SUMMARY

§ 1 is concerned with the term model of the $\lambda$-calculus. It is proved that Church's $\delta$ is not definable and that the definable functions into the numerals are constant. In § 2 it is proved that for several $\lambda$-algebras the range of a representable function is either a singleton or infinite. In § 3 it is examined in which $\lambda$-algebras the local representability of external functions implies the global representability.

INTRODUCTION

Let $\mathcal{M} = \langle M, \cdot \rangle$ be a $\lambda$-algebra (i.e. a model of the $\lambda$-calculus). Elements of $M$ are thought of as functions. Arbitrary $f : M \to M$ are called external functions. Such a function is representable (by an element $a \in M$) if $\forall b \in M \ f(b) = a \cdot b$. A function $f$ is definable in $\mathcal{M}$ if $f$ is representable by $[[F]]^\mathcal{M}$ for some closed term $F$. Here $[[F]]^\mathcal{M}$ denotes the value of $F$ in the model $\mathcal{M}$.

Other notations:

- $x, y, \ldots$ denote variables of the $\lambda$-calculus.
- $a, b, \ldots$ denote variables ranging over the elements of a $\lambda$-algebra.
- $F, G, \ldots$ denote $\lambda$-terms.

The numerals $0, 1, 2, \ldots$ denote some adequate representation of the natural numbers as $\lambda$-terms e.g. those of Church: $n = \lambda f. f^n(x)$.

If $\mathcal{M} = \langle M, \cdot \rangle$ is a $\lambda$-algebra, then $\mathcal{M}^0$ is the sub-$\lambda$-algebra $\langle M^0, \cdot \rangle$ where $M^0 = \{ [[F]]^\mathcal{M} \in M \mid F$ closed term $\}.

If $T$ is a consistent extension of the $\lambda$-calculus, $\mathcal{M}(T)$ is the term-model of $T$, i.e. the set of all $\lambda$-terms modulo provable equality in $T$. The closed term-model of $T$, notation $\mathcal{M}^0(T)$, is defined as $(\mathcal{M}(T))^0$. A $\lambda$-algebra $\mathcal{M}$ is hard if $\mathcal{M} = \mathcal{M}^0$. In such an $\mathcal{M}$ a function is representable iff it is definable.

For other terminology see Barendregt [1976].

The three sections of the paper treat different aspects of the notion of representability.

In § 1 attention is restricted to the standard extensional term model $\mathcal{M} = \mathcal{M}(\lambda\eta)$. 25 Indagationes
Church's $\delta$ is an external function satisfying
\[
(\star) \quad \delta MM = 0 \text{ if } M \text{ is a closed normal form (nf)}
\]
\[
\delta MM' = \mathbb{I} \text{ if } M, M' \text{ are different closed nf's.}
\]

In Böhm [1972] it is proved that $\forall N_1 \ldots N_n$ different $\beta\eta$-nf's $\exists F \vdash FN_i \ldots = i$. As a consequence it follows that for every finite set $A$ of nf's there is a term $\delta$ satisfying $(\star)$ for $M, M' \in A$.

At the Orléans logic conference (1972) the question was raised whether the general Church's $\delta$ is definable as a $\lambda$-term.

We will give a negative answer which was already established in Barendregt [1972] and independently in Wadsworth [1972] (see also Hindley and Mitschke [1975]). All three proofs of the non-existence of $\delta$ are different.

Furthermore it is proved that the only definable functions from the terms into the numerals are the constant functions.

In § 2 it will be proved that definable functions in various $\lambda$-algebras have a range of cardinality 1 or $\aleph_0$. For representable functions this is not true in $D_\infty$ and $P_\omega$.

Two external functions $f$ and $g$ on $\mathcal{M}$ are dual, notation $f \sim_\mathcal{M} g$, if $f(a) \cdot b = g(b) \cdot a$ for all $a, b \in \mathcal{M}$. In that case for each $b$ the map $\lambda a. f(a) \cdot b$ is representable and $f$ is said to be locally representable, similarly for $g$.

A model $\mathcal{M}$ is rich if for all $f, g$:

$$f \sim_\mathcal{M} g \Rightarrow f \text{ and } g \text{ are representable in } \mathcal{M}.$$  

The results of § 3 are: $D_\infty$ and $\mathcal{M}(\lambda \eta)$ are rich; rich models are extensional; hard sensible models (e.g. the interior of $D_\infty$) are not rich.

