SUMMARY

§ 1 is concerned with the term model of tho λ-calculus. It is proved that Church's δ is not definable and that the definable functions into the numerals are constant. In § 2 it is proved that for several λ-algebras the range of a representable function is either a singleton or infinite. In § 3 it is examined in which λ-algebras the local representability of external functions implies the global representability.

INTRODUCTION

Let $\mathcal{M} = \langle M, \cdot \rangle$ be a λ-algebra (i.e. a model of the λ-calculus). Elements of $M$ are thought of as functions. Arbitrary $f: M \to M$ are called external functions. Such a function is representable (by an element $a \in M$) if $\forall b \in M f(b) = a \cdot b$. A function $f$ is definable in $\mathcal{M}$ if $f$ is representable by $\lfloor F \rfloor_{\mathcal{M}}$ for some closed term $F$. Here $\lfloor F \rfloor_{\mathcal{M}}$ denotes the value of $F$ in the model $\mathcal{M}$.

Other notations:

$x, y, \ldots$ denote variables of the λ-calculus.

$a, b, \ldots$ denote variables ranging over the elements of a λ-algebra.

$F, G, \ldots$ denote λ-terms.

The numerals $0, 1, 2, \ldots$ denote some adequate representation of the natural numbers as λ-terms e.g. those of Church: $n = \lambda x. f^n(x)$.

If $\mathcal{M} = \langle M, \cdot \rangle$ is a λ-algebra, then $\mathcal{M}^0$ is the sub-λ-algebra $\langle M^0, \cdot \rangle$ where $M^0 = \{ \lfloor F \rfloor_{\mathcal{M}} \in M \mid F$ closed term $\}$. If $T$ is a consistent extension of the λ-calculus, $\mathcal{M}(T)$ is the term-model of $T$, i.e. the set of all λ-terms modulo provable equality in $T$. The closed term-model of $T$, notation $\mathcal{M}^0(T)$, is defined as $(\mathcal{M}(T))^0$. A λ-algebra $\mathcal{M}$ is hard if $\mathcal{M} = \mathcal{M}^0$. In such an $\mathcal{M}$ a function is representable iff it is definable.

For other terminology see Barendregt [1976].

The three sections of the paper treat different aspects of the notion of representability.

In § 1 attention is restricted to the standard extensional term model $\mathcal{M} = \mathcal{M}(\lambda \eta)$. 25 Indagationes
Church's \( \delta \) is an external function satisfying
\[
\delta MM = 0 \quad \text{if } M \text{ is a closed normal form (nf)}
\]
\[
\delta MM' = 1 \quad \text{if } M, M' \text{ are different closed nf's.}
\]

In Böhm [1972] it is proved that \( FN_1 \ldots N_n \) different \( \beta \eta \)-nf's \( \mathcal{A}F \models FN_i = i \). As a consequence it follows that for every finite set \( A \) of nf's there is a term \( \delta \) satisfying (★) for \( M, M' \in A \).

At the Orléans logic conference (1972) the question was raised whether the general Church's \( \delta \) is definable as a \( \lambda \)-term.

We will give a negative answer which was already established in Barendregt [1972] and independently in Wadsworth [1972] (see also Hindley and Mitschke [1975]). All three proofs of the non-existence of \( \delta \) are different.

Furthermore it is proved that the only definable functions from the terms into the numerals are the constant functions.

In § 2 it will be proved that definable functions in various \( \lambda \)-algebras have a range of cardinality 1 or \( \aleph_0 \). For representable functions this is not true in \( D_\infty \) and \( P_0 \).

Two external functions \( f \) and \( g \) on \( \mathcal{M} \) are dual, notation \( f \sim \mathcal{M} g \), if \( f(a) \cdot b = g(b) \cdot a \) for all \( a, b \in \mathcal{M} \). In that case for each \( b \) the map \( \lambda a. f(a) \cdot b \) is representable and \( f \) is said to be locally representable, similarly for \( g \).

A model \( \mathcal{M} \) is rich if for all \( f, g \):
\[
f \sim \mathcal{M} g \Rightarrow f \text{ and } g \text{ are representable in } \mathcal{M}.
\]

The results of § 3 are: \( D_\infty \) and \( \mathcal{M}(\lambda \eta) \) are rich; rich models are extensional; hard sensible models (e.g. the interior of \( D_\infty \)) are not rich.

