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§0. Introduction. The theorem proved in this paper answers some transitivity questions (in the geometric sense) for the type free λ-calculus: Which objects can be mapped on all other objects? How much can an object do by applying it to other objects (see footnote 2)?

The main result is that, for closed terms of the λI-calculus, the following conditions are equivalent:

(a) M has a normal form.
(b) FM = I for some λI-term F.
(c) MN₁⋯Nₙ = I for some λI-terms N₁, ⋯, Nₙ.

By the same method it follows that if M is a closed term of the λK-calculus having a normal form, then for some λI-terms (sic) N₁, ⋯, Nₙ, MN₁⋯Nₙ = I is provable in the λK-calculus.

The theorem of Böhm [2] states that if M₁, M₂ are terms of the λK-calculus having different βη-normal forms, then ∀A₁, A₂ ∃N₁, ⋯, NₙM₁N₁⋯Nₙ = A₁ is provable in the λK-βη-calculus for i = 1, 2. As a consequence of this it was shown (implicitly) in [1, 3.2.20 1/2 (1)] that if M has a normal form, then for some λK-terms N₁, ⋯, Nₙ, MN₁⋯Nₙ = I is provable in the λK-calculus.

It was not clear that this also could be proved for the λI-calculus since the proof of the theorem of Böhm essentially made use of λK-terms.

We conjecture that, using the results of this paper, the full theorem of Böhm can be proved for the λI-calculus.²

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§1. Preliminaries. We assume familiarity with the λI- and the λK-calculus as treated e.g. in [4, Chapter 3] or [3, Chapters II, V].

1.1. Notation. L_I (L_K) is the language of the λI-calculus (λK-calculus). [x/N]M is the result of substituting N for the free occurrences of x in M. FV(M) is the set of free variables of M.

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2 Professor Böhm has informed us that, using Corollary 2.15, one can prove also for the λI-calculus his generalized theorem: Let M₁, ⋯, Mₙ be terms having different βη-normal forms, then

∀A₁⋯Aₙ ∃N₁⋯Nₙ λη ⊢ MₙN₁⋯Nₙ = Aᵢ, 1 ≤ i ≤ n.
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The \( \lambda \eta \)-calculus (\( \lambda K \eta \)-calculus) is the extensional theory containing \( \eta \)-reduction. When in a certain context \( L \), \( \lambda \) or \( \lambda \eta \) is used, \( L \), \( \lambda \) and \( \lambda \eta \) should be replaced throughout that context by \( L_\eta \), \( \lambda \eta \) and \( \lambda K \eta \) (theorems stated for \( L \), etc. hold for both versions).

"normal form" will be abbreviated by n.f.

\( MN \sim n \) is \( MN \cdots N \) (\( N \) appearing \( n \) times). \( \lambda(\eta) \vdash \) denotes provability in \( \lambda(\eta) \). \( \geq \) is the reduction relation, \( = \) the convertibility relation and \( \equiv \) the relation of syntactic identity.

1.2. Definition. Let \( M \) be a term \( \in L_\eta \). \( M \) is \( I(\eta)\)-solvable iff \( \exists N_1 \cdots N_n \in L_\eta \lambda(\eta) \vdash MN_1 \cdots N_n = I. \) \( M \) is \( K(\eta)\)-solvable iff \( \exists N_1 \cdots N_n \in L K \lambda K(\eta) \vdash MN_1 \cdots N_n = I. \)

By the following lemma there is no need to make a distinction between \( I(\eta)\)-solvable or \( K(\eta)\)-solvable.

1.3. Lemma. The \( \lambda K(\eta)\)-calculus is a conservative extension of the \( \lambda I(\eta)\)-calculus.

Proof. Show first \( [\lambda K(\eta) \vdash M \geq N \text{ and } M \in L_\eta] \Rightarrow [N \in L_\eta \text{ and } \lambda(\eta) \vdash M \geq N] \); then use the well-known Church-Rosser theorem (see e.g. [4, Chapter 4]) for \( \lambda K(\eta) \).

1.4. Lemma. Let \( M \) be a term \( \in L_\eta \). \( M \) has a \( \beta\)-n.f. \( \Rightarrow \text{ M has a } \beta\eta\)-n.f.

Proof. \( \Rightarrow \): Each \( \beta\)-n.f. has a \( \beta\eta\)-n.f. by contracting some \( \eta\)-redexes.

\( \Leftarrow \): See [5, Chapter HE, Lemma 13.1].

1.5. Lemma. \( M \) is \( I\)-solvable \( \Rightarrow \) \( M \) is \( I(\eta)\)-solvable;

\( M \) is \( K\)-solvable \( \Rightarrow \) \( M \) is \( K(\eta)\)-solvable.

Proof. (Same proof for both cases.) \( \Rightarrow \): Trivial.

