Chapter 12
Topos theory and quantum logic

The topos-theoretic approach to quantum mechanics (also known as ***quantum toposophy***) has the same origin as the quantum logic programme initiated by Birkhoff and von Neumann, namely the feeling that classical logic is inappropriate for quantum theory and needs to be replaced by something else. For example, Schrödinger's Cat serves as an "intuition pump" for this feeling (at least in the naive view—dispensed with in Chapter 11—that it is neither alive nor dead). However, we feel that the quantum logic proposed by Birkhoff and von Neumann is:

- *too radical* in giving up distributivity (rendering it problematic to interpret the logical operations $\wedge$ and $\vee$ as conjunction and disjunction, respectively);
- *not radical enough* in keeping the law of excluded middle, which is precisely what intuition pumps like Schrödinger's cat and the like challenge.

Thus it would be preferable to have a quantum logic with exactly the opposite features, i.e., one that is distributive but drops the law of excluded middle: this suggest the use of *intuitionistic logic*. It is interesting to note that Birkhoff and von Neumann (who had earlier corresponded with Brouwer about possible intuitionistic aspects of game theory, notably chess) actually considered intuitionistic logic, but rejected it:

> 'The models for propositional calculi which have been considered in the preceding sections are also interesting from the standpoint of pure logic. Their nature is determined by quasi-physical and technical reasoning, different from the introspective and philosophical considerations which have had to guide logicians hitherto. Hence it is interesting to compare the modifications which they introduce into Boolean algebra, with those which logicians on "intuitionist" and related grounds have tried introducing. The main difference seems to be that whereas logicians have usually assumed that properties L71–L73 [i.e. $(a')' = a$, $a \cap a' = \perp$, $a \cup a' = \top$, and $a \subset b$ implies $a' \supset b'$] of negation were the ones least able to withstand a critical analysis, the study of mechanics points to the *distributive identities*as the weakest link in the algebra of logic. (...) Our conclusion agrees perhaps more with those critiques of logic, which find most objectionable the assumption that $a' \cup b = \top$ implies $a \subset b$ (or, dually, the assumption that $a \cap b' = \perp$ implies $b \supset a$—the assumption that to deduce an absurdity from the conjunction of a and not b, justifies one in inferring that a implies b).' (Birkhoff & von Neumann, 1936, p. 837).

As already made clear, then, our view is exactly the opposite. It is perhaps more striking that our position on (quantum) logic also differs from Bohr's:

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Fundamental Theories of Physics 188, DOI 10.1007/978-3-319-51777-3_12

> 'All departures from common language and ordinary logic are entirely avoided by reserving the word "phenomenon" solely for reference to unambiguously communicable information, in the account of which the word "measurement" is used in its plain meaning of standardized comparison.' (Bohr, 1996, p. 393)

Rather than *postulate* the logical structure of quantum mechanics, our goal is to *derive* it from our Bohrification ideology, more specifically, from the poset $\mathscr{C}(A)$ of all unital commutative C*-subalgebras of a unital C*-algebra A, ordered by inclusion. One may think of this poset as a mathematical home for Bohr's notion of ***Complementarity***, in that each $C \in \mathscr{C}(A)$ represents some classical or experimental context, which has been decoupled from the others, *except for the inclusion relations, which relate* compatible *experiments* (in general there seem to be no preferred *pairs* of complementary subalgebras $C, C' \in \mathscr{C}(A)$ that jointly generate A, although Bohr typically seems to have had such pairs in mind, e.g. position and momentum).

Quantum toposophy also accommodates the feeling that quantum mechanics is so radical that not just the actors of classical mechanics, but its whole stage must be replaced. This need is well expressed by the following quotation from Grothendieck, who created topos theory (but never witnessed its application to quantum theory):

> 'Passer de la mécanique de Newton à celle d'Einstein doit être un peu, pour le mathématicien, comme de passer du bon vieux dialecte provençal à l'argot parisien dernier cri. Par contre, passer à la mécanique quantique, j'imagine, c'est passer du français au chinois.' (Grothendieck, 1986, p. 61).[1]

Indeed, topos theory replaces even set theory, seen as the stage of classical mathematics and physics, by some other stage: each topos provides a "universe of discourse" in which to do mathematics. One major difference with set theory, then, is that logic in most toposes (including the ones we will use) is ... intuitionistic!

This chapter presupposes familiarity with §C.11 on the logical side of the Gelfand isomorphism for commutative C*-algebras, Appendix D on lattice theory and logic, and Appendix E on topos theory. Since this material is off the beaten track, as in Chapter 6 it may be helpful to provide a very brief guided tour through this chapter.

In §12.1 we first define the "quantum mechanical" topos $\mathsf{T}(A)$ that will act as the mathematical stage for the remainder of the chapter; it depends some given (unital) C*-algebra A only via the poset $\mathscr{C}(A)$. We then define C*-algebras internal to any topos T (in which the natural numbers and hence the rationals can be defined), which notion we then apply to $\mathsf{T} = \mathsf{T}(A)$, so as to define an internal C*-algebra $\underline{A}$, which turns out to be *commutative*. Following an interlude on constructive Gelfand spectra in §12.2, in §12.3 we then compute the internal Gelfand spectrum of $\underline{A}$ for $A = M_n(\mathbb{C})$, and derive our intuitionistic logic of quantum mechanics from this, given by eqs. (12.95) - (12.96) and (12.103) - (12.107). We also discuss its (Kripke) semantics. In §12.4 we generalize these computations to arbitrary (unital) C*-algebras A, culminating in Corollary 12.22. Finally, in §12.5 we relate this material to both the Kochen–Specker Theorem (which provided the original motivation for quantum toposophy), as well as to an attempt at ontology called "Daseinisation."

[1] 'For a mathematician, switching from Newton's mechanics to Einstein's must to some extent be like switching from a good old provincial dialect to Paris slang. In contrast, I imagine that switching to quantum mechanics amounts to switching to Chinese.' Translation by the author.

12.1 C*-algebras in a topos

Let A be a unital C*-algebra (in Sets), with associated poset $\mathscr{C}(A)$ of all unital commutative C*-subalgebras $C \subset A$ ordered by inclusion. Regarding $\mathscr{C}(A)$ as a (posetal) category, in which there is a unique arrow $C \to D$ iff $C \subseteq D$ and there are no other arrows, we obtain the topos $\mathsf{T}(A)$ of functors $\underline{F} : \mathscr{C}(A) \to \mathsf{Sets}$ (F underlined!), i.e.,

$$\mathsf{T}(A) = [\mathscr{C}(A), \mathsf{Sets}]. \tag{12.1}$$

Since for any poset X we have an isomorphism of categories $[X, \mathsf{Sets}] \simeq \mathsf{Sh}(X)$, where X is endowed with the Alexandrov topology, see (E.84), we may alternatively write

$$\mathsf{T}(A) \simeq \mathsf{Sh}(\mathscr{C}(A)). \tag{12.2}$$

This alternative description will turn out to be very useful in computing the Gelfand spectrum of the internal commutative C*-algebra $\underline{A}$ to be defined shortly. Since we occasionally switch between $\mathsf{T}(A)$ and the topos Sets, we underline objects (i.e., functors $\underline{F} : \mathscr{C}(A) \to \mathsf{Sets}$) of the former. In order to do some kind of Analysis in $\mathsf{T}(A)$, we need real numbers. In many toposes this is a tricky concept, but:

Proposition 12.1. *In* $\mathsf{T}(A)$*, the* **Dedekind reals** *are given by the constant functor*

$$\underline{\mathbb{R}}_0 : C \mapsto \mathbb{R}, \tag{12.3}$$

where $C \in \mathscr{C}(A)$*, with associated frame given by the functor*

$$\mathscr{O}(\underline{\mathbb{R}})_0 : C \mapsto \mathscr{O}((\uparrow C) \times \mathbb{R}). \tag{12.4}$$

Similarly, we have complex numbers $\underline{\mathbb{C}}$ and their frame $\mathscr{O}(\underline{\mathbb{C}})$ in $\mathsf{T}(A)$.

Proof. In a general sheaf topos $\mathsf{Sh}(X)$, the Dedekind real numbers object is the sheaf (E.150), with frame (E.149). The point now is that each continuous function $f \in C(\mathscr{C}(A), \mathbb{R})$ on $X = \mathscr{C}(A)$ with the Alexandrov topology is locally constant.

To see this, suppose $C \leq D$ in U, and take $V \subseteq \mathbb{R}$ open with $f(C) \in V$. Then $C \in f^{-1}(V)$ and $f^{-1}(V)$ is open by continuity of f. But the smallest open set containing C is $\uparrow C$, which contains D, so that $f(D) \in V$. Taking $V = (f(C) - \varepsilon, \infty)$ gives the inequality $f(D) > f(C) - \varepsilon$ for all $\varepsilon > 0$, whence $f(D) \geqslant f(C)$, whereas $V = (-\infty, f(C) + \varepsilon)$ yields $f(D) \leq f(C)$. Hence $f(C) = f(D)$.

Thus we obtain (12.3) - (12.4) as special cases of (E.150) - (E.149). □

Other objects of interest in $\mathsf{T}(A)$ that we will steadily use are:

- The ***terminal object*** $\underline{\mathbf{1}}$, i.e., the constant functor $C \mapsto *$, where $*$ is a singleton.
- The ***truth object*** $\underline{\Omega}$, which according to (E.86) - (E.87) is given by

$$\underline{\Omega}_0(C) = \text{Upper}(C); \tag{12.5}$$
$$\underline{\Omega}_1(C \subseteq D) = (-) \cap (\uparrow D), \tag{12.6}$$

where $\mathrm{Upper}(C)$ is the set of all upper sets above C (i.e., $S \in \mathrm{Upper}(C)$ iff $S \subset \mathscr{C}(A)$ such that: (i) $C \subseteq D$ for each $D \in S$, and (ii) $D \in S$ and $D \subseteq E$ imply $E \in S$).

- The ***subobject classifier*** $t : \underline{\mathbf{1}} \to \underline{\Omega}$, which is a natural transformation whose components t_C are given, according to (E.88), as

$$t_C(*) = \uparrow C, \tag{12.7}$$

i.e., the set of all $D \supseteq C$ in $\mathscr{C}(A)$; this is the maximal element of $\mathrm{Upper}(C)$.

Furthermore, exponentials in $\mathsf{T}(A)$ have the following straightforward description:

$$\underline{F}_0^{\underline{G}}(C) = \mathrm{Nat}(\underline{G}_{\uparrow C}, \underline{F}_{\uparrow C}) \quad (C \in \mathscr{C}(A)), \tag{12.8}$$

where $\underline{F}_{\uparrow C}$ is the restriction of the functor $\underline{F} : \mathscr{C}(A) \to \mathsf{Sets}$ to $\uparrow C \subseteq \mathscr{C}(A)$, and $\mathrm{Nat}(-,-)$ denotes the set of natural transformations between the functors in question. In particular, since $\mathbb{C} \cdot 1$ is the bottom element of the poset $\mathscr{C}(A)$, one has

$$\underline{F}^{\underline{G}}(\mathbb{C} \cdot 1) = \mathrm{Nat}(\underline{G}, \underline{F}). \tag{12.9}$$

One way to derive (12.8) is to start from general sheaf toposes $\mathsf{Sh}(X)$, where

$$F_0^G(U) = \mathrm{Nat}(G_{|U}, F_{|U}), \tag{12.10}$$

both restricted to $\mathscr{O}(U)$ (i.e. defined on each open $V \subseteq U$ instead of all $V \in \mathscr{O}(X)$), and use (E.84). Combining these observations, one has

$$\underline{\Omega}^{\underline{F}}(C) \cong \mathrm{Sub}(\underline{F}_{\uparrow C}), \tag{12.11}$$

i.e., the set of subfunctors of $\underline{F}_{\uparrow C}$. In particular, like in (12.9), we find

$$\Omega^{\underline{F}}(\mathbb{C} \cdot 1) \cong \mathrm{Hom}(\underline{F}, \Omega) \cong \mathrm{Sub}(\underline{F}), \tag{12.12}$$

the set of subfunctors of $\underline{F}$ itself. Recall that, as explained after Lemma E.16, a subfunctor $\underline{Z} \in \mathrm{Sub}(\underline{F})$ is a functor $\underline{Z} : \uparrow\mathscr{C}(A) \to \mathsf{Sets}$ for which $\underline{Z}_0(C) \subseteq \underline{F}_0(C)$ for all $C \in \mathscr{C}(A)$ and $\underline{Z}_1$ is the restriction of $\underline{F}_1$. If $C \subseteq D$, then the set-theoretic map $\Omega^{\underline{F}}(C) \to \Omega^{\underline{F}}(D)$ defined by $\Omega^{\underline{F}}$, identified with a map $\mathrm{Sub}(\underline{F}_{\uparrow C}) \to \mathrm{Sub}(\underline{F}_{\uparrow D})$, is simply given by restricting a given subfunctor of $\underline{F}_{\uparrow C}$ to $\uparrow D$.

Using either the internal language of a topos (see §E.5) or direct object-arrow constructions, one can copy standard definitions in set theory so as to define mathematical objects "internal" to any given topos, *as long as these definitions make sense in first-order intuitionistic logic* (which roughly speaking means that they are "constructive", in not using the axiom of choice or the law of the excluded middle).