We would like to draw the proof of 3.6 to the reader's attention. There variables of the $\lambda$-calculus are not just used in the usual way, but also serve as separate entities.

§ 1. NON-DEFINABILITY RESULTS

The main tool in this section is the "Böhm out" technique 1.4. This result is also of use in § 2.

1.1. DEFINITION. Let $BT(M)$ be the Böhm tree of $M$, see Barendregt [1976], § 6. $x \in BT(M)$ iff $x \in FV(M^k)$ for some $k$, where $M^k$ is the $k$th approximate normal form of $M$.

1.2. DEFINITION. (i) A selector is a term of the form

$$U \equiv \lambda x_1 \ldots x_n \cdot x_i, \quad 1 < i < n.$$  

A permutator is a term of the form

$$C \equiv \lambda x_1 \ldots x_n \cdot x_{\pi(1)} \ldots x_{\pi(n)}$$

for some permutation $\pi$.

(ii) Simple terms are inductively defined by: Any variable, selector or permutator is a simple term. If $P, Q$ are simple terms, so is $PQ$.  

\[
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\]
1.3. **Lemma.** Simple terms have a normal form (nf).

**Proof.** Realize that each simple term is of the form $x\overrightarrow{P}$, $U\overrightarrow{P}$, $C\overrightarrow{P}$ with $\overrightarrow{P}$ simple, $U$ a selector and $C$ a permutator. Then it can be shown by induction on the term length that they have a nf. ■

1.4. **Theorem.** Let $FV(M)=\{x\}$ and $x \in BT(M)$. Then

(i) For some $\overrightarrow{P}, \overrightarrow{Q}$, with $x \notin FV(\overrightarrow{P})$, $\lambda \vdash \overrightarrow{M}=x\overrightarrow{Q}$ ("$x$ is Böhmed out").

(ii) Moreover $\overrightarrow{P}$ can be chosen as a sequence of simple terms.

**Proof.** Let $x$ occur in $BT(M)$ at depth $k > 0$. By a similar construction as in Barendregt [1976] 6.14, 6.15, for some Böhm-transformation $\pi_1, x$ occurs in $BT(M^{\pi_1})$ at depth $k - 1$. Iterating this leads to $M^{\pi_2} = \lambda y \cdot x\overrightarrow{Q}$, hence $M^{\pi_2}y = x\overrightarrow{Q}$, for a Böhme transformation $\pi_2$.

Checking the details of the construction of $\pi_2$ one verifies that  

$$M^{\pi_2}y \equiv M \ldots x_i \ldots [x_j/Ux_j] \ldots [x_k/Ux_k] \ldots y \equiv M\overrightarrow{P}$$

for some simple terms $\overrightarrow{P}$ with $x \notin FV(\overrightarrow{P})$ (where $C$ is a permutator and $U$ a selector). ■

1.5. **Lemma.** Let $F$ be a closed $\lambda$-term such that $F$ is not constant, i.e. $\lambda \not\vdash FX_1 = FX_2$ for some $X_1, X_2$, and suppose that for some closed $\lambda$-term $M$, $FM$ has a nf. Then $x \in BT(Fx)$ for all $x$.

**Proof.** Note that if $P, P'$ have equal finite $\Omega$-free Böhm-trees, then $\lambda \vdash P = P'$. Now suppose $x \notin BT(Fx)$ for some $x$. Then for all $k$, $x \notin FV((Fx)^k)$ ($N^k$ is the $k$-th approximate normal form of $N$, cf. Barendregt [1976] 7.4 (iv)). Hence $(FM)^k \equiv (Fx)^k [x/M] \equiv (Fx)^k$ for all $k$, and it follows that $BT(FM) = BT(Fx)$. But since $FM$ has a nf, $BT(FM)$ is finite and $\Omega$-free and therefore $\lambda \vdash FM = Fx$. Since $F, M$ are closed it follows that for all $\lambda$-terms $N$, $\lambda \vdash FN = FM$, i.e. $F$ is constant, a contradiction. ■

**Remark.** 1.5 also holds for $F, M$ not necessarily closed.

1.6. **Definition.** $\emptyset = I$, $n+1 = K\; n$.

1.7. **Lemma.** The function $sg$ is not $\lambda$-definable with respect to $\{n|n \in \omega\}$, i.e. for no closed $\lambda$-term $F \vdash F \emptyset = 0$, $\vdash F\; n+1 = I$.

