

# Enumerators of lambda terms are reducing constructively

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Dedicated in friendship to my teacher  
*Dirk van Dalen*  
on the occasion of his 60<sup>th</sup> birthday

## **Abstract**

A closed  $\lambda$ -term  $E$  is called an *enumerator* if

$$\forall M \in \Lambda^\circ \exists n \in \mathbb{N} E^{\ulcorner n \urcorner} =_\beta M.$$

Here  $\Lambda^\circ$  is the set of closed  $\lambda$ -terms,  $\mathbb{N}$  is the set of natural numbers and the  $\ulcorner n \urcorner$  are the Church's numerals  $\lambda f x. f^n x$ . Such an  $E$  is called *reducing* if moreover

$$\forall M \in \Lambda^\circ \exists n \in \mathbb{N} E^{\ulcorner n \urcorner} \rightarrow_\beta M.$$

In 1983 I conjectured that every enumerator is reducing. An ingenious recursion theoretic proof of this conjecture by Statman is presented in Barendregt [1992]. The proof is not intuitionistically valid, however. Dirk van Dalen has encouraged me to find intuitionistic proofs whenever possible. In the lambda calculus this is usually not difficult. In this paper an intuitionistic version of Statman's proof will be given. It took me somewhat longer to find it than in other cases.

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## 1. Introduction

If we have proved in Heytings arithmetic **HA** that **E** is an enumerator, then by Statmans result we can prove in Peano's arithmetic **PA** that **E** is reducing. The statement that a combinator is a reducing enumerator is  $\Pi_2^0$ . Therefore, by a well-known result, see e.g. Troelstra and van Dalen [1988], proposition 3.3.5 (ii), it follows that also in **HA** one can prove that **E** is reducing. So the reader may wonder why we give an intuitionistic proof of Statmans theorem. The first reason is that there is a difference between knowing that a statement  $A$  can be proved intuitionistically and having an intuitionistic proof. By Kreisels result we have a general recipe for transforming any proof  $D_{PA}$  in **PA** of a  $\Pi_2^0$ -statement into a proof  $D_{HA}$  in **HA**. But in order to obtain  $D_{HA}$  in this way, we first have to write down a formalized proof of  $A$  and then apply the recipe. The result is a formal proof but may not be understandable. The second reason is that by using Kreisels general recipe one only obtains the validity of the rule

$$\vdash_{HA} \mathbf{E} \text{ is an enumerator} \Rightarrow \vdash_{HA} \mathbf{E} \text{ is a reducing enumerator.}$$

A concrete **HA** proof of a statement  $A$  may be such that it also shows the implication within **HA**:

$$\vdash_{HA} \mathbf{E} \text{ is an enumerator} \rightarrow \mathbf{E} \text{ is a reducing enumerator.}$$

Indeed our constructive proof will yield the validity of this direct implication.

Statmans result is stronger than just stated. He showed in **PA** the following. Let  $\mathcal{A} \subseteq \Lambda^\circ$  be an r.e. set. Suppose

$$\forall M \in \Lambda^\circ \exists N \in \mathcal{A} N =_\beta M. \quad (1)$$

Then

$$\forall M \in \Lambda^\circ \exists N \in \mathcal{A} N \rightarrow_\beta M. \quad (2)$$

By applying this to the set  $\mathcal{A} = \{\mathbf{E} \uparrow n \mid n \in \mathbf{N}\}$  one obtains his result concerning enumerators **E**. We will prove

$$\vdash_{HA} (1) \rightarrow (2).$$

## 2. Statmans proof

We use lambda calculus notation from Barendregt [1984] and recursion theoretic notations from Rogers [1967]. In particular if  $\psi$  is a partial recursive function, then  $\psi(n) \downarrow$  means that  $\psi(n)$  is defined and  $\psi(n) \uparrow$  means that  $\psi(n)$  is undefined. A set  $A \subseteq \mathbf{N}$  is called recursively enumerable (r.e.) if for some partial recursive  $\psi: \mathbf{N} \rightarrow \mathbf{N}$  one has  $A = \text{dom}(\psi)$ , i.e.  $\forall n \in \mathbf{N} [n \in A \Leftrightarrow \psi(n) \downarrow]$ . In the following the reader is supposed to know some elementary properties of r.e. sets. For example, that if  $A$  and its complement are both r.e., then  $A$  is recursive; moreover, that there exists a set  $K \subseteq \mathbf{N}$  that is r.e. but not recursive.