As a case in point, let us now define ***internal C*-algebras*** in $\mathsf{T}(A)$ (this may be done even more generally in any topos T in which at least the natural numbers $\mathbb{N}$, and hence the rationals $\mathbb{Q}$, are defined). Vector spaces (over $\underline{\mathbb{R}}$ or $\underline{\mathbb{C}}$) and (commutative) *-algebras may be defined in $\mathsf{T}(A)$ through straightforward object-arrow translations of the usual constructions in Sets, i.e., one has an object $\underline{A}$ and arrows:

$$\cdot : \underline{\mathbb{C}} \times \underline{A} \to \underline{A} \quad \text{(scalar multiplication);} \tag{12.13}$$
$$+ : \underline{A} \times \underline{A} \to \underline{A} \quad \text{(addition);} \tag{12.14}$$
$$\times : \underline{A} \times \underline{A} \to \underline{A} \quad \text{(multiplication);} \tag{12.15}$$
$$* : \underline{A} \to \underline{A} \quad \text{(involution),} \tag{12.16}$$

subject to the usual axioms. Syntactically, a ***unit (internal)*** in $\underline{A}$ is a constant

$$1_{\underline{A}} : \underline{\mathbf{1}} \to \underline{A},$$

with $\underline{\mathbf{1}}$ the terminal object in $\mathsf{T}(A)$, such that

$$\left(\underline{A} \xrightarrow{\cong} \mathbf{1} \times \underline{A} \xrightarrow{(1_{\underline{A}}, \mathrm{id}_{\underline{A}})} \underline{A} \times \underline{A} \xrightarrow{\times} \underline{A}\right) = \left(\underline{A} \xrightarrow{\mathrm{id}_{\underline{A}}} \underline{A}\right). \tag{12.17}$$

The notions of norm and completeness are less easily defined internally, and hence one starts reinterpreting the notion of a ***seminorm*** in Sets as a subset

$$N \subset A \times \mathbb{Q}^+, \tag{12.18}$$

for which

$$(a,q) \in \underline{N} \text{ iff } \|a\| < q. \tag{12.19}$$

In our topos $\mathsf{T}(A)$, we interpret $\underline{N} \subset \underline{A} \times \underline{\mathbb{Q}}^+$ as a subfunctor $\underline{N} \to \underline{A} \times \underline{\mathbb{Q}}^+$ (or, equivalently by λ-conversion (E.153), as an arrow $\mathbf{1} \to \underline{\Omega}^{\underline{A} \times \underline{\mathbb{Q}}^+}$), subject to the axioms:

$$\forall_p\, p > 0 \to (0,p) \in \underline{N}; \tag{12.20}$$
$$\exists_q\, q > 0 \wedge (a,q) \in \underline{N}; \tag{12.21}$$
$$\forall_a \forall_p\, (a,p) \in \underline{N} \to (a^*,p) \in \underline{N}; \tag{12.22}$$
$$\forall_a \forall_q\, ((a,q) \in \underline{N} \leftrightarrow \exists_p p < q \wedge (a,p) \in \underline{N}); \tag{12.23}$$
$$\forall_a \forall_p\, ((a,p) \in \underline{N} \wedge (b,q) \in \underline{N} \to (a+b, p+q) \in \underline{N}); \tag{12.24}$$
$$\forall_a \forall_p\, ((a,p) \in \underline{N} \wedge (b,q) \in \underline{N} \to (a \cdot b, p \cdot q) \in \underline{N}); \tag{12.25}$$
$$\forall_a \forall_p \forall_z ((a,p) \in \underline{N} \wedge (|z| < q) \to (z \cdot a, p \cdot q) \in \underline{N}). \tag{12.26}$$

Here a,b are variables of type $\underline{A}$, p and q are variables of type $\underline{\mathbb{Q}}$, z is a variable of type $\underline{\mathbb{C}}$, 0 is the zero constant in $\underline{A}$, etc. For a unital *-algebra (whose internal definition we leave to the reader), with unit denoted by 1_A as usual, we also require

$$\Vdash \forall_a \forall_p p > 1 \to (1_A, p) \in \underline{N}. \tag{12.27}$$

If the seminorm relation furthermore satisfies

$$(a^* \cdot a, q^2) \in \underline{N} \leftrightarrow \ (a,q) \in \underline{N} \tag{12.28}$$

for all $a \in A$ and $q \in \mathbb{Q}^+$, then A is said to be a ***pre-semi-C*-algebra***.

To proceed to a C*-algebra, one requires $a = 0$ whenever $(a,q) \in N$ for all q in $\mathbb{Q}^+$, making the seminorm into a norm, and subsequently this normed space should be complete. The latter condition is quite complicated, since in a topos one has no Cauchy sequences in the usual sense, because $\underline{A}$ may not have global elements (in the sense of arrows $\mathbf{1} \to \underline{A}$). Indeed, our algebra $\underline{A}$ defined below only has trivial global elements, namely multiples of the the unit operator.

Hence one needs a generalization of Cauchy sequences in the general spirit of topos theory, where global elements are replaced by general elements.

Definition 12.2. *With $\underline{\mathbb{N}}$ the natural numbers object in $\mathsf{T}(A)$ (which is simply the constant functor $C \mapsto \mathbb{N}$), a* **Cauchy approximation** *in $\underline{A}$ is an arrow $s : \underline{\mathbb{N}} \to \underline{\Omega}^{\underline{A}}$ (or, equivalently, by λ-conversion* (E.153)*, an arrow $\chi : \underline{\mathbb{N}} \times \underline{A} \to \underline{\Omega}$, which in turn is the same as a subobject $\underline{S}$ of $\underline{\mathbb{N}} \times \underline{A}$) such that:*

$$\forall_n \exists_a \, a \in s_n; \tag{12.29}$$

$$\forall_k \exists_m \forall_n \forall_{n'} (n > m, n' > m, a \in s_n, a' \in s_{n'}) \to (a - a', 1/k) \in \underline{N}. \tag{12.30}$$

Here (for brevity) the first three comma's (but not the last!) stand for $\wedge$, and $a \in s_n$ denotes $(n,a) \in \underline{S}$, where $\underline{S}$ is the above subobject of $\underline{\mathbb{N}} \times \underline{A}$ classified by χ (we use the notation explained in item 9 at the end of §E.5, where the variable $x : X$ is now the pair (n,a) of type $\underline{\mathbb{N}} \times \underline{A}$). Moreover, a Cauchy approximation **converges** *to b if:*

$$\forall_k \exists_m \forall_n \, (n > m, a \in s_n) \to (a - b, 1/k) \in \underline{N}, \tag{12.31}$$

and we call A **complete** *if each Cauchy approximation in A converges.*

Finally, a **C*-algebra in** $\mathsf{T}(A)$ *(and similarly in any topos with natural numbers) is a complete pre-semi-C*-algebra in which the semi-norm is a norm.*

Homomorphisms and isomorphisms between such (internal) C*-algebras may be defined in the usual way, bijections in set theory being replaced by isomorphisms of objects. We only consider internal C*-algebras with unit, so that we may define internal categories $\underline{\mathsf{CA}}_1$ (and $\underline{\mathsf{CCA}}_1$) of (commutative) unital C*-algebras in $\mathsf{T}(A)$ in the obvious way (where the homomorphisms are required to preserve the unit).

We now come to the basic construction that underlies "quantum toposophy".

Theorem 12.3. *Let A be a unital C*-algebra. Define a functor $\underline{A} \in \mathsf{T}(A)$ by*

$$\underline{A} : \mathscr{C}(A) \to \mathsf{Sets}; \tag{12.32}$$

$$\underline{A}_0(C) = C; \tag{12.33}$$

$$\underline{A}_1(C \subseteq D) = (C \hookrightarrow D). \tag{12.34}$$

Then $\underline{A}$ is an internal unital commutative *C*-algebra under pointwise operations.*

Here A is meant to be an "ordinary" unital C*-algebra, i.e., defined in Sets. Note that the symbol C in (12.33) changes character from left to right: on the left-hand side it is a *point* in $\mathscr{C}(A)$, whereas on the right-hand side it is a *subset* of A. Nonetheless, one might describe $\underline{A}$ as the ***tautological functor*** in $[\mathscr{C}(A), \mathsf{Sets}]$.

The pointwise operations in $\underline{A}$ are the obvious natural transformations that are ultimately defined by the corresponding operations in each commutative C*-algebra C. For exampe, addition $+ : \underline{A} \times \underline{A} \to \underline{A}$ is a natural transformation with components $+_C : C \times C \to C$ defined in C, etc. Commutativity of $\underline{A}$ then trivially follows from commutativity of each *commutative* C*-subalgebra C.

As already mentioned, the unit $1_{\underline{A}}$ is syntactically a constant $1_{\underline{A}} : \underline{\mathbf{1}} \to \underline{A}$, whose components $(1_{\underline{A}})_C : * \to C$ are just the units 1_C in each C (recall that elements of our poset $\mathscr{C}(A)$ were defined as *unital* commutative C*-subalgebras of A!).

Finally, we regard the (semi) norm $\underline{N}$ as a subobject of $\underline{A} \times \underline{\mathbb{R}}^+$ (or $\underline{A} \times \underline{\mathbb{Q}}^+$), hence as a natural transformation, with components $\underline{N}_C \subset C \times \mathbb{R}^+$ defined by

$$(c,q) \in \underline{N}_C \text{ iff } \|c\| < q, \tag{12.35}$$

where $\|\cdot\|$ is the norm in C (which of course is inherited from A).

Proof. The proof is a straightforward verification, expect perhaps for completeness. First, the above subobject $\underline{S}$ of $\underline{\mathbb{N}} \times \underline{A}$, realized as a subfunctor as usual, looks as follows: for each $C \in \mathscr{C}(A)$ we have a subset $\underline{S}_C \subset \mathbb{N} \times C$, regarded as a sequence (C_n) of subsets of C through the identification $(n,c) \in \underline{S}_C$ iff $c \in C_n$, such that $C_n \subset D_n$ whenever $C \subset D$. Unfolding axiom (12.29) using the Kripke–Joyal semantics rules listed at the end of §E.5, we find that this axiom holds iff:

$$\forall_{C \in \mathscr{C}(A)} \forall_{n \in \mathbb{N}} \exists_{c \in C} \forall_{D \supseteq C}\, c \in D_n, \tag{12.36}$$

which is satisfied iff each of the above subsets $C_n \subseteq C$ is non-empty. By a similar analysis, axiom (12.30) is satisfied iff for each $\varepsilon > 0$ there is $m \in \mathbb{N}$ such that for all $n, n, > m$ and all $c \in C_n$, $c' \in C_{n'}$ one has $\|c - c'\| < \varepsilon$ in C. This simply means that any choice (c_n) where $c_n \in C_n$ is a Cauchy sequence in C. Accordingly, $\underline{A}$ is complete provided each such sequence converges, i.e., iff each $C \in \mathscr{C}(A)$ is complete. Since these C's are C*-subalgebras of C, this is simply true by construction. □

In a similar way, one easily proves the following generalization of Theorem 12.3:

Theorem 12.4. *Let* C *be a small category. Any internal C*-algebra in the associated presheaf topos* $[\mathsf{C}^{\mathrm{op}}, \mathsf{Sets}]$ *is given by a contravariant functor* $\underline{A} : \mathsf{C} \to \mathsf{CA}$*, where* CA *is the category that has C*-algebras as objects and homomorphims as arrows. Moreover,* $\underline{A}$ *is unital/commutative iff each C*-algebra* $\underline{A}(C)$ *is unital/commutative.*

It should be mentioned that internal C*-algebras on sheaf toposes $\mathsf{T} = \mathsf{Sh}(X)$ are not covered by this theorem (except in the somewhat degenerate case we use, namely $X = \mathscr{C}(A)$ with the Alexandrov topology). As a case in point, we just mention the beautiful fact that internal C*-algebras in $\mathsf{Sh}(X)$ correspond to continuous bundles of C*-algebras over X (in Sets).

12.2 The Gelfand spectrum in constructive mathematics

In this chapter we rely on a particular construction of the frame $\mathscr{O}(\Sigma(A))$ (cf. §C.11) that can be generalized to topos theory (in which the Gelfand spectrum $\Sigma(A)$ of an internal commutative C*-algebra A is a locale). We start with some lattice lore.

Definition 12.5. *Let L be a* distributive *lattice with top $\top$ and bottom $\perp$.*

1. *A* **lower set** *in L is a subset $S \subseteq L$ such that if $x \in S$ and $y \leq x$, then $y \in S$. We denote the poset of all lower subsets of L, ordered by inclusion, by* $\mathrm{D}(L)$*.*
2. *An* **ideal** *in a lattice L is a lower set I in L such that $x, y \in I$ implies $x \vee y \in I$. The poset all ideals in a lattice L, ordered by inclusion, is denoted by* $\mathrm{Idl}(L)$*.*
3. *We say that $x \ll y$ (in words: "x* **is well inside** *y" or "x* **is rather below** *y") iff there exists z such that $x \wedge z = \perp$ and $y \vee z = \top$. Note that $x \ll y$ implies $x \leq y$, as*

$$x = x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) = x \wedge y \leq y. \tag{12.37}$$

4. *An ideal $I \in \mathrm{Idl}(L)$ is* **regular** *if the condition $I \supseteq \{y \in L \mid y \ll x\}$ implies $x \in I$. The poset of regular ideals in L, ordered by inclusion, is called* $\mathrm{RIdl}(L)$*, i.e.,*

$$\mathrm{RIdl}(L) = \{I \in \mathrm{Idl}(L) \mid (\forall_{y \in L}\, y \ll x \Rightarrow y \in I) \Rightarrow x \in I\}. \tag{12.38}$$

The posets $\mathrm{D}(L)$, $\mathrm{Idl}(L)$ and $\mathrm{RIdl}(L)$ are easily seen to be frames. Any ideal $I \in \mathrm{Idl}(L)$ can be *regularized*, i.e., turned into a regular ideal $\mathscr{A}(I)$, by means of the restriction to $\mathrm{Idl}(L) \subset \mathrm{D}(L)$ of the "closure" map $\mathscr{A} : \mathrm{D}(L) \to \mathrm{D}(L)$ defined by

$$\mathscr{A}(I) = \{x \in L \mid \forall_{y \in L}\, y \ll x \Rightarrow y \in I\}. \tag{12.39}$$

In terms of $\mathscr{A}$, the canonical map $x \mapsto \downarrow\! x$ from L to $\mathrm{Idl}(L)$ "regularizes" to a map

$$f : L \to \mathrm{RIdl}(L); \tag{12.40}$$
$$x \mapsto \mathscr{A}(\downarrow\! x). \tag{12.41}$$

For $I \in \mathrm{RIdl}(L)$ we obviously have $\mathscr{A}(I) = I$, and hence we may write

$$\mathrm{RIdl}(L) = \{I \in \mathrm{Idl}(L) \mid \mathscr{A}(I) = I\}. \tag{12.42}$$

Definition 12.6. *1. A frame $\mathscr{O}(X)$ with top element $\top$ is called* **compact** *if every subset $S \subset \mathscr{O}(X)$ with $\bigvee S = \top$ has a finite subset $F \subset S$ with $\bigvee F = \top$.*
2. A frame $\mathscr{O}(X)$ is called **regular** *if each $V \in \mathscr{O}(X)$ satisfies*

$$V = \bigvee \{U \in \mathscr{O}(X) \mid U \ll V\}. \tag{12.43}$$

When $\mathscr{O}(X)$ is the topology of some space X, the frame $\mathscr{O}(X)$ is compact (regular) iff X is compact (regular) as a space. Furthermore, X is compact and Hausdorff iff it is compact and regular, and hence the Gelfand spectrum $\Sigma(A)$ of a commutative unital C*-algebra A will be a compact and regular frame; see Theorem 12.8 below.