\( \Leftarrow \): Suppose that \( \exists N_1 \cdots N_n \lambda(\eta) \vdash MN_1 \cdots N_n = I. \) Then \( MN_1 \cdots N_n \) has a \( \beta\eta\)-n.f., hence by 1.4, a \( \beta\)-n.f. \( M' \). \( M' \) has the properties: \( \lambda \vdash MN_1 \cdots N_n = M' \) and \( \lambda(\eta) \vdash M' \geq I \) (by the Church-Rosser theorem for \( \lambda(\eta) \)). Since \( M' \) is in \( \beta\)-n.f., \( M' \geq I \) is a pure \( \eta\)-reduction, say with the number of \( \eta\)-contractions \( q \). By induction on \( q \) it follows that \( M' \) must be of the form \( M' \equiv \lambda x_1 \cdots x_m . x_1 M_2 \cdots M_m \), where \( M_i \geq x_i \) (\( 2 \leq i \leq m \)) by an \( \eta\)-reduction and \( F \lor (M_i) = \{ x_i \} \). By induction on \( q \) it now follows that \( M' \) is solvable. If \( q = 0 \) this is clear. If \( q > 0 \), then \( m \geq 2 \) and \( M_i \geq x_i \) by an \( \eta\)-reduction of less than \( q \) steps. Hence also \( [x_i/I]M_i \geq I \) by an \( \eta\)-reduction of less than \( q \) steps. By the induction hypothesis,

\[ \exists N_{i_1} \cdots N_{i_k} \in L \lambda(\eta) \vdash [x_i/I]M_{i_1} \cdots N_{i_k} \geq I, \quad 2 \leq i \leq m. \]

Then

\[ \lambda \vdash M'_1 \cdots L_m = I, \]

where

\[ L_1 \equiv \lambda y_1 \cdots y_m . (y_2 N_{21} \cdots N_{2k_2}) \cdots (y_m N_{m1} \cdots N_{mk_m}), \quad L_2 \equiv \cdots \equiv L_m \equiv I. \]

Hence \( \lambda \vdash MN_1 \cdots N_{L_1} \cdots N_{L_m} = M'_1 \cdots L_m = I; \) i.e., \( M \) is solvable.

1.6. Lemma. If \( M \in L_\eta \) and has a n.f., then every subterm of \( M \) has a n.f.

Proof. See [3, p. 27, Theorem 7 XXII].

1.7. Example. Let \( \Omega \equiv (\lambda x . xx)(\lambda x . xx) \). Then \( \Xi \equiv \lambda x . x f \Omega \) is a term which is \( K\)-solvable but not \( I\)-solvable; \( \lambda K \vdash \Xi K \) is \( I \), but \( \Xi \) cannot be solved by \( \lambda\)-terms as follow from 1.6.

§2. Proof of the main theorem.

2.1. Theorem. (i) If \( M \) is a closed term of \( L_\eta \), the following are equivalent:
(a) \( M \) has a n.f.
(b) \( \exists F \in L_I, \lambda F FM = I \).
(c) \( M \) is I-solvable.

(ii) If \( M \) is a closed term of \( L_K \), then \( M \) has a n.f. \( \Rightarrow \) \( M \) is I-solvable.

Proof. (i) We show (c) \( \Rightarrow \) (b) \( \Rightarrow \) (a) \( \Rightarrow \) (c). (c) \( \Rightarrow \) (b): If \( M \) is I-solvable, then 
\[
\xi M N_1 \cdots N_n = I
\]
for some \( N_1, \ldots N_n \in L_I \). Take \( F = \lambda x.xN_1 \cdots N_n \). (b) \( \Rightarrow \) (a):
If \( \lambda F FM = I \) for some \( F \in L_I \), then \( FM \) has a n.f. Hence, by 1.6, \( M \) has a n.f.
(a) \( \Rightarrow \) (c): The proof of this fact occupies 2.3-2.13.

(ii) This will be a corollary to the proof of (a) \( \Rightarrow \) (c) of (i).

2.2. The converse of 2.1(ii) is false: Let \( M = \lambda x.xKQ \), where \( D \).

2.3. Definition. \( S \)-indices (integers) are defined inductively as follows:
0 is an \( S \)-index.
If \( s \) is an \( S \)-index, then, for integers all \( n \geq 1, m \geq 0, <a^\circ, m, s> \) is an \( S \)-index.

2.4. Definition. \( S \)-polynomials \( P \) and their depth \( d(P) \):
\( Os \) is an \( S \)-polynomial for every \( S \)-index \( s \); \( d(Os) = l(s) \).
If \( P_1, P_2 \) are \( S \)-polynomials, so is \( (P_1P_2) \); \( d(P_1P_2) = d(P_1) + d(P_2) \).