We would like to draw the graph of 3.6 to the reader's attention. There variables of the \( \lambda \)-calculus are not just used in the usual way, but also serve as separate entities.

§ 1. NON-DEFINABILITY RESULTS

The main tool in this section is the "Böhm out" technique 1.4. This result is also of use in § 2.

1.1. Definition. Let \( BT(M) \) be the Böhm tree of \( M \), see Barendregt [1976], § 6. \( x \in BT(M) \) iff \( x \in FV(M^k) \) for some \( k \), where \( M^k \) is the \( k \)th approximate normal form of \( M \).

1.2. Definition. (i) A selector is a term of the form
\[
U \equiv \lambda x_1 \ldots x_n \cdot x_i, \quad 1 < i < n.
\]

A permutator is a term of the form
\[
C \equiv \lambda x_1 \ldots x_n \cdot x_{\pi(1)} \ldots x_{\pi(n)}
\]
for some permutation \( \pi \).

(ii) Simple terms are inductively defined by: Any variable, selector or permutator is a simple term. If \( P, Q \) are simple terms, so is \( PQ \).
1.3. Lemma. Simple terms have a normal form (nf).

Proof. Realize that each simple term is of the form $x\overrightarrow{P}$, $U\overrightarrow{P}$, $C\overrightarrow{P}$ with $\overrightarrow{P}$ simple, $U$ a selector and $C$ a permutator. Then it can be shown by induction on the term length that they have a nf. ■

1.4. Theorem. Let $FV(M) = \{x\}$ and $x \in BT(M)$. Then

(i) For some $\overrightarrow{P}, \overrightarrow{Q}$, with $x \notin FV(\overrightarrow{P})$, $\lambda \vdash \overrightarrow{MP} = x\overrightarrow{Q}$ ("$x$ is Böhm-ed out").
(ii) Moreover $\overrightarrow{P}$ can be chosen as a sequence of simple terms.

Proof. Let $x$ occur in $BT(M)$ at depth $k > 0$. By a similar construction as in Barendregt [1976] 6.14, 6.15, for some Böhm-transformation $\pi_1$, $x$ occurs in $BT(M_{\pi_1})$ at depth $k - 1$. Iterating this leads to $M^{\pi_2} = \lambda y \cdot x\overrightarrow{Q}$, hence $M^{\pi_2}y = x\overrightarrow{Q}$, for a Böhm transformation $\pi_2$.

Checking the details of the construction of $\pi_2$ one verifies that

$$M^{\pi_2}y \equiv M ... x_i ... [x_j/Cx_j] ... [x_k/Ux_k] ... y \equiv M\overrightarrow{P}$$

for some simple terms $\overrightarrow{P}$ with $x \notin FV(\overrightarrow{P})$ (where $C$ is a permutator and $U$ a selector). ■

1.5. Lemma. Let $F$ be a closed $\lambda$-term such that $F$ is not constant, i.e. $\lambda \not\vdash FX_1 = FX_2$ for some $X_1, X_2$, and suppose that for some closed $\lambda$-term $M$, $FM$ has a nf. Then $x \in BT(Fx)$ for all $x$.

Proof. Note that if $P, P'$ have equal finite $\Omega$-free Böhm-trees, then $\lambda \vdash P = P'$. Now suppose $x \notin BT(Fx)$ for some $x$. Then for all $k$, $x \notin FV((Fx)_k)$ ($N_k$ is the $k$-th approximate normal form of $N$, cf. Barendregt [1976] 7.4 (iv)). Hence $(FM)^k = (Fx)^k [x/M] = (Fx)^k$ for all $k$, and it follows that $BT(FM) = BT(Fx)$. But since $FM$ has a nf, $BT(FM)$ is finite and $\Omega$-free and therefore $\lambda \vdash FM = Fx$. Since $F, M$ are closed it follows that for all $\lambda$-terms $N, \lambda \vdash FN = FM$, i.e. $F$ is constant, a contradiction. ■

Remark. 1.5 also holds for $F, M$ not necessarily closed.

1.6. Definition. $\varnothing = I$, $n + 1 = K n$.

1.7. Lemma. The function $sg$ is not $\lambda$-definable with respect to $\{n | n \in \omega\}$, i.e. for no closed $\lambda$-term $F \vdash \varnothing = 0, \vdash F n + 1 = 1$.

