminus G) \subseteq X_n$, so $\pi f(\Sigma_{p_i} \setminus G)$ is a single point by $(4)_n$. Let $f(x_i) = x_j$. It now easily follows that $f(\Sigma_{p_i} \setminus G) \subseteq \{x_j\} \cup \Sigma_{p_j}$. Since Σ_{p_i} is anti-Hausdorff, $\Sigma_{p_i} \setminus G$ is connected and an application of Theorem 2 and the fact that x_i is in the closure of $\Sigma_{p_i} \setminus G$ now yield that

$$f(\Sigma_{p_i} \setminus G) = \{x_j\}.$$

Since $\Sigma_{p_i} \setminus G$ is dense in $\Sigma_{p_i} \cup \{\infty\}$, it also follows that $f(\infty) = x_j$. There is a neighborhood H of ∞ such that $f(H) \subseteq E$. Fix $k \neq j$. Then H contains $\Sigma_{p_k} \setminus A$, for certain finite $A \subseteq \Sigma_{p_k}$. Argueing as above yields that

$$f(\Sigma_{p_k} \setminus A) = \{x_j\}.$$

Since $\Sigma_{p_k} \setminus A$ is dense in $\Sigma_{p_k} \cup \{x_k\}$ it also follows that $f(x_k) = x_j$. We conclude that $f(x_k) = x_j$ for every $k \neq j$. By connectivity of X_n it follows that x_j is in the closure of $X_n \setminus \{x_j\}$. From this we conclude that $f(x_j) = x_j$, i.e. $f \restriction X_n$ is constant. □

LEMMA 6. $(4)_{n+1}$ holds.

PROOF. Let $k > n+1$ and $p \in A_k$. In addition, let $U \subseteq \Sigma_p$ be nonempty and open and let $f: U \to X_{n+1}$ be continuous. Suppose first that for some $x \in U$ we have $f(x) \in \bigcup_{i \in \mathbb{N}} \Sigma_{p_i}$. Then there are a nonempty open $V \subseteq U$ and an $i \in \mathbb{N}$ such that $f(V) \subseteq \Sigma_{p_i}$. By Theorem 2, $f(V)$ is a single point. Since U is anti-Hausdorff, V is dense in U, whence $f(U)$ is a single point. We may therefore assume that $f(U) \cap \bigcup_{i \in \mathbb{N}} \Sigma_{p_i} = \phi$. Now, by connectivity of $f(U)$, either $f(U) \subseteq X_n$ or $f(U) = \{\infty\}$. If $f(U) \subseteq X_n$ then f is constant by $(4)_n$. In addition, if $f(u) = \{\infty\}$ then f is trivially constant. □

LEMMA 7. $(5)_{n+1}$ holds.

PROOF. Let $k > n+1$ and $p \in A_k$. In addition, let $U \subseteq X_{n+1}$ be *connected* and open and let $f: U \to \Sigma_p$ be continuous. Take $x \in U$ arbitrarily and put $a = f(x)$. Let $V = \{y \in U : f(y) = a\}$. Then V is a closed subset of U since Σ_p is T_1. We will show that V is open from which follows that $V = U$ since U is connected.

Take $y \in V$ arbitrarily.

Case 1. $y = \infty$.

Since U is open, there is a finite set $F \subseteq X_{n+1}$ such that

$$E = \{\infty\} \cup \bigcup_{i \in \mathbb{N}} \Sigma_{p_i} \setminus F) \subseteq U.$$

Fix $i \in \mathbb{N}$. By Theorem 2, $f(\Sigma_{p_i} \setminus F)$ is single point. Since ∞ is in the closure of $\Sigma_{p_i} \setminus F$ we have $f(\Sigma_{p_i} \setminus F) = \{a\}$. We conclude that $E \subseteq V$.

Case 2. $y \in \Sigma_{p_i}$ for certain $i \in \mathbb{N}$.

Let $E \subseteq \Sigma_{p_i}$ be a neighborhood of y such that $E \subseteq U$. By Theorem 2, $f \restriction E$ is constant, whence $f(E) = \{a\}$, i.e. $E \subseteq V$.

Case 3. $y \in X_n$, say $y = x_i$.

