SUMMARY
§ 1 is concerned with the term model of the \( \lambda \)-calculus. It is proved that Church's \( \delta \) is not definable and that the definable functions into the numerals are constant. In § 2 it is proved that for several \( \lambda \)-algebras the range of a representable function is either a singleton or infinite. In § 3 it is examined in which \( \lambda \)-algebras the local representability of external functions implies the global representability.

INTRODUCTION
Let \( \mathcal{M} = \langle M, \cdot \rangle \) be a \( \lambda \)-algebra (i.e. a model of the \( \lambda \)-calculus). Elements of \( M \) are thought of as functions. Arbitrary \( f: M \to M \) are called external functions. Such a function is representable (by an element \( a \in M \)) if \( \forall b \in M \ f(b) = a \cdot b \). A function \( f \) is definable in \( \mathcal{M} \) if \( f \) is representable by \( \llbracket F \rrbracket^\mathcal{M} \) for some closed term \( F \). Here \( \llbracket F \rrbracket^\mathcal{M} \) denotes the value of \( F \) in the model \( \mathcal{M} \).

Other notations:
\( x, y, \ldots \) denote variables of the \( \lambda \)-calculus.
\( a, b, \ldots \) denote variables ranging over the elements of a \( \lambda \)-algebra.
\( F, G, \ldots \) denote \( \lambda \)-terms.

The numerals 0, 1, 2, \ldots denote some adequate representation of the natural numbers as \( \lambda \)-terms e.g. those of Church: \( n = \lambda fx. f^n(x) \).

If \( \mathcal{M} = \langle M, \cdot \rangle \) is a \( \lambda \)-algebra, then \( \mathcal{M}^0 \) is the sub-\( \lambda \)-algebra \( \langle M^0, \cdot \rangle \) where \( M^0 = \{ \llbracket F \rrbracket^\mathcal{M} \in M \mid F \ \text{closed term} \} \).

If \( T \) is a consistent extension of the \( \lambda \)-calculus, \( \mathcal{M}(T) \) is the term-model of \( T \), i.e. the set of all \( \lambda \)-terms modulo provable equality in \( T \). The closed term-model of \( T \), notation \( \mathcal{M}^0(T) \), is defined as \( (\mathcal{M}(T))^0 \). A \( \lambda \)-algebra \( \mathcal{M} \) is hard if \( \mathcal{M} = \mathcal{M}^0 \). In such an \( \mathcal{M} \) a function is representable iff it is definable.

For other terminology see Barendregt [1976].
The three sections of the paper treat different aspects of the notion of representability.
In § 1 attention is restricted to the standard extensional term model \( \mathcal{M} = \mathcal{M}(\lambda \eta) \).
Church's $\delta$ is an external function satisfying
\[(\star) \quad \delta MM = 0 \text{ if } M \text{ is a closed normal form (nf)}
\]
\[\delta MM' = 1 \text{ if } M, M' \text{ are different closed nf's.}\]

In Böhm [1972] it is proved that $\forall N_1 \ldots N_n$ different $\beta\eta$-nf's $\mathcal{N} \vdash FN_i = i$. As a consequence it follows that for every finite set $A$ of nf's there is a term $\delta$ satisfying $(\star)$ for $M, M' \in A$.

At the Orleans logic conference (1972) the question was raised whether the general Church's $\delta$ is definable as a $\lambda$-term.

We will give a negative answer which was already established in Barendregt [1972] and independently in Wadsworth [1972] (see also Hindley and Mitschke [1975]). All three proofs of the non-existence of $\delta$ are different.

Furthermore it is proved that the only definable functions from the terms into the numerals are the constant functions.

In § 2 it will be proved that definable functions in various $\lambda$-algebras have a range of cardinality 1 or $\aleph_0$. For representable functions this is not true in $D_\infty$ and $P_0$.

Two external functions $f$ and $g$ on $\mathcal{M}$ are dual, notation $f \sim_\mathcal{M} g$, if $f(a) \cdot b = g(b) \cdot a$ for all $a, b \in \mathcal{M}$. In that case for each $b$ the map $\lambda a. f(a).b$ is representable and $f$ is said to be locally representable, similarly for $g$.

A model $\mathcal{M}$ is rich if for all $f, g$:

\[f \sim_\mathcal{M} g \Rightarrow f \text{ and } g \text{ are representable in } \mathcal{M}.\]

The results of § 3 are: $D_\infty$ and $\mathcal{M}(\lambda\eta)$ are rich; rich models are extensional; hard sensible models (e.g. the interior of $D_\infty$) are not rich.

