SUMMARY
§ 1 is concerned with the term model of the $\lambda$-calculus. It is proved that Church's $\delta$ is not definable and that the definable functions into the numerals are constant. In § 2 it is proved that for several $\lambda$-algebras the range of a representable function is either a singleton or infinite. In § 3 it is examined in which $\lambda$-algebras the local representability of external functions implies the global representability.

INTRODUCTION
Let $\mathcal{M} = \langle M, \cdot \rangle$ be a $\lambda$-algebra (i.e. a model of the $\lambda$-calculus). Elements of $M$ are thought of as functions. Arbitrary $f : M \to M$ are called external functions. Such a function is representable (by an element $a \in M$) if $\forall b \in M \ f(b) = a \cdot b$. A function $f$ is definable in $\mathcal{M}$ if $f$ is representable by $[F]_{\mathcal{M}}$ for some closed term $F$. Here $[F]_{\mathcal{M}}$ denotes the value of $F$ in the model $\mathcal{M}$.

Other notations:
$x, y, \ldots$ denote variables of the $\lambda$-calculus.
$a, b, \ldots$ denote variables ranging over the elements of a $\lambda$-algebra.
$F, G, \ldots$ denote $\lambda$-terms.

The numerals $0, 1, 2, \ldots$ denote some adequate representation of the natural numbers as $\lambda$-terms e.g. those of Church: $n = \lambda f x. f^n(x)$.

If $\mathcal{M} = \langle M, \cdot \rangle$ is a $\lambda$-algebra, then $\mathcal{M}^0$ is the sub-$\lambda$-algebra $\langle M^0, \cdot \rangle$ where $M^0 = \{ [F]_{\mathcal{M}} \in M \mid F \text{ closed term} \}$.

If $T$ is a consistent extension of the $\lambda$-calculus, $\mathcal{M}(T)$ is the term-model of $T$, i.e. the set of all $\lambda$-terms modulo provable equality in $T$. The closed term-model of $T$, notation $\mathcal{M}_c(T)$, is defined as $(\mathcal{M}(T))^0$. A $\lambda$-algebra $\mathcal{M}$ is hard if $\mathcal{M} = \mathcal{M}^0$. In such an $\mathcal{M}$ a function is representable iff it is definable.

For other terminology see Barendregt [1976].

The three sections of the paper treat different aspects of the notion of representability.

In § 1 attention is restricted to the standard extensional term model $\mathcal{M} = \mathcal{M}(\lambda\eta)$. 

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Church's $\delta$ is an external function satisfying

\begin{align*}
(\star) \quad & \delta M \bar{M} = 0 \text{ if } M \text{ is a closed normal form (nf)} \\
& \delta M \bar{M}' = 1 \text{ if } M, M' \text{ are different closed nf's.}
\end{align*}

In Böhm [1972] it is proved that $\forall N_1 \ldots N_n \text{ different } \beta\eta\text{-nf's}$ $\exists F \models FN_1 = i$. As a consequence it follows that for every finite set $A$ of nf's there is a term $\delta$ satisfying $(\star)$ for $M, M' \in A$.

At the Orléans logic conference (1972) the question was raised whether the general Church's $\delta$ is definable as a $\lambda$-term.

We will give a negative answer which was already established in Barendregt [1972] and independently in Wadsworth [1972] (see also Hindley and Mitschke [1975]). All three proofs of the non-existence of $\delta$ are different.

Furthermore it is proved that the only definable functions from the terms into the numerals are the constant functions.

In § 2 it will be proved that definable functions in various $\lambda$-algebras have a range of cardinality 1 or $\aleph_0$. For representable functions this is not true in $D_\infty$ and $P_0$.

Two external functions $f$ and $g$ on $\mathcal{M}$ are dual, notation $f \sim_{\mathcal{M}} g$, if $f(a) \cdot b = g(b) \cdot a$ for all $a, b \in \mathcal{M}$. In that case for each $b$ the map $\lambda a. f(a).b$ is representable and $f$ is said to be locally representable, similarly for $g$.

A model $\mathcal{M}$ is rich if for all $f, g$:

$$f \sim_{\mathcal{M}} g \Rightarrow f \text{ and } g \text{ are representable in } \mathcal{M}.$$ 

The results of § 3 are: $D_\infty$ and $\mathcal{M}(\lambda \eta)$ are rich; rich models are extensional; hard sensible models (e.g. the interior of $D_\infty$) are not rich.