2.1. LEMMA. Let  $M \in \Lambda$ . Then there is an  $M_1 \in \Lambda$  in  $\beta$ -nf such that  $M_1 \downarrow \twoheadrightarrow_\beta M$  and  $\text{FV}(M) = \text{FV}(M_1)$ . Here  $\downarrow \equiv \lambda x.x$ .

**Proof.** By induction on the structure of  $M$  we define  $M_1$  in the following table.

$M$	$M_1$
$x$	$\lambda z.zx$
$PQ$	$\lambda z.z(zP_1z)(zQ_1z)$
$\lambda x.P$	$\lambda zx.zP_1z$

Then by induction it follows that  $M_1 \downarrow \twoheadrightarrow_\beta M$  and  $\text{FV}(M) = \text{FV}(M_1)$ . ■

Remember that a term  $M \in \Lambda$  is of order 0 if for no  $P \in \Lambda$  one has  $M =_\beta \lambda x.P$ . For example  $(\lambda x.xx)(\lambda x.xx)$  is of order 0.

2.2. LEMMA. (i) For every partial recursive function  $\psi$  there is a term  $F \in \Lambda^\circ$  such that for all  $n \in \mathbb{N}$  one has

$$\begin{aligned} \psi(n) \downarrow &\Rightarrow F \ulcorner n \urcorner =_\beta \ulcorner \psi(n) \urcorner \\ \psi(n) \uparrow &\Rightarrow F \ulcorner n \urcorner \text{ is of order } 0. \end{aligned}$$

(ii) Let  $K \subseteq \mathbb{N}$  be an r.e. set. Then for some  $P_K \in \Lambda^\circ$  one has for all  $n \in \mathbb{N}$

$$\begin{aligned} n \in K &\Rightarrow P_K \ulcorner n \urcorner \twoheadrightarrow_\beta \downarrow; \\ n \notin K &\Rightarrow P_K \ulcorner n \urcorner \text{ is of order } 0. \end{aligned}$$

**Proof.** (i) Inspection of the usual proof of the  $\lambda$ -definability of the partial recursive functions shows that in case the function is undefined on an argument the representing  $\lambda$ -term is of order 0 on the corresponding numeral. For another proof due to Statman, see Barendregt [1992a].

(ii) Let  $K = \text{dom}(\psi)$ . Let  $\psi$  be  $\lambda$ -defined by  $F$ . Then take  $P_K \equiv \lambda c.Fc \downarrow$ , noting that for Church's numerals one has  $\ulcorner n \urcorner \downarrow =_\beta \downarrow$ . ■

2.3. THEOREM (Statman [1987]). Let  $\mathcal{A} \subseteq \Lambda^\circ$  (after coding) be an r.e. set. Suppose

$$\forall M \in \Lambda^\circ \exists N \in \mathcal{A} N =_\beta M. \quad (3)$$

Then

$$\forall M \in \Lambda^\circ \exists N \in \mathcal{A} N \twoheadrightarrow_\beta M. \quad (4)$$

**Proof.** Assume (3). Suppose towards a contradiction that (4) does not hold, i.e. for some  $M_0 \in \Lambda^\circ$

$$\forall N \in \mathcal{A} N \not\twoheadrightarrow_\beta M_0. \quad (5)$$

Using lemma 2 construct a term  $M_1$  in  $\beta$ -nf such that  $M_1 \dashv\rightarrow_\beta M_0$ . Let  $P = P_K$  as in lemma 2 for some non-recursive r.e. set  $K$ . Define a predicate  $R$  on  $\mathbb{N}$  as follows:

$$R(n) \iff \exists N \in \mathcal{A} \exists Q \in \Lambda [P \ulcorner n \urcorner \dashv\rightarrow_\beta Q \ \& \ N \dashv\rightarrow_\beta Q M_1].$$

Note that  $R$  is an r.e. predicate. Claim

$$R(n) \iff n \notin K.$$

As to  $(\Rightarrow)$ , suppose  $R(n)$ , i.e. for some  $N \in \mathcal{A}$  and  $Q \in \Lambda$  one has

$$P \ulcorner n \urcorner \dashv\rightarrow_\beta Q \ \& \ N \dashv\rightarrow_\beta Q M_1.$$