We would like to draw the proof of 3.6 to the reader’s attention. There variables of the $\lambda$-calculus are not just used in the usual way, but also serve as separate entities.

§ 1. NON-DEFINABILITY RESULTS

The main tool in this section is the "Böhm out" technique 1.4. This result is also of use in § 2.

1.1. Definition. Let $BT(M)$ be the Böhm tree of $M$, see Barendregt [1976], § 6. $x \in BT(M)$ iff $x \in FV(M^k)$ for some $k$, where $M^k$ is the $k^{th}$ approximate normal form of $M$.

1.2. Definition. (i) A selector is a term of the form

\[U \equiv \lambda x_1 \ldots x_n \cdot x_i, \quad 1 < i \leq n.\]

A permutator is a term of the form

\[C \equiv \lambda x_1 \ldots x_n \cdot x_{\pi(1)} \ldots x_{\pi(n)}\]

for some permutation $\pi$.

(ii) Simple terms are inductively defined by: Any variable, selector or permutator is a simple term. If $P, Q$ are simple terms, so is $PQ$. 

1.3. Lemma. Simple terms have a normal form (nf).

Proof. Realize that each simple term is of the form \( x\vec{P}, U\vec{P}, C\vec{P} \) with \( \vec{P} \) simple, \( U \) a selector and \( C \) a permutator. Then it can be shown by induction on the term length that they have a nf.

1.4. Theorem. Let \( FV(M) = \{x\} \) and \( x \in BT(M) \). Then

(i) For some \( \vec{P}, \vec{Q} \), with \( x \notin FV(\vec{P}) \), \( \lambda \models \vec{M}\vec{P} = x\vec{Q} \) ("\( x \) is Böhmed out").

(ii) Moreover \( \vec{P} \) can be chosen as a sequence of simple terms.

Proof. Let \( x \) occur in \( BT(M) \) at depth \( k > 0 \). By a similar construction as in Barendregt [1976] 6.14, 6.15, for some Böhm-transformation \( \pi_1, x \) occurs in \( BT(M^{\pi_1}) \) at depth \( k-1 \). Iterating this leads to \( M^{\pi_2} = \lambda \vec{y} \cdot x\vec{Q} \), hence \( M^{\pi_2} y = x\vec{Q} \), for a Böhm transformation \( \pi_2 \).

Checking the details of the construction of \( \pi_2 \) one verifies that

\[
M^{\pi_2} y = M \ldots x_i \ldots [x_j/Cx_j] \ldots [x_k/Ux_k] \ldots y = \vec{M}\vec{P}
\]

for some simple terms \( \vec{P} \) with \( x \notin FV(\vec{P}) \) (where \( C \) is a permutator and \( U \) a selector).

1.5. Lemma. Let \( F \) be a closed \( \lambda \)-term such that \( F \) is not constant, i.e. \( \lambda \models FX_1 = FX_2 \) for some \( X_1, X_2 \), and suppose that for some closed \( \lambda \)-term \( M \), \( FM \) has a nf. Then \( x \in BT(Fx) \) for all \( x \).

Proof. Note that if \( P, P' \) have equal finite \( \Omega \)-free Böhm-trees, then \( \lambda \models P = P' \). Now suppose \( x \notin BT(Fx) \) for some \( x \). Then for all \( k \), \( x \notin BT((Fx)^k) \) (\( N^k \) is the \( k \)-th approximate normal form of \( N \), cf. Barendregt [1976] 7.4 (iv)). Hence \( (FM)^k \equiv (Fx)^k [x/M] \equiv (Fx)^k \) for all \( k \), and it follows that \( BT(FM) = BT(Fx) \). But since \( FM \) has a nf, \( BT(FM) \) is finite and \( \Omega \)-free and therefore \( \lambda \models FM = Fx \). Since \( F, M \) are closed it follows that for all \( \lambda \)-terms \( N \), \( \lambda \models FN = FM \), i.e. \( F \) is constant, a contradiction.

Remark. 1.5 also holds for \( F, M \) not necessarily closed.

1.6. Definition. \( 0 = I, \ n+1 = K n \).

1.7. Lemma. The function \( sq \) is not \( \lambda \)-definable with respect to \( \{n|n \in \omega\} \), i.e. for no closed \( \lambda \)-term \( F \models F \ 0 = 0, \models F \ n+1 = 1 \).