**Proof.** Suppose $F$ exists. Then by 1.5 $x \in BT(Fx)$. Hence by 1.4 $Fx\overrightarrow{P} = x\overrightarrow{Q}$ for some $\overrightarrow{P}, \overrightarrow{Q} = Q_1 \ldots Q_m$. But then for all $n > m$,

$$\vdash \overrightarrow{P} = F\; n\; \overrightarrow{P} = n\; Q_1 \ldots Q_m = n-m$$

contradicting the Church-Rosser theorem since the $k$ are different nf's. ■
1.8. **Definition.** A system of terms \{M_n| n \in \omega\} is an **adequate system of numerals** iff

(i) Each \(M_n\) has a \textit{nf}.
(ii) Each recursive function can be \(\lambda\)-defined with respect to the \(M_n\).

In Barendregt [1977] is shown that the second condition can be replaced by (ii'): The successor, predecessor and \(sg\) functions can be \(\lambda\)-defined with respect to the \(M_n\).

The following corollary was proved independently by Barendregt [1972] and Wadsworth [1972].

1.9. **Corollary.**

(i) \(\{n\ | n \in \omega\}\) is not an adequate system of numerals. (ii) Church's \(\delta\) is not \(\lambda\)-definable.

**Proof.** (i) Immediate. (ii) If \(\delta\) were \(\lambda\)-definable, then so would be \(sg\), viz. by \(\lambda x \cdot \delta x \equiv 0 \equiv 1\).

**Remark.** (i) Although not definable, \(\delta\) can consistently be added to the \(\lambda\)-calculus, see Church [1941].
(ii) Contrary to this, the corresponding \(\delta\) for \textit{open} \(\lambda\)-terms would be inconsistent at once. For let \(x \not= y\), then

\[(\lambda y \cdot \delta xy(KK)S)x = (\lambda y \cdot 1(KK)S)x = (\lambda y \cdot KKS)x = KKS = K\]

but also

\[(\lambda y \cdot \delta xy(KK)S)x = \delta xx(KK)S = \theta(KK)S = S.\]

(iii) One could also consider the definability of a \(\delta\) for \textit{all} closed terms, i.e.: \(\delta MM = \theta\) for \(M\) closed

\[\delta MN = \overline{1}\] for \(M, N\) closed such that \(\not\vdash M = N\).

But then the following version of the Russell paradox would result.
Define \(\neg X = \delta \overline{1}\). If \(\not\vdash \theta = \overline{1}\), then \(\not\vdash X = \overline{1} \iff \neg \neg X = \overline{1}\).
Now let \(A = FP\) (i.e. the fixed point of \(\neg\cdot\neg\): \(\neg A = \neg \neg A\)).
Then \(\not\vdash A = \overline{1} \iff \neg A = \overline{1}\). Thus \(\not\vdash \theta = \overline{1}\).

To see the relation with the Russell paradox, note that \(A = BB\) with \(B = \lambda x. \neg (xx)\). (In illative combinatory logic \(MN\) is interpreted as \(N \in M\) and \(\lambda x \cdot P\) as \(\{x|P\}\).)

1.10. **Theorem.** Let \(\omega = \{n| n \in \omega\}\) be an adequate system of numerals and let \(f\) be a map into \(\omega\) definable by \(F\). Then \(f\) is constant.

**Proof.** First assume \(\omega\) is Church's system of numerals.
Suppose \(f\) is not constant, then by 1.5 \(x \in BT(Fx)\). Hence for some simple \(\overline{P}\) and \(\overline{Q}, \lambda \vdash Fx\overline{P} = x\overline{Q}\).