Recall that the self-adjoint part A_{sa} of any C*-algebra A is partially ordered by putting $a \leq b$ iff $b - a \in A^+$, cf. §C.7. This partial order is, of course, inherited by the positive cone $A^+ \subset A_{\mathrm{sa}}$. If A is commutative, this partial ordering makes A_{sa} a lattice; for example, if $A = C(X)$ the lattice operations are $a \vee b = \max\{a,b\}$ and $a \wedge b = \min\{a,b\}$ (taken pointwise). In general, one may then compute $\vee$ and $\wedge$ from the Gelfand isomorphism $A \cong C(X)$, but they are intrinsically defined via $\leq$.

Let A be a commutative unital C*-algebra. For $a, b \in A^+$, define $a \preccurlyeq b$ iff there exists $n \in \mathbb{N}$ such that $a \leq nb$. Define $a \approx b$ iff $a \preccurlyeq b$ and $b \preccurlyeq a$. This is an equivalence relation. Moreover, $\approx$ is a ***congruence***, that is, an equivalence relation $\sim$ on a lattice L that is compatible with $\wedge$ and $\vee$ in the sense that $x \sim y$ and $x' \sim y'$ imply $x \wedge x' \sim y \wedge y'$ and $x \vee x' \sim y \vee y'$. Given some congruence $\sim$ on L, one may define $\wedge$ and $\vee$ on $L/\sim$ by $[x] \wedge [y] = [x \wedge y]$ and $[x] \vee [y] = [x \vee y]$, respectively, so that the set-theoretic quotient $L/\sim$ inherits the lattice structure of L and hence is a lattice in its own right.

This quotient construction by a congruence preserves distributivity, so that

$$L_A = A^+/\approx . \tag{12.44}$$

is a distributive lattice. We will use the elements $\mathsf{D}_a \equiv [a^+]$ of L_A (indexed by $a \in A_{\mathrm{sa}}$), where $[a^+]$ is the equivalence class in L_A of the positive part a^+ in the canonical decomposition $a = a^+ - a^-$, with $a^{\pm} \geq 0$ and $a_+ a_- = 0$; lattice-theoretically, one has $a_+ = a \vee 0$ and $a_- = a \wedge 0$. This gives a lattice homomorphism $A_{\mathrm{sa}} \to L_A$, $a \mapsto \mathsf{D}_a$, whose restriction to A^+ is just the canonical projection $A^+ \to L_A$. These D_a satisfy:

$$\mathsf{D}_1 = \top; \tag{12.45}$$
$$\mathsf{D}_a \wedge \mathsf{D}_{-a} = \bot; \tag{12.46}$$
$$\mathsf{D}_a = \bot \;\; (a \leq 0); \tag{12.47}$$
$$\mathsf{D}_{a+b} \leqslant \mathsf{D}_a \vee \mathsf{D}_b; \tag{12.48}$$
$$\mathsf{D}_a \wedge \mathsf{D}_b \leqslant \mathsf{D}_{ab}; \tag{12.49}$$
$$\mathsf{D}_{ab} \leqslant \mathsf{D}_a \vee \mathsf{D}_{-b}, \tag{12.50}$$

where the inequalities may also be written as equalities, since $x \leq y$ iff $x = x \wedge y$. These relations are easy to check for $A = C(X)$, and hence they are true for any A. The elements D_a obviously exhaust A^+, and eqs. (12.45) - (12.50) imply:

$$a \leq b \;\Rightarrow\; \mathsf{D}_a \leqslant \mathsf{D}_b; \tag{12.51}$$
$$\mathsf{D}_a = \mathsf{D}_{a^+}; \tag{12.52}$$
$$\mathsf{D}_{na} = \mathsf{D}_a \;\; (n \in \mathbb{N}); \tag{12.53}$$
$$\mathsf{D}_{ab} = (\mathsf{D}_a \wedge \mathsf{D}_b) \vee (\mathsf{D}_a \wedge \mathsf{D}_{-b}; \tag{12.54}$$
$$\mathsf{D}_a \wedge \mathsf{D}_b = \mathsf{D}_{a \wedge b}. \tag{12.55}$$

For the Gelfand spectrum we need the frame $\mathrm{RIdl}(L_A)$, and hence the relation $\ll$.

Lemma 12.7. *For all $\mathsf{D}_a, \mathsf{D}_b \in L_A$, we have (with both $q \in \mathbb{Q}^+$ and $q \in \mathbb{R}^+$):*

$$\mathsf{D}_b \ll \mathsf{D}_a \textit{ iff } \exists_{q>0} \mathsf{D}_b \leqslant \mathsf{D}_{a-q}. \tag{12.56}$$

Proof. From right to left, just choose $\mathsf{D}_c = \mathsf{D}_{q-a}$. Conversely, if $A = C(X)$, it is easy to see that if there exists $\mathsf{D}_c \in L_A$ such that $\mathsf{D}_c \vee \mathsf{D}_a = \top$ and $\mathsf{D}_c \wedge \mathsf{D}_b = \bot$, then there exists $q > 0$ such that $\mathsf{D}_{c-q} \vee \mathsf{D}_{a-q} = \top$. Hence $\mathsf{D}_c \vee \mathsf{D}_{a-q} = \top$, so that

$$\mathsf{D}_b = \mathsf{D}_b \wedge (\mathsf{D}_c \vee \mathsf{D}_{a-q}) = \mathsf{D}_b \wedge \mathsf{D}_{a-q} \leqslant \mathsf{D}_{a-q}. \qquad \square$$

Note that by construction the map f in (12.40) is given by

$$f(\mathsf{D}_a) = \{\mathsf{D}_c \in L_A \mid \forall_{\mathsf{D}_b \in L_A} \mathsf{D}_b \ll \mathsf{D}_c \Rightarrow \mathsf{D}_b \leqslant \mathsf{D}_a\}, \qquad (12.57)$$

and, by Lemma 12.7, satisfies

$$f(\mathsf{D}_a) \leqslant \bigvee \{f(\mathsf{D}_{a-q}) \mid q > 0\}. \qquad (12.58)$$

For later use, also note that (12.57) implies

$$f(\mathsf{D}_a) = \top \Leftrightarrow \mathsf{D}_a = \top. \qquad (12.59)$$

Theorem 12.8. *The topology $\mathcal{O}(\Sigma(A))$ of the Gelfand spectrum $\Sigma(A)$ of a commutative unital C*-algebra A is isomorphic to the frame of all regular ideals of L_A:*

$$\mathcal{O}(\Sigma(A)) \cong \mathrm{RIdl}(L_A); \qquad (12.60)$$

$$\{\omega \in \Sigma(A) \mid \omega(a) > 0\} \leftrightarrow \mathsf{D}_a, \qquad (12.61)$$

or, equivalently, for the opens $(r,s) \in \mathcal{O}(\mathbb{R})$ with ensuing opens $\hat{a}^{-1}(r,s)$ in $\mathcal{O}(\Sigma(A))$,

$$\hat{a}^{-1}(r,s) \equiv \{\omega \in \Sigma(A) \mid \omega(a) \in (r,s)\} \leftrightarrow f(\mathsf{D}_{s-a} \wedge \mathsf{D}_{a-r}) \ (r < s). \qquad (12.62)$$

Moreover, on this isomorphism, $\mathcal{O}(\Sigma(A))$ is a compact regular frame.

The proof of this theorem is unfortunately beyond our reach; instead, we now give an alternative descriptions of the frame $\mathrm{RIdl}(L_A)$, which will be useful for computational purposes in topos theory. This again requires some more background in lattice theory. Let $(L, \leqslant)$ be a meet semilattice (i.e., a poset in which any pair of elements has an infimum; in most of our applications $(L, \leqslant)$ is actually a distributive lattice).

Definition 12.9. *A* covering relation *on L is a relation $\lhd \subseteq L \times \mathcal{P}(L)$—equivalently, a function $L \to \mathcal{P}(\mathcal{P}(L))$—written $x \lhd U$ when $(x,U) \in \lhd$, such that:*

1. *If $x \in U$ then $x \lhd U$.*
2. *If $x \lhd U$ and $U \lhd V$ (i.e., $y \lhd V$ for all $y \in U$) then $x \lhd V$.*
3. *If $x \lhd U$ then $x \wedge y \lhd U$.*
4. *If $x \in U$ and $x \in V$, then $x \lhd U \wedge V$ (where $U \wedge V = \{x \wedge y \mid x \in U, y \in V\}$).*

For example, if $(L, \leqslant) = (\mathcal{O}(X), \subseteq)$ one may take $x \lhd U$ iff $x \leqslant \bigvee U$, i.e., iff U covers x. Also here we have a closure operation $\mathscr{A} : \mathrm{D}(L) \to \mathrm{D}(L)$, given by

$$\mathscr{A}U = \{x \in L \mid x \lhd U\}. \qquad (12.63)$$

This operation has the following properties:

$$\downarrow U \subseteq \mathscr{A}U; \tag{12.64}$$

$$U \subseteq \mathscr{A}V \Rightarrow \mathscr{A}U \subseteq \mathscr{A}V; \tag{12.65}$$

$$\mathscr{A}U \cap \mathscr{A}V \subseteq \mathscr{A}(\downarrow U \cap \downarrow V). \tag{12.66}$$

The frame $\mathscr{F}(L,\lhd)$ generated by such a structure is then defined by

$$\mathscr{F}(L,\lhd) = \{U \in \mathrm{D}(L) \mid \mathscr{A}U = U\} = \{U \in \mathscr{P}(L) \mid x \lhd U \Rightarrow x \in U\}; \tag{12.67}$$

the second equality follows because firstly the property $\mathscr{A}U = U$ guarantees that $U \in \mathrm{D}(L)$, and secondly one has $\mathscr{A}U = U$ iff $x \lhd U$ implies $x \in U$. Defining

$$U \sim V \text{ iff } U \lhd V \text{ and } V \lhd U, \tag{12.68}$$

an equivalent description of the frame $\mathscr{F}(L,\lhd)$ that is occasionally useful is

$$\mathscr{F}(L,\lhd) \cong \mathscr{P}(L)/\sim. \tag{12.69}$$

Indeed, the map $U \mapsto [U]$ from $\mathscr{F}(L,\lhd)$ (as defined in (12.67)) to $\mathscr{P}(L)/\sim$ is a frame map with inverse $[U] \mapsto \mathscr{A}U$. The idea behind the isomorphism (12.69) is that the map $\mathscr{A}$ picks a unique representative in the equivalence class $[U]$, namely $\mathscr{A}U$. As in (12.40) - (12.71), also here we have a canonical map

$$f : L \to \mathscr{F}(L,\lhd); \tag{12.70}$$

$$x \mapsto \mathscr{A}(\downarrow x), \tag{12.71}$$

which satisfies $f(x) \leqslant \bigvee f(U)$ if $x \lhd U$. In fact, f is universal with this property, in that any homomorphism $g : L \to \mathscr{G}$ of meet semilattices into a frame $\mathscr{G}$ such that $g(x) \leqslant \bigvee g(U)$ whenever $x \lhd U$ has a factorisation $g = \varphi \circ f$ for some unique frame map $\varphi : \mathscr{F}(L,C) \to \mathscr{G}$. This may suggest the following result:

Proposition 12.10. *Suppose one has a frame $\mathscr{F}$ and a meet semilattice L with a map $f : L \to \mathscr{F}$ of meet semilattices that generates $\mathscr{F}$ in the sense that for each $U \in \mathscr{F}$ one has $U = \bigvee\{f(x) \mid x \in L, f(x) \leq U\}$. Define a cover relation $\lhd$ on L by*

$$x \lhd U \textit{ iff } f(x) \leqslant \bigvee f(U). \tag{12.72}$$

Then one has a frame isomorphism $\mathscr{F} \cong \mathscr{F}(L,\lhd)$.

We now turn to maps between frames, from the point of view of coverings.

Definition 12.11. *Let $(L,\lhd)$ and $(M,\blacktriangleleft)$ be meet semilattices with covering relation as above, and let $f^* : L \to \mathscr{P}(M)$ be such that:*

1. *$f^*(L) = M$;*
2. *$f^*(x) \wedge f^*(y) \blacktriangleleft f^*(x \wedge y)$;*
3. *$x \lhd U \Rightarrow f^*(x) \blacktriangleleft f^*(U)$ (where $f^*(U) = \bigcup_{u \in U} f(U)$).*

If L and M have top elements $\top_L$ and $\top_M$, respectively, then the first condition may be replaced by $f^*(\top_L) = \top_M$. Define two such maps f_1^*, f_2^* to be equivalent if $f_1^*(x) \sim f_2^*(x)$ (i.e., $f_1^*(x) \blacktriangleleft f_2^*(x)$ and $f_2^*(x) \blacktriangleleft f_1^*(x)$) for all $x \in L$. A continuous map $f : (M, \blacktriangleleft) \to (L, \lhd)$ is an equivalence class of such maps $f^* : L \to \mathscr{P}(M)$.