Proof. Suppose \( F \) exists. Then by 1.5 \( x \in BT(Fx) \). Hence by 1.4 \( Fx\vec{P} = x\vec{Q} \) for some \( \vec{P}, \vec{Q} = Q_1 \ldots Q_m \). But then for all \( n > m \),

\[
\models I\vec{P} = F \ n \ \vec{P} = \ n \ Q_1 \ldots Q_m = n - m
\]

contradicting the Church-Rosser theorem since the \( k \) are different nf's.
1.8. Definition. A system of terms \( \{M_n \mid n \in \omega \} \) is an adequate system of numerals iff

(i) Each \( M_n \) has a \( \text{nf} \).
(ii) Each recursive function can be \( \lambda \)-defined with respect to the \( M_n \).

In Barendregt [1977] is shown that the second condition can be replaced by (ii'): The successor, predecessor and \( \text{sg} \) functions can be \( \lambda \)-defined with respect to the \( M_n \).

The following corollary was proved independently by Barendregt [1972] and Wadsworth [1972].

1.9. Corollary.

(i) \( \{n \mid n \in \omega \} \) is not an adequate system of numerals. (ii) Church’s \( \delta \) is not \( \lambda \)-definable.

Proof. (i) Immediate. (ii) If \( \delta \) were \( \lambda \)-definable, then so would be \( \text{sg} \), viz. by \( \lambda x . \delta x \equiv 0 \to 0 \lor I \).

Remark. (i) Although not definable, \( \delta \) can consistently be added to the \( \lambda \)-calculus, see Church [1941].

(ii) Contrary to this, the corresponding \( \delta \) for open \( \lambda \)-terms would be inconsistent at once. For let \( x \not \equiv y \), then

\[
(\lambda y . \delta xy(KK)S)x = (\lambda y . I(KK)S)x = (\lambda y . KKKS)x = KKKS = K
\]

but also

\[
(\lambda y . \delta xy(KK)S)x = \delta xx(KK)S = 0(KK)S = S.
\]

(iii) One could also consider the definability of a \( \delta \) for all closed terms, i.e.: \( \delta MM = \theta \) for \( M \) closed

\[ \delta MN = I \] for \( M, N \) closed such that \( \not \vdash M = N. \]

But then the following version of the Russell paradox would result.
Define \( \neg X = \delta X I \). If \( \not \vdash \theta = I \) then \( \not \vdash X = I \iff \neg \neg X = I \).

Now let \( A = FP \) (i.e. the fixed point of \( \neg \): \( \neg A = \neg \neg A \)).

Then \( \not \vdash A = I \iff \neg A = I \). Thus \( \neg \theta = I \).

To see the relation with the Russell paradox, note that \( A = BB \) with \( B = \lambda x . \neg (xx) \). (In illative combinatory logic \( MN \) is interpreted as \( N \in M \) and \( \lambda x . P \) as \( \{x|P\} \).)

1.10. Theorem. Let \( \omega = \{n \mid n \in \omega \} \) be an adequate system of numerals and let \( f \) be a map into \( \omega \) definable by \( F \). Then \( f \) is constant.

Proof. First assume \( \omega \) is Church’s system of numerals.
Suppose \( f \) is not constant, then by 1.5 \( x \in BT(Fx) \). Hence for some simple \( \vec{P} \) and \( \vec{Q} \), \( \lambda x . Fx \vec{P} = x \vec{Q} \).

Hence \( \lambda x . FM \vec{P} = M \vec{Q} \) for all \( M \). But \( M \vec{Q} \) can take arbitrary values and not \( FM \vec{P} \), since \( \vec{P} = P_1 \ldots (P_2) \ldots P_k \) always has a \( \text{nf} \) by 1.3.
Now let \( \omega \) be an arbitrary system of numerals. It is well-known how to define a term \( G \) such that \( G\eta = \eta \).

Suppose a non-constant \( f : \text{terms} \to \omega \) would be definable, then \( G \circ f \) were a definable non-constant mapping into \( \omega \).

First alternative proof (due to the referee).

Suppose \( F \) is not constant, i.e. let \( n_1 \neq n_2 \in \text{Ra}(F) \). Define \( G \) as the \( \lambda \)-defining term of the recursive function
\[
g(x) = \begin{cases} 
0 & \text{if } x = n_1, \\
1 & \text{else}.
\end{cases}
\]

Then the range of \( G \circ F \) is \( \{0, 1\} \) contrary to 2.3.