Hence \(\lambda \vdash FM\overline{P} = M\overline{Q}\) for all \(M\). But \(M\overline{Q}\) can take arbitrary values and not \(FM\overline{P}\), since \(\forall \overline{P} = P_1^n(P_2)P_3\ldots P_k\) always has a \(nf\) by 1.3.
Now let $\omega$ be an arbitrary system of numerals. It is well-known how to define a term $G$ such that $Gn = n$
Suppose a non-constant $f$: terms $\rightarrow \omega$ would be definable, then $G \circ f$ were a definable non-constant mapping into $\omega$. ■

First alternative proof (due to the referee).
Suppose $F$ is not constant, i.e. let $n_1 \neq n_2 \in Ra(F)$. Define $G$ as the $\lambda$-defining term of the recursive function
$$g(x) = \begin{cases} 0 & \text{if } x = n_1, \\ 1 & \text{else.} \end{cases}$$
Then the range of $G \circ F$ is $\{0, 1\}$ contrary to 2.3. ■

Second alternative proof. By Barendregt's lemma in de Boer [1975] it follows that if $\Omega$ is unsolvable and $N = nf$, then $F \Omega = N = Fx = N$ for all $x$. (General genericity lemma; see also Barendregt [1977a] for a proof.) Now if the values of $F$ are numerals it follows that $F \Omega$ has a $nf$, i.e. $F$ is constant. ■

1.11. Corollary. There is no $F$ such that
$$FM = 0 \text{ if } M \text{ is a numeral (i.e. } \vdash M = n \text{ for some } n)$$
$$1 \text{ else}$$
for any adequate system.

1.12. Question. Is there a term $F$ such that
$$FM \text{ has a } nf \text{ (is solvable) if } M \text{ is a numeral}$$
$$\text{has no } nf \text{ (is unsolvable) else.}$$

§ 2. THE RANGE PROPERTY

2.1. Definition. Let $\mathcal{M} = \langle M, \cdot \rangle$ be a $\lambda$-algebra. For each $f \in M$, we define $Ra^\mathcal{M}(f)$, the range of $f$ in $\mathcal{M}$, as follows:
$$Ra^\mathcal{M}(f) = \{f \cdot x | x \in M\}.$$**Notation.** $Ra^\mathcal{M}(F) = Ra^\mathcal{M}([F]^\mathcal{M})$ for terms $F$.
When possible, the superscript $\mathcal{M}$ will be dropped in $Ra^\mathcal{M}$.

2.2. Definition. A $\lambda$-algebra $\mathcal{M}$ satisfies the range property if for all $f \in M$, the cardinality of $Ra^\mathcal{M}(f)$ is 1 or $\aleph_0$.

2.3. Range theorem. (Barendregt; Myhill). Let $T$ be a r.e. $\lambda$-theory. Then $\mathcal{M}(T)$ (and also $\mathcal{M}^0(T)$) has the range property.

**Proof.** Suppose $f \in M$ and $Ra(f) = \{m_0, \ldots, m_k\}$, $k > 0$. Define
$$N_1 = \{x \cdot f \cdot x = m_i \} \subseteq M.$$
Every such \( N_t \) is r.e. Therefore \( N = \bigcup_t N_t \), the complement of \( N_0 \) is also r.e. Hence \( N_0 \) is recursive.

On the other hand \( N_0 \) is non-trivial and closed under equality, which contradicts Scott’s theorem (Barendregt [1976] 2.21).

The proof for \( A^0(T) \) is the same.

2.4. Corollary. \( \mathcal{M}(\lambda) \), \( \mathcal{M}(\lambda_0) \), \( \mathcal{M}(\lambda) \) and \( \mathcal{M}(\lambda_0) \) have the range property.

The range property, however, is not satisfied in every \( \lambda \)-algebra.

2.5. Theorem. \( P_\omega \) and \( D_\infty \) do not satisfy the range property.

Proof. Since the proof is similar in both cases, let \( \mathcal{S} = (S, \prec) \) denote either \( (P_\infty, \subseteq) \) or \( (D_\infty, \subseteq) \). We define the following function \( f: S \to S \) by \( f(x) = \top \) if \( x \neq \bot \) else \( \bot \) (\( \top \) and \( \bot \) are the largest respectively smallest element of \( S \)).

Claim: \( f \) is continuous. Then by Scott [1972], [1975] \( f \) is representable and since \( f \) has range of cardinality two we are done.

For open \( O \) in \( S \) one has: \( x \in O \) and \( x \prec y \Rightarrow y \in O \).


Hence for open \( O \), \( \bot \in O \Rightarrow O = S \), and \( O \neq \emptyset \Rightarrow \top \in O \).

Now for every open set \( O \), \( f^{-1}(O) \) is open:

Case 1. \( \bot \in O \). Then \( O = S \) so \( f^{-1}(S) = S \) which is open.