We would like to draw the proof of 3.6 to the reader's attention. There variables of the $\lambda$-calculus are not just used in the usual way, but also serve as separate entities.

§ 1. NON-DEFINABILITY RESULTS

The main tool in this section is the "Böhm out" technique 1.4. This result is also of use in § 2.

1.1. DEFINITION. Let $BT(M)$ be the Böhm tree of $M$, see Barendregt [1976], § 6. $x \in BT(M)$ iff $x \in FV(M^k)$ for some $k$, where $M^k$ is the $k^{th}$ approximate normal form of $M$.

1.2. DEFINITION. (i) A selector is a term of the form

$$U \equiv \lambda x_1 \ldots x_n. x_i, \quad 1 < i < n.$$ 

A permutator is a term of the form

$$C \equiv \lambda x_1 \ldots x_n. x_{\pi(1)} \ldots x_{\pi(n)}$$

for some permutation $\pi$.

(ii) Simple terms are inductively defined by: Any variable, selector or permutator is a simple term. If $P, Q$ are simple terms, so is $PQ$. 

1.3. Lemma. Simple terms have a normal form (nf).

Proof. Realize that each simple term is of the form $x\overrightarrow{P}$, $U\overrightarrow{P}$, $C\overrightarrow{P}$ with $\overrightarrow{P}$ simple, $U$ a selector and $C$ a permutator. Then it can be shown by induction on the term length that they have a nf. ■

1.4. Theorem. Let $FV(M) = \{x\}$ and $x \in BT(M)$. Then

(i) For some $\overrightarrow{P}, \overrightarrow{Q}$, with $x \notin FV(\overrightarrow{P})$, $\lambda \vdash M\overrightarrow{P} = x\overrightarrow{Q}$ ("$x$ is Böhmed out").

(ii) Moreover $\overrightarrow{P}$ can be chosen as a sequence of simple terms.

Proof. Let $x$ occur in $BT(M)$ at depth $k > 0$. By a similar construction as in Barendregt [1976] 6.14, 6.15, for some Böhm-transformation $\pi_1, x$ occurs in $BT(M^{\pi_1})$ at depth $k - 1$. Iterating this leads to $M^{\pi_2} = \overrightarrow{\lambda y} \cdot x\overrightarrow{Q}$, hence $M^{\pi_2}y = x\overrightarrow{Q}$, for a Böhm transformation $\pi_2$.

Checking the details of the construction of $\pi_2$ one verifies that

$$M^{n_k}y = M \ldots x_1 \ldots [x_j/Cx_j] \ldots [x_k/Ux_k] \ldots y = M\overrightarrow{P}$$

for some simple terms $\overrightarrow{P}$ with $x \notin FV(\overrightarrow{P})$ (where $C$ is a permutator and $U$ a selector). ■

1.5. Lemma. Let $F$ be a closed $\lambda$-term such that $F$ is not constant, i.e. $\lambda \not\vdash FX_1 = FX_2$ for some $X_1, X_2$, and suppose that for some closed $\lambda$-term $M$, $FM$ has a nf. Then $x \in BT(Fx)$ for all $x$.

Proof. Note that if $P, P'$ have equal finite $\Omega$-free Böhm-trees, then $\lambda \vdash P = P'$. Now suppose $x \notin BT(Fx)$ for some $x$. Then for all $k$, $x \notin F((Fx)^k)$ ($N^k$ is the $k$-th approximate normal form of $N$, cf. Barendregt [1976] 7.4 (iv)). Hence $(FM)^k \equiv (Fx)^k [x/M] \equiv (Fx)^k$ for all $k$, and it follows that $BT(FM) = BT(Fx)$. But since $FM$ has a nf, $BT(FM)$ is finite and $\Omega$-free and therefore $\lambda \vdash FM = Fx$. Since $F, M$ are closed it follows that for all $\lambda$-terms $N$, $\lambda \vdash FN = FM$, i.e. $F$ is constant, a contradiction. ■

Remark. 1.5 also holds for $F, M$ not necessarily closed.

