A CHARACTERIZATION OF TERMS OF THE $\lambda I$-CALCULUS HAVING A NORMAL FORM

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§0. Introduction. The theorem proved in this paper answers some transitivity questions (in the geometric sense) for the type free $\lambda$-calculus: Which objects can be mapped on all other objects? How much can an object do by applying it to other objects (see footnote 2)?

The main result is that, for closed terms of the $\lambda I$-calculus, the following conditions are equivalent:

(a) $M$ has a normal form.
(b) $FM = I$ for some $\lambda I$-term $F$.
(c) $M N_1 \cdots N_n = I$ for some $\lambda I$-terms $N_1, \ldots, N_n$.

By the same method it follows that if $M$ is a closed term of the $\lambda K$-calculus having a normal form, then for some $\lambda I$-terms (sic) $N_1, \ldots, N_n$, $MN_1 \cdots N_n = I$ is provable in the $\lambda K$-calculus.

The theorem of Böhm [2] states that if $M_1, M_2$ are terms of the $\lambda K$-calculus having different $\beta\eta$-normal forms, then $\forall A_1, A_2 \exists N_1, \ldots, N_n M N_1 \cdots N_n = A_i$ is provable in the $\lambda K$-$\beta\eta$-calculus for $i = 1, 2$. As a consequence of this it was shown (implicitly) in [1, 3.2.20 1/2 (1)] that if $M$ has a normal form, then for some $\lambda K$-terms $N_1, \ldots, N_n$, $MN_1 \cdots N_n = I$ is provable in the $\lambda K$-calculus.

It was not clear that this also could be proved for the $\lambda I$-calculus since the proof of the theorem of Böhm essentially made use of $\lambda K$-terms.

We conjecture that, using the results of this paper, the full theorem of Böhm can be proved for the $\lambda I$-calculus.²

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§1. Preliminaries. We assume familiarity with the $\lambda I$- and the $\lambda K$-calculus as treated e.g. in [4, Chapter 3] or [3, Chapters II, V].

1.1. Notation. $L_I$ ($L_K$) is the language of the $\lambda I$-calculus ($\lambda K$-calculus). $[x/N]M$ is the result of substituting $N$ for the free occurrences of $x$ in $M$. $FV(M)$ is the set of free variables of $M$.

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2 Professor Böhm has informed us that, using Corollary 2.15, one can prove also for the $\lambda I$-calculus his generalized theorem: Let $M_1, \ldots, M_n$ be terms having different $\beta\eta$-normal forms, then

$$\forall A_1 \cdots A_n \exists N_1 \cdots N_m \lambda \eta \vdash M_i N_1 \cdots N_m = A_i, \quad 1 \leq i \leq n.$$
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The $\lambda\eta$-calculus ($\lambda K\eta$-calculus) is the extensional theory containing $\eta$-reduction. When in a certain context $L$, $\lambda$ or $\lambda\eta$ is used, $L$, $\lambda$ and $\lambda\eta$ should be replaced throughout that context by $L_\lambda$, $\lambda\eta$ and $L_{\lambda\eta}$ (theorems stated for $L$, etc. hold for both versions).

“normal form” will be abbreviated by n.f.

$MN^n$ is $MN\cdots N^\cdot (N$ appearing $n$ times). $\lambda(\eta) \vdash$ denotes provability in $\lambda(\eta)$. $\ge\$ is the reduction relation, $\equiv$ the convertibility relation and $\equiv$ the relation of syntactic identity.

1.1. Definition. Let $M$ be a term $\in L$. $M$ is I-solvable iff $\exists N_1\cdots N_n\in L_\lambda$ $\lambda(\eta) \vdash MN_1\cdots N_n = I$. $M$ is K-solvable iff $\exists N_1\cdots N_n\in L_K$ $\lambda K(\eta) \vdash MN_1\cdots N_n = I$.

By the following lemma there is no need to make a distinction between $I(\eta)$-solvable in $\lambda I(\eta)$ or in $\lambda K(\eta)$.