There is a connected neighborhood W of x_i in X_n and a finite $F \subseteq X_{n+1}$ such that

$$E = (W \cup \bigcup\{\Sigma_{p_j} : x_j \in W\}) \setminus F \subseteq U.$$

Since W is connected, by $(5)_n$ we find that $f \upharpoonright W$ is constant, whence $f(W) = \{a\}$. Argueing as in Case 1 now yields that $f(E) = \{a\}$, i.e. $E \subseteq V$. □

This completes the construction of the spaces X_n, $n \in \mathbb{N}$. Let $X = \bigcup_{n \in \mathbb{N}} X_n$ and define a topology τ on X by

$$U \in \tau \Leftrightarrow U \cap X_n \text{ is open in } X_n, \text{ for each } n \in \mathbb{N}.$$

It is clear that τ is indeed a topology. Observe that $V \subseteq X$ is closed if and only if $V \cap X_n$ is closed in X_n, for each $n \in \mathbb{N}$. Since X_n is closed in X_{n+1}, this implies that each X_n is closed in X. Also, $\tau \upharpoonright X_n$ is the original topology on X_n. Observe that X is T_1.

The space X is in fact the direct limit of the spaces X_n, with inclusions as bonding maps. In the framework of direct limits the following lemma is well-known: the simple proof will be included for completeness sake.

LEMMA 8. If $K \subseteq X$ is compact then $K \subseteq X_n$ for certain $n \in \mathbb{N}$.

PROOF. Let $E = \{n \in \mathbb{N} : K \cap (X_n \setminus X_{n-1}) \neq \phi\}$ and for each $n \in E$ pick a point $x_n \in K \cap (X_n \setminus X_{n-1})$. Let $F = \{x_n : n \in E\}$. Let H be an arbitrary subset of F. Then $H \cap X_n$ is finite for every $n \in \mathbb{N}$, so H is closed in X. We conclude that F is discrete, i.e. every subset of F is closed, or equivalently, open. Since K is compact, so is F. Consequently, F is finite, whence E is finite. □

THEOREM 3. If $f : X \to X$ is continuous then either f is constant or f is the identity on X.

PROOF. We distinguish two cases.

Case 1. $(\exists n \in \mathbb{N})(f \upharpoonright X_n : X_n \to X$ is constant$)$.

Let $f(X_n) = \{a\}$. Choose $m > n$ such that $a \in X_m$. Suppose that there is an $l \geq m$ such that $f(y) \notin X_l$ for certain $y \in X_l$. By lemma 8 there is $k > l$ such that $f(X_l) \subseteq X_k$. By induction hypothesis $(2)_k$ it follows that $f \upharpoonright X_l$ is constant, so $f(X_l) = \{b\}$ for certain $b \notin X_l$. Since $X_n \subseteq X_l$ and $f(X_n) = \{a\} \subseteq X_m \subseteq X_l$, this is a contradiction. We therefore conclude $f(x_l) \subseteq X_l$ for every $l \geq m$. Consider $m+1$. We have $f(X_{m+1}) \subseteq X_{m+1}$ and $f(X_m) \subseteq X_m$. By induction hypothesis $(3)_{m+1}$ we conclude that either $f \upharpoonright X_m$ is constant or $f \upharpoonright X_m$ is the identity on X_m. Since $f \upharpoonright X_n$ is constant on X_n and $X_n \subseteq X_m$ this implies that $f \upharpoonright X_m$ is constant. Continuation of this process yields that $f \upharpoonright X_1$ is constant for every $l \geq m$, i.e. f is constant.

Case 2. $(\forall n \in \mathbb{N})(f \upharpoonright X_n : X_n \to X$ is not constant$)$.

Suppose that for certain $n \in \mathbb{N}$ and $x \in X_n$. We have $f(x) \notin X_n$. By Lemma 8 there is $m > n$ such that $f(X_n) \subseteq X_m$. By induction hypothesis $(2)_m$ we conclude that $f \upharpoonright X_n$ is constant, contradiction. Consequently, $f(X_n) \subseteq X_n$ for every $n \in \mathbb{N}$. Applying induction hypothesis $(3)_n$ for every $n \in \mathbb{N}$ and using the fact that $f \upharpoonright X_n$ is not constant, we conclude that $f \upharpoonright X_n$ is the identity on X_n for every $n \in \mathbb{N}$, i.e. f is the identity on X. □

It remains to construct the spaces Σ_n, $n \in \mathbb{N}$, of Theorem 2. This turns out to be surprisingly complicated.

Let $p \subseteq P(\mathbb{N})$ be a free ultrafilter. Λ_p is the space with underlying set $\mathbb{N}$ and topology $p \cup \{\phi\}$, i.e. $U \subseteq \mathbb{N}$ is open in Λ_p if and only if $U \in p \cup \{\phi\}$. The following lemma is left as an exercise to the reader.