Second alternative proof. By Barendregt's lemma in de Boer [1975] it follows that if \( \Omega \) is unsolvable and \( N \) a \( nf \), then \( F\Omega = N \Rightarrow Fx = N \) for all \( x \). (General genericity lemma; see also Barendregt [1977a] for a proof.) Now if the values of \( F \) are numerals it follows that \( F\Omega \) has a \( nf \), i.e. \( F \) is constant.

1.11. COROLLARY. There is no \( F \) such that
\[
FM = \begin{cases} 
\emptyset & \text{if } M \text{ is a numeral (i.e. } \vdash M = n \text{ for some } n), \\
1 & \text{else}
\end{cases}
\]

for any adequate system.

1.12. QUESTION. Is there a term \( F \) such that \( FM \) has a \( nf \) (is solvable) if \( M \) is a numeral has no \( nf \) (is unsolvable) else.

§ 2. THE RANGE PROPERTY

2.1. DEFINITION. Let \( \mathcal{M} = \langle M, \cdot \rangle \) be a \( \lambda \)-algebra. For each \( f \in M \), we define \( \text{Ra}\mathcal{M}(f) \), the range of \( f \) in \( \mathcal{M} \), as follows:
\[
\text{Ra}\mathcal{M}(f) = \{ f \cdot x \mid x \in M \}.
\]

NOTATION. \( \text{Ra}\mathcal{M}(F) = \text{Ra}\mathcal{M}(\lfloor F \rfloor^\mathcal{M}) \) for terms \( F \).

When possible, the superscript \( \mathcal{M} \) will be dropped in \( \text{Ra}\mathcal{M} \).

2.2. DEFINITION. A \( \lambda \)-algebra \( \mathcal{M} \) satisfies the range property if for all \( f \in M \), the cardinality of \( \text{Ra}\mathcal{M}(f) \) is 1 or \( \aleph_0 \).

2.3. RANGE THEOREM. (Barendregt; Myhill). Let \( T \) be a r.e. \( \lambda \)-theory. Then \( \mathcal{M}(T) \) (and also \( \mathcal{M}^0(T) \)) has the range property.

PROOF. Suppose \( f \in M \) and \( \text{Ra}(f) = \{ m_0, \ldots, m_k \} \). Define
\[
N_f = \{ x \mid f \cdot x = m_i \} \subseteq M.
\]
Every such \( N_t \) is r.e. Therefore \( N = \bigcup_{i=1}^k N_t \), the complement of \( N_0 \) is also r.e. Hence \( N_0 \) is recursive.

On the other hand \( N_0 \) is non-trivial and closed under equality, which contradicts Scott’s theorem (Barendregt [1976] 2.21).

The proof for \( \mathcal{M}_0(T) \) is the same. ■

2.4. Corollary. \( \mathcal{M}(\lambda), \mathcal{M}_0(\lambda), \mathcal{M}(\lambda \eta) \) and \( \mathcal{M}_0(\lambda \eta) \) have the range property.

The range property, however, is not satisfied in every \( \lambda \)-algebra.

2.5. Theorem. \( P^0 \) and \( D_\infty \) do not satisfy the range property.

Proof. Since the proof is similar in both cases, let \( \mathcal{S} = (S, \preceq) \) denote either \( (P_\infty, \subseteq) \) or \( (D_\infty, \subseteq) \). We define the following function \( f \colon S \to S \) by \( f(x) = \top \) if \( x \neq \bot \) else \( \bot \) (\( \top \) and \( \bot \) are the largest respectively smallest element of \( S \)).

Claim: \( f \) is continuous. Then by Scott [1972], [1975] \( f \) is representable and since \( f \) has range of cardinality two we are done.

For open \( O \) in \( S \) one has: \( x \in O \) and \( x \preceq y \Rightarrow y \in O \).


Hence for open \( O, \bot \in O \Rightarrow O = S \), and \( O \neq \emptyset \Rightarrow \top \in O \).

Now for every open set \( O \), \( f^{-1}(O) \) is open:

Case 1. \( \bot \in O \). Then \( O = S \) so \( f^{-1}(S) = S \) which is open.

Case 2. \( \bot \notin O \). If \( O = \emptyset \), then we are done. Else \( \top \in O \) and hence \( f^{-1}(O) = S - \{ \bot \} = \{ x \mid x \notin \bot \} \) def \( U_\bot \).

\( U_\bot \) is open in \( D_\infty \), see e.g. Barendregt [1976] 1.2.

\( U_\bot \) is open in \( P^0 \): Let \( O_k = \{ x \mid e_k \subseteq x \} \). Note \( e_0 = \emptyset = \bot \) and that the \( O_k \) form a base for the topology on \( P^0 \).