1.2. Lemmas. The $\lambda K(\eta)$-calculus is a conservative extension of the $\lambda I(\eta)$-calculus.

Proof. Show first $[\lambda K(\eta) \vdash M \ge N$ and $M \in L_\lambda] \Rightarrow [N \in L_\lambda and \lambda I(\eta) \vdash M \ge N]$; then use the well-known Church-Rosser theorem (see e.g. [4, Chapter 4]) for $\lambda K(\eta)$.

1.3. Lemma. Let $M$ be a term $\in L$. $M$ has a $\beta$-n.f. $\iff$ $M$ has a $\beta\eta$-n.f.

Proof. $\Rightarrow$: Each $\beta$-n.f. has a $\beta\eta$-n.f. by contracting some $\eta$-redexes.

$\Leftarrow$: See [5, Chapter 11E, Lemma 13.1].

1.4. Lemma. $M$ is I-solvable $\iff$ $M$ is I-\eta-solvable;

$M$ is K-solvable $\iff$ $M$ is K-\eta-solvable.

Proof. (Same proof for both cases.) $\Rightarrow$: Trivial.

$\Leftarrow$: Suppose that $\exists N_1\cdots N_n\lambda\eta \vdash MN_1\cdots N_n = I$. Then $MN_1\cdots N_n$ has a $\beta\eta$-n.f., hence by 1.4, a $\beta$-n.f. $M'$. $M'$ has the properties: $\lambda \vdash MN_1\cdots N_n = M'$ and $\lambda(\eta) \vdash M' \ge I$ (by the Church-Rosser theorem for $\lambda(\eta)$). Since $M'$ is in $\beta$-n.f., $M' \ge I$ is a pure $\eta$-reduction, say with the number of $\eta$-contractions $q$. By induction on $q$ it follows that $M'$ must be of the form $M' \equiv \lambda x_1\cdots x_m.x_1M_2\cdots M_n$, where $M_i \ge x_i$ ($2 \le i \le m$) by an $\eta$-reduction and $F \vee (M_i) = \{x_i\}$. By induction on $q$ it now follows that $M'$ is solvable. If $q = 0$ this is clear. If $q > 0$, then $m \ge 2$ and $M_i \ge x_i$ by an $\eta$-reduction of less than $q$ steps. Hence also $[x_i/I]M_i \ge I$ by an $\eta$-reduction of less than $q$ steps. By the induction hypothesis,

$\exists N_{11}\cdots N_{ik}\in L$ $\lambda \vdash [x_i/I]M_iN_{11}\cdots N_{ik} \ge I, \ 2 \le i \le m$.

Then

$\lambda \vdash M'L_1\cdots L_m = I$,

where

$L_1 \equiv \lambda y_2\cdots y_m.(y_2N_{21}\cdots N_{2k_2})\cdots(y_mN_{m1}\cdots N_{mk_m}), \ L_2 = \cdots = L_m \equiv I.$

Hence $\lambda \vdash MN_1\cdots N_{L_1}\cdots L_m = M'L_1\cdots L_m = I$; i.e., $M$ is solvable.

1.5. Lemma. If $M \in L_\lambda$ and has a n.f., then every subterm of $M$ has a n.f.

Proof. See [3, p. 27, Theorem 7 XXII].

1.6. Lemma. If $M \in L_\lambda$ and has a n.f., then every subterm of $M$ has a n.f.

Proof. See [3, p. 27, Theorem 7 XXII].

1.7. Example. Let $\Omega \equiv (\lambda x.xx)(\lambda x.xx)$. Then $\Xi \equiv \lambda x.xI\Omega$ is a term which is $K$-solvable but not $I$-solvable; $\lambda K \vdash \Xi K = I$, but $\Xi$ cannot be solved by $\lambda I$-terms as follow from 1.6.

§2. Proof of the main theorem.

2.1. Theorem. (i) If $M$ is a closed term of $L_\lambda$, the following are equivalent:
(a) \( M \) has a n.f.
(b) \( \exists F \in L_1 \quad \lambda \beta \ F M = I \).
(c) \( M \) is I-solvable.

(ii) If \( M \) is a closed term of \( L_K \), then \( M \) has a n.f. \( \Rightarrow \) \( M \) is I-solvable.