Proof. Suppose $F$ exists. Then by 1.5 $x \in BT(Fx)$. Hence by 1.4 $Fx\overrightarrow{P} = x\overrightarrow{Q}$ for some $\overrightarrow{P}, \overrightarrow{Q} = Q_1 ... Q_m$. But then for all $n > m$,

$$\vdash I\overrightarrow{P} = F n \overrightarrow{P} = n Q_1 ... Q_m = n - m$$

contradicting the Church-Rosser theorem since the $k$ are different nf's. ■
1.8. Definition. A system of terms \( \{ M_n | n \in \omega \} \) is an adequate system of numerals iff
(i) Each \( M_n \) has a \( \text{nf} \).
(ii) Each recursive function can be \( \lambda \)-defined with respect to the \( M_n \).

In Barendregt [1977] is shown that the second condition can be replaced by (ii'): The successor, predecessor and \( sg \) functions can be \( \lambda \)-defined with respect to the \( M_n \).

The following corollary was proved independently by Barendregt [1972] and Wadsworth [1972].

1.9. Corollary.
(i) \( \{ n | n \in \omega \} \) is not an adequate system of numerals. (ii) Church's \( \delta \) is not \( \lambda \)-definable.

Proof. (i) Immediate. (ii) If \( \delta \) were \( \lambda \)-definable, then so would be \( sg \), viz. by \( \lambda x \cdot \delta x = 0 \oplus 1 \).

Remark. (i) Although not definable, \( \delta \) can consistently be added to the \( \lambda \)-calculus, see Church [1941].
(ii) Contrary to this, the corresponding \( \delta \) for open \( \lambda \)-terms would be inconsistent at once. For let \( x \neq y \), then
\[
(\lambda y \cdot \delta xy(KK)S)x = (\lambda y \cdot \underline{I}(KK)S)x = (\lambda y \cdot KKS)x = KKS = K
\]
but also
\[
(\lambda y \cdot \delta xy(KK)S)x = \delta xx(KK)S = \underline{0}(KK)S = S.
\]
(iii) One could also consider the definability of a \( \delta \) for all closed terms, i.e.: \( \delta MM = \underline{0} \) for \( M \) closed
\( \delta MN = \underline{I} \) for \( M, N \) closed such that \( \not\vdash M = N \).

But then the following version of the Russell paradox would result. Define \( \rightarrow X = \delta X \underline{I} \). If \( \not\vdash \underline{0} = \underline{I} \) then \( \not\vdash X = \underline{I} \iff \not\vdash \rightarrow X = \underline{I} \).

Now let \( A = FP \rightarrow \) (i.e. the fixed point of \( \rightarrow : \vdash A = \rightarrow A \)).
Then \( \not\vdash A = \underline{I} \iff \vdash A = \underline{I} \). Thus \( \not\vdash \underline{0} = \underline{I} \).

To see the relation with the Russell paradox, note that \( A = BB \) with \( B = \lambda x. \rightarrow (xx) \). (In illative combinatory logic \( MN \) is interpreted as \( N \in M \) and \( \lambda x \cdot P \) as \( \{ x | P \} \).)

1.10. Theorem. Let \( \omega = \{ n | n \in \omega \} \) be an adequate system of numerals and let \( f \) be a map into \( \omega \) definable by \( F \). Then \( f \) is constant.

Proof. First assume \( \omega \) is Church's system of numerals.
Suppose \( f \) is not constant, then by 1.5 \( x \in BT(Fx) \). Hence for some simple \( \tilde{P} \) and \( \tilde{Q} \), \( \lambda \vdash Fx \tilde{P} = x \tilde{Q} \).

Hence \( \lambda \vdash FM \tilde{P} = M \tilde{Q} \) for all \( M \). But \( M \tilde{Q} \) can take arbitrary values and not \( FM \tilde{P} \), since \( n \tilde{P} = P_1^n(P_2)P_3 \ldots P_k \) always has a \( \text{nf} \) by 1.3.
Now let \( \omega \) be an arbitrary system of numerals. It is well-known how to define a term \( G \) such that \( Gn = n \).

Suppose a non-constant \( f : \text{terms} \to \omega \) would be definable, then \( G \circ f \) were a definable non-constant mapping into \( \omega \).