1.6. Definition. $\emptyset = I$, $n + 1 = K n$.

1.7. Lemma. The function $sg$ is not $\lambda$-definable with respect to $\{n | n \in \omega\}$, i.e. for no closed $\lambda$-term $F \vdash F \emptyset = \emptyset$, $\vdash F n + 1 = I$.

Proof. Suppose $F$ exists. Then by 1.5 $x \in BT(Fx)$. Hence by 1.4 $Fx\overrightarrow{P} = x\overrightarrow{Q}$ for some $\overrightarrow{P}, \overrightarrow{Q} = Q_1 \ldots Q_m$. But then for all $n > m$,

$$\vdash \overrightarrow{P} = F \emptyset \overrightarrow{P} = \emptyset \overrightarrow{Q} = Q_1 \ldots Q_m = n - m$$

contradicting the Church-Rosser theorem since the $k$ are different nf's. ■
1.8. Definition. A system of terms \( \{M_n | n \in \omega \} \) is an adequate system of numerals iff

(i) Each \( M_n \) has a \( \text{nf} \).

(ii) Each recursive function can be \( \lambda \)-defined with respect to the \( M_n \).

In Barendregt [1977] is shown that the second condition can be replaced by (ii'): The successor, predecessor and \( sg \) functions can be \( \lambda \)-defined with respect to the \( M_n \).

The following corollary was proved independently by Barendregt [1972] and Wadsworth [1972].

1.9. Corollary.

(i) \( \{n | n \in \omega \} \) is not an adequate system of numerals. (ii) Church's \( \delta \) is not \( \lambda \)-definable.

Proof. (i) Immediate. (ii) If \( \delta \) were \( \lambda \)-definable, then so would be \( sg \), viz. by \( \lambda x \cdot \delta x \, 0 \, 0 \, 1. \)

Remark. (i) Although not definable, \( \delta \) can consistently be added to the \( \lambda \)-calculus, see Church [1941].

(ii) Contrary to this, the corresponding \( \delta \) for open \( \lambda \)-terms would be inconsistent at once. For let \( x \neq y \), then

\[
(\lambda y \cdot \delta xy(KK)S)x = (\lambda y \cdot I(KK)S)x = (\lambda y \cdot KKS)x = KKS = K
\]

but also

\[
(\lambda y \cdot \delta xy(KK)S)x = \delta xx(KK)S = \delta (KK)S = S.
\]

(iii) One could also consider the definability of a \( \delta \) for all closed terms, i.e.:

\[
\delta MM = 0 \quad \text{for } M \text{ closed}
\]

\[
\delta MN = I \quad \text{for } M, N \text{ closed such that } \not\vdash M = N.
\]

But then the following version of the Russell paradox would result.

Define \( \neg X = \delta X I \). If \( \not\vdash 0 = I \), then \( \not\vdash X = I \iff \neg \neg X = I \).

Now let \( A = FP \rightarrow \neg (i.e. \text{the fixed point of } \neg): \not\vdash A = \neg A \).

Then \( \not\vdash A = I \iff \not\vdash A = I. \) Thus \( \not\vdash 0 = I \).

To see the relation with the Russell paradox, note that \( A = BB \) with \( B = \lambda x. \neg (xz) \). (In illative combinatory logic \( MN \) is interpreted as \( N \in M \) and \( \lambda x \cdot P \) as \( \{x|P\} \).)

1.10. Theorem. Let \( \omega = \{n | n \in \omega \} \) be an adequate system of numerals and let \( f \) be a map into \( \omega \) definable by \( F \). Then \( f \) is constant.

Proof. First assume \( \omega \) is Church's system of numerals.

Suppose \( f \) is not constant, then by 1.5 \( x \in BT(Fx) \). Hence for some simple \( \overrightarrow{P} \) and \( \overrightarrow{Q}, \lambda \vdash Fx\overrightarrow{P} = x\overrightarrow{Q}. \)

Hence \( \lambda \vdash FM\overrightarrow{P} = MQ \) for all \( M \). But \( M\overrightarrow{Q} \) can take arbitrary values and not \( FM \overrightarrow{P} \), since \( \vdash \overrightarrow{P} = P_1^n(P_2)P_3 \ldots P_k \) always has a \( \text{nf} \) by 1.3.
Now let $\omega$ be an arbitrary system of numerals. It is well-known how to define a term $G$ such that $Gn = n$.