Proof. (i) We show (c) \( \Rightarrow \) (b) \( \Rightarrow \) (a) \( \Rightarrow \) (c).
(c) \( \Rightarrow \) (b): If \( M \) is /-solvable, then
\( Xx.b M N_1 \cdots N_n = I \) for some \( N_1 \cdots N_n \in L_1 \). Take \( F = Xx.xN_1 \cdots N_n . \) (b) \( \Rightarrow \) (a):
If \( \lambda \beta \ F M = I \) for some \( F \in L_1 \), then \( FM \) has a n.f. Hence, by 1.6, \( M \) has a n.f.
(a) \( \Rightarrow \) (c): The proof of this fact occupies 2.3-2.13.
(ii) This will be a corollary to the proof of (a) \( \Rightarrow \) (c) of (i).

2.2. The converse of 2.1(ii) is false: Let \( M = Xx.x \). Then
\( M \) is /-solvable: \( Xx.b M SI \) = \( I \). But \( M \) has no n.f.

2.3. Definition. \( S \)-indices (integers) are defined inductively as follows:

0 is an \( S \)-index.
If \( s \) is an \( S \)-index, then, for integers all \( n > 1, m > 0, \sigma, m, s \) is an \( S \)-index.

(w = \( \langle x, y, z \rangle \) is a coding of triples of integers with projections \( x = (w)_0, y = (w)_1, z = (w)_2 \) such that \( (w)_i < w, i = 0, 1, 2. \)

2.4. Definition. For every \( S \)-index \( s \) we define a closed term \( O_s \) of \( L_7, \quad O_0 = /, \) \( O_{\langle n, m, s \rangle} = \cdot \cdot \cdot a_n (a_s O_s^{-m}). \)

2.5. Definition. For \( S \)-indices \( s \) we define inductively a length \( l(s) \):
\( l(0) = 0, \quad l(\langle n, m, s \rangle) = n.m.l(s) + 1. \)

2.6. Definition. We define simultaneously the class of \( S \)-polynomials \( P \) and their depth \( d(P) \):
\( O_s \) is an \( S \)-polynomial for every \( S \)-index \( s \); \( d(O_s) = l(s). \)
If \( P_1, P_2 \) are \( S \)-polynomials, so is \( (P_1 P_2) \); \( d(P_1 P_2) = d(P_1) + d(P_2). \)

2.7. Lemma. Each \( S \)-polynomial \( P \) is I-solvable (using only \( I \)'s).
Proof. Induction on \( d(P) \). If \( d(P) = 0 \), then \( P \) is a combination of \( I \)'s and hence I-solvable. Suppose \( d(P) = n > 0 \). By contracting several \( I \)'s, \( \lambda \beta \ F P = O_s P_1 \cdots P_p \), \( P \neq 0, p \geq 0 \) and \( d(O_s P_1 \cdots P_p) = d(P) \). If \( P < (s_0) \), then
\( \lambda \beta \ F P_1 \cdots P_p I^{-m} = P' \) where \( P' = (P_1 O_s^{-m}) \cdots (P_p O_s^{-m}) (O_x O_s^{-m}) \cdots (O_x O_s^{-m}) I^{-m} \) \( m = (s)_1 \) and \( s' = (s)_2 \). We have \( d(P') = d(P_1 \cdots P_p) + m \cdot d(O_s) \cdot (s)_0 < d(P_1 \cdots P_p) + l(s) = d(O_s P_1 \cdots P_p) = d(P) \). If \( P > (s_0) \), then a similar argument shows that once more \( \lambda \beta \ F P' I^{-m} = I \) for some \( m \). Hence \( \lambda \beta \ F P' I^{-m} I^{-m} = I \).

2.8. Lemma. The class of \( L_1 \) terms in \( \beta \)-n.f. has the following inductive definition:
\( x \in \beta \)-n.f. \[ [M \in \beta \)-n.f. and \( x \in FV(M) ] \Rightarrow \lambda x.M \in \beta \)-n.f. \]
\( M_1, \cdots, M_k \in \beta \)-n.f. \( \Rightarrow x M_1 \cdots M_k \in \beta \)-n.f.
Proof. The terms obtained by this inductive definition are clearly in \( \beta \)-n.f. Conversely, every term has one of the three following forms: \( x, x M_1 \cdots M_k \) and \( (x_1.M_1) M_2 \cdots M_k \). The only \( \beta \)-n.f.'s among those are \( x, x M_1 \cdots M_k \) and \( \lambda x.M_1 \), if \( M_1, \cdots, M_k \) are in \( \beta \)-n.f.