Case 2. \( \bot \notin O \). If \( O \neq \emptyset \), then we are done. Else \( \top \in O \) and hence \( f^{-1}(O) = S - \{ \bot \} = \{ x \mid x \notin \bot \} \) is open.

Now:

\[
x \in U_\bot \iff x \notin O \iff \bigcup_{k=0}^\infty e_k \subseteq x \iff x \in \bigcup_{k=0}^\infty O_k
\]

which is, as a union of elements of a base, indeed open.

The following theorem was announced in Wadsworth [1973] for the \( D_\infty \) case.

2.6. Theorem. Let \( \mathcal{S} \) be \( D_\infty^0 \) or \( P_\omega^0 \). Then \( \mathcal{S} \) satisfies the range property.

Proof. Let \( F \) be a closed term. Consider \( BT(Fx) \).

Case 1. \( x \notin BT(Fx) \). Then \( BT(FM) = BT(FM') \) for all \( M, M' \). Since terms with equal Böhm trees are equal in \( \mathcal{S} \) (see Barendregt [1976], Hyland [1976]), it follows that \( Ra^\mathcal{S}(F) \) has cardinality 1.

Case 2. \( x \in BT(Fx) \). Then by 1.4 \( \lambda \vdash Fx\vec{F} = x\vec{Q} \).

Since \( [N\vec{Q}]^\mathcal{S} \) can take arbitrary values in \( \mathcal{S} \) when \( N \) ranges over the closed terms, \( Ra^\mathcal{S}(F) \) is infinite.
2.7. Conjecture. \( \mathcal{M}(\mathcal{H}) \) satisfies the range property.

2.8. Question. Does every hard \( \lambda \)-algebra \( \mathcal{M} \) (i.e. \( \mathcal{M} = \mathcal{M}^0 \)) satisfy the range theorem?

§ 3. Duality

3.1. Definition. Let \( f, g \) be two external functions on a \( \lambda \)-algebra \( \mathcal{M} = \langle M, \cdot \rangle \).

\( f, g \) are dual iff \( \forall a, b \in M : f(a) \cdot b = g(b) \cdot a \). Notation \( f \sim \mathcal{M} \ g \), or simply \( f \sim g \).

3.2. Definition. \( \mathcal{M} \) is rich iff all dual functions on \( \mathcal{M} \) are representable in \( \mathcal{M} \).

Remarks. (i) Let \( \mathcal{M} \) be an external function on \( \mathcal{M} \). \( \mathcal{M} \) is locally representable iff for each \( b \in M \) the function \( h \) defined by \( h(a) = f(a) \cdot b \) is representable. Then \( f \) is locally representable iff \( f \) has a dual. A model is rich iff all locally representable functions are representable.

(ii) If \( f \) is representable (by \( f_0 \in M \), say), then \( f \) has a dual \( g \) which is also representable (by \( g_0 = \lambda ab \cdot f_0 ba \)).

(iii) Let \( \mathcal{M} \) be extensional. Then \( f \) has at most one dual. Hence if \( f \sim \mathcal{M} \ g \) and \( f \) is representable, then by (ii) \( g \) is representable.

3.3. Theorem. If \( \mathcal{M} \) is rich, then \( \mathcal{M} \) is extensional.

Proof. Suppose \( \mathcal{M} \) is not extensional. Then there exist \( b, b' \in M \) such that for all \( c \in M \) \( b \cdot c = b' \cdot c \) and \( b \neq b' \).

Define

\[
f(a) = \begin{cases} 
  b' & \text{if } a = b \\
  b & \text{else.}
\end{cases}
\]

and

\[
g = [...K(\lambda y \cdot K(b)y) ...]_{\mathcal{M}},
\]

then for all \( a, a' \in M : f(a) \cdot a' = b \cdot a' = g(a') \cdot a \), hence \( f \sim g \). But \( f \) cannot be representable since it has no fixed point. Thus \( \mathcal{M} \) is not rich.

3.4. Corollary. The following \( \lambda \)-algebras are not rich: \( P\omega; P^0\omega; \mathcal{M}(\lambda); \mathcal{M}^0(\lambda); \mathcal{M}^0(\lambda^0) \).

Proof.

1. \( P\omega \) is not extensional:

Take for example \( a = \{(0, 0)\} \) and \( b = \{(0, 0), (1, 0)\} \).

Then \( \forall c \in P\omega \ a \cdot c = b \cdot c \) but \( a \neq b \).