Now:

\[ x \in U_\bot \iff x \notin O \iff \exists k \neq 0 \quad e_k \subseteq x \iff x \in \bigcup_{k=0}^{\infty} O_k \]

which is, as a union of elements of a base, indeed open. ■

The following theorem was announced in Wadsworth [1973] for the \( D_\infty \) case.

2.6. Theorem. Let \( \mathcal{S} \) be \( D^0_\infty \) or \( P^{0_\infty} \). Then \( \mathcal{S} \) satisfies the range property.

Proof. Let \( F \) be a closed term. Consider \( BT(Fx) \).

Case 1. \( x \notin BT(Fx) \). Then \( BT(FM) = BT(FM') \) for all \( M, M' \). Since terms with equal Böhm trees are equal in \( \mathcal{S} \) (see Barendregt [1976], Hyland [1976]), it follows that \( Ra^{\mathcal{S}}(F) \) has cardinality 1.

Case 2. \( x \in BT(Fx) \). Then by 1.4 \( \lambda \vdash Fx = \tilde{x}Q \).

Since \( [N\tilde{Q}]^{\mathcal{S}} \) can take arbitrary values in \( \mathcal{S} \) when \( N \) ranges over the closed terms, \( Ra^{\mathcal{S}}(F) \) is infinite. ■
2.7. **Conjecture.** $\mathcal{M}(\mathcal{H})$ satisfies the range property.

2.8. **Question.** Does every hard $\lambda$-algebra $\mathcal{M}$ (i.e. $\mathcal{M} = \mathcal{M}_0$) satisfy the range theorem?

§ 3. **Duality**

3.1. **Definition.** Let $f, g$ be two external functions on a $\lambda$-algebra $\mathcal{M} = \langle M, \cdot \rangle$.

$f, g$ are **dual** iff $V a, b \in M : f(a) \cdot b = g(b) \cdot a$. Notation $f \sim_\mathcal{M} g$, or simply $f \sim g$.

3.2. **Definition.** $\mathcal{M}$ is **rich** iff all dual functions on $\mathcal{M}$ are representable in $\mathcal{M}$.

**Remarks.**
(i) Let $f$ be an external function on $\mathcal{M}$. $f$ is **locally representable** iff for each $b \in M$ the function $h$ defined by $h(a) = f(a) \cdot b$ is representable. Then $f$ is locally representable iff $f$ has a dual. A model is rich iff all locally representable functions are representable.

(ii) If $f$ is representable (by $f_0 \in M$, say), then $f$ has a dual $g$ which is also representable (by $g_0 = \lambda a b f_0 a b$).

(iii) Let $\mathcal{M}$ be extensional. Then $f$ has at most one dual. Hence if $f \sim_\mathcal{M} g$ and $f$ is representable, then by (ii) $g$ is representable.

3.3. **Theorem.** If $\mathcal{M}$ is rich, then $\mathcal{M}$ is extensional.

**Proof.** Suppose $\mathcal{M}$ is not extensional. Then there exist $b, b' \in M$ such that for all $c \in M$ $b \cdot c = b' \cdot c$ and $b \neq b'$.

Define

$$f(a) = \begin{cases} b' \text{ if } a = b \\ b \text{ else.} \end{cases}$$

and

$$g = [\lambda y \cdot K(by)]^\mathcal{M},$$

then for all $a, a' \in M$ : $f(a) \cdot a' = b \cdot a' = g(a') \cdot a$, hence $f \sim g$. But $f$ cannot be representable since it has no fixed point. Thus $\mathcal{M}$ is not rich.

3.4. **Corollary.** The following $\lambda$-algebras are not rich: $P_\omega; P^{0_\omega}; \mathcal{M}(\lambda); \mathcal{M}^0(\lambda); \mathcal{M}^{0_\lambda}(\lambda)$.

**Proof.**
1. $P_\omega$ is not extensional:

Take for example $a = \{(0, 0)\}$ and $b = \{(0, 0), (1, 0)\}$.

Then $V c \in P_\omega a \cdot c = b \cdot c$ but $a \neq b$.

2. $P^{0_\omega}$ is not extensional: Let $1 = \lambda x y \cdot x y$, then $P^{0_\omega} \models I x y = 1 \cdot x y$, but $P^{0_\omega} \models I = 1$ for otherwise $P_\omega \models I = 1$, so $P_\omega \models V x x = \lambda y \cdot x y$ which implies that $P_\omega$ were extensional.