Suppose a non-constant $f$: terms $\rightarrow \omega$ would be definable, then $G \circ f$ were a definable non-constant mapping into $\omega$.

First alternative proof (due to the referee).
Suppose $F$ is not constant, i.e. let $n_1 \neq n_2 \in Ra(F)$. Define $G$ as the $\lambda$-defining term of the recursive function

$$g(x) = \begin{cases} 0 & \text{if } x = n_1, \\ 1 & \text{else}. \end{cases}$$

Then the range of $G \circ F$ is $\{0, 1\}$ contrary to 2.3.

Second alternative proof. By Barendregt’s lemma in de Boer [1975] it follows that if $\Omega$ is unsolvable and $N$ a $nf$, then $F\Omega = N \Rightarrow Fx = N$ for all $x$. (General genericity lemma; see also Barendregt [1977a] for a proof.) Now if the values of $F$ are numerals it follows that $F\Omega$ has a $nf$, i.e. $F$ is constant.

1.11. COROLLARY. There is no $F$ such that

- $FM = 0$ if $M$ is a numeral (i.e. $\vdash M = n$ for some $n$)
- $1$ else

for any adequate system.

1.12. QUESTION. Is there a term $F$ such that

- $FM$ has a $nf$ (is solvable) if $M$ is a numeral
- has no $nf$ (is unsolvable) else.

§ 2. THE RANGE PROPERTY

2.1. DEFINITION. Let $\mathcal{M} = \langle M, \cdot \rangle$ be a $\lambda$-algebra. For each $f \in M$, we define $Ra^\mathcal{M}(f)$, the range of $f$ in $\mathcal{M}$, as follows:

$$Ra^\mathcal{M}(f) = \{ f \cdot x | x \in M \}.$$  

**NOTATION.** $Ra^\mathcal{M}(F) = Ra^\mathcal{M}(\lambda F \cdot ^x)$ for terms $F$.

When possible, the superscript $\mathcal{M}$ will be dropped in $Ra^\mathcal{M}$.

2.2. DEFINITION. A $\lambda$-algebra $\mathcal{M}$ satisfies the range property if for all $f \in M$, the cardinality of $Ra^\mathcal{M}(f)$ is 1 or $\aleph_0$.

2.3. RANGE THEOREM. (Barendregt; Myhill). Let $T$ be a r.e. $\lambda$-theory. Then $\mathcal{M}(T)$ (and also $\mathcal{N}(T)$) has the range property.

**PROOF.** Suppose $f \in M$ and $Ra(f) = \{m_0, \ldots, m_k\}$, $k > 0$. Define $N_f = \{x | f \cdot x = m_i \} \subseteq M$. 

Every such $N_t$ is r.e. Therefore $N = \bigcup_1^k N_t$, the complement of $N_0$ is also r.e. Hence $N_0$ is recursive.

On the other hand $N_0$ is non-trivial and closed under equality, which contradicts Scott's theorem (Barendregt [1976] 2.21).

The proof for $\mathcal{M}^0(T)$ is the same.

2.4. Corollary. $\mathcal{M}(\lambda), \mathcal{M}^0(\lambda), \mathcal{M}(\lambda\eta)$ and $\mathcal{M}^0(\lambda\eta)$ have the range property.

The range property, however, is not satisfied in every $\lambda$-algebra.

2.5. Theorem. $P\omega$ and $D_{\infty}$ do not satisfy the range property.

Proof. Since the proof is similar in both cases, let $\mathcal{S} = (S, \subset)$ denote either $(P\omega, \subseteq)$ or $(D_{\infty}, \subseteq)$. We define the following function $f : S \to S$ by $f(x) = \top$ if $x \neq \bot$ else $\bot$ ($\top$ and $\bot$ are the largest respectively smallest element of $S$.)

Claim: $f$ is continuous. Then by Scott [1972], [1975] $f$ is representable and since $f$ has range of cardinality two we are done.

For open $O$ in $S$ one has: $x \in O$ and $x < y \Rightarrow y \in O$.


Hence for open $O$, $\bot \in O \Rightarrow O = S$, and $O \neq \emptyset \Rightarrow \top \in O$.

Now for every open set $O$, $f^{-1}(O)$ is open:

Case 1. $\bot \in O$. Then $O = S$ so $f^{-1}(S) = S$ which is open.