2.7. Lemma. Each \( S \)-polynomial \( P \) is I-solvable (using only I's).
Proof. Induction on \( d(P) \). If \( d(P) = 0 \), then \( P \) is a combination of I's and hence I-solvable. Suppose \( d(P) = n > 0 \). By contracting several I's, \( \lambda F P = OsP_1 \cdots P_p \), with \( s \neq p, p \geq 0 \) and \( d(OsP_1 \cdots P_p) = d(P) \). If \( p < (s)_0 \), then \( \lambda F \), \( = OsP_1 \cdots P_p I^<(s)_0 = P' \) where \( P' \equiv (P_1O_{s}^{-m}) \cdots (P_pO_{s}^{-m})(IO_{s}^{-m}) \cdots (IO_{s}^{-m})I^{-p} \) and \( m = (s)_1 \) and \( s' = (s)_2 \). We have \( d(P') = d(P_1 \cdots P_p) + m \cdot d(O_{s}) \cdot (s)_0 < d(P_1 \cdots P_p) + l(s) = d(OsP_1 \cdots P_p) = d(P) \). If \( p \geq (s)_0 \), then a similar argument shows that once more \( \lambda F P' = P' \) where \( P' \) is an \( S \)-polynomial with \( d(P') < d(P) \). By the induction hypothesis, \( P' \) is I-solvable, using only I's; thus \( \lambda F P' = P' \) for some \( m \). Hence \( \lambda F P' = P' \) for some \( m \).

2.8. Lemma. The class of \( L_I \) terms in \( \beta \)-n.f. has the following inductive definition:
\( x \in \beta \)-n.f. if and only if \( \lambda x.M \in \beta \)-n.f.
\( M_1, \ldots, M_k \in \beta \)-n.f. \( \Rightarrow xM_1 \cdots M_k \in \beta \)-n.f.
Proof. The terms obtained by this inductive definition are clearly in \( \beta \)-n.f. Conversely, every term has one of the three following forms: \( x, xM_1 \cdots M_k \) and \( (\lambda x.M_1)M_2 \cdots M_k \). The only \( \beta \)-n.f.'s among those are \( x, xM_1 \cdots M_k \) and \( \lambda x.M_1 \), if \( M_1, \ldots, M_k \) are in \( \beta \)-n.f.
$s \subseteq s'$ iff $[s = 0 \lor (s)_0 \leq (s')_0 \land (s)_1 \leq (s')_1 \land (s)_2 \leq (s')_2]$,
\[
s \cap s' = s' \text{ if } s = 0,
\]
\[
= \langle \operatorname{Max}(s)_0, \operatorname{Max}(s')_0, (s)_1, (s')_1, (s)_2, (s')_2 \rangle \text{ else.}
\]
$s/n = 0$ if $s = 0,$
\[
= \langle (s)_0, m, (s)_1/m \rangle \text{ else}
\]
Then $\subseteq$ is transitive, $s \cap s' \supseteq s, s \cap s' = s', s \cap s' \supseteq (s)_2 \supseteq (s')_2$ if $s' \neq 0$ and
\[
\langle n, m, s'/m' = \langle n, m', s/m' \rangle.
\]
2.10. Notation. We write $M(x_1, \ldots, x_p)$ to indicate that $FV(M) \subseteq \{x_1, \ldots, x_p\}$ and the $x_i$ are distinct. If $N_1, \ldots, N_p$ are closed terms, then $M(N_1, \ldots, N_p)$ is
\[
[x_1/N_1] \cdots [x_p/N_p] M.
\]
2.11. Lemma. For every term $M(x_1, \ldots, x_p)$ of $L_1$ in $\beta$-n. f.
\[
\exists s \forall t_0 \supseteq s, \ldots, t_p \supseteq s \exists n \forall m \geq n M(O_{1/m}, \ldots, O_{t_p/m}) O_{t_0/m}^n
\]
is provably equal (in $\lambda t$) to an $S$-polynomial.

Proof. Induction on the definition of $\beta$-n. f. ‘s given in 2.8. We write $s_M, n_{M,t}$ to indicate the dependence of $s$ and $n$ on $M, t$ ($t = t_0, \ldots, t_p$).

$M \equiv x$. Let $x = x_{t_0}$. Take $s_M = 0$, $n_{M,t} = 0$. Let $t, m$ be given. Then
\[
M(O_{t_1/m}, \ldots, O_{t_p/m}) O_{t_0/m}^n = \langle t_0, \ldots, t_p \rangle O_{t_0/m}^{n-1}
\]
is an $S$-polynomial.