2. \( P^0\omega \) is not extensional: Let \( 1 = \lambda xy \cdot xy \), then \( P^0\omega \models Ixy = 1xy \), but \( P^0\omega \models I = 1 \) for otherwise \( P\omega \models I = 1 \), so \( P\omega \models Vx \ x = \lambda y \cdot xy \) which implies that \( P\omega \) were extensional.

3. By the Church Rosser property \( \lambda \not\models I = 1 \). So \( \mathcal{M}(\lambda), \mathcal{M}^0(\lambda) \) are not extensional.
4. $\mathcal{M}^0(\lambda\eta)$ is not extensional because the $\lambda$-calculus is $\omega$-incomplete, see Plotkin [1974].

3.5. **Theorem.** $D_\infty$ is rich.

**Proof.** Suppose that $f, g$ are dual i.e.:

$$Va, b \in D_\infty: f(a) \cdot b = g(b) \cdot a.$$ 

We have to show that $f, g$ are representable.

It is sufficient to show that $f, g$ are continuous. Take a directed $X \subseteq D_\infty$.

For all $b \in D_\infty$

$$f(\sqcup X) \cdot b = g(b) \cdot \sqcup X = \sqcup \{g(b) \cdot a | a \in X\} = \sqcup \{f(a) \cdot a | a \in X\} \cdot b$$

by the duality condition and the continuity of application.

Thus by extensionality in $D_\infty$: for all directed $X \subseteq D_\infty$:

$$f(\sqcup X) = \sqcup \{f(a) | a \in X\}$$

i.e. $f$ is continuous. The proof for $g$ is dual. ■

3.6. **Theorem.** $\mathcal{M}(\lambda\eta)$ is rich.

**Proof.** Define

$$M = \lambda\eta N \text{ iff } \lambda\eta \vdash M = N,$$

$$x \in_{\lambda\eta} M \text{ iff for all } M' = \lambda\eta M \text{ one has } x \in FV(M').$$

Let $f, g$ be dual functions on $\mathcal{M}(\lambda\eta)$.

3.6.0. **Lemma.**

(i) $x \in_{\lambda\eta} M \Leftrightarrow FV[\lambda\eta \vdash M \rightarrow N \Rightarrow x \in FV(N)].$

(ii) Let $M' \equiv M[z/y]$ and $\lambda \vdash M' \rightarrow N'$. Then $\mathcal{M}N \lambda \vdash M \rightarrow N$ and $N' \equiv N[z/y]$.

(iii) $x \in_{\lambda\eta} M \Rightarrow x \in_{\lambda\eta} M[z/y]$, for $z \neq x$.

**Proof.**

(i) $\Rightarrow$ Trivial. $\Leftarrow$ Suppose $M = \lambda\eta M'$. By the Church-Rosser theorem $\lambda\eta \vdash M \rightarrow N$, $M' \rightarrow N'$ for some $N$. By assumption $x \in FV(N)$. But then $x \in FV(M')$.

(ii) Induction on the length of proof of $M' \rightarrow N'$. In the case that $M' \equiv (\lambda a \cdot P)Q$, $N' \equiv P[a/Q]$ it may be assumed that $a \neq z, y$. Therefore one can apply the well-known substitution lemma

$$A[u/B][v/C] = A[v/C][u/B[v/C]] \text{ if } u \neq v \text{ and } u \notin FV(C).$$

(iii) Suppose $\lambda\eta \vdash P[z/y] \rightarrow R'$. By (ii) for some $R \lambda\eta \vdash P \rightarrow R$ and $R' \equiv R[z/y]$. By assumption and (i), $x \in FV(R)$. Since $x \neq z$ also $x \in FV(R')$. Therefore by (i) $x \in_{\lambda\eta} P[z/y]$. ■3.6.0
3.6.1. **Lemma.** (i) If \( x \in \lambda y \cdot P \) then \( x \in \lambda y P \) and \( x \neq y \).
(ii) If \( x \neq y \), then \( x \in \lambda y M \iff x \in \lambda y My \).