Our main interest in continuous maps lies in the following result:

Proposition 12.12. *Each continuous map $f : (M, \blacktriangleleft) \to (L, \lhd)$ is equivalent to a frame map $\mathscr{F}(f) : \mathscr{F}(L, \lhd) \to \mathscr{F}(M, \blacktriangleleft)$, given by*

$$\mathscr{F}(f) : U \mapsto \mathscr{A} f^*(U). \tag{12.73}$$

We may now equip L_A with the covering relation defined by (12.72), given (12.60) and the ensuing map (12.57). Consequently, by Proposition 12.10 one has

$$\mathscr{O}(\Sigma) \cong \mathscr{F}(L_A, \lhd), \tag{12.74}$$

which yields the following expression for the constructive Gelfand spectrum:

$$\mathscr{O}(\Sigma) \cong \{U \in \mathrm{D}(L_A) \mid x \lhd U \Rightarrow x \in U\}. \tag{12.75}$$

This lattice becomes computable through a lemma that is crucial for what follows:

Lemma 12.13. *In any topos, the covering relation $\lhd$ on L_A defined by* (12.72) *with* (12.60) *and* (12.57)*, is given by $\mathsf{D}_a \lhd U$ iff for all $q > 0$ there exists a (Kuratowski) finite $U_0 \subseteq U$ such that $\mathsf{D}_{a-q} \leqslant \bigvee U_0$. If U is directed, this means that there exists $\mathsf{D}_b \in U$ such that $\mathsf{D}_{a-q} \leqslant \mathsf{D}_b$.*

Proof. The easy part is the "$\Leftarrow$" direction: from (12.58) and the assumption we have $f(\mathsf{D}_a) \leqslant \bigvee f(U)$ and hence $\mathsf{D}_a \lhd U$ by definition of the covering relation.

In the opposite direction, assume $\mathsf{D}_a \lhd U$ and take some $q > 0$. From (the proof of) Lemma 12.7, $\mathsf{D}_a \vee \mathsf{D}_{q-a} = \top$, hence $\bigvee f(U) \vee f(\mathsf{D}_{q-a}) = \top$. Since $\mathscr{O}(\Sigma)$ is compact, there is a finite $U_0 \subset U$ for which $\bigvee f(U_0) \vee f(\mathsf{D}_{q-a}) = \top$, so that by (12.59) we have $\mathsf{D}_b \vee \mathsf{D}_{q-a} = \top$, with $\mathsf{D}_b = \bigvee U_0$. By (12.46) we have

$$\mathsf{D}_{a-q} \wedge \mathsf{D}_{q-a} = \bot, \tag{12.76}$$

and hence

$$\mathsf{D}_{a-q} = \mathsf{D}_{a-q} \wedge \top = \mathsf{D}_{a-q} \wedge (\mathsf{D}_b \vee \mathsf{D}_{q-a}) = \mathsf{D}_{a-q} \wedge \mathsf{D}_b \leqslant \mathsf{D}_b = \bigvee U_0. \qquad \square$$

If A is finite-dimensional, L_A is a finite lattice. In that case, since $\mathsf{D}_{a-q} = \mathsf{D}_a$ for small enough q, one simply has $x \lhd U$ iff $x \leq \bigvee U$, and the condition $x \lhd U \Rightarrow x \in U$ in (12.75) holds iff U is a (principal) down set, i.e. $U = \downarrow x$ for some $x \in L_A$ (not the same x as the placeholder x in (12.75)). Hence for finite-dimensional A we obtain

$$\mathscr{O}(\Sigma(A)) \cong \mathrm{Idl}(L_A) = \{\downarrow x \mid x \in L_A\}. \tag{12.77}$$

12.3 Internal Gelfand spectrum and intuitionistic quantum logic

We are now going to combine the (*a priori* independent) material in the previous two sections. The point of the above description of the topology $\mathscr{O}(\Sigma(A))$ of the Gelfand spectrum $\Sigma(A)$ of a unital commutative C*-algebra A is that it may be "internalized" to any topos (with natural number object, i.e., in which C*-algebras may be defined internally in the first place). The key to the ensuing generalization of Gelfand duality is that in topos theory (and more generally in constructive mathematics) the *space* $\Sigma(A)$ in set theory needs to be replaced by the corresponding *frame* $\mathscr{O}(\Sigma(A))$, or preferably by its associated *locale*, which confusingly is denoted by $\Sigma(A)$, even though it is the same thing as $\mathscr{O}(\Sigma(A))$ and neither may be spatial (in being the topology of some space); see §C.11 and §E.4 for this bizarre notation. Similarly, we write $f : X \to Y$ for a map between locales, which is essentially the same as the frame map $f^{-1} : \mathscr{O}(Y) \to \mathscr{O}(X)$, but seen as a map in the opposite direction (where once again nothing is assumed about possible spatiality of the frames in question).

Using this notation, the ***constructive Gelfand isomorphism*** (which is valid in any topos T in which commutative C*-algebras make sense) states:

Theorem 12.14. *For each (internal) commutative unital C*-algebra A in T there exists a compact regular locale $\Sigma(A)$ such that one has a Gelfand isomorphism*

$$A \cong C(\Sigma(A), \mathbb{C}). \tag{12.78}$$

Furthermore, the locale $\Sigma(A)$ is uniquely determined by A up to isomorphism and its corresponding frame is given by Theorem 12.8 (or, more explicitly, by (12.75) *in conjunction with Lemma 12.13, all of which makes sense internally).*

Here $\cong$ denotes (internal) isomorphism of (commutative) C*-algebras, and the notation $C(\Sigma(A), \mathbb{C})$ stands for the object of all frame maps from $\mathscr{O}(\mathbb{C})$ to $\mathscr{O}(\Sigma(A))$ (which object turns out to be a commutative C*-algebra in any case). As usual, we denote the Gelfand transform $A \to C(\Sigma(A), \mathbb{C})$ by $a \mapsto \hat{a}$, where, as explained above, the locale map $\hat{a} : \Sigma(A) \to \mathbb{C}$ is really the reverse reading of the frame map

$$\hat{a}^{-1} : \mathscr{O}(\mathbb{C}) \to \mathscr{O}(\Sigma(A)). \tag{12.79}$$

Note that in Sets, the latter is given by its literal meaning, given $\hat{a} : \omega \mapsto \omega(a)$.

We will shortly apply this formalism to our internal C*-algebra $\underline{A}$ in the topos $\mathsf{T}(A)$, but since these computations are a bit involved, as a warm-up we first apply our machinery to a very simple case, namely $A = \mathbb{C}^n$ in Sets. Recall (12.44) etc.

For $A = \mathbb{C}^n$ we have $A^+ = (\mathbb{R}^n)^+$, in which $(r_1, \ldots, r_n) \approx (s_1, \ldots, s_n)$ just in case $r_i = 0$ iff $s_i = 0$ for all $i = 1, \ldots n$. Hence each equivalence class under $\approx$ has a unique representative of the form $[k_1, \ldots, k_n]$ with $k_i = 0$ or $k_i = 1$; the pre-images of such an element of L_A in A^+ under the natural projection $A^+ \to A^+/\approx$ are the diagonal matrices whose i'th entry is zero if $k_i = 0$ and any nonzero positive number if $k_i = 1$. The partial order in L_A is pointwise, i.e. $[k_1, \ldots, k_n] \leq [l_1, \ldots, l_n]$ iff $k_i \leq l_i$ for all i. Hence $L_{\mathbb{C}^n}$ is isomorphic as a distributive lattice to the lattice $\mathscr{P}(D_n(\mathbb{C})) \equiv \mathscr{P}(\mathbb{C}^n)$ of projections in $D_n(\mathbb{C})$, i.e. the lattice of diagonal projections in $M_n(\mathbb{C})$.

Under this isomorphism, $[k_1,\ldots,k_n]$ corresponds to the matrix $\mathrm{diag}(k_1,\ldots,k_n)$. If we equip $\mathscr{P}(\mathbb{C}^n)$ with the usual partial ordering of projections on the Hilbert space $\mathbb{C}^n$, viz. $e \leq f$ whenever $e\mathbb{C}^n \subseteq f\mathbb{C}^n$ (which coincides with their ordering as element of positive cone of the C*-algebra $M_n(\mathbb{C})$), then this is even a lattice isomorphism. Hence by (12.77), the frame $\mathscr{O}(\Sigma(\mathbb{C}^n))$ consists of all sets of the form $\downarrow e$, $e \in \mathscr{P}(\mathbb{C}^n)$, partially ordered by inclusion. This means that

$$\mathscr{O}(\Sigma(\mathbb{C}^n)) \cong \mathscr{P}(\mathbb{C}^n), \tag{12.80}$$

under the further identification of $\downarrow p \subset \mathscr{P}(\mathbb{C}^n)$ with $p \in \mathscr{P}(\mathbb{C}^n)$. This starts out just as an isomorphism of posets, and turns out to be one of frames (which in the case at hand happen to be Boolean). To draw the connection with the usual spectrum $\hat{\mathbb{C}}^n = \{1,2,\ldots,n\}$ of $\mathbb{C}^n$, we note that the right-hand side of (12.80) is isomorphic to the discrete topology $\mathscr{O}(\hat{\mathbb{C}}^n)$ of $\hat{\mathbb{C}}^n$ (i.e. its power set) under the frame isomorphism

$$\begin{aligned} \mathscr{P}(\mathbb{C}^n) &\overset{\cong}{\to} \mathscr{O}(\hat{\mathbb{C}}^n); \\ \mathrm{diag}(k_1,\ldots,k_n) &\mapsto \{i \in \{1,2,\ldots,n\} \mid k_i = 1\}. \end{aligned} \tag{12.81}$$

We now describe the Gelfand transform (12.78) - (12.79) for self-adjoint a, so that one has a (locale) map $A_{\mathrm{sa}} \to C(\Sigma(A),\mathbb{R})$. Let $a = (a_1,\ldots,a_n) \in \mathbb{C}^n_{\mathrm{sa}} = \mathbb{R}^n$. With $\Sigma(\mathbb{C}^n)$ realized as $\hat{\mathbb{C}}^n$, this just reads $\hat{a}(i) = a_i$, for $\hat{a} : \hat{\mathbb{C}}^n \to \mathbb{C}$. The induced frame map $\hat{a}^{-1} : \mathscr{O}(\mathbb{C}) \to \mathscr{O}(\hat{\mathbb{C}}^n)$ is given by $U \mapsto \{i \in \{1,2,\ldots,n\} \mid a_i \in U\}$, and by (12.81), this is equivalent to

$$\begin{aligned} \hat{a}^{-1} : \mathscr{O}(\mathbb{R}) &\to \mathscr{P}(\mathbb{C}^n); \\ U &\mapsto 1_U(a), \end{aligned} \tag{12.82}$$

where $U \in \mathscr{O}(\mathbb{R})$, and the right-hand side denotes the spectral projection $1_U(a)$ defined by the self-adjoint operator a on the Hilbert space $\mathbb{C}^n$.

After this warm-up, we now compute the Gelfand spectrum $\mathscr{O}(\underline{\Sigma}(\underline{A}))$ in our topos $\mathsf{T}(A)$, for the special case $A = M_n(\mathbb{C})$ (which is still an exercise for the general case). For simplicity we write $\underline{L}$ for the lattice $L_{\underline{A}}$ in $\mathsf{T}(A)$; similarly, $\underline{\Sigma}$ stands for $\underline{\Sigma}(\underline{A})$.

First, for arbitrary A, the lattice functor $\underline{L}$ can be computed "locally", in the sense that $\underline{L}_0(C) = L_C$, see Proposition 12.17 in §12.4 below, so that by (12.44) one has

$$\underline{L}_0(C) = C^+/\approx . \tag{12.83}$$

Let $\mathscr{P}(C)$ be the (Boolean) lattice of projections in C, and consider the functor

$$\underline{\mathscr{P}}_0(C) = \mathscr{P}(C); \tag{12.84}$$
$$\underline{\mathscr{P}}_1(C \subseteq D) = (\mathscr{P}(C) \hookrightarrow \mathscr{P}(D)). \tag{12.85}$$

As in the case $A = \mathbb{C}^n$ just discussed, it follows that we may identify $\underline{L}_0(C)$ with $\mathscr{P}(C)$ and hence we may and will identify the functor $\underline{L}$ with the functor $\underline{\mathscr{P}}$.