First alternative proof (due to the referee).
Suppose \( F \) is not constant, i.e. let \( n_1 \neq n_2 \in Ra(F) \). Define \( G \) as the \( \lambda \)-defining term of the recursive function

\[
g(x) = \begin{cases} 
0 & \text{if } x = n_1, \\
1 & \text{else}.
\end{cases}
\]

Then the range of \( G \circ F \) is \( \{0, 1\} \) contrary to 2.3.

Second alternative proof. By Barendregt's lemma in de Boer [1975] it follows that if \( \Omega \) is unsolvable and \( N \) a \( nf \), then \( F\Omega = N \Rightarrow Fx = N \) for all \( x \). (General genericity lemma; see also Barendregt [1977a] for a proof.) Now if the values of \( F \) are numerals it follows that \( F\Omega \) has a \( nf \), i.e. \( F \) is constant.

1.11. Corollary. There is no \( F \) such that

\[
FM = 0 \quad \text{if } M \text{ is a numeral (i.e. } \vdash M = n \text{ for some } n)
\]

\[ I \quad \text{else}
\]

for any adequate system.

1.12. Question. Is there a term \( F \) such that

\[
FM \text{ has a } nf \text{ (is solvable) if } M \text{ is a numeral has no } nf \text{ (is unsolvable) else.}
\]

§ 2. The Range Property

2.1. Definition. Let \( M = \langle M, \cdot \rangle \) be a \( \lambda \)-algebra. For each \( f \in M \), we define \( Ra^M(f) \), the range of \( f \) in \( M \), as follows:

\[
Ra^M(f) = \{ f \cdot x | x \in M \}.
\]

Notation. \( Ra^M(F) = Ra^M([F]^M) \) for terms \( F \).

When possible, the superscript \( M \) will be dropped in \( Ra^M \).

2.2. Definition. A \( \lambda \)-algebra \( M \) satisfies the range property if for all \( f \in M \), the cardinality of \( Ra^M(f) \) is 1 or \( \aleph_0 \).

2.3. Range theorem. (Barendregt; Myhill). Let \( T \) be a r.e. \( \lambda \)-theory. Then \( M(T) \) (and also \( M^0(T) \)) has the range property.

Proof. Suppose \( f \in M \) and \( Ra(f) = \{ m_0, ..., m_k \}, \ k > 0 \). Define

\[
N_t = \{ x | f \cdot x = m_t \} \subseteq M.
\]
Every such \( N_t \) is r.e. Therefore \( N = \bigcup_1^k N_t \), the complement of \( N_0 \) is also r.e. Hence \( N_0 \) is recursive.

On the other hand \( N_0 \) is non-trivial and closed under equality, which contradicts Scott's theorem (Barendregt [1976] 2.21).

The proof for \( M^0(T) \) is the same.

2.4. COROLLARY. \( M(\lambda), M^0(\lambda), M(\lambda \eta) \) and \( M^0(\lambda \eta) \) have the range property.

The range property, however, is not satisfied in every \( \lambda \)-algebra.

2.5. THEOREM. \( P_\omega \) and \( D_\infty \) do not satisfy the range property.

PROOF. Since the proof is similar in both cases, let \( S = (S, \prec) \) denote either \((P_\infty, \subseteq) \) or \((D_\infty, \subseteq) \). We define the following function \( f : S \to S \) by 
\[
f(x) = T \quad \text{if} \quad x \neq _T \quad \text{else} \quad _T \quad (T \quad \text{and} \quad _T \quad \text{are the largest respectively smallest element of} \quad S.)
\]
Claim: \( f \) is continuous. Then by Scott [1972], [1975] \( f \) is representable and since \( f \) has range of cardinality two we are done.

For open \( O \) in \( S \) one has: \( x \in O \) and \( x \prec y \Rightarrow y \in O \).


Hence for open \( O, \_ \in O \Rightarrow O = S, \) and \( O \neq 0 \Rightarrow T \in O \).

Now for every open set \( O, f^{-1}(O) \) is open:

Case 1. \( \_ \in O \). Then \( O \supset S \) so \( f^{-1}(S) = S \) which is open.

Case 2. \( \_ \notin O \). If \( O = \emptyset \), then we are done. Else \( T \in O \) and hence \( f^{-1}(O) = S - \{\_\} = \{x | x \nless _T\} \defeq U_\perp . \)

\( U_\perp \) is open in \( D_\infty \), see e.g. Barendregt [1976] 1.2.