**Proof.** (i) Since \( x \in FV(\lambda y \cdot P) \) clearly \( x \neq y \). Suppose \( P = \lambda y \cdot N \), then \( \lambda y \cdot P = \lambda y \cdot N \). By assumption \( x \in FV(\lambda y \cdot N) \subset FV(N) \). Thus \( x \in \lambda y P \).
(ii) \( \Rightarrow \) Suppose \( \lambda y \vdash My \rightarrow N \) in order to prove \( x \in FV(N) \).
Case 1. \( N = M' \cdot y \) with \( \lambda y \vdash M \rightarrow M' \). Since \( x \in \lambda y M \), also \( x \in FV(M') \subset C FV(N) \).
Case 2. \( M \rightarrow \lambda z \cdot M_1 \) and \( \lambda y \vdash My \rightarrow (\lambda z \cdot M_1) y \rightarrow M_1[z/y] \rightarrow N \).
Since \( x \in \lambda y M \), also \( x \in \lambda y \lambda z \cdot M_1 \) and by (i) \( x \in \lambda y M_1 \) and \( z \neq x \), so by 3.6.0. (iii) \( x \in \lambda y M_1[z/y] \). Therefore \( x \in FV(N) \).

3.6.2. **Lemma.** If \( y \neq x \) \( x \in \lambda y f(y) \), then \( V y \neq x \) \( x \in \lambda y g(y) \) (and hence \( V y \neq x \) \( x \in \lambda y f(y) \)).

**Proof.** Suppose \( x \in \lambda y f(y) \), \( y \neq x \). Let \( y' \neq x \). Then by 3.6.1. (ii) \( x \in \lambda y f(y) \cdot y' = \lambda y g(y') \cdot y \). Hence, by 3.6.1. (ii), \( x \in \lambda y g(y') \). (The rest follows by applying the statement to \( x \in \lambda y g(y) \)). \( \Box \)

3.6.3. **Main Lemma.** There is a variable \( x \) such that for all terms \( M : f(x)[x/M] = f(M) \).

**Proof.** Let \( v \) be any variable. Choose \( x \neq v \) such that \( x \neq \lambda h \cdot f(v) \). Then \( x \neq \lambda h \cdot g(z) \) for all \( z \neq v \), by the dual of 3.6.2.

Given \( M \), one can find a \( y \) such that \( y \neq \lambda h \cdot M, f(M), x, f(x) \). Hence \( x \neq \lambda h \cdot g(y) \). Now since \( y \neq x \) and \( x \neq \lambda h \cdot g(y) \), \( (f(x)[x/M])[y] = (f(x) \cdot y)[x/M] = (g(y) \cdot x)[x/M] = g(y) \cdot M = f(M) \cdot y \).

Since \( y \neq f(x) \), \( M, f(M) \), extensionality yields \( f(x)[x/M] = f(M) \). \( \Box \)

Now it follows by 3.6.3. that \( f \) can be represented by the term \( \lambda x \cdot f(x) \) and similarly for \( g \). \( \Box \)

The following construction is needed for the proof of 3.10.

3.7. **Definition.** Let \( \neq \) be a Gödel numbering of terms. \( \langle M \rangle \) is the numeral \( \neq M \). A sequence of terms \( M_n \) is recursive if \( \lambda n \cdot \neq M_n \) is a recursive function.

3.8. **Lemma.** (Coding of infinite sequences). Let \( \{M_n\} \) be a recursive sequence of terms such that \( FV(M_n) \subset \{x\} \) for all \( n \). Then there exists a term \( X \) such that \( p^i X = M_i \), for all \( i \), where \( p \) is some fixed closed term. Par abus de langage we write \( \langle M_n \rangle \) \( \in \omega \) for \( X \).

**Proof.**
(1) As in Curry et al. [1972], 13 B3 there is a term \( E \) which enumerates all terms with \( x \) as only free variable:

\[
E('M^n') = M, \text{ for } M \text{ with } FV(M) = \{x\}.
\]
(2) Let \([M, N]\) be a pairing of terms defined by \(\lambda z \cdot z M N\). Then 
\([M, N]K = M\) and \([M, N](KI) = N\). Define ordered tuples as follows:

\([M] = M, [M_1, ..., M_{n+1}] = [M_1, [M_2, ..., M_{n+1}]]\).