$M \equiv \lambda x. N$. Let $FV(M) \subseteq \{x_1, \ldots, x_p\}$ and $x \equiv x_{t_p+1}$. Then $FV(N) \subseteq \{x_1, \ldots, x_{t_p+1}\}$. Take $s_M = s_N$ and $n_{M,t_0, \ldots, t_p} = n_{N,t_0, \ldots, t_p} + 1$. Then
\[
\lambda t M(O_{t_1/m}, \ldots, O_{t_p/m}) O_{t_0/m}^n = (\lambda x_{t_p+1} \cdot N(O_{t_1/m}, \ldots, O_{t_p/m}, x_{t_p+1})) O_{t_0/m}^n
\]
\[
= N(O_{t_1/m}, \ldots, O_{t_p/m}, O_{t_0/m}^n)
\]
Since $t_i \supseteq s_M = s_N$ and $t_{p+1}$ this is provably equal to an $S$-polynomial by the induction hypothesis.

$M \equiv \lambda x_1 \cdots x_k. M_k$. Let $x \equiv x_{t_0}$. Take $s_M = s_n \cup \ldots \cup s_{M_k}$, $s_M = s_1 \cup \langle k + 1, 0, s_i \rangle$ and $n_{M,t_0, \ldots, t_p} = \max_{1 \leq i \leq k} \{n_{M_i, (t_0)} t_0, \ldots, t_p \}$, $n_{M,t_0, \ldots, t_p} = \max \{s_{i, t_0, \ldots, t_p} \}$. Then
\[
\lambda t M(O_{t_1/m}, \ldots, O_{t_p/m}) O_{t_0/m}^n = M_{k_1}^* M_{k_1}^{*2} \cdots M_{k_1}^{n_{M,t_0, \ldots, t_p}} = M_{k_1}^* M_{k_1}^{*2} \cdots M_{k_1}^{n_{M,t_0, \ldots, t_p}} O_{t_0/m}^n
\]
where $M_{k_1}^* = M(O_{t_1/m}, \ldots, O_{t_p/m})$ and $\cdots$ consists of $S$-polynomials (in this step it is used that $n \geq (t_{i_0})_0 \geq (s_M)_0 \geq k$). Since, for $j = 1, \ldots, p, t_j \supseteq s_M \supseteq s_M, t_{i_0})_0 \geq (s_M)_0 \supseteq t_{i_0})_0 \geq n_{M,t_1, \ldots, t_p} \geq n_{M,t_1, \ldots, t_p}$, the induction hypothesis each $M_{k_1}^* O_{t_0/m}^n = M(O_{t_1/m}, \ldots, O_{t_p/m}) O_{t_0/m}^n$ is provably equal to an $S$-polynomial. Hence $M(O_{t_1/m}, \ldots, O_{t_p/m}) O_{t_0/m}^n$ is provably equal to an $S$-polynomial.

2.12. Corollary. If $M$ is a closed $L_1$ term in $\beta$-n. f., then $M$ is $I$-solvable.

Proof. By the theorem, $\lambda t M O_{t_0/m}^n = P$ for some $s, n$ and $S$-polynomial $P$. Hence, by 2.7, $M$ is $I$-solvable.

2.13. (a) $\Rightarrow$ (c) of 2.1(i) follows immediately from 2.12. 2.1(ii) follows by repeating the proofs of 2.8, 2.11, 2.12 for the $\lambda K$-calculus.

The following corollary shows that a finite number of terms can be solved in a uniform way.

2.14. Corollary. If $M_1, \cdots, M_k$ are closed terms having a normal form, then, for some $s, n, m$,
\[
\lambda t M_i O_{t_0/m}^n I^{-m} = I, \quad i = 1, \ldots, k.
\]
PROOF. For $L_\beta$: Let $s' = s_{M_1} \cup \cdots \cup s_{M_k}$. Take $n = \text{Max}\{n_{M_i}; s\}$. Then $M_iO_{s'/n}$ is provably equal to an $S$-polynomial. Hence, by 2.7, $\lambda I \vdash M_iO_{s'/n}I^m = I$ for $m$ big enough, where $s = s'/n$. The proof for $L_K$ is similar, following the proof of 2.1(ii).

It follows that for a finite set of terms having a normal form $K$ can be simulated in the $\lambda I$-calculus.

2.15. Corollary. Let $X \subseteq L_\beta$ be a finite set of terms having a normal form. Then there is a $K^* \in L_\beta$ such that $\lambda I \vdash K^*MN = M$ for all $M \in L_\beta$ and all $N \in X$.

Proof. Let $X = \{M_1, \ldots, M_k\}$. By 2.14, $\lambda I \vdash M_iN_1 \cdots N_p = I$, $1 \leq i \leq k$, for some closed terms $N_1, \ldots, N_p \in L_\beta$. Define $K^* \equiv \lambda xy.yN_1 \cdots N_p x$. Then $\lambda I \vdash K^*MN = NN_1 \cdots N_p M = IM = M$ provided $N \in X$.

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