3. By the Church Rosser property $\lambda \not\models I = 1$. So $\mathcal{M}(\lambda), \mathcal{M}^0(\lambda)$ are not extensional.
4. \( M^0(\lambda\eta) \) is not extensional because the \( \lambda \)-calculus is \( \omega \)-incomplete, see Plotkin [1974].

3.5. **Theorem.** \( D_\omega \) is rich.

**Proof.** Suppose that \( f, g \) are dual i.e.:

\[ \forall a, b \in D_\omega : f(a) \cdot b = g(b) \cdot a. \]

We have to show that \( f, g \) are representable.

It is sufficient to show that \( f, g \) are continuous. Take a directed \( X \subseteq D_\omega \).

For all \( b \in D_\omega \)

\[ f(\bigcup X) \cdot b = g(b) \cdot \bigcup X = \bigcup \{ g(b) \cdot a | a \in X \} = \bigcup \{ f(a) \cdot b | a \in X \} = \bigcup \{ f(a) | a \in X \} \cdot b \]

by the duality condition and the continuity of application.

Thus by extensionality in \( D_\omega \): for all directed \( X \)

\[ f(\bigcup X) = \bigcup \{ f(a) | a \in X \} \]

i.e. \( f \) is continuous. The proof for \( g \) is dual. ■

3.6. **Theorem.** \( M(\lambda\eta) \) is rich.

**Proof.** Define

\[ M = \lambda_n N \text{ iff } \lambda_n \vdash M = N, \]

\[ x \in \lambda_n M \text{ iff for all } M' = \lambda_n M \text{ one has } x \in FV(M'). \]

Let \( f, g \) be dual functions on \( M(\lambda\eta) \).

3.6.0. **Lemma.** (i) \( x \in \lambda_n M \iff VN[\lambda_n \vdash M = N \Rightarrow x \in FV(N)] \).

(ii) Let \( M' \equiv M[z/y] \) and \( \lambda \vdash M' \rightarrow N' \). Then \( \exists N \lambda \vdash M \rightarrow N \) and \( N' \equiv N[z/y] \).

\[ \begin{array}{c}
 M \rightarrow \rightarrow N \\
 [z/y] \downarrow \quad \downarrow \quad [z/y] \\
 M' \rightarrow \rightarrow N'
\end{array} \]

(iii) \( x \in \lambda_n M \Rightarrow x \in \lambda_n M[z/y] \), for \( z \not\equiv x \).

**Proof.** (i) \( \Rightarrow \) Trivial. \( \Leftarrow \) Suppose \( M = \lambda_n M' \). By the Church-Rosser theorem \( \lambda_n \vdash M \rightarrow N, M' \rightarrow N' \) for some \( N \). By assumption \( x \in FV(N) \). But then \( x \in FV(M') \).

(ii) Induction on the length of proof of \( M' \rightarrow N' \). In the case that \( M' \equiv (\lambda \cdot P)Q, N' \equiv P[a/Q] \) it may be assumed that \( a \not\equiv z, y \). Therefore one can apply the well-known substitution lemma

\[ A[u/B][v/C] = A[v/C][u/B[v/C]] \text{ if } u \not\equiv v \text{ and } u \not\in FV(C). \]

(iii) Suppose \( \lambda_n \vdash P[z/y] \rightarrow R' \). By (ii) for some \( R \lambda_n \vdash P \rightarrow R \) and \( R' \equiv R[z/y] \). By assumption and (i), \( x \in FV(R) \). Since \( x \not\equiv z \) also \( x \in FV(R') \). Therefore by (i) \( x \in \lambda_n P[z/y] \). ■
3.6.1. **Lemma.** (i) If \( x \in_{I_n} \lambda y \cdot P \) then \( x \in_{I_n} P \) and \( x \neq y \).
(ii) If \( x \neq y \), then \( x \in_{I_n} M \iff x \in_{I_n} My \).