2.9. Definition. By course of value recursion, the following number-theoretic predicate and functions are defined.
s < s' iff \([s = 0 \lor ((s)_0 \leq (s')_0 \land (s)_1 \leq (s')_1 \land (s)_2 \leq (s')_2)]\),

\(s \cup s' = s'\) if \(s = 0\),

\[= \langle \text{Max}(s)_0, \text{Max}(s')_0, (s)_1, (s')_1, (s)_2 \cup (s')_2 \rangle\] else.

\(s/m = 0\) if \(s = 0\),

\[= \langle (s)_0, m, (s)_1/m \rangle\] else

Then \(\preceq\) is transitive, \(s \cup s' \models s, s \cup s' \models s', s \models s \models (s')_2 \models (s')_2\) (if \(s' \neq 0\)) and \(\langle n, m, s/m' = \langle n, m', s/m' \rangle\).

2.10. **Notation.** We write \(M(x_1, \ldots, x_p)\) to indicate that \(FV(M) = \{x_1, \ldots, x_p\}\) and the \(x_i\) are distinct. If \(N_1, \ldots, N_p\) are closed terms, then \(M(N_1, \ldots, N_p)\) is \([x_1/N_1]\cdots[x_p/N_p]M\).

2.11. **Lemma.** *For every term \(M(x_1, \ldots, x_p)\) of \(L_1\) in \(\beta\text{-n.f.}\)*

\[\exists s \forall t_0 \models s, \ldots, t_p \models s \exists n \forall m \geq nM(O_{t_1/m}, \ldots, O_{t_p/m})O_{t_0/m}^{-n}\]

is provably equal (in \(\lambda I\)) to an \(S\)-polynomial.

**Proof.** Induction on the definition of \(\beta\text{-n.f.}'s\) given in 2.8. We write \(s_M, n_M, t_{~}t\) to indicate the dependence of \(s\) and \(n\) on \(M, t_{~}t\) (\(t_{~}t = t_{0}, \ldots, t_{p}\)).

\(M \equiv x\). Let \(x \equiv x_{t_0}\). Take \(s_M = 0\), \(n_M, t_{~}t = 0\). Let \(t_{~}t, m\) be given. Then \(M(O_{t_1/m}, \ldots, O_{t_p/m})O_{t_0/m}^{-n} \equiv O_{t_1/m}^{-n},\) which is an \(S\)-polynomial.

\(M \equiv \lambda x N\). Let \(FV(M) = \{x_1, \ldots, x_p\}\) and \(x \equiv x_{p+1}\). Then \(FV(N) = \{x_1, \ldots, x_{p+1}\}\). Take \(s_M = s_N\) and \(n_{M; t_0}, \ldots, t_p = n_{N; t_0}, \ldots, t_p + 1\). Then

\[\lambda I \vdash M(O_{t_1/m}, \ldots, O_{t_p/m})O_{t_0/m}^{-n} = (\lambda x_{p+1} N(O_{t_1/m}, \ldots, O_{t_p/m}, x_{p+1})O_{t_0/m}^{-n} = N(O_{t_1/m}, \ldots, O_{t_p/m}, O_{t_0/m}O_{t_0/m}^{-n}).\]

Since \(t_{1} \models s_M = s_{N}, m \geq n_{M; t_0}, \ldots, t_p > n_{N; t_0}, \ldots, t_p, t_0\) and \(n = n_{N; t_0}, \ldots, t_p, t_0\), this is provably equal to an \(S\)-polynomial by the induction hypothesis.