\( U_\perp \) is open in \( P_\omega \): Let \( O_k = \{x | e_k \subseteq x\} \). Note \( e_0 = 0 = _T \) and that the \( O_k \) form a base for the topology on \( P_\omega \).

Now:
\[
x \in U_\perp \Rightarrow x \notin O \Rightarrow \exists k \quad e_k \subseteq x \Rightarrow x \in \bigcup_{k=0}^\infty O_k
\]
which is, as a union of elements of a base, indeed open.

The following theorem was announced in Wadsworth [1973] for the \( D_\infty \) case.

2.6. THEOREM. Let \( \mathcal{S} \) be \( D_\infty^0 \) or \( P_\omega^0 \). Then \( \mathcal{S} \) satisfies the range property.

PROOF. Let \( F \) be a closed term. Consider \( BT(Fx) \).

Case 1. \( x \notin BT(Fx) \). Then \( BT(FM) = BT(FM') \) for all \( M, M' \). Since terms with equal Böhm trees are equal in \( \mathcal{S} \) (see Barendregt [1976], Hyland [1976]), it follows that \( Ra_{\mathcal{S}}(F) \) has cardinality 1.

Case 2. \( x \in BT(Fx) \). Then by 1.4 \( \lambda \vdash Fx F = x Q \).

Since \( [NQ]_{\mathcal{S}} \) can take arbitrary values in \( \mathcal{S} \) when \( N \) ranges over the closed terms, \( Ra_{\mathcal{S}}(F) \) is infinite.
2.7. CONJECTURE. \( \mathcal{M}(\mathcal{H}) \) satisfies the range property.

2.8. QUESTION. Does every hard \( \lambda \)-algebra \( \mathcal{M} \) (i.e. \( \mathcal{M} = \mathcal{M}^0 \)) satisfy the range theorem?

§ 3. DUALITY

3.1. DEFINITION. Let \( f, g \) be two external functions on a \( \lambda \)-algebra \( \mathcal{M} = \langle M, \cdot \rangle \).

\( f, g \) are dual iff \( \forall a, b \in M : f(a) \cdot b = g(b) \cdot a. \) Notation \( f \sim_{\mathcal{M}} g, \) or simply \( f \sim g. \)

3.2. DEFINITION. \( \mathcal{M} \) is rich iff all dual functions on \( \mathcal{M} \) are representable in \( \mathcal{M}. \)

REMARKS. (i) Let \( f \) be an external function on \( \mathcal{M}. \) \( f \) is locally representable iff for each \( b \in M \) the function \( h \) defined by \( h(a) = f(a) \cdot b \) is representable. Then \( f \) is locally representable iff \( f \) has a dual. A model is rich iff all locally representable functions are representable.

(ii) If \( f \) is representable (by \( f_0 \in M, \) say), then \( f \) has a dual \( g \) which is also representable (by \( g_0 = \lambda a b \cdot f_0 b a). \)

(iii) Let \( \mathcal{M} \) be extensional. Then \( f \) has at most one dual. Hence if \( f \sim_{\mathcal{M}} g \) and \( f \) is representable, then by (ii) \( g \) is representable.

3.3. THEOREM. If \( \mathcal{M} \) is rich, then \( \mathcal{M} \) is extensional.

PROOF. Suppose \( \mathcal{M} \) is not extensional. Then there exist \( b, b' \in M \) such that for all \( c \in M \) \( b \cdot c = b' \cdot c \) and \( b \neq b'. \)

Define

\[
 f(a) = \begin{cases} 
 b' \text{ if } a = b \\
 b \text{ else.} 
\end{cases}
\]

and

\[
 g = [\lambda y \cdot K(by)]^\mathcal{M},
\]

then for all \( a, a' \in M : f(a) \cdot a' = b \cdot a' = g(a') \cdot a, \) hence \( f \sim g. \) But \( f \) cannot be representable since it has no fixed point. Thus \( \mathcal{M} \) is not rich. \( \blacksquare \)

3.4. COROLLARY. The following \( \lambda \)-algebras are not rich: \( P\omega \); \( P^0\omega \); \( \mathcal{M}(\lambda) \); \( \mathcal{M}^0(\lambda) \); \( \mathcal{M}^{00}(\lambda). \)

PROOF.