Second, whereas in Sets eq. (12.77) makes $\mathscr{O}(\Sigma)$ a subset of L, in the topos $\mathsf{T}(A)$ the frame $\mathscr{O}(\underline{\Sigma})$ is a subobject $\mathscr{O}(\underline{\Sigma}) \rightarrowtail \underline{\Omega^L}$. It then follows from (12.11) that $\mathscr{O}(\underline{\Sigma})(C)$ is a subset of $\mathrm{Sub}(\underline{\mathscr{P}}_{\uparrow C})$, the set of subfunctors of the functor $\underline{\mathscr{P}} : \mathscr{C}(A) \to$ Sets restricted to $\uparrow C \subset \mathscr{C}(A)$. To see which subset, define

$$\mathrm{Sub}_d(\underline{\mathscr{P}}_{\uparrow C}) = \{\tilde{S} \in \mathrm{Sub}(\underline{\mathscr{P}}_{\uparrow C}) \mid \forall D \supseteq C\, \exists x_D \in \mathscr{P}(D) : \tilde{S}(D) = \downarrow x_D\}. \quad (12.86)$$

Thus $\mathrm{Sub}_d(\underline{\mathscr{P}}_{\uparrow C})$ consists of subfunctors S of $\underline{\mathscr{P}}_{\uparrow C}$ that are locally down-sets. It then follows from (12.77) and the local interpretation of the relation $\lhd$ in $\mathsf{T}(A)$ (see Lemma 12.18 in §12.4 below) that the subobject $\mathscr{O}(\underline{\Sigma}) \rightarrowtail \underline{\Omega^L}$ in $\mathsf{T}(A)$ is the functor

$$\mathscr{O}(\underline{\Sigma})_0(C) = \mathrm{Sub}_d(\underline{\mathscr{P}}_{\uparrow C}); \quad (12.87)$$

$$\mathscr{O}(\underline{\Sigma})_1(C \subseteq D) = (\mathscr{O}(\underline{\Sigma})(C) \hookrightarrow \mathscr{O}(\underline{\Sigma})(D)), \quad (12.88)$$

where $\mathscr{O}(\underline{\Sigma})_1$ is inherited from $\underline{\Omega^L}$ (of which $\mathscr{O}(\underline{\Sigma})$ is a subobject), and hence is just given by restricting an element of $\mathscr{O}(\underline{\Sigma})(C)$ to $\uparrow D$. Writing

$$\mathrm{Sub}_d(\underline{\mathscr{P}}) = \{\tilde{S} \in \mathrm{Sub}(\underline{\mathscr{P}}) \mid \forall D \in \mathscr{C}(A)\, \exists x_D \in \mathscr{P}(D) : \tilde{S}(D) = \downarrow x_D\}, \quad (12.89)$$

it is convenient to embed $\mathrm{Sub}_d(\underline{\mathscr{P}}_{\uparrow C}) \subseteq \mathrm{Sub}_d(\underline{\mathscr{P}})$ by requiring elements of the left-hand side to vanish whenever D does not contain C. We also note that if $\tilde{S}$ is to be a subfunctor of $\underline{\mathscr{P}}_{\uparrow C}$, one must have $\tilde{S}(D) \subseteq \tilde{S}(E)$ whenever $D \subseteq E$, and that $\downarrow x_D \subseteq \downarrow x_E$ iff $x_D \leq x_E$ in $\mathscr{P}(E)$. Thus one may simply describe elements of $\mathscr{O}(\underline{\Sigma})(C)$ via maps $S : \mathscr{C}(A) \to \mathscr{P}(A)$ such that:

$$S(D) \in \mathscr{P}(D); \quad (12.90)$$

$$S(D) = 0 \text{ if } D \notin \uparrow C \text{ (i.e. } C \nsubseteq D); \quad (12.91)$$

$$S(D) \leq S(E) \text{ if } C \subseteq D \subseteq E. \quad (12.92)$$

The corresponding element $\tilde{S}$ of $\mathscr{O}(\underline{\Sigma})(C)$ is then given by

$$\tilde{S}(D) = \downarrow S(D), \quad (12.93)$$

seen as a subset of $\mathscr{P}(D)$. Hence it is convenient to introduce the notation

$$\mathscr{O}(\Sigma)_{\uparrow C} = \{S : \uparrow C \to \mathscr{P}(A) \mid S(D) \in \mathscr{P}(D), S(D) \leq S(E) \text{ if } D \subseteq E\}, \quad (12.94)$$

of which we single out the case $C = \mathbb{C} \cdot 1_A$, which will be of great importance:

$$\mathscr{O}(\Sigma) = \{S : \mathscr{C}(A) \to \mathscr{P}(A) \mid S(C) \in \mathscr{P}(C), S(C) \leq S(D) \text{ if } C \subseteq D\}. \quad (12.95)$$

Both are posets and even frames in the pointwise partial order with respect to the usual ordering of projections (which algebraically means $e \leq f$ iff $ef = e$), i.e.,

$$S \leq T \Leftrightarrow S(C) \leq T(C) \text{ for all } C \in \mathscr{C}(A). \quad (12.96)$$

In terms of (12.94) - (12.95), we then have isomorphisms

$$\mathscr{O}(\underline{\Sigma})_0(\mathbb{C}\cdot 1) \cong \mathscr{O}(\Sigma); \tag{12.97}$$
$$\mathscr{O}(\underline{\Sigma})(C)_0 \cong \mathscr{O}(\Sigma)_{\uparrow C}. \tag{12.98}$$

More importantly, the frame $\mathscr{O}(\Sigma)$ in Sets is the key to the *external description* of the *internal frame* $\mathscr{O}(\underline{\Sigma})$ in $\mathsf{T}(A)$; see the end of §E.4. Since $\mathscr{C}(A)$ carries the Alexandrov topology, by (E.84) this description is given by the frame map

$$\pi_\Sigma^{-1} : \mathscr{O}(\mathscr{C}(A)) \to \mathscr{O}(\Sigma), \tag{12.99}$$

given on the basic opens $\uparrow D \in \mathscr{O}(\mathscr{C}(A))$ by

$$\pi_\Sigma^{-1}(\uparrow D) = \chi_{\uparrow D} : E \mapsto 1 \ (E \supseteq D); \quad E \mapsto 0 \ (E \not\supseteq D). \tag{12.100}$$

As explained before, even in Sets, in principle $\mathscr{O}(\Sigma)$ is just a notation for a frame, without suggesting that there exists an underlying space Σ whose topology it is. In this case, however, there is such a space (as we shall show in the next section), and also (12.99) is in fact the inverse image map to a genuine map $\pi_\Sigma : \Sigma \to \mathscr{C}(A)$ between spaces (as opposed to the formal notation used for a locale map).

We now state the Heyting algebra structure of $\mathscr{O}(\Sigma)$. First, top and bottom are

$$\top(C) = 1 \ \text{ for all } C; \tag{12.101}$$
$$\bot(C) = 0 \ \text{ for all } C. \tag{12.102}$$

The logical operations on $\mathscr{O}(\Sigma)$ may be computed from the partial order as

$$(S \wedge T)(C) = S(C) \wedge T(C); \tag{12.103}$$
$$(S \vee T)(C) = S(C) \vee T(C); \tag{12.104}$$
$$(S \dashrightarrow T)(C) = \bigwedge_{D \supseteq C}^{\mathscr{P}(C)} S(D)^\perp \vee T(D); \tag{12.105}$$
$$(\neg S)(C) = \bigwedge_{D \supseteq C}^{\mathscr{P}(C)} S(D)^\perp; \tag{12.106}$$
$$(\neg\neg S)(C) = \bigwedge_{D \supseteq C}^{\mathscr{P}(C)} \bigvee_{E \supseteq D}^{\mathscr{P}(D)} S(E), \tag{12.107}$$

where the right-hand side of (12.105) (and similarly (12.106) - (12.107)) is short for

$$\bigwedge_{D \supseteq C}^{\mathscr{P}(C)} S(D)^\perp \vee T(D) \equiv \bigvee \{e \in \mathscr{P}(C) \mid e \leq S(D)^\perp \vee T(D) \, \forall D \supseteq C\}. \tag{12.108}$$

Recall that a Heyting algebra is Boolean iff $\neg\neg S = S$ for each S. One sees from (12.107) that (at least if $n > 1$) the property $\neg\neg S = S$ only holds iff S is either $\top$ or $\perp$, so that the Heyting algebra $\mathscr{O}(\Sigma) \equiv CO(\Sigma(A))$ is properly intuitionistic.

Since from both a physical and a logical point of view the Heyting algebra $\mathscr{O}(\Sigma(A))$ has vast advantages over the projection lattice $\mathscr{P}(A)$ of Birkhoff and von Neumann, we propose it as a candidate for a new quantum logic. Let us explain why.

Physically, in von Neumann's approach each projection $e \in \mathscr{P}(A)$ defines an elementary proposition, whereas in Bohr's (where the classical context C is crucial) an elementary proposition is a *pair* (C,e), where $e \in \mathscr{P}(C)$ is a proposition à la von Neumann (who lost sight of the context C). If for each such pair (C,e) we define

$$S_{(C,e)} : \mathscr{C}(A) \to \mathscr{P}(A); \tag{12.109}$$

$$D \mapsto e \ \ (C \subseteq D); \tag{12.110}$$

$$D \mapsto \perp \ \text{ otherwise}, \tag{12.111}$$

we see that each pair (C,e) injectively defines an element of $\mathscr{O}(\Sigma)$. Furthermore, each element S of $\mathscr{O}(\Sigma)$ is a disjunction over such elementary propositions, since

$$S = \bigvee_{C \in \mathscr{C}(A)} S_{(C,S(e))}. \tag{12.112}$$

In contrast to traditional quantum logic, both logical connectives $\wedge$ and $\vee$ on $\mathscr{O}(\Sigma)$ are physically meaningful, as they only involve *local* conjunctions $S(C) \wedge T(C)$ and disjunctions $S(C) \vee T(C)$, for which $S(C) \in \mathscr{P}(C)$ and $T(C) \in \mathscr{P}(C)$ commute.

Logically, the absence of an implication arrow in quantum logic has always been worrying; this has now been put straight in $\mathscr{O}(\Sigma)$, where $\dashrightarrow$ belongs to the defining structure and behaves well logically. Truth attribution in quantum logic is equally suspicious: for any state ω on A one declares a proposition $e \in \mathscr{P}(A)$ *true* iff $\omega(e) = 1$, and *false* iff $\omega(e) = 0$, with no verdict otherwise (except probabilistically).

We, however, define a natural ***Kripke semantics*** (cf. §D.3) on $P = \mathscr{C}(A)$ by

$$V_\omega : \mathscr{O}(\Sigma) \to \text{Upper}(\mathscr{C}(A)) = \mathscr{O}(\mathscr{C}(A)); \tag{12.113}$$

$$V_\omega(S) = \{C \in \mathscr{C}(A) \mid \omega(S(C)) = 1\}, \tag{12.114}$$

where $\mathscr{C}(A)$ carries the Alexandrov topology as usual. Note that $V_\omega(S)$ indeed defines an upper set in $\mathscr{C}(A)$, for if $C \subseteq D$ then $S(C) \leq S(D)$, so that $\omega(S(C)) \leq \omega(S(D))$ by positivity of states, and hence $\omega(S(D)) = 1$ whenever $\omega(S(C)) = 1$ (given that $\omega(S(D)) \leq 1$, which is true since $0 \leq \omega(e) \leq 1$ for any projection e).

As explained in §D.3, a proposition $S \in \mathscr{O}(\Sigma)$ is ***true*** in a state ω if $V_\omega(S) = \mathscr{C}(A)$, i.e. the top element of the frame $\mathscr{O}(\mathscr{C}(A))$; we also declare it ***false*** if $V_\omega(S) = \emptyset$, i.e. the bottom element of $\mathscr{O}(\mathscr{C}(A))$. Then $\neg S$ is true iff S is false, and $S \vee T$ is true iff either S or T is true (since $V_\omega(S) = \mathscr{C}(A)$ iff $S(\mathbb{C} \cdot 1) = 1$, which forces $S(C) = 1$ for all C). Consequently, (12.114) simply lists the contexts C in which $S(C)$ is true.

12.4 Internal Gelfand spectrum for arbitrary C*-algebras

In this section we compute the internal Gelfand spectrum $\underline{\Sigma}(\underline{A}) \equiv \underline{\Sigma}$ in $\mathsf{T}(A)$ for an arbitrary unital C*-algebra A. Recall Definition D.6 (in §D.1) of a free lattice $\mathscr{L}_S$ on a set S, and its refinement in quotienting by a congruence on $\mathscr{L}_S$ explained after that definition. According to Definition E.21, lattices can be defined in any topos. The following "locality lemma" shows that the construction of a free lattice on some object makes sense in functor toposes, and so does its refinement just mentioned, at least as long as the congruence in question is defined through equalities.

Lemma 12.15. *Let $\mathsf{T} = [\mathsf{C}, \mathsf{Sets}]$ be any functor topos (where C is some category).*

1. *There exists a free distributive lattice $\underline{\mathscr{L}_S} \in \mathsf{T}$ on any object $\underline{S} \in \mathsf{T}$, which can be computed locally: the object part of $\underline{\mathscr{L}_S}$ is given by*
$$(\underline{\mathscr{L}_S})_0(C) = \mathscr{L}_{\underline{S}_0(C)}, \tag{12.115}$$
where $\mathscr{L}_{\underline{S}_0(C)}$ defined in Sets, and the arrow part is defined as follows. If $f : C \to D$, then $(\underline{\mathscr{L}_S})_1(f)$ is the unique arrow making the following diagram commute:
$$\begin{tikzcd} \underline{S}_0(C) \arrow[r, "\underline{S}_1(f)"] \arrow[d, "\mathscr{L}"'] & \underline{S}_0(D) \arrow[d, "\mathscr{L}"] \\ \mathscr{L}_{\underline{S}_0(C)} \arrow[r, "(\underline{\mathscr{L}_S})_1(f)"] & \mathscr{L}_{\underline{S}_0(D)} \end{tikzcd} \tag{12.116}$$
2. *The same is true if $\underline{\mathscr{L}_S}$ is subject to relations defined by* equalities *among elements of $\underline{\mathscr{L}_S}$ (as long as these equalities generate a congruence).*

Proof. The proof is an elaborate verification, which may be summarized as follows.

1. Existence and uniqueness of the arrow $(\underline{\mathscr{L}_S})_1(f)$ in (12.116) follows from the universal property of the free distributive lattice $\mathscr{L}_{\underline{S}_0(C)}$ in Sets; just consider the function $\mathscr{L} \circ \underline{S}_1(f) : \underline{S}_0(C) \to \mathscr{L}_{\underline{S}_0(D)}$. The claim follows from the fact that $\underline{\mathscr{L}_S}$ (defined locally) has the required universal property (as can be established locally, from the corresponding property of each $(\underline{\mathscr{L}_S})_0(C)$) and hence is unique.
2. This is proved in a similar way, since also a free distributive lattice $\mathscr{L}_S/\sim$ on generators S with relations given by equalities has a universal property, cf. the final part of §D.1. This works locally in a functor topos by rule no. 7 of Kripke–Joyal semantics, cf. §E.5 (which states that equalities are enforced locally). □

We will apply this lemma to $\mathsf{T} = \mathsf{T}(A)$, as in (12.1), with $\mathsf{C} = \mathscr{C}(A)$. This hinges on a lemma of independent interest, which we first state for Sets, i.e., for "ordinary" commutative unital C*-algebras A, to be subsequently internalized to our topos $\mathsf{T}(A)$.

Lemma 12.16. *The lattice L_A in* (12.44) *is (constructively) isomorphic to the lattice L'_A freely generated by the symbols D_a, $a \in A_{\mathrm{sa}}$ and the relations* (12.45) - (12.50).