**Proof.** (i) Since \( x \in FV(\lambda y \cdot P) \) clearly \( x \neq y \). Suppose \( P =_{I_n} N \), then \( \lambda y \cdot P =_{I_n} \lambda y \cdot N \). By assumption \( x \in FV(\lambda y \cdot N) \subset FV(N) \). Thus \( x \in_{I_n} P \).
(ii) \( \Rightarrow \) Suppose \( \lambda y \vdash My \rightarrow N \) in order to prove \( x \in FV(N) \).
Case 1. \( N = M' y \) with \( \lambda y \vdash M \rightarrow M' \). Since \( x \in_{I_n} M \), also \( x \in FV(M') \subset FV(N) \).
Case 2. \( M \rightarrow \lambda z \cdot M_1 \) and \( \lambda y \vdash My \rightarrow (\lambda z \cdot M_1)y \rightarrow M_1[z/y] \rightarrow N \).
Since \( x \in_{I_n} M \), also \( x \in_{I_n} \lambda z \cdot M_1 \) and by (i) \( x \in_{I_n} M_1 \) and \( z \neq x \), so by 3.6.0. (iii) \( x \in_{I_n} M_1[z/y] \). Therefore \( x \in FV(N) \). ■

3.6.2. **Lemma.** If \( \forall y \neq x \ x \in_{I_n} f(y) \), then \( \forall y \neq x \ x \in_{I_n} g(y) \) (and hence \( \forall y \neq x \ x \in_{I_n} f(y) \)).

**Proof.** Suppose \( x \in_{I_n} f(y) \), \( y \neq x \). Let \( y' \neq x \). Then by 3.6.1. (ii) \( x \in_{I_n} f(y') \cdot y' =_{I_n} g(y') \cdot y \). Hence, by 3.6.1. (ii), \( x \in_{I_n} g(y') \). (The rest follows by applying the statement to \( x \in_{I_n} g(y) \)). ■

3.6.3. **Main Lemma.** There is a variable \( x \) such that for all terms \( M : \{ x \} \)\([x]/M\) = \( f(M) \).

**Proof.** Let \( v \) be any variable. Choose \( x \neq v \) such that \( x \notin_{I_n} f(v) \). Then \( x \notin_{I_n} g(z) \) for all \( z \neq x \), by the dual of 3.6.2.

Given \( M \), one can find a \( y \) such that \( y \notin_{I_n} M, f(M), x, f(x) \). Hence \( x \notin_{I_n} g(y) \). Now since \( y \neq x \) and \( x \notin_{I_n} g(y) \), \( (f(x)[x]/M]) \cdot y = (f(x) \cdot y)[x]/M = (g(y) \cdot x)[x]/M = g(y) \cdot M = f(M) \cdot y \).

Since \( y \notin f(x), M, f(M) \), extensionality yields \( f(x)[x]/M = f(M) \). ■

Now it follows by 3.6.3. that \( f \) can be represented by the term \( \lambda x \cdot f(x) \) and similarly for \( g \). ■

The following construction is needed for the proof of 3.10.

3.7. **Definition.** Let \( \neq \) be a Gödel numbering of terms. \( \{ M \} \) is the numeral \( \neq M \). A sequence of terms \( M_n \) is recursive if \( \lambda n \cdot \neq M_n \) is a recursive function.

3.8. **Lemma.** (Coding of infinite sequences). Let \( \{ M_n \} \) be a recursive sequence of terms such that \( FV(M_n) \subset \{ x \} \) for all \( n \). Then there exists a term \( X \) such that \( p_i X = M_i \), for all \( i \), where \( p \) is some fixed closed term. Par abus de langage we write \( \langle M_n \rangle_{n \in \omega} \) for \( X \).

**Proof.**
(1) As in Curry et al. [1972], 13 B3 there is a term \( E \) which enumerates all terms with \( x \) as only free variable:

\[
E(\{ M \}) = M, \text{ for } M \text{ with } FV(M) = \{ x \}.
\]
2. Let \([M, N]\) be a pairing of terms defined by \(\lambda z \cdot z MN\). Then \([M, N]K = M\) and \([M, N](KI) = N\). Define ordered tuples as follows:

\[
[M] = M, \quad [M_1, \ldots, M_{n+1}] = [M_1, [M_2, \ldots, M_{n+1}]].
\]

3. Let \(M_n\) with \(FV(M_n) \subseteq \{x\}\) be a recursive sequence of terms.

We want to code the sequence \([M_n]\) as a \(\lambda\)-term. Let \(S^+\) be such that \(S^+n \xrightarrow{\beta} n + 1\) and let \(b = \lambda xy \cdot [E(Fy), (x(S^+y))]\), where \(F\) \(\lambda\)-defines \(f\), and \(B = FP b\) (i.e. the fixed point of \(b\)). Then

\[
Bn \xrightarrow{\beta} bBn \xrightarrow{\beta} [E(Fn), Bn + 1] \xrightarrow{\beta} [M_n, Bn + 1].
\]

So \(B0 = [M_0, B_1] = [M_0, M_1, B_2] = \ldots\) Hence by setting \(\langle M_n \rangle_{n \in \omega} = B0\) we have a coding for infinite sequences of terms with one fixed free variable.