1. \( P\omega \) is not extensional:

Take for example \( a = \{(0, 0)\} \) and \( b = \{(0, 0), (1, 0)\}. \)

Then \( \forall c \in P\omega \ a \cdot c = b \cdot c \) but \( a \neq b. \)

2. \( P^0\omega \) is not extensional: Let \( 1 = \lambda x y \cdot x y, \) then \( P^0\omega \models I x y = 1 x y, \) but \( P^0\omega \models \neg I = 1 \) for otherwise \( P\omega \models I = 1, \) so \( P\omega \models \forall x \ x = \lambda y \cdot x y \) which implies that \( P\omega \) were extensional.

3. By the Church Rosser property \( \lambda \models \neg I = 1. \) So \( \mathcal{M}(\lambda), \mathcal{M}^0(\lambda) \) are not extensional.
4. $\mathcal{M}^0(\lambda\eta)$ is not extensional because the $\lambda$-calculus is $\omega$-incomplete, see Plotkin [1974].

3.5. Theorem. $D_\infty$ is rich.

Proof. Suppose that $f, g$ are dual i.e.:

$$Va, b \in D_\infty: f(a) \cdot b = g(b) \cdot a.$$ 

We have to show that $f, g$ are representable.

It is sufficient to show that $f, g$ are continuous. Take a directed $X \subset D_\infty$.

For all $b \in D_\infty$

$$f(\bigsqcup X) \cdot b = g(b) \cdot \bigsqcup X = \bigsqcup \{g(b) \cdot a | a \in X\} = \bigsqcup \{f(a) \cdot a | a \in X\} \cdot b$$

by the duality condition and the continuity of application.

Thus by extensionality in $D_\infty$: for all directed $X f(\bigsqcup X) = \bigsqcup \{f(a) | a \in X\}$

i.e. $f$ is continuous. The proof for $g$ is dual. ■

3.6. Theorem. $\mathcal{M}(\lambda\eta)$ is rich.

Proof. Define

$$M = \lambda\eta N \text{ iff } \lambda\eta \vdash M = N,$$

$$x \in_{\lambda\eta} M \text{ iff for all } M' = \lambda\eta M \text{ one has } x \in FV(M').$$

Let $f, g$ be dual functions on $\mathcal{M}(\lambda\eta)$.

3.6.0. Lemma. (i) $x \in_{\lambda\eta} M \iff V N[\lambda\eta \ldots M = N, x \in FV(N)]$.

(ii) Let $M' \equiv M[z/y]$ and $\lambda \vdash M' \rightarrow N'$. Then $\forall N \lambda \vdash M \rightarrow N$ and $N' \equiv N[z/y]$.

(iii) $x \in_{\lambda\eta} M \rightarrow x \in_{\lambda\eta} M[z/y]$, for $z \neq x$.

Proof. (i) $\Rightarrow$ Trivial. $\Leftarrow$ Suppose $M = \lambda\eta M'$. By the Church-Rosser theorem $\lambda\eta \vdash M \rightarrow N, M' \rightarrow N'$ for some $N$. By assumption $x \in FV(N)$. But then $x \in FV(M')$.

(ii) Induction on the length of proof of $M' \rightarrow N'$. In the case that $M' \equiv (\lambda a \cdot P)Q$, $N' \equiv P[a/Q]$ it may be assumed that $a \neq z, y$. Therefore one can apply the well-known substitution lemma

$$A[u/B][v/C] = A[v/C][u/B[v/C]] \text{ if } u \neq v \text{ and } u \notin FV(C).$$

(iii) Suppose $\lambda\eta \vdash P[z/y] \rightarrow R'$. By (ii) for some $R \lambda\eta \vdash P \rightarrow R$ and $R' \equiv R[z/y]$. By assumption and (i), $x \in FV(R)$. Since $x \neq z$ also $x \in FV(R')$. Therefore by (i) $x \in_{\lambda\eta} P[z/y]$. ■
3.6.1. **Lemma.** (i) If \( x \in \lambda y \cdot P \) then \( x \in \lambda y \cdot P \) and \( x \neq y \).
(ii) If \( x \neq y \), then \( x \in \lambda y \cdot M \iff x \in \lambda y \cdot My \).

**Proof.** (i) Since \( x \in FV(\lambda y \cdot P) \) clearly \( x \neq y \). Suppose \( P = \lambda y \cdot N \), then
\[
\lambda y \cdot P = \lambda y \cdot \lambda y \cdot N.
\]
By assumption \( x \in FV(\lambda y \cdot N) \subset FV(N) \). Thus \( x \in \lambda y \cdot P \).
(ii) Suppose \( \lambda y \mid \lambda y \cdot N \rightarrow N \) in order to prove \( x \in FV(N) \).