Proof. The point is that the map $a \mapsto \mathsf{D}_a$ from A_{sa} to L'_A is surjective; this follows from the relations (12.45) - (12.50) through their consequences (12.51) - (12.55). The pertinent isomorphism $L'_A \cong L_A$ is then given by mapping $\mathsf{D}_a \leftrightarrow [a^+]$ on generators (note that in the original discussion of L_A following (12.44) this map was the *definition* of D_a; this time, these play an independent role as generators of the lattice L'_A, and in the present proof they are *related* to the elements $[a^+] \in L_A$). □

Now let A be a (not necessarily commutative) unital C*-algebra (in Sets), with ensuing internal commutative C*-algebra $\underline{A}$ in the functor topos $\mathsf{T}(A)$, cf. Theorem 12.3. Our goal is to apply the constructive definition of the Gelfand spectrum $\Sigma(A)$, or rather of its topology $\mathscr{O}(\Sigma(A))$ (seen as a frame, so that $\Sigma(A)$ is seen as a locale) in §12.2 to $\underline{A}$. The first step concerns the lattice L_A, which in $\mathsf{T}(A)$ is denoted by $\underline{L_A}$. Here and in what follows, we try to avoid notational confusion by writing D_a for the formal variable indexed by a (which is a variable of type $\underline{A}$ in $\mathsf{T}(A)$), whilst writing D_c for the actual element $[c^+]$ of L_C if we apply (12.44) etc. to $C \in \mathscr{C}(A)$.

Proposition 12.17. *For each $C \in \mathscr{C}(A)$ one has*

$$\underline{L_A}(C) = L_C, \tag{12.117}$$

where L_C is defined in Sets *through* (12.44) *(with $A \rightsquigarrow C$), where it may be computed through Lemma 12.16. Furthermore, if $C \subseteq D$, then the map $\underline{L_A}(C) \to \underline{L_A}(D)$ given by the functoriality of $\underline{L_A}$, i.e., $L_C \to L_D$, maps each generator D_c in L_C (where $c \in C_{\mathrm{sa}}$) to the same generator in L_D. This is well defined, because $c \in D_{\mathrm{sa}}$, and this inclusion preserves the relations* (12.45) - (12.50). *We write this as $L_C \hookrightarrow L_D$.*

Proof. Internalizing Lemma Lemma 12.16 to our functor topos $\mathsf{T}(A)$, it follows that the internal lattice $\underline{L_A}$ in $\mathsf{T}(A)$ is isomorphic to a distributive lattice freely generated by generators and relations given by equalities. Hence Lemma 12.15 applies to it. □

The next step is to move from $\underline{L_A}$ to the corresponding frame of regular ideals, cf. Theorem 12.8. Abbreviating $\mathscr{O}(\Sigma(A)) \equiv \mathscr{O}(\Sigma)$, we first rewrite (12.60) as

$$\mathscr{O}(\Sigma) \cong \{U \in \mathrm{Idl}(L_A) \mid \forall_{q>0} \mathsf{D}_{a-q} \in U \Rightarrow \mathsf{D}_a \in U\}. \tag{12.118}$$

To apply this to our functor topos $\mathsf{T}(A)$, we apply Kripke–Joyal semantics for the internal language of the topos $\mathsf{T}(A)$ (which is reviewed §E.5) to the formula $\mathsf{D}_a \lhd U$. This is a formula φ with two free variables, namely D_a of type L_A, and U of type

$$\mathscr{P}(L_A) \equiv \Omega^{L_A}. \tag{12.119}$$

Hence in the forcing statement $C \Vdash \varphi(\alpha)$ in $\mathsf{T}(A)$, we have to insert

$$\alpha \in (\underline{L_A} \times \underline{\Omega}^{\underline{L_A}})(C) \cong L_C \times \mathrm{Sub}(\underline{L_A}_{|\uparrow C}),$$

where $\underline{L_A}_{|\uparrow C}$ is the restriction of the functor

$$\underline{L_A} : \mathscr{C}(A) \to \mathsf{Sets} \tag{12.120}$$

to $\uparrow C \subset \mathscr{C}(A)$. Here we have used (12.117), as well as the isomorphism (12.11). Consequently, we have

$$\alpha = (D_c, \underline{U}), \tag{12.121}$$

where $D_c \in L_C$ for some $c \in C_{\mathrm{sa}}$, and $\underline{U} : \uparrow C \to \mathsf{Sets}$ is a subfunctor of $\underline{L}_{\underline{A}|\uparrow C}$. In particular, $\underline{U}(D) \subseteq L_D$ is defined whenever $D \supseteq C$, and the subfunctor condition on $\underline{U}$ simply boils down to $\underline{U}(D) \subseteq \underline{U}(E)$ whenever $C \subseteq D \subseteq E$.

Lemma 12.18. *In the topos $\mathsf{T}(A)$, the cover $\lhd$ of Lemma 12.13 may be computed locally, in the sense that for any $C \in \mathscr{C}(A)$, $D_c \in L_C$ and $\underline{U} \in \mathrm{Sub}(\underline{L}_{\underline{A}|\uparrow C})$, one has*

$$C \Vdash \mathsf{D}_a \lhd U(D_c, \underline{U}) \textit{ iff } D_c \lhd_C \underline{U}(C),$$

in that for all $q > 0$ there exists a finite $U_0 \subseteq \underline{U}(C)$ such that $D_{c-q} \leqslant \bigvee U_0$.

Proof. We assume that $\bigvee U_0 \in U$, so that we may replace U_0 by $\mathsf{D}_b = \bigvee U_0$; the general case is analogous. We then have to inductively analyze the formula $\mathsf{D}_a \lhd U$, which, under the stated assumption, in view of Lemma 12.13 may be taken to mean

$$\forall_{q>0} \exists_{\mathsf{D}_b \in L_A} (\mathsf{D}_b \in U \wedge \mathsf{D}_{a-q} \leqslant \mathsf{D}_b). \tag{12.122}$$

We now infer from the rules for Kripke–Joyal semantics in a functor topos that

$$C \Vdash (\mathsf{D}_a \in U)(D_c, \underline{U}) \tag{12.123}$$

iff for all $D \supseteq C$ one has $D_c \in \underline{U}(D)$; since $\underline{U}(C) \subseteq \underline{U}(D)$, this happens to be the case iff $D_c \in \underline{U}(C)$. Furthermore,

$$C \Vdash (\mathsf{D}_b \leqslant \mathsf{D}_a)(D_{c'}, D_c) \tag{12.124}$$

iff $D_{c'} \leqslant D_c$ in L_C. Also,

$$C \Vdash (\exists_{\mathsf{D}_b \in L_A} \mathsf{D}_b \in U \wedge \mathsf{D}_{a-q} \leqslant \mathsf{D}_b)(D_c, \underline{U}) \tag{12.125}$$

iff there is $D_{c'} \in \underline{U}(C)$ such that $D_{c-q} \leqslant D_{c'}$. Finally,

$$C \Vdash (\forall_{q>0} \exists_{\mathsf{D}_b \in L_A} \mathsf{D}_b \in U \wedge \mathsf{D}_{a-q} \leqslant \mathsf{D}_b)(D_c, \underline{U}) \tag{12.126}$$

iff for all $D \supseteq C$ and all $q > 0$ there is $D_d \in \underline{U}(D)$ such that $D_{c-q} \leqslant D_d$, where $D_c \in L_C$ is seen as an element of L_D through the injection $L_C \hookrightarrow L_D$ of Proposition 12.17, and $\underline{U} \in \mathrm{Sub}(\underline{L}_{\underline{A}|\uparrow C})$ is seen as an element of $\mathrm{Sub}(\underline{L}_{\underline{A}|\uparrow D})$ by restriction. This, however, is true at all $D \supseteq C$ iff it is true at C, because $\underline{U}(C) \subseteq \underline{U}(D)$ and hence one can take $D_d = D_{c'}$ for the $D_{c'} \in L_C$ that makes the condition true at C. $\square$

Lemma 12.19. *The spectrum $\mathscr{O}(\underline{\Sigma})$ of $\underline{A}$ in $\mathsf{T}(A)$ may be computed as follows:*

1. *At $C \in \mathscr{C}(A)$, the set $\mathscr{O}(\underline{\Sigma})(C)$ consists of those subfunctors $\underline{U} \in \mathrm{Sub}(\underline{L}_{\underline{A}|\uparrow C})$ such that for all $D \supseteq C$ and all $D_d \in L_D$ one has:*

$$D_d \lhd_D \underline{U}(D) \Rightarrow D_d \in \underline{U}(D).$$

2. *At* $\mathbb{C}\cdot 1$*, the set* $\mathcal{O}(\underline{\Sigma})(\mathbb{C}\cdot 1)$ *consists of those subfunctors* $\underline{U}\in \mathrm{Sub}(\underline{L_A})$ *such that for all* $C\in\mathcal{C}(A)$ *and all* $D_c\in L_C$ *one has:*

$$D_c \lhd_C \underline{U}(C) \Rightarrow D_c \in \underline{U}(C).$$

3. *The condition that* $\underline{U}=\{\underline{U}(C)\subseteq L_C\}_{C\in\mathcal{C}(A)}$ *be a subfunctor of* $\underline{L_A}$ *comes down to the requirement that:*

$$C\subseteq D \Rightarrow \underline{U}(C)\subseteq \underline{U}(D).$$

4. *The map* $\mathcal{O}(\underline{\Sigma})(C)\to\mathcal{O}(\underline{\Sigma})(D)$ *given by the functoriality of* $\mathcal{O}(\underline{\Sigma})$ *whenever* $C\subseteq D$ *is given by truncating an element* $\underline{U}:\uparrow C\to \mathsf{Sets}$ *of* $\mathcal{O}(\underline{\Sigma})(C)$ *to* $\uparrow D$*.*
5. *The external description of* $\mathcal{O}(\underline{\Sigma})$ *is the frame map*

$$\pi_{\Sigma}^{*}:\mathcal{O}(\mathcal{C}(A))\to\mathcal{O}(\underline{\Sigma})(\mathbb{C}\cdot 1), \tag{12.127}$$

given on the basic opens $\uparrow D\in\mathcal{O}(\mathcal{C}(A))$ *by*

$$\begin{aligned}\pi_{\Sigma}^{*}(\uparrow D)=\chi_{\uparrow D}: E&\mapsto \top\ (E\supseteq D);\\ E&\mapsto \bot\ (E\not\supseteq D),\end{aligned} \tag{12.128}$$

where the top and bottom $\top,\bot$ *at* E *are given by* $\{L_E\}$ *and* $\emptyset$*, respectively.*

Proof. By (12.75), $\mathcal{O}(\underline{\Sigma})$ is the subobject of $\underline{\Omega}^{\underline{L_A}}$ defined by the formula φ given by

$$\forall_{\mathsf{D}_a\in L_A}\,\mathsf{D}_a\lhd U\Rightarrow \mathsf{D}_a\in U, \tag{12.129}$$

whose interpretation in $\mathsf{T}(A)$ is an arrow from $\underline{\Omega}^{\underline{L_A}}$ to $\underline{\Omega}$. In view of (12.11), we may identify an element $\underline{U}\in\mathcal{O}(\underline{\Sigma})(C)$ with a subfunctor of $\underline{L_A}_{|\uparrow C}$, and by (12.129) and Kripke–Joyal semantics in functor topoi, we have $\underline{U}\in\mathcal{O}(\underline{\Sigma})(C)$ iff $C\Vdash\varphi(\underline{U})$, with φ given by (12.129). Unfolding this using Kripke–Joyal semantics and using Lemma 12.18 (including part 1 of its proof), we find that $\underline{U}\in\mathcal{O}(\underline{\Sigma})(C)$ iff

$$\forall_{D\supseteq C}\forall_{D_d\in L_D}\forall_{E\supseteq D}\,D_d\lhd_E \underline{U}(E)\Rightarrow D_d\in\underline{U}(E), \tag{12.130}$$

where D_d is regarded as an element of L_E. This condition, however, is equivalent to the apparently weaker condition

$$\forall_{D\supseteq C}\forall_{D_d\in L_D}\,D_d\lhd_D \underline{U}(D)\Rightarrow D_d\in\underline{U}(D); \tag{12.131}$$

indeed, condition (12.130) clearly implies (12.131), but the latter applied at $D=E$ actually implies the first, since $D_d\in L_D$ also lies in L_E.

Clauses 2 to 4 should now be obvious. Clause 5 follows by the explicit prescription for the external description of frames (which has been recalled in the previous section, after its initial description the end of §E.4). Note that each $\mathcal{O}(\underline{\Sigma})(C)$ is a frame in Sets, inheriting the frame structure of the ambient frame $\mathrm{Sub}(\underline{L_A}_{|\uparrow C})$. $\square$

We now present the computation of $\mathscr{O}(\underline{\Sigma}) \equiv \mathscr{O}(\underline{\Sigma}(\underline{A}))$ for general unital C*-algebras A. To explain the final formula, topologize the disjoint union

$$\Sigma^A = \bigsqcup_{C \in \mathscr{C}(A)} \Sigma(C), \tag{12.132}$$

where $\Sigma(C)$ is the Gelfand spectrum of $C \in \mathscr{C}(A)$, as follows, abbreviating

$$\mathscr{U}_C \equiv \mathscr{U} \cap \Sigma(C). \tag{12.133}$$

One has $\mathscr{U} \in \mathscr{O}(\Sigma^A)$ iff the following two conditions are satisfied for all $C \in \mathscr{C}(A)$:

1. $\mathscr{U}_C \in \mathscr{O}(\Sigma(C))$.
2. For all $D \supseteq C$, if $\lambda \in \mathscr{U}_C$ and $\lambda' \in \Sigma(D)$ such that $\lambda'_{|C} = \lambda$, then $\lambda' \in \mathscr{U}_D$.