\(M \equiv xM_{1} \cdots M_{k}\). Let \(x \equiv x_{t_0}\). Take \(s_1 = s_{M_1} \cup \cdots \cup s_{M_k}\), \(s_M = s_1 \cup \langle k + 1, 0, s_1\rangle\) and \(n_{1; t_0}, \ldots, t_p = \text{Max}_{1 \leq i \leq k}(n_{M_i; (t_1, \ldots, t_p)}), n_{M; t_0}, \ldots, t_p = \text{Max}(\{t_{i_0}, n_{1; t_0}, t_p\})\).

Then

\[\lambda I \vdash M(O_{t_1/m}, \ldots, O_{t_p/m})O_{t_0/m}^{-n} = O_{t_0/m}M_{1}^* \cdots M_{k}O_{t_0/m}^{-n} = (M_{1}^*O_{t_1/m}O_{t_1/m}^{-n}) \cdots (M_{k}^*O_{t_p/m}O_{t_p/m}^{-n}),\]

where \(M_{i}^* = M(O_{t_1/m}, \ldots, O_{t_p/m})\) and \(\cdots\) consists of \(S\)-polynomials (in this step it is used that \(n \geq (t_{i_0})_0 \geq (s_{M})_0\) \(\geq k\)). Since, for \(j = 1, \ldots, p, t_j \models s_M \geq s_{M_j}, (t_{i_0})_2 \geq s_2 \models s_M, m \geq n_{M; t_1} \geq n_{M_1; (t_0), t_1, t_2}, \ldots, t_p\), by the induction hypothesis each \(M_{i}^*O_{t_1/m}O_{t_1/m}^{-n} \equiv M_{i}(O_{t_1/m}, \ldots, O_{t_p/m})O_{t_0/m}O_{t_0/m}^{-n}\) is provably equal to an \(S\)-polynomial. Hence \(M(O_{t_1/m}, \ldots, O_{t_p/m})O_{t_0/m}O_{t_0/m}^{-n}\) is provably equal to an \(S\)-polynomial.

2.12. **Corollary.** *If \(M\) is a closed \(L_1\) term in \(\beta\text{-n.f.}, then \(M\) is \(I\text{-solvable}.\)*

**Proof.** By the theorem, \(\lambda I \vdash M_{0}O_{t_0/m}^{-n} = P\) for some \(s, n\) and \(S\)-polynomial \(P\). Hence, by 2.7, \(M\) is \(I\)-solvable.

2.13. (a) \(\Rightarrow\) (c) of 2.1(i) follows immediately from 2.12. 2.1(ii) follows by repeating the proofs of 2.8, 2.11, 2.12 for the \(\lambda K\)-calculus.

The following corollary shows that a finite number of terms can be solved in a uniform way.

2.14. **Corollary.** *If \(M_{1}, \ldots, M_{k}\) are closed terms having a normal form, then, for some \(s, n, m,\)*

\[\lambda I \vdash M_{i}O_{t_0/m}^{-n}I_{t_0/m}^{-n} = I, \quad i = 1, \ldots, k.\]
Proof. For $L_i$: Let $s' = s_{M_1} \cup \cdots \cup s_{M_k}$. Take $n = \text{Max}\{n_{M_i}; s_i\}$. Then $M_iO_{s'/n}$ is provably equal to an S-polynomial. Hence, by 2.7, $\lambda I \vdash M_iO_{s'/n}I^{-m} = I$ for $m$ big enough, where $s = s'/n$. The proof for $L_K$ is similar, following the proof of 2.1(ii).

It follows that for a finite set of terms having a normal form $K$ can be simulated in the $\lambda I$-calculus.

2.15. Corollary. Let $X \subseteq L_I$ be a finite set of terms having a normal form. Then there is a $K^* \in L_I$ such that $\lambda I \vdash K^*MN = M$ for all $M \in L_I$ and all $N \in X$.

Proof. Let $X = \{M_1, \ldots, M_k\}$. By 2.14, $\lambda I \vdash M_iN_1 \cdots N_p = I$, $1 \leq i \leq k$, for some closed terms $N_1, \ldots, N_p \in L_I$. Define $K^* \equiv \lambda xy.xN_1 \cdots N_px$. Then $\lambda I \vdash K^*MN = NN_1 \cdots N_pM = IM = M$ provided $N \in X$.

REFERENCES