If  $n \in K$ , then  $\ulcorner \ \urcorner =_\beta P \ulcorner n \urcorner =_\beta Q$ , so by the Church-Rosser theorem  $Q \dashv\rightarrow_\beta \ulcorner \ \urcorner$  and therefore  $N \dashv\rightarrow_\beta \ulcorner M_1 \urcorner \dashv\rightarrow_\beta M_0$ , contradicting (5). Therefore  $n \notin K$  and we are done. As to  $(\Leftarrow)$ , suppose  $n \notin K$ . Then  $P \ulcorner n \urcorner$  is of order 0. By (3) there is an  $N \in \mathcal{A}$  such that  $N =_\beta P \ulcorner n \urcorner M_1 \urcorner$ . By the Church-Rosser theorem there is a common reduct  $L$  of  $N$  and  $P \ulcorner n \urcorner M_1 \urcorner$ . Since  $P \ulcorner n \urcorner$  is of order 0 and  $M_1, \ulcorner \ \urcorner$  are in nf one must have  $L \equiv Q M_1 \urcorner$  with  $P \ulcorner n \urcorner \dashv\rightarrow_\beta Q$ . Therefore  $R(n)$ .

From the claim it follows that the complement of  $K$  is r.e., hence recursive (since  $K$  is itself r.e.) contradicting the choice of  $K$ . ■

What is happening here? Given  $\mathcal{A}$  and a term  $M$ , we want to construct a term  $N \in \mathcal{A}$  such that  $N \dashv\rightarrow M$ . We know that there is a term  $N_n = P_n M_1 \urcorner$ , with  $P_n \equiv P_K \ulcorner n \urcorner$ . Now

$$\begin{aligned} n \in K &\Rightarrow P_n \dashv\rightarrow \ulcorner \ \urcorner; \\ n \notin K &\Rightarrow N_n \text{ is of order 0} \\ &\Rightarrow N_n \dashv\rightarrow P'_n M_1 \urcorner, \end{aligned}$$

for some  $P'_n \leftarrow P_n$ . If—in some ‘dialectic’ way—one would have  $n \in K \ \& \ n \notin K$  we would be done. Indeed, then

$$N_n \dashv\rightarrow P'_n M_1 \urcorner \dashv\rightarrow \ulcorner M_1 \urcorner \rightarrow M_1 \urcorner \dashv\rightarrow M.$$

This is impossible of course. But for some  $e$  and  $P'_e \leftarrow P_e$  one has

$$e \in K \ \& \ N_e \dashv\rightarrow P'_e M_1 \urcorner,$$

because otherwise  $\mathbb{N} - K = \{n \mid \exists P'_n \leftarrow P_n \ N_n \dashv\rightarrow P'_n M_1 \urcorner\}$ ; since the latter set is r.e., the negation theorem implies that  $K$  is recursive, contrary to the choice of  $K$ . Therefore one has for this  $e$

$$N_e \dashv\rightarrow P'_e M_1 \urcorner \dashv\rightarrow \ulcorner M_1 \urcorner \dashv\rightarrow M.$$

### 3. The intuitionistic proof

The difficulty making this reasoning constructive is the following. The  $e$  to be constructed is found via the unsolvability of the halting problem. So let  $K = \{n \mid \phi_n(n)\downarrow\}$  and  $R$  be an r.e. set such that  $\mathbf{N}-K \subseteq R$ . We want to construct an  $e$  such that  $e \in R \cap K$ . Now let  $R = W_e = \{n \mid \phi_e(n)\downarrow\}$ . Then

$$e \notin R \Rightarrow e \notin W_e \Rightarrow e \in \mathbf{N}-K \Rightarrow e \in R.$$

Therefore by reductio ad absurdum  $e \in R = W_e$  and hence also  $e \in K$ . Intuitionistically one has only  $\neg\neg(e \in R \cap K)$ . By analysing why  $\mathbf{N}-K \subseteq R$  we can nevertheless prove that  $e \in R$  and hence  $e \in R \cap K$ .

3.1. LEMMA. *The following is provable in HA. Let  $K$  be an r.e. set. Then for some  $P = P_K \in \Lambda^\circ$  one has for all  $n \in \mathbf{N}$*

$$\begin{aligned} n \in K &\Rightarrow P \ulcorner n \urcorner \twoheadrightarrow_\beta \! \downarrow; \\ P \ulcorner n \urcorner \twoheadrightarrow \lambda x.M &\Rightarrow n \in K. \end{aligned}$$

In particular,  $n \notin K \Rightarrow P \ulcorner n \urcorner$  is of order 0.