In fact, $\mathscr{O}(\Sigma^A)$ is simply the weakest topology making the canonical projection

$$\pi : \Sigma^A \to \mathscr{C}(A); \tag{12.134}$$

$$\pi(\sigma) = C \;\; (\sigma \in \Sigma(C) \subset \Sigma^A), \tag{12.135}$$

continuous with respect to the Alexandrov topology on $\mathscr{C}(A)$. For $U \in \mathscr{O}(\mathscr{C}(A))$,

$$\Sigma^A_U = \bigsqcup_{C \in U} \Sigma(C) \tag{12.136}$$

is a subset of Σ^A, with relative topology inherited from Σ^A. In particular, for the basic opens $U = \uparrow C$ of the Alexandrov topology on $\mathscr{C}(A)$ we have

$$\Sigma^A_{\uparrow C} = \bigsqcup_{D \supseteq C} \Sigma(D). \tag{12.137}$$

Theorem 12.20. *Let A be a unital C*-algebra A. The internal Gelfand spectrum $\mathscr{O}(\underline{\Sigma}(\underline{A}))$ of our internal commutative C*-algebra $\underline{A}$ in the topos $\mathsf{T}(A)$ is the functor*

$$\mathscr{O}(\underline{\Sigma}(\underline{A}))_0 : C \mapsto \mathscr{O}(\Sigma^A_{\uparrow C}), \tag{12.138}$$

i.e., the frame (in Sets) of the open sets of $\Sigma^A_{\uparrow C}$ in the topology defined after (12.132)*; if $C \subseteq D$, the arrow-part of the functor in question is given by*

$$\mathscr{O}(\underline{\Sigma}(\underline{A}))_1 : \mathscr{O}(\Sigma^A_{\uparrow C}) \to \mathscr{O}(\Sigma^A_{\uparrow D}); \tag{12.139}$$

$$\mathscr{U} \mapsto \mathscr{U} \cap \uparrow D. \tag{12.140}$$

Similarly, in the description of $\mathsf{T}(A)$ as the category of sheaves $\mathsf{Sh}(\mathscr{C}(A))$, cf. (E.84)*, the Gelfand spectrum is given by the sheaf (where $U \subseteq V$ in* (12.142)*):*

$$\mathscr{O}(\underline{\Sigma}(\underline{A}))_0 : U \mapsto \mathscr{O}(\Sigma^A_U) \;\; (U \in \mathscr{O}(\mathscr{C}(A))); \tag{12.141}$$

$$\mathscr{O}(\underline{\Sigma}(\underline{A}))_1 : \mathscr{U} \mapsto \mathscr{U} \cap \Sigma^A_U \;\; (\mathscr{U} \in \mathscr{O}(\Sigma^A_V)). \tag{12.142}$$

Proof. The proof is based on Lemma 12.19, which implies that the internal frame $\underline{\mathrm{RIdl}}(\underline{L_A})$ in $\mathsf{T}(A)$ is given by the functor

$$\underline{\mathrm{RIdl}}(\underline{L_A}) : C \mapsto \{\underline{F} \in \mathrm{Sub}(\underline{L_A}_{|\uparrow C}) \mid \underline{F}(D) \in \mathrm{RIdl}(L_D) \text{ for all } D \supseteq C\}. \quad (12.143)$$

Here, since D is a commutative unital C*-algebra in Sets, according to (12.60) the set $\mathrm{RIdl}(L_D)$ may be identified with the topology $\mathscr{O}(\Sigma(D))$, where $\Sigma(D)$ is the Gelfand spectrum of D in the usual sense. We will make this identification in the following step, which is the last step of the proof of Theorem 12.20.

Lemma 12.21. *The transformation* $\theta : \underline{\mathrm{RIdl}}(\underline{L_A}) \to \mathscr{O}(\underline{\Sigma}(\underline{A}))$ *with components*

$$\theta_C : \{\underline{F} \in \mathrm{Sub}(\underline{L_A}_{|\uparrow C}) \mid \underline{F}(D) \in \mathscr{O}(\Sigma(D)) \,\text{for all}\, D \supseteq C\} \to \mathscr{O}(\Sigma^A_{\uparrow C});$$
$$\underline{F} \mapsto \bigsqcup_{D \supseteq C} \underline{F}(D), \quad (12.144)$$

is a natural isomorphism of functors—i.e., an isomorphism of objects in $\mathsf{T}(A)$.

Since $\underline{\mathrm{RIdl}}(\underline{L_A})$ and $\mathscr{O}(\underline{\Sigma})$ are internal frames in $\mathsf{T}(A)$, it suffices to prove that each θ_C is an isomorphism of frames in Sets. Unfortunately, even this proof is a very lengthy (though straightforward) affair, for which we refer to the literature. □

Corollary 12.22. *The external description (in* Sets*) of the internal locale* $\underline{\Sigma}(\underline{A})$ *(in* $\mathsf{T}(A)$*) is given by the canonical projection* (12.134).

Note that both Σ^A and $\mathscr{C}(A)$ are topological spaces, so that (12.134) is a *bona fide* continuous map between spaces. This is worth stressing, since in general, an external description of an internal locale in a sheaf topos, though defined in Sets, is a map between locales (or, equivalently, between frames) that are not necessarily topological spaces. But in the case (12.134) at hand they are, so at least this time there is no confusion between $\mathscr{O}(X)$ as both formal notation for a frame (not necessarily coming from a topology) and notation for the topology of a space X; see §C.11.

Note that (12.95) is a special case of Theorem 12.20 or Corollary 12.22, for

$$A = M_n(\mathbb{C}). \quad (12.145)$$

To see this, we identify $\mathscr{U} = \bigsqcup_{C \in \mathscr{C}(A)} \mathscr{U}_C$ as an element of $\mathscr{O}(\Sigma^A)$ with

$$S : \mathscr{C}(A) \to \mathscr{P}(A)$$

on the right-hand side of (12.95), where $S(C) \in \mathscr{P}(C)$ is the image of $\mathscr{U}_C \in \mathscr{O}(\Sigma(C))$ under the isomorphism $\mathscr{O}(\Sigma(C)) \to \mathscr{P}(C)$ between the (discrete) topology of the (finite) Gelfand spectrum of C and the (Boolean) projection lattice of C derived earlier, see (12.80). Similarly, for $U \in \mathscr{O}(\mathscr{C}(A))$, the frame $\mathscr{O}(\Sigma^A_U)$ may be identified with maps

$$S : U \to \mathscr{P}(A)$$

satisfying the conditions in (12.95). Of course, the special case (12.145) leading to (12.95) is very appealing, and was well worth treating in its own right!

Theorem 12.20 and Corollary 12.22 also give an explicit description of the general internal Gelfand isomorphism (12.78), whose real part in $\mathsf{T}(A)$ reads

$$\underline{A}_{\mathrm{sa}} \cong C(\underline{\Sigma}, \underline{\mathbb{R}}) \equiv \mathrm{Frm}(\mathscr{O}(\underline{\mathbb{R}}), \mathscr{O}(\underline{\Sigma})), \tag{12.146}$$

where the right-hand side, which denotes the object of frame homomrphisms from $\mathscr{O}(\underline{\mathbb{R}})$ to $\mathscr{O}(\underline{\Sigma})$ within $\mathsf{T}(A)$, is the *definition* of the middle term (which is just a notation). To understand the situation in $\mathsf{T}(A)$, one has to distinguish between:

1. The *object* $\mathrm{Frm}(\mathscr{O}(\underline{\mathbb{R}}), \mathscr{O}(\underline{\Sigma}))$ in $\mathsf{T}(A)$, defined as the subobject of the exponential $\mathscr{O}(\underline{\Sigma})^{\mathscr{O}(\underline{\mathbb{R}})}$ consisting of (internal) frame maps from $\mathscr{O}(\underline{\mathbb{R}})$ to $\mathscr{O}(\underline{\Sigma})$.
2. The *set* $\mathrm{Hom}_{\mathrm{Frm}}(\mathscr{O}(\underline{\mathbb{R}}), \mathscr{O}(\underline{\Sigma}))$ of internal frame maps from the frame $\mathscr{O}(\underline{\mathbb{R}})$ of (Dedekind) real numbers in $\mathsf{T}(A)$ to the frame $\mathscr{O}(\underline{\Sigma})$ (i.e., the set of those arrows from $\mathscr{O}(\underline{\mathbb{R}})$ to $\mathscr{O}(\underline{\Sigma})$ that happen to be frame maps as seen from within $\mathsf{T}(A)$).

The connection between 1. and 2. is given by λ-conversion, i.e., the bijective correspondence between $C \to B^A$ and $A \times C \to B$, cf. (E.153). Taking $C = \underline{\mathbf{1}}$ (i.e. the terminal object in $\mathsf{T}(A)$), we see that an *element* of the set $\mathrm{Hom}(A, B)$ corresponds to an *arrow* $1 \to B^A$. Eq. (12.8) yields

$$\mathrm{Frm}(\mathscr{O}(\underline{\mathbb{R}}), \mathscr{O}(\underline{\Sigma}))(C) = \mathrm{Nat}_{\mathrm{Frm}}(\mathscr{O}(\underline{\mathbb{R}})_{\uparrow C}, \mathscr{O}(\underline{\Sigma})_{\uparrow C}), \tag{12.147}$$

the set of all natural transformations between the functors $\mathscr{O}(\underline{\mathbb{R}})$ and $\mathscr{O}(\underline{\Sigma})$, both restricted to $\uparrow C \subset \mathscr{C}(A)$, that are frame maps. This set can be computed from the external description of frames and frame maps in §E.4. Recall (12.4) etc. The frame $\mathscr{O}(\underline{\mathbb{R}})_{\uparrow C}$ has external description

$$\pi_{\mathbb{R}}^{-1} : \mathscr{O}(\uparrow C) \to \mathscr{O}(\uparrow C \times \mathbb{R}), \tag{12.148}$$

where $\pi_{\mathbb{R}} : \uparrow C \times \mathbb{R} \to \uparrow C$ is projection on the first component. The special case $C = \mathbb{C} \cdot 1$ yields the external description of $\mathscr{O}(\underline{\mathbb{R}})$ itself, namely

$$\pi_{\mathbb{R}}^{-1} : \mathscr{O}(\mathscr{C}(A)) \to \mathscr{O}(\mathscr{C}(A) \times \mathbb{R}), \tag{12.149}$$

where this time (with abuse of notation) the projection is $\pi_{\mathbb{R}} : \mathscr{C}(A) \times \mathbb{R} \to \mathscr{C}(A)$. By Corollary 12.22, the Gelfand frame $\mathscr{O}(\underline{\Sigma})_{\uparrow C}$ has external description

$$\pi_{\Sigma}^{-1} : \mathscr{O}(\uparrow C) \to \mathscr{O}(\Sigma)_{\uparrow C}, \tag{12.150}$$

given by (12.128), with the understanding that $D \supseteq C$ (the special case $C = \mathbb{C} \cdot 1$ then recovers the external description (12.99) of $\mathscr{O}(\underline{\Sigma})$ itself). It follows that there is a bijective correspondence between two classes of frame maps:

$$\underline{\varphi}_C^{-1} : \mathscr{O}(\underline{\mathbb{R}})_{\uparrow C} \to \mathscr{O}(\underline{\Sigma})_{\uparrow C} \ \text{ (in } \mathsf{T}(A)); \tag{12.151}$$

$$\varphi_C^{-1} : \mathscr{O}(\uparrow C \times \mathbb{R}) \to \mathscr{O}(\Sigma)_{\uparrow C} \ \text{ (in } \mathsf{Sets}), \tag{12.152}$$

where φ_C must satisfy the condition that for any $D \supseteq C$,

$$\varphi_C^{-1}(\uparrow D \times \mathbb{R}) = \chi_{\uparrow D}. \tag{12.153}$$

Indeed, such a map φ_C^{-1} defines an element $\underline{\varphi}_C^{-1}$ of $\mathrm{Nat}(\mathscr{O}(\underline{\mathbb{R}})_{\uparrow C}, \mathscr{O}(\underline{\Sigma})_{\uparrow C})$ in the obvious way: for $D \in \uparrow C$, the components

$$\underline{\varphi}_C^{-1}(D) : \mathscr{O}(\underline{\mathbb{R}})(D) \to \mathscr{O}(\underline{\Sigma})(D) \tag{12.154}$$

of the natural transformation $\underline{\varphi}_C^{-1}$, i.e.

$$\underline{\varphi}_C^{-1}(D) : \mathscr{O}(\uparrow D \times \mathbb{R}) \to \mathscr{O}(\Sigma)_{\uparrow D}, \tag{12.155}$$

are simply given by the restriction of φ_C^{-1} to $\mathscr{O}(\uparrow D \times \mathbb{R}) \subset \mathscr{O}(\uparrow C \times \mathbb{R})$; cf. (E.147). This is consistent, because (12.153) implies that for any $U \in \mathscr{O}(\mathbb{R})$ and $C \subseteq D \subseteq E$,

$$\varphi_C^{-1}(\uparrow E \times U)(F) \leq \varphi_C^{-1}(\uparrow D \times \mathbb{R})(F), \tag{12.156}$$

which by (12.153) vanishes whenever $F \not\supseteq D$. Consequently,

$$\varphi_C^{-1}(\uparrow E \times U)(F) = 0 \, \mathrm{if} \, F \not\supseteq D, \tag{12.157}$$

so that $\underline{\varphi}_C^{-1}(D)$ actually takes values in $\mathscr{O}(\Sigma)_{\uparrow D}$ (rather than in $\mathscr{O}(\Sigma)_{\uparrow C}$, as might be expected). Denoting the set of frame maps (12.152) that satisfy (12.153) by $\mathrm{Frm}'(\mathscr{O}(\uparrow C \times \mathbb{R}), \mathscr{O}(\Sigma)_{\uparrow C})$, we obtain a functor

$$\mathrm{Frm}'(\mathscr{O}(\uparrow(-) \times \mathbb{R}), \mathscr{O}(\Sigma)_{\uparrow -}) : \mathscr{C}(A) \to \mathsf{Sets}, \tag{12.158}$$

with the stipulation that for $C \subseteq D$ the induced map

$$\mathrm{Frm}'(\mathscr{O}(\uparrow C \times \mathbb{R}), \mathscr{O}(\Sigma)_{\uparrow C}) \to \mathrm{Frm}'(\mathscr{O}(\uparrow D \times \mathbb{R}), \mathscr{O}(\Sigma)_{\uparrow D})$$

is given by restricting an element of the left-hand side to $\mathscr{O}(\uparrow D \times \mathbb{R}) \subset \mathscr{O}(\uparrow C \times \mathbb{R})$; this is consistent by the same argument (12.157).