LEMMA 9. Let $p \subseteq P(\mathbb{N})$ be a free ultrafilter. Then

(1) Λ_p is T_1 and anti-Hausdorff,

(2) if $V \subseteq \Lambda_p$ then the following statements are equivalent:

  (a) V is connected and $|v| > 1$,

  (b) V is open and $V \neq \phi$,

  (c) $V \in p$,

(3) if $V \subseteq \Lambda_p$ is open and nonempty then $\Lambda_p \setminus V$ is relatively discrete, i.e. every point of $\Lambda_p \setminus V$ is open in the inherited subspace topology on $\Lambda_p \setminus V$.

Let $\beta\mathbb{N}$ be the set of all ultrafilters on $\mathbb{N}$. If $p, q \in \beta\mathbb{N}$ define

$$p \leq q \Leftrightarrow (\exists f : \mathbb{N} \to \mathbb{N})(p = \{B \subseteq \mathbb{N} : (\exists A \in q)(f(A) \subseteq B)\}).$$

(This is the Rudin-Keisler (pre)order on $\beta\mathbb{N}$).

The following non-trivial result is due to Kunen [3].

THEOREM 4. There is an uncountable set $A \subseteq \beta\mathbb{N}$ such that for all distinct $p, q \in A$ we have $p \not\leq q$ and $q \not\leq p$.

Let A be as in Theorem 4.

LEMMA 10. If $p, q \in A$ are distinct then Λ_p then Λ_q are orthogonal.

PROOF. Let $U \subseteq \Lambda_p$ be open and nonempty and let $f : U \to \Lambda_q$ be continuous. Take $V \in p$. If $|f(U \cup V)| = 1$ then $|f(U)| = 1$ since $U \cap V$ is dense in U and Λ_q is T_1, lemma 9(1). We can therefore assume that $|f(U \cap V)| > 1$ for every $V \in p$. Define $g : \mathbb{N} \to \mathbb{N}$ by

$$\begin{cases} g \upharpoonright U = f, \\ g \upharpoonright (\mathbb{N} \setminus U) \equiv 1. \end{cases}$$

Take $V \in p$ arbitrarily. Then $f(U \cap V) \subseteq g(V)$. Since $f(U \cap V)$ is non-degenerate and connected, lemma 9(1), it follows that $f(U \cap V) \in q$, lemma 9(2), from which follows that $g(V) \in q$. From this we find that $\{g(V) : V \in p\} \subseteq q$ and this easily implies that

$$q = \{B \subseteq \mathbb{N} : (\exists V \in p)(g(V) \subseteq B)\},$$

i.e. $q \leq p$, a contradiction. □

Now let ∞_p a point not in Λ_p, for certain $p \in A$. Let $\Sigma_p = \Lambda_p \cup \{\infty_p\}$ and define a topology on Σ_p as follows: every open subset of Λ_p is open Σ_p and a basic neighborhood of ∞_p has the form

$$\Sigma_p \setminus F,$$

where $F \subseteq \Lambda_p$ is finite.

It is clear that Σ_p is a compact T_1-space and that for the proof of Theorem 2 it suffices to verify the following

LEMMA 11. If $p, q \in A$ are distinct then Σ_p and Σ_q are orthogonal.

PROOF. Let $U \subseteq \Sigma_p$ be open and nonempty and let $f : U \to \Sigma_q$ be continuous. Put $U' = U \cap \Lambda_p$. If $f(U') = \{\infty_q\}$ then $f(U) = \{\infty_q\}$ since U' is dense in U and Σ_q is T_1. Therefore, assume that $f(U') \neq \{\infty_q\}$, say $f(x) \neq \infty_q$ for the point $x \in U'$. By continuity there is a neighborhood V of x in Σ_p such that $\infty_q \notin f(V)$. We may assume that $V \subseteq \Lambda_p$. By lemma 10, $f \upharpoonright V$ is constant. Since Σ_p is anti-Hausdorff, V is dense in Σ_p. Consequently, f is constant.

REFERENCES

1. Barendregt, H.P. – The lambda calculus, North Holland, Amsterdam (1984).
2. Gierz, G., J.D. Lawson, K.H. Hofmann, M. Milstone, K. Keimel and D.S. Scott – A compendium of continuous lattices, Springer, Berlin (1980).
3. Kunen, K. – Ultrafilters and independent sets, Trans. Am. Math. Soc., **172**, 299–306 (1972).
4. Lévy, A. – Basic set theory, Springer, Berlin (1979).