**Proof.** Let  $\mathbf{E}$  be a reducing self-interpreter, e.g. the one constructed by P. de Bruin, see Barendregt [1992]. Using lemma 2 let  $\mathbf{E}_1$  be a  $\beta$ -nf such that  $\mathbf{E}_1 \! \downarrow \twoheadrightarrow \mathbf{E}$ . Let  $t$  be a recursive predicate such that

$$n \in K \iff \exists k t(n, k).$$

Let  $t$  be  $\lambda$ -defined by  $T \in \Lambda^\circ$ . By the second fixed-point theorem, see Barendregt [1984], there exists a term  $H \in \Lambda^\circ$  such that

$$Hxy \twoheadrightarrow Txy(\mathbf{K}^4 \! \downarrow) \langle \! \downarrow \rangle \mathbf{E}_1 \ulcorner H \urcorner x(\mathbf{S}^+ y),$$

where  $\langle M \rangle = \lambda x.xM$  and  $\mathbf{S}^+$   $\lambda$ -defines the successor function. We set  $P \equiv \lambda x.Hx \ulcorner 0 \urcorner$ . In order to show that  $P$  satisfies the requirements, define

$$\begin{aligned} A_k^n &\equiv \! \downarrow && \text{if } \exists k' < k t(n, k'); \\ &\equiv H \ulcorner n \urcorner \ulcorner k \urcorner && \text{else.} \end{aligned}$$

Claim  $A_k^n \twoheadrightarrow A_{k+1}^n$ . If  $A_k^n \equiv \! \downarrow$  because  $\exists k' < k t(n, k')$ , then also  $A_{k+1}^n \equiv \! \downarrow$  and we are done. Otherwise  $A_k^n \equiv H \ulcorner n \urcorner \ulcorner k \urcorner$  because  $\neg \exists k' < k t(n, k')$ . Then we have the following.

Case 1.  $t(n, k)$  holds. Then  $T \ulcorner n \urcorner \ulcorner k \urcorner \twoheadrightarrow \text{true}$  and

$$\begin{aligned} H \ulcorner n \urcorner \ulcorner k \urcorner &\twoheadrightarrow T \ulcorner n \urcorner \ulcorner k \urcorner (\mathbf{K}^4 \! \downarrow) \langle \! \downarrow \rangle \mathbf{E}_1 \ulcorner H \urcorner \ulcorner n \urcorner (\mathbf{S}^+ \ulcorner k \urcorner) \\ &\twoheadrightarrow \text{true}(\mathbf{K}^4 \! \downarrow) \langle \! \downarrow \rangle \mathbf{E}_1 \ulcorner H \urcorner \ulcorner n \urcorner (\ulcorner k \urcorner + \! \downarrow) \\ &\twoheadrightarrow_{gk} \mathbf{K}^4 \! \downarrow \mathbf{E}_1 \ulcorner H \urcorner \ulcorner n \urcorner \ulcorner k \urcorner + \! \downarrow \\ &\twoheadrightarrow \! \downarrow \equiv A_{k+1}^n. \end{aligned}$$

Case 2.  $t(n, k)$  does not hold. Then  $T^{\ulcorner n \urcorner k^{\urcorner}} \rightarrow \text{false}$  and

$$\begin{aligned}
H^{\ulcorner n \urcorner k^{\urcorner}} &\rightarrow T^{\ulcorner n \urcorner k^{\urcorner}}(K^4) \langle \text{!} \rangle E_1^{\ulcorner H^{\ulcorner n \urcorner} \urcorner} (S^{\ulcorner k \urcorner}) \\
&\rightarrow \text{false}(K^4) \langle \text{!} \rangle E_1^{\ulcorner H^{\ulcorner n \urcorner} \urcorner} (\ulcorner k + 1 \urcorner) \\
\rightarrow_{gk} &\langle \text{!} \rangle E_1^{\ulcorner H^{\ulcorner n \urcorner} \urcorner} \ulcorner k + 1 \urcorner \\
&\rightarrow E_1^{\ulcorner H^{\ulcorner n \urcorner} \urcorner} \ulcorner k + 1 \urcorner \\
&\rightarrow E^{\ulcorner H^{\ulcorner n \urcorner} \urcorner} \ulcorner k + 1 \urcorner \\
&\rightarrow H^{\ulcorner n \urcorner} \ulcorner k + 1 \urcorner \equiv A_{k+1}^n.
\end{aligned}$$