Case 1. \( N = \lambda y \cdot \lambda y \cdot M \) with \( \lambda y \mid \lambda y \cdot M \rightarrow \lambda y \cdot M \).
Since \( x \in \lambda y \cdot M \), also \( x \in FV(\lambda y \cdot M) \subset FV(N) \).

Case 2. \( \lambda y \mid \lambda y \cdot \lambda z \cdot M \) and \( \lambda y \mid \lambda y \cdot \lambda z \cdot M \rightarrow \lambda y \cdot (\lambda z \cdot M) \).
Therefore \( x \in FV(N) \). □

3.6.2. **Lemma.** If \( \lambda y \neq x \), then \( x \in \lambda y \cdot f(y) \) (and hence \( \lambda y \neq x \), \( x \in \lambda y \cdot g(y) \)).

**Proof.** Suppose \( x \in \lambda y \cdot f(y) \), \( y \neq x \). Let \( y' \neq x \). Then by 3.6.1. (ii) \( x \in \lambda y \cdot f(y \cdot y' = \lambda y \cdot g(y') \cdot y \). Hence, by 3.6.1. (ii), \( x \in \lambda y \cdot g(y') \). (The rest follows by applying the statement to \( x \in \lambda y \cdot g(y') \)). □3.6.2

3.6.3. **Main Lemma.** There is a variable \( x \) such that for all terms \( M : f(x)[x/M] = f(M) \).

**Proof.** Let \( v \) be any variable. Choose \( x \neq v \) such that \( x \neq \lambda y \cdot f(v) \). Then \( x \neq \lambda y \cdot g(z) \) for all \( z \neq x \), by the dual of 3.6.2.

Given \( M \), one can find a \( y \) such that \( y \neq \lambda y \cdot M \), \( f(M), x, f(x) \). Hence \( x \neq \lambda y \cdot g(y) \). Now since \( y \neq x \) and \( x \neq \lambda y \cdot g(y), (f(x)[x/M]) \cdot y = (f(x) \cdot y)[x/M] = (g(y) \cdot x)[x/M] = g(y) \cdot M = f(M) \cdot y \).

Since \( y \neq f(x), M, f(M), \) extensionality yields \( f(x)[x/M] = f(M) \). □3.6.3

Now it follows by 3.6.3. that \( f \) can be represented by the term \( \lambda x \cdot f(x) \) and similary for \( g \). □3.6

The following construction is needed for the proof of 3.10.

3.7. **Definition.** Let \( \neq \) be a Gödel numbering of terms. \( [M] \) is the numeral \( \neq M \). A sequence of terms \( M_n \) is recursive if \( \lambda n \cdot \neq M_n \) is a recursive function.

3.8. **Lemma.** (Coding of infinite sequences). Let \( \{M_n\} \) be a recursive sequence of terms such that \( FV(M_n) \subset \{x\} \) for all \( n \). Then there exists a term \( X \) such that \( p_i X = M_i \), for all \( i \), where \( p \) is some fixed closed term. Par abus de langage we write \( \langle M_n \rangle_{n \in \omega} \) for \( X \).

**Proof.** (1) As in Curry et al. [1972], 13 B3 there is a term \( E \) which enumerates all terms with \( x \) as only free variable:
\[
E(\langle M \rangle) = M, \text{ for } M \text{ with } FV(M) = \{x\}.
\]
(2) Let \([M, N]\) be a pairing of terms defined by \(\lambda z \cdot z M N\). Then \([M, N]K = M\) and \([M, N](KI) = N\). Define ordered tuples as follows:

\[
[M] = M, \quad [M_1, \ldots, M_{n+1}] = [M_1, [M_2, \ldots, M_{n+1}]].
\]

(3) Let \(M_n\) with \(FV(M_n) \subseteq \{x\}\) be a recursive sequence of terms. We want to code the sequence \([M_n]\) as a \(\lambda\)-term. Let \(S^+ = \lambda x y \cdot [E(Fy), (x(S^+ y))]\), where \(F\) \(\lambda\)-defines \(f\), and \(B = FP b\) (i.e. the fixed point of \(b\)). Then

\[
B_n \xrightarrow{\beta} b B_n \xrightarrow{\beta} [E(F_n)\), \(B_{n+1}] \xrightarrow{\beta} [M_n, B_{n+1}].
\]

So \(B_0 = [M_0, B_1] = [M_0, M_1, B_2] = \ldots\) Hence by setting \(\langle M_n \rangle_{n \in \omega} = B_0\) we have a coding for infinite sequences of terms with one fixed free variable.