(3) Let \(M_n\) with \(FV(M_n) \subseteq \{x\}\) be a recursive sequence of terms. 
We want to code the sequence \([M_n]\) as a \(\lambda\)-term. Let \(S^+\) be such that 
\(S^+ n \xrightarrow{\beta} n + 1\) and let \(b = \lambda y \cdot [E(Fy), (x(S^+)y)]\), where \(F\ \lambda\)-defines \(f\), and 
\(B = FP\ b\) (i.e. the fixed point of \(b\)). Then

\[B_n \xrightarrow{\beta} bB_n \xrightarrow{\beta} [E(Fn), B_{n+1}] \xrightarrow{\beta} [M_n, B_{n+1}]\]

So \(B_0 = [M_0, B_1] = [M_0, M_1, B_2] = \ldots\). Hence by setting \(\langle M_n \rangle_{n \in \omega} = B_0\) we have a coding for infinite sequences of terms with one fixed free variable.

(4) It is easy to construct a term \(p\) such that \(pm \langle M \rangle_n = \langle M_n \rangle\) (take e.g. \(p(x) = \text{if zero } x \text{ then } aK\) else \(p(x - 1)(a(KI))\), using the fixed point theorem).

3.9. Lemma. For all closed \(Z\) there is an \(n\) such that \(Z\Omega^n = Z\Omega\). 
\(Z\Omega^n\) is short for \(Z\Omega\ldots\Omega\) 
\(n\) times

**Proof.**

Case 1. \(Z\) is unsolvable; then \(Z = \not\Omega\), so \(n = 0\).

Case 2. \(Z\) is solvable; then \(Z\) has a \(\lambda n\), \(Z = \lambda x \cdot x_1 A_1 \ldots A_m\ \langle x_i \in \Omega\rangle\).

Take \(n = i\), so \(Z\Omega^i = \lambda x^i \cdot \Omega A_1 \ldots A_m = \not\Omega\).

3.10. Theorem. If \(\mathcal{M}\) is hard and sensible, then \(\mathcal{M}\) is not rich.

**Proof.** If \(\mathcal{M}\) is hard, then \(\mathcal{M}\) is isomorphic to \(\mathcal{M}^0(T)\), where \(T = Th(\mathcal{M})\). We reason in \(\mathcal{M}^0(T)\). Since \(\mathcal{M}\) is sensible, \(\mathcal{M} \subseteq T\).

Let \(h: \omega \to \omega\) be a function not definable in \(\mathcal{M}\). Such an \(h\) exists since a hard model is countable.

Let \(A_n(x, y)\) be the term \(x\Omega^n(y\Omega^n(hn))\), \(n \in \omega\). For closed \(M\) the sequence 
\(A_0(M, y), A_1(M, y), \ldots\) is by 3.9 
\(M(y(hn)), M\Omega(y\Omega(hn)), \ldots, M\Omega^n(y\Omega^n(hn))\), \(\Omega, \Omega, \ldots\), 
where \(n\) is such that \(M\Omega^{n+1} = \Omega\). Thus \(\lambda n \cdot A_n(M, y)\) is up to convertibility 
a recursive sequence containing one fixed free variable and hence representable as a term. Define \(f(M) = \lambda y \cdot \langle A_n(M, y)\rangle_{n \in \omega}\). Similarly for closed \(N\) \(\lambda n \cdot A_n(x, N)\) is recursive and it is possible to define \(g(N) = \lambda x \cdot \langle A_n(x, N)\rangle_{n \in \omega}\).

Then for all closed \(M, N\): 
\(f(M)\) and \(g(N)\) are well defined and \(f(M) \cdot N = g(N) \cdot M = \langle A_n(M, N)\rangle_{n \in \omega}\) by construction. So \(f\) and \(g\) are dual.

Suppose now that \(\mathcal{M}\) is rich, i.e. \(f\) were representable by some closed \(F\). Then for all closed \(M, N\): 
\(FMN = f(M)N = \langle A_n(M, N)\rangle_{n \in \omega}\).

But then \(p(M)(F(K^I)(K^I)) = p(M)(\langle h^I\rangle)_{n \in \omega} = h^I\), hence \(h\) were definable, contradiction. Thus \(\mathcal{M}\) is not rich.
3.11. **Corollary.** $D^\omega_\infty$ and $\mathcal{M}_\eta(T)$ for $T \supseteq \mathcal{H}$ are not rich.

3.12. **Questions.** (i) Is every extensional term model $\mathcal{M}(T)$ rich?
(ii) Is $\mathcal{M}_\eta(\lambda\omega)$ rich?

Here $\lambda\omega$ is the $\lambda$-theory obtained by adding the $\omega$-rule to the theory, see Barendregt [1974].

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