Case 2. $\bot \notin O$. If $O = \emptyset$, then we are done. Else $\top \in O$ and hence $f^{-1}(O) = S - \{\bot\} = \{x | x \leq \bot\} \overset{\text{def}}{=} U_\bot$.

$U_\bot$ is open in $D_{\infty}$, see e.g. Barendregt [1976] 1.2.

$U_\bot$ is open in $P\omega$: Let $O_k = \{x | e_k \subseteq x\}$. Note $e_0 = \emptyset = \bot$ and that the $O_k$ form a base for the topology on $P\omega$.

Now:

$x \in U_\bot \iff x \notin O \iff \forall k \exists e_k \subseteq x \iff x \in \bigcup_{k=0}^\infty O_k$

which is, as a union of elements of a base, indeed open.

The following theorem was announced in Wadsworth [1973] for the $D_{\infty}$ case.

2.6. Theorem. Let $\mathcal{S}$ be $D_0^\infty$ or $P^0\omega$. Then $\mathcal{S}$ satisfies the range property.

Proof. Let $F$ be a closed term. Consider $BT(Fx)$.

Case 1. $x \notin BT(Fx)$. Then $BT(FM) = BT(FM')$ for all $M, M'$. Since terms with equal Böhm trees are equal in $\mathcal{S}$ (see Barendregt [1976], Hyland [1976]), it follows that $Ra^\mathcal{S}(F)$ has cardinality 1.

Case 2. $x \in BT(Fx)$. Then by 1.4 $\lambda \vdash F \vec{x} \overset{\nu}{=} x \vec{Q}$.

Since $[NQ]_{\mathcal{S}}$ can take arbitrary values in $\mathcal{S}$ when $N$ ranges over the closed terms, $Ra^\mathcal{S}(F)$ is infinite.
2.7. Conjecture. \( \mathcal{M}(\mathcal{H}) \) satisfies the range property.

2.8. Question. Does every hard \( \lambda \)-algebra \( \mathcal{M} \) (i.e. \( \mathcal{M} = \mathcal{M}^0 \)) satisfy the range theorem?

§ 3. Duality

3.1. Definition. Let \( f, g \) be two external functions on a \( \lambda \)-algebra \( \mathcal{M} = \langle M, \cdot \rangle \).

\( f, g \) are dual iff \( Va, b \in M: f(a) \cdot b = g(b) \cdot a. \) Notation \( f \sim \mathcal{M} g \), or simply \( f \sim g \).

3.2. Definition. \( \mathcal{M} \) is rich iff all dual functions on \( \mathcal{M} \) are representable in \( \mathcal{M} \).

Remarks. (i) Let \( f \) be an external function on \( \mathcal{M} \). \( f \) is locally representable iff for each \( b \in M \) the function \( h \) defined by \( h(a) = f(a) \cdot b \) is representable. Then \( f \) is locally representable iff \( f \) has a dual. A model is rich iff all locally representable functions are representable.

(ii) If \( f \) is representable (by \( f_0 \in M \), say), then \( f \) has a dual \( g \) which is also representable (by \( g_0 = \lambda a b \cdot f_0 b a \)).

(iii) Let \( \mathcal{M} \) be extensional. Then \( f \) has at most one dual. Hence if \( f \sim \mathcal{M} g \) and \( f \) is representable, then by (ii) \( g \) is representable.

3.3. Theorem. If \( \mathcal{M} \) is rich, then \( \mathcal{M} \) is extensional.

Proof. Suppose \( \mathcal{M} \) is not extensional. Then there exist \( b, b' \in M \) such that for all \( c \in M \) \( b \cdot c = b' \cdot c \) and \( b \neq b' \).

Define

\[
 f(a) = \begin{cases} 
 b' & \text{if } a = b \\
 b & \text{else}.
\end{cases}
\]

and

\[
 g = [\lambda y \cdot K(by)]^\mathcal{M},
\]

then for all \( a, a' \in M \): \( f(a) \cdot a' = b \cdot a' = g(a') \cdot a \), hence \( f \sim g \). But \( f \) cannot be representable since it has no fixed point. Thus \( \mathcal{M} \) is not rich.