(4) It is easy to construct a term \(p\) such that \(p_m(M_n) = M_n\) (take e.g. \(pxa = \text{if zero } x \text{ then } aK\) else \(p(x - 1)(a(KI))\), using the fixed point theorem).

3.9. Lemma. For all closed \(Z\) there is an \(n\) such that \(Z \mathcal{Q}^n = \mathcal{Z}\). \((Z \mathcal{Q}^n)\) is short for \(\underbrace{Z \mathcal{Q} \mathcal{Q} \ldots \mathcal{Q}}_n\).

Proof. Case 1. \(Z\) is unsolvable; then \(Z = \mathcal{Z}\), so \(n = 0\).

Case 2. \(Z\) is solvable; then \(Z\) has a \(\lambda\)nf, \(Z = \lambda x \cdot x_1 A_1 \ldots A_m \langle x_i e \rangle\).

Take \(n = i\), so \(Z \mathcal{Q}^i = \lambda x^i \cdot \Omega A_1 \ldots A_m = \mathcal{Z}\).

3.10. Theorem. If \(\mathcal{U}\) is hard and sensible, then \(\mathcal{U}\) is not rich.

Proof. If \(\mathcal{U}\) is hard, then \(\mathcal{U}\) is isomorphic to \(\mathcal{U}^0(T)\), where \(T = Th(\mathcal{U})\). We reason in \(\mathcal{U}^0(T)\). Since \(\mathcal{U}\) is sensible, \(\mathcal{U} \subseteq T\).

Let \(h : \omega \rightarrow \omega\) be a function not definable in \(\mathcal{U}\). Such an \(h\) exists since a hard model is countable.

Let \(A_n(x, y)\) be the term \(x \mathcal{Q}^n(y \mathcal{Q}^n((h y)))\), \(n \in \omega\). For closed \(M\) the sequence \(A_0(M, y), A_1(M, y), \ldots\) is by 3.9

\[
M(y(h y)), \Omega(y \Omega(h y)), \ldots, \Omega \mathcal{Q}^n(y \mathcal{Q}^n((h y))), \Omega, \Omega, \ldots,
\]

where \(n\) is such that \(\Omega \mathcal{Q}^{n+1} = \Omega\). Thus \(\lambda n \cdot A_n(M, y)\) is up to convertibility a recursive sequence containing one fixed free variable and hence representable as a term. Define \(f(M) = \lambda y \cdot \langle A_n(M, y) \rangle_{n \in \omega}\). Similarly for closed \(N\)

\(\lambda n \cdot A_n(x, N)\) is recursive and it is possible to define \(g(N) = \lambda x \cdot \langle A_n(x, N) \rangle_{n \in \omega}\). Then for all closed \(M, N:\) \(f(M)\) and \(g(N)\) are well defined and \(f(M) \cdot N = g(N) \cdot M = \langle A_n(M, N) \rangle_{n \in \omega}\) by construction. So \(f\) and \(g\) are dual.

Suppose now that \(\mathcal{U}\) is rich, i.e. \(f\) were representable by some closed \(F\). Then for all closed \(M, N:\)

\(FMN = f(M)N = \langle A_n(M, N) \rangle_{n \in \omega}\).

But then \(p_M(F(K^n I)(K^n I)) = p_M(\langle h(y) \rangle_{n \in \omega} = h(y)\), hence \(h\) were definable, contradiction. Thus \(\mathcal{U}\) is not rich.
3.11. Corollary. \( D_\infty \) and \( \mathcal{M}(T) \) for \( T \subseteq \mathcal{H} \) are not rich.

3.12. Questions. (i) Is every extensional term model \( \mathcal{M}(T) \) rich?
(ii) Is \( \mathcal{M}(\lambda \omega) \) rich?

Here \( \lambda \omega \) is the \( \lambda \)-theory obtained by adding the \( \omega \)-rule to the theory, see Barendregt [1974].

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