In the above  $\rightarrow_{gk}$  means that the reduction involves at least one gk-step of completely developing all present redexes in a term. Therefore we have that

$$\sigma : P^{\ulcorner n \urcorner} \rightarrow A_0^n \rightarrow A_1^n \rightarrow \dots \rightarrow A_k^n \rightarrow \dots$$

is a quasi-Gross-Knuth reduction path, hence by Barendregt [1984] thm.13.2.11, a cofinal reduction sequence starting with  $P^{\ulcorner n \urcorner}$ . The reasoning can be carried out in **HA**.

Now suppose that  $n \in K$ . Then  $t(n, k)$  for some  $k$ . Therefore

$$P^{\ulcorner n \urcorner} \rightarrow A_{k+1}^n \equiv \text{!}.$$

Suppose on the other hand that  $P^{\ulcorner n \urcorner} \rightarrow \lambda x.M$ . Then by the cofinality of  $\sigma$  it follows that  $\lambda x.M \rightarrow A_k^n$  for some  $k$ . But then  $A_k^n \equiv \text{!}$  is the only possibility; therefore  $n \in K$ . ■

Now we can give the proof of the main theorem.

**3.2. THEOREM** (Constructive version of 2). *The following is provable in **HA**. Let  $\mathcal{A} \subseteq \Lambda^\circ$  be an r.e. set. Suppose*

$$\forall M \in \Lambda^\circ \exists N \in \mathcal{A} N =_\beta M. \quad (6)$$

*Then*

$$\forall M \in \Lambda^\circ \exists N \in \mathcal{A} N \rightarrow_\beta M. \quad (7)$$

**Proof.** Suppose we have (6). Given  $M \in \Lambda^\circ$  we want to construct an  $N \in \mathcal{A}$  such that  $N \rightarrow M$ . Let  $K = \{n \in \mathbb{N} \mid \phi_n(n) \downarrow\}$  and  $P = P_K$  as in lemma 3. Define

$$R = \{n \mid \exists Q \in \Lambda^\circ \exists N \in \mathcal{A} N \rightarrow Q M_1 \ \& \ P^{\ulcorner n \urcorner} \rightarrow Q\}.$$

Clearly  $R$  is an r.e. set. Let  $R = W_e$  in the notation of Rogers [1967]. By the assumption there exists an  $N \in \mathcal{A}$  such that  $N =_\beta P^{\ulcorner e \urcorner} M_1 \text{!}$ . Therefore by the Church-Rosser theorem for some  $L \in \Lambda^\circ$  one has

$$N \rightarrow L \leftarrow P^{\ulcorner e \urcorner} M_1 \text{!}.$$

Case 1. In the given reduction  $P^{\lceil e^{\lceil} M_1 \rceil} \twoheadrightarrow L$  the head  $P^{\lceil e^{\lceil}$  is never reduced to a term of the form  $\lambda x.T$ . Then  $L \equiv Q M_1 \rceil$  for some  $Q \leftarrow P^{\lceil e^{\lceil}$ . Then  $e \in R = W_e$ , so  $e \in K$ , hence  $P^{\lceil e^{\lceil} = \perp$  and therefore  $Q \twoheadrightarrow \perp$ . But then

$$N \twoheadrightarrow L \equiv Q M_1 \rceil \twoheadrightarrow M_1 \rceil \twoheadrightarrow M.$$

Case 2. In the given reduction  $P^{\lceil e^{\lceil} M_1 \rceil} \twoheadrightarrow L$  the head  $P^{\lceil e^{\lceil}$  is reduced to a term of the form  $\lambda x.T$ . Then by lemma 3 it follows that  $e \in K$  so  $e \in W_e = R$  and therefore  $N' \twoheadrightarrow Q' M_1 \rceil$  for some  $N' \in \mathcal{A}$  and  $Q' \leftarrow P^{\lceil e^{\lceil}$ . Since  $e \in K$  again we have  $Q' \twoheadrightarrow \perp$  and hence  $N' \twoheadrightarrow M$ . ■

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