3.4. Corollary. The following \( \lambda \)-algebras are not rich: \( \text{P}_\omega; \text{P}^0\omega; \mathcal{M}(\lambda); \mathcal{M}^0(\lambda); \mathcal{M}^0(\lambda) \).

Proof.

1. \( \text{P}_\omega \) is not extensional:
Take for example \( a = \{(0, 0)\} \) and \( b = \{(0, 0), (1, 0)\} \).
Then \( Vc \in \text{P}_\omega \ a \cdot c = b \cdot c \) but \( a \neq b \).

2. \( \text{P}^0\omega \) is not extensional: Let \( 1 = \lambda xy \cdot xy \), then \( \text{P}^0\omega \models Ixy = 1xy \), but \( \text{P}^0\omega \not\models I = 1 \) for otherwise \( \text{P}_\omega \models I = 1 \), so \( \text{P}_\omega \models Vx \ x = \lambda y \cdot xy \) which implies that \( \text{P}_\omega \) were extensional.

3. By the Church Rosser property \( \lambda \not\models I = 1 \). So \( \mathcal{M}(\lambda), \mathcal{M}^0(\lambda) \) are not extensional.
4. \( \mathcal{M}^0(\lambda \eta) \) is not extensional because the \( \lambda \)-calculus is \( \omega \)-incomplete, see Plotkin [1974].

3.5. **Theorem.** \( D_\infty \) is rich.

**Proof.** Suppose that \( f, g \) are dual i.e.:

\[
Va, b \in D_\infty: f(a) \cdot b = g(b) \cdot a.
\]

We have to show that \( f, g \) are representable.

It is sufficient to show that \( f, g \) are continuous. Take a directed \( X \subset D_\infty \).

For all \( b \in D_\infty \)

\[
f(\bigsqcup X) \cdot b = \bigsqcup \{g(b) \cdot a | a \in X\} = \bigsqcup \{f(a) \cdot a | a \in X\} \cdot b
\]

by the duality condition and the continuity of application.

Thus by extensionality in \( D_\infty \): for all directed \( X \subset D_\infty \)

\[
f(\bigsqcup X) = \bigsqcup \{f(a) | a \in X\}
\]

i.e. \( f \) is continuous. The proof for \( g \) is dual. ■

3.6. **Theorem.** \( \mathcal{M}(\lambda \eta) \) is rich.

**Proof.** Define

\[
M = \lambda \eta N \iff \lambda \eta \vdash M = N,
\]

\[
x \in_{\lambda \eta} M \iff \text{for all } M' = \lambda \eta M, \text{ one has } x \in \text{FV}(M').
\]

Let \( f, g \) be dual functions on \( \mathcal{M}(\lambda \eta) \).

3.6.0. **Lemma.** (i) \( x \in_{\lambda \eta} M \iff \lambda \eta \vdash M = N \iff x \in \text{FV}(N) \).

(ii) Let \( M' = M[z/y] \and \lambda \vdash M' \rightarrow N' \). Then \( \forall N \and \lambda \vdash M \rightarrow N \) and \( N' = N[z/y] \).

\[
\begin{array}{c}
M \longrightarrow N \\
[z/y] \\
M' \longrightarrow N'
\end{array}
\]

(iii) \( x \in_{\lambda \eta} M \Rightarrow x \in_{\lambda \eta} M[z/y], \text{ for } z \not= x \).

**Proof.** (i) \( \Rightarrow \) Trivial. \( \Leftarrow \) Suppose \( M = \lambda \eta M' \). By the Church-Rosser theorem \( \lambda \eta \vdash M \rightarrow N, M' \rightarrow N' \) for some \( N \). By assumption \( x \in \text{FV}(N) \).

But then \( x \in \text{FV}(M') \).

(ii) Induction on the length of proof of \( M \rightarrow N' \). In the case that \( M' = (\lambda \eta \cdot P)Q, N' = P[a/Q] \) it may be assumed that \( a \not= z, y \). Therefore one can apply the well-known substitution lemma

\[
A[u/B][v/C] = A[v[C][u/B][v/C]] \text{ if } u \not= v \text{ and } u \notin \text{FV}(C).
\]

(iii) Suppose \( \lambda \eta \vdash P[z/y] \rightarrow R' \). By (ii) for some \( R \and \lambda \eta \vdash P \rightarrow R \) and \( R' = R[z/y] \). By assumption and (i), \( x \in \text{FV}(R) \). Since \( x \not= z \) also \( x \in \text{FV}(R') \). Therefore by (i) \( x \in_{\lambda \eta} P[z/y] \). ■
3.6.1. Lemma. (i) If \( x \in \lambda y \cdot P \) then \( x \in \eta P \) and \( x \neq y \).