4. It is easy to construct a term \(p\) such that \(pM \langle N \rangle = M\) (take \(e.g.\) \(pxa = \text{if zero } x \text{ then } a \text{ else } p(x - 1)(a(KI))\), using the fixed point theorem).

3.9. Lemma. For all closed \(Z\) there is an \(n\) such that \(ZQ^n = \emptyset\). \(ZQ^n\) is short for \(ZQQ \ldots QQ\) \(n\) times.

Proof.

Case 1. \(Z\) is unsolvable; then \(Z = \emptyset\), so \(n = 0\).

Case 2. \(Z\) is solvable; then \(Z\) has a \(hn\), \(Z = \lambda x \cdot xA_1 \ldots A_m (x_i \in \emptyset)\).

Take \(n = i\), so \(ZQ^n = \lambda x^i \cdot QA_1 \ldots A_m = \emptyset\).  \(\Box\)

3.10. Theorem. If \(\mathcal{M}\) is hard and sensible, then \(\mathcal{M}\) is not rich.

Proof. If \(\mathcal{M}\) is hard, then \(\mathcal{M}\) is isomorphic to \(\mathcal{M}^0(T)\), where \(T = Th(\mathcal{M})\). We reason in \(\mathcal{M}^0(T)\). Since \(\mathcal{M}\) is sensible, \(\emptyset \subseteq T\).

Let \(h: \omega \rightarrow \omega\) be a function not definable in \(\mathcal{M}\). Such an \(h\) exists since a hard model is countable.

Let \(A_n(x, y)\) be the term \(xQ^n(yQ^n(hn)), n \in \omega\). For closed \(M\) the sequence \(A_0(M, y), A_1(M, y), \ldots\) is by 3.9

\[
M(y(h\eta)), M\Omega(yQ^n(h\eta)), \ldots, M\Omega^n(yQ^n(h\eta)), \Omega, \Omega, \ldots,
\]

where \(n\) is such that \(M\Omega^{n+1} = \Omega\). Thus \(\lambda n \cdot A_n(M, y)\) is up to convertibility a recursive sequence containing one fixed free variable and hence representable as a term. Define \(f(M) = \lambda y \cdot \langle A_n(M, y) \rangle_{n \in \omega}\). Similarly for closed \(N\) \(\lambda n \cdot A_n(x, N)\) is recursive and it is possible to define \(g(N) = \lambda x \cdot \langle A_n(x, N) \rangle_{n \in \omega}\).

Then for all closed \(M, N: f(M) = g(N)\) and \(g(N)\) are well defined and \(f(M) \cdot N = g(N) \cdot M = \langle A_n(M, N) \rangle_{n \in \omega}\) by construction. So \(f\) and \(g\) are dual.

Suppose now that \(\mathcal{M}\) is rich, i.e. \(f\) were representable by some closed \(F\).

Then for all closed \(M, N: FMN = f(M)N = \langle A_n(M, N) \rangle_{n \in \omega}\).

But then \(pM \langle F(K^n I)(K^n I) \rangle = pM \langle h(y) \rangle_{n \in \omega} = h(y)\), hence \(h\) were definable, contradiction. Thus \(\mathcal{M}\) is not rich.  \(\Box\)
3.11. **Corollary.** \( D^\omega \) and \( \mathcal{M}(T) \) for \( T \supset \mathcal{H} \) are not rich.

3.12. **Questions.** (i) Is every extensional term model \( \mathcal{M}(T) \) rich? 
(ii) Is \( \mathcal{M}(\lambda \omega) \) rich?

Here \( \lambda \omega \) is the \( \lambda \)-theory obtained by adding the \( \omega \)-rule to the theory, see Barendregt [1974].

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**REFERENCES**

Barendregt, H. P. - The undefinability of Church's \( \delta \), unpublished manuscript, Utrecht. (1972).


Boer, S. de - De ondefinieerbaarheid van Church's \( \delta \)-functie in de \( \lambda \)-calculus en Barendregts lemma, unpublished, T.H. Eindhoven (1975).

Böhm, C. - An interpolation theorem in the \( \lambda \)-calculus, mimeographed, Torino. (1972).


Plotkin, G. - The \( \lambda \)-calculus is \( \omega \)-incomplete, J. Symbolic Logic, 39, 313-317 (1974).


Wadsworth, C. P. - Letter to one of the authors. 1972.