(ii) If \( x \neq y \), then \( x \in \eta M \iff x \in \eta My \).

Proof. (i) Since \( x \in FV(\lambda y \cdot P) \) clearly \( x \neq y \). Suppose \( P = \eta N \), then \( \lambda y \cdot P = \eta \lambda y \cdot N \). By assumption \( x \in FV(\lambda y \cdot N) \subseteq FV(N) \). Thus \( x \in \eta P \).

(ii) \( \Rightarrow \) Suppose \( \eta y \vdash My \to N \) in order to prove \( x \in FV(N) \).

Case 1. \( N = M' y \) with \( \eta y \vdash M \to M' \). Since \( x \in \eta M \), also \( x \in FV(M') \subseteq FV(N) \).

Case 2. \( M \to \lambda z . M_1 \) and \( \eta y \vdash My \to (\lambda z . M_1)y \to M_1[z/y] \to N \).

Since \( x \in \eta M \), also \( x \in \eta \lambda z . M_1 \) and by (i) \( x \in \eta M_1 \) and \( z \neq x \), so by 3.6.0. (iii) \( x \in \eta M_1[z/y] \). Therefore \( x \in FV(N) \).

3.6.2. Lemma. If \( \forall y \neq x \ x \in \eta f(y) \), then \( \forall y \neq x \ x \in \eta g(y) \) (and hence \( \forall y \neq x \ x \in \eta f(y) \)).

Proof. Suppose \( x \in \eta f(y) \), \( y \neq x \). Let \( y' \neq x \). Then by 3.6.1. (ii) \( x \in \eta f(y) \cdot y' = \eta g(y') \cdot y \). Hence, by 3.6.1. (ii), \( x \in \eta g(y') \). (The rest follows by applying the statement to \( x \in \eta g(y') \)).

3.6.3. Main Lemma. There is a variable \( x \) such that for all terms \( M : f(x)[x/M] = f(M) \).

Proof. Let \( v \) be any variable. Choose \( x \neq v \) such that \( x \notin \eta f(v) \). Then \( x \notin \eta g(z) \) for all \( z \neq x \), by the dual of 3.6.2.

Given \( M \), one can find a \( y \) such that \( y \notin \eta M \), \( f(M) \), \( x \notin f(x) \). Hence \( x \notin \eta g(y) \). Now since \( y \neq x \) and \( x \notin \eta g(y) \), \((f(x)[x/M]) \cdot y = (f(x) \cdot y)[x/M] = (g(y) \cdot x)[x/M] = g(y) \cdot M = f(M) \cdot y \).

Since \( y \notin f(x) \), \( f(M) \), extensionality yields \( f(x)[x/M] = f(M) \).

Now it follows by 3.6.3. that \( f \) can be represented by the term \( \lambda x . f(x) \) and similar for \( g \).

The following construction is needed for the proof of 3.10.

3.7. Definition. Let \( \# \) be a Gödel numbering of terms. \( [M] \) is the numeral \( \# M \). A sequence of terms \( M_n \) is recursive if \( \lambda n . \# M_n \) is a recursive function.

3.8. Lemma. (Coding of infinite sequences). Let \( \{M_n\} \) be a recursive sequence of terms such that \( FV(M_n) \subseteq \{x\} \) for all \( n \). Then there exists a term \( X \) such that \( p_i X = M_i \), for all \( i \), where \( p \) is some fixed closed term. Par abus de langage we write \( \langle M_n \rangle_{n \in \omega} \) for \( X \).

Proof. (1) As in Curry et al. [1972], 13 B3 there is a term \( E \) which enumerates all terms with \( x \) as only free variable:

\[ E(\langle M \rangle) = M, \text{ for } M \text{ with } FV(M) = \{x\} \]
Let \([M, N]\) be a pairing of terms defined by \(\lambda z \cdot zMN\). Then \([M, N]K = M\) and \([M, N](KI) = N\). Define ordered tuples as follows:

\([M] = M, [M_1, \ldots, M_{n+1}] = [M_1, [M_2, \ldots, M_{n+1}]]\).

Let \(M_n\) with \(FV(M_n) \subseteq \{x\}\) be a recursive sequence of terms. We want to code the sequence \([M_n]\) as a \(\lambda\)-term. Let \(S^+\) be such that \(S^+ n \cdot \beta \rightarrow n + 1\) and let \(b = \lambda xy \cdot [E(Fy), (x(S^+y))]\), where \(F\) \(\lambda\)-defines \(f\), and \(B = FP\) \(b\) (i.e. the fixed point of \(b\)). Then

\[B_n \cdot \beta \rightarrow bB_n \cdot \beta \rightarrow [E(Fy), B_n + 1] \cdot \beta \rightarrow [M_n, B_n + 1].\]

So \(B_0 = [M_0, B_1] = [M_0, M_1, B_2] = \ldots\) Hence by setting \(\langle M_n \rangle_{n \in \omega} = B_0\) we have a coding for infinite sequences of terms with one fixed free variable.

It is easy to construct a term \(p\) such that \(p_M = 3\) \(\langle I_n \rangle = 31\) \(\langle M, N\rangle = \langle A_n(x, y), A_1(y) \rangle_{n \in \omega}\). 3.9. LEMMA. For all closed \(Z\) there is an \(n\) such that \(Z\Omega^n = \not\in \Omega\). \(Z\Omega^n\) is short for \(\underbrace{Z \Omega \cdots \Omega}_n\).

PROOF. 3.10. THEOREM. If \(\mathcal{H}\) is hard and sensible, then \(\mathcal{H}\) is not rich.

PROOF. If \(\mathcal{H}\) is hard, then \(\mathcal{H}\) is isomorphic to \(\mathcal{H}^0(T)\), where \(T = Th(\mathcal{H})\). We reason in \(\mathcal{H}^0(T)\). Since \(\mathcal{H}\) is sensible, \(\mathcal{H} \subseteq T\).

Let \(h : \omega \rightarrow \omega\) be a function not definable in \(\mathcal{H}\). Such an \(h\) exists since \(\mathcal{H}\) is first-order countable.

Let \(A_n(x, y)\) be the term \(x\Omega^n(y\Omega^n(hn))\), \(n \in \omega\). For closed \(M\) the sequence \(A_0(M, y), A_1(M, y), \ldots\) is by 3.9

\[M(y(hn)), M\Omega(y\Omega(hl)), \ldots, M\Omega^n(y\Omega^n(hn)), \Omega, \Omega, \ldots,\]

where \(n\) is such that \(M\Omega^{n+1} = \Omega\). Thus \(\lambda n \cdot A_n(M, y)\) is up to convertibility a recursive sequence containing one fixed free variable and hence representable as a term. Define \(f(M) = \lambda y \cdot \langle A_n(M, y) \rangle_{n \in \omega}\). Similarly for closed \(N\) \(\lambda n \cdot A_n(x, N)\) is recursive and it is possible to define \(g(N) = \lambda x \cdot \langle A_n(x, N) \rangle_{n \in \omega}\).

Then for all closed \(M, N\):

\[f(M) \cdot N = \langle A_n(M, N) \rangle_{n \in \omega}\]

by construction. So \(f\) and \(g\) are dual.

Suppose now that \(\mathcal{H}\) is rich, i.e. \(f\) were representable by some closed \(F\). Then for all closed \(M, N\): \(F(MN) = f(M)N = \langle A_n(M, N) \rangle_{n \in \omega}\).

But then \(p_M(F(K^nI)(K^nI)) = p_M\langle h(y) \rangle_{n \in \omega} = h(y)\), hence \(h\) were definable, contradiction. Thus \(\mathcal{H}\) is not rich.
3.11. Corollary. $D^\omega_\infty$ and $\mathcal{M}^\omega(T)$ for $T \supset H$ are not rich.

3.12. Questions. (i) Is every extensional term model $\mathcal{M}(T)$ rich?  
(ii) Is $\mathcal{M}^\omega(\lambda\omega)$ rich?

Here $\lambda\omega$ is the $\lambda$-theory obtained by adding the $\omega$-rule to the theory, see Barendregt [1974].

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