The Gelfand isomorphism (12.78) is therefore a natural transformation

$$\underline{A} \xrightarrow{\cong} \mathrm{Frm}'(\mathscr{O}(\uparrow - \times \mathbb{R}), \mathscr{O}(\Sigma)_{\uparrow -}), \tag{12.159}$$

which means that one has a compatible (i.e. natural) family of isomorphisms

$$\begin{aligned} C &\xrightarrow{\cong} \mathrm{Frm}'(\mathscr{O}(\uparrow C \times \mathbb{R}), \mathscr{O}(\Sigma)_{\uparrow C}); \\ a &\mapsto \hat{a}^{-1} : \mathscr{O}(\uparrow C \times \mathbb{R}) \to \mathscr{O}(\Sigma)_{\uparrow C}. \end{aligned} \tag{12.160}$$

On basic opens $\uparrow D \times U \in \mathscr{O}(\uparrow C \times \mathbb{R})$, with $D \supseteq C$, we obtain

$$\begin{aligned} \hat{a}^{-1}(\uparrow D \times U) : E &\mapsto 1_U(a) \text{ if } E \supseteq D; \\ E &\mapsto 0 \text{ if } E \not\supseteq D. \end{aligned} \tag{12.161}$$

Here $1_U(a)$ is the spectral projection of a in U, cf. (12.82); as it lies in $\mathscr{P}(C)$ and $C \subseteq D \subseteq E$, the projection $1_U(a)$ certainly lies in $\mathscr{P}(E)$, as required. Furthermore, we need to extend $\hat{a}^{-1}$ to general opens in $\uparrow C \times \mathbb{R}$ by the frame map property, and note that (12.153) for $\varphi_C^{-1} = \hat{a}^{-1}$ is satisfied.

This analysis also holds in the topos $\mathsf{Sh}(\mathscr{C}(A))$ of sheaves in $\mathscr{C}(A)$ (as always, equipped with the Alexandrov topology, cf. (E.84). It then follows from (12.159) and (12.141) that as a sheaf,

$$C(\underline{\Sigma}, \underline{\mathbb{C}}) : U \mapsto C(\Sigma^A_U, \mathbb{C}), \tag{12.162}$$

where Σ^A_U is given by (12.136); if $U \subseteq V$, the map $C(\Sigma^A_V, \mathbb{C}) \to C(\Sigma^A_U, \mathbb{C})$ is given by the pullback of the inclusion $\Sigma^A_U \hookrightarrow \Sigma^A_V$ (that is, by restriction). It then follows from (12.162) that the isomorphism (12.146) is given by its components

$$\underline{A}(U) \cong C(\Sigma^A_U, \mathbb{C}). \tag{12.163}$$

In particular, the component of the natural isomorphism in (12.146) at $U = \uparrow C$ is

$$C \cong C(\Sigma^A_{\uparrow C}, \mathbb{C}). \tag{12.164}$$

A glance at the topology of Σ^A shows that the so-called ***Hausdorffication***, which for a general compact space may be defined either directly, or C*-algebraically by $X^H = \Sigma(C(X))$, and coincides with the left adjoint of the forgetful functor from the category of compact Hausdorff spaces (and continuous maps) to the category of compact spaces (and continuous maps), is given by $(\Sigma^A_{\uparrow C})^H \cong \Sigma(C)$, so that

$$C(\Sigma^A_{\uparrow C}, \mathbb{C}) \cong C(\Sigma(C), \mathbb{C}), \tag{12.165}$$

where the isomorphism is given by restricting $f \in C(\Sigma^A_{\uparrow C}, \mathbb{C})$ to $\Sigma(C) \subset \Sigma^A_{\uparrow C}$.

Corollary 12.23. *The internal Gelfand isomorphism*

$$\underline{A} \xrightarrow{\cong} C(\underline{\Sigma}, \underline{\mathbb{C}}), \tag{12.166}$$

which is a natural isomorphism between functors $\mathscr{C}(A) \to \mathsf{Sets}$, is given at each $C \in \mathscr{C}(A)$ by the usual Gelfand isomorphism for the commutative C-algebra C:*

$$\underline{A}_0(C) = C \xrightarrow{\cong} C(\Sigma(C), \mathbb{C}) \cong C(\underline{\Sigma}, \underline{\mathbb{C}})_0(C). \tag{12.167}$$

At the end of the day, the Gelfand isomorphism (12.146) therefore turns out to simply assemble all isomorphisms (12.167) for the commutative C*-subalgebras C of A into a single sheaf-theoretic construction. Incidentally, taking $C = \mathbb{C} \cdot 1$ in (12.164) shows that $(\Sigma^A)^H$ is a point, which is also obvious from the fact that any open set containing the point $\Sigma(\mathbb{C} \cdot 1)$ of Σ^A must be all of Σ^A.

12.5 "Daseinisation" and Kochen–Specker Theorem

The internal Gelfand transform (12.166) constructed in the previous section acts on each commutative subalgebra $A \in \mathscr{C}(A)$. What about A itself? There is a more subtle transform, inspired by the remarkable "Daseinisation" construction of Döring and Isham (whose name has unfortunately been inspired by the controversial German philosopher Heidegger), which turns self-adjoint elements a of A into continuous functions $\delta(a)$ on the topos-theoretical phase space Σ^A, whose range is the so-called *interval domain* $\mathbb{IR}$ (which is a fuzzy version of $\mathbb{R}$). Hence we will define a map

$$\delta : A_{\mathrm{sa}} \to C(\Sigma^A, \mathbb{IR}), \tag{12.168}$$

which, alas, is defined only if A is a von Neumann algebra; we shall therefore assume this throughout this section. Similarly, the notation $\mathscr{C}(A)$ will now stand for the poset of abelian *von Neumann* subalgebras of A (as opposed to abelian C^*-subalgebras of A, as in the remainder of this book).

"Daseinisation" requires two slightly unusual concepts, the first of which is the said ***interval domain*** $\mathbb{IR}$. To motivate its definition, consider Brouwer's approximation of real numbers by nested intervals with endpoints in $\mathbb{Q}$. For example, the real number π can be described by specifying the sequence

$$[3,4], [3.1,3.2], [3.14,3.15], [3.141,3.142], \ldots$$

This description of the reals is formalized by $\mathbb{IR}$, defined as the poset whose elements are *compact* intervals $[a,b]$ in $\mathbb{R}$ (including singletons $[a,a] = \{a\}$), ordered by *reverse* inclusion (for a smaller interval means that we have more information about the real number that the ever smaller intervals converge to). This poset is a so-called ***dcpo*** (for ***directed complete partial order***); directed suprema are simply intersections. As such, it carries the ***Scott topology***, whose open sets are upper subsets U of $\mathbb{IR}$ with the additional property that for every directed set D with $\bigvee D \in U$ the intersection $D \cap U$ is nonempty. This means that each open interval (p,q) in $\mathbb{R}$ (with $p = -\infty$ and $q = +\infty$ allowed) corresponds to a Scott open

$$U_{(p,q)} = \{[a,b] \mid p < a \leq b < q\}. \tag{12.169}$$

Indeed, these opens form a basis of the Scott topology $\mathscr{O}_{\mathrm{Scott}}(\mathbb{IR}) \equiv \mathscr{O}(\mathbb{IR})$ of $\mathbb{IR}$. This topology is, of course, a frame, so far defined in Sets. However, this frame is easily internalized to any (pre)sheaf topos, similar to the Dedekind reals (12.3) - (E.149); in particular, in $\mathsf{T}(A)$ we have

$$\mathscr{O}(\underline{\mathbb{IR}})_0 : C \mapsto \mathscr{O}((\uparrow C) \times \mathbb{IR}), \tag{12.170}$$

with external description as a locale (see §E.4) given by the canonical projection

$$\pi_1 : \mathscr{C}(A) \times \mathbb{IR} \to \mathscr{C}(A). \tag{12.171}$$

The second ingredient of "Daseinisation" is the *spectral order* on A_{sa}. The partial order $\leq$ defined in §C.7 (in which $a \leq b$ iff $\omega(a) \leq \omega(b)$ for all states ω on A) has good linearity properties in that it makes A_+ a convex cone in the real vector space A_{sa} (cf. Definition C.50), but it is terrible from a lattice point of view (unless A is abelian): for example, for $A = B(H)$, suprema $a \vee b$ and infima $a \wedge b$ exist iff either $a \leq b$ or $b \leq a$ (and indeed A_{sa} is a lattice with respect to $\leq$ iff A is abelian). However, there is a different order on A_{sa} that turns it into a ***conditionally*** (or ***boundedly***)***complete lattice***, i.e., a poset X with the property that if some subset $S \subseteq X$ has an upper bound (i.e., there is $x \in X$ such that $s \leq x$ for each $s \in S$), then it has a *lowest* upper bound (i.e., $\bigvee S$ exists), and similarly for (greatest) lower bounds.

Definition 12.24. *For $a, b \in A_{\mathrm{sa}}$ we say that $a \leq_s b$ (i.e., a is less or equal than b in the* **spectral order***) iff $a^n \leq b^n$ for each $n \in \mathbb{N}$.*

It can be shown that $a \leq_s b$ iff $e^{(b)}_{(\lambda)} \leq e^{(a)}_{(\lambda)}$ for each $\lambda \in \mathbb{R}$ (note the change of order), where $e^{(a)}_{(\lambda)}$ is the spectral projection $1_{(-\infty,\lambda]\cap\sigma(a)}(a)$, etc. This, in turn, implies, that

$$a \leq_s b \text{ iff } \mu_\omega(a \leq \lambda) \geq \mu_\omega(b \leq \lambda), \tag{12.172}$$

for each (normal) state ω on A and each $\lambda \in \mathbb{R}$, where

$$\mu_\omega(a \leq \lambda) = \omega(1_{(-\infty,\lambda]\cap\sigma(a)}(a)) \tag{12.173}$$

is the Born probability for the outcome $a \leq \lambda$ in state ω (and similarly for b). Furthermore, if a and b commute, or if a and b are both projections, the $a \leq_s b$ iff $a \leq b$, i.e., $\leq_s$ coincides with the usual partial order $\leq$ iff A is abelian, and $\leq_s$ restricts to $\leq$ on the projection lattice $\mathscr{P}(A)$ of A. For each $a \in A_{\mathrm{sa}}$ and $C \in \mathscr{C}(A)$, we define

$$\delta^i_C(a) = \bigvee\{b \in C_{\mathrm{sa}} \mid b \leq_s a\}; \tag{12.174}$$

$$\delta^o_C(a) = \bigwedge\{b \in C_{\mathrm{sa}} \mid a \leq_s b\}, \tag{12.175}$$

called the ***inner*** and ***outer Daseinisation*** of a with respect to C, respectively; those objecting to Heidegger might prefer to simply call these the inner and outer ***localizations*** of a with respect to C. For projections, these expressions simplify to

$$\delta^i_C(e) = \bigvee\{f \in \mathscr{P}(C) \mid f \leq_s e\}; \tag{12.176}$$

$$\delta^o_C(e) = \bigwedge\{f \in \mathscr{P}(C) \mid e \leq_s f\}, \tag{12.177}$$

and in fact one has a very nice categorical description, in that $\delta^i_C : \mathscr{P}(A) \to \mathscr{P}(C)$ and $\delta^o_C : \mathscr{P}(A) \to \mathscr{P}(C)$ are the right and left adjoint, respectively, of the inclusion functor $\mathscr{P}(C) \hookrightarrow \mathscr{P}(A)$ in the category of complete orthomodular lattices.

We are now in a position to define the map (12.168): for $a \in A_{\mathrm{sa}}$ we put

$$\delta(a) : (C, \omega) \mapsto [\omega(\delta^i_C(a)), \omega(\delta^o_C(a))], \tag{12.178}$$

where (as the notation indicates) the point $(C, \omega) \in \Sigma(C) \subset \Sigma^A$ is just $\omega \in \Sigma(C)$.

It is easily checked that the right-hand side of (12.178) makes sense, since positivity of states and (12.174) - (12.175) obviously imply $\omega(\delta^i_C(a)) \leq \omega(\delta^o_C(a))$. Also, $\delta(a)$ is continuous, so that δ is well defined. If we define a closely related map

$$\hat{\delta}(a) : \Sigma^A \to \mathscr{C}(A) \times \mathbb{IR}; \tag{12.179}$$

$$\hat{\delta}(a)(C,\omega) = (C, \delta(a)(C,\omega)), \tag{12.180}$$

then $\hat{\delta}(a)$ is the external description of an internal locale map

$$\underline{\delta}(a) : \underline{\Sigma}(\underline{A}) \to \underline{\mathbb{IR}}. \tag{12.181}$$

In view of this, we may regard (12.168) as a hybrid (i.e. “category mistake”) map

$$\underline{\delta} : A_{\mathrm{sa}} \to C(\underline{\Sigma}(\underline{A}), \underline{\mathbb{IR}}); \tag{12.182}$$

see the text below (12.146), with $\mathbb{R} \rightsquigarrow \mathbb{IR}$, for the meaning of the right-hand side.

The relationship between δ and the Gelfand transform (12.166) is as follows. For $a \in A_{\mathrm{sa}}$, let $W^*(a)$ be the unital commutative von Neumann algebra generated by $a = a^*$ and 1_A within A. Using (12.164), we then have a Gelfandish isomorphism

$$W^*(a)_{\mathrm{sa}} \xrightarrow{\cong} C(\Sigma^A_{\uparrow W^*(a)}, \mathbb{R}); \tag{12.183}$$

$$c \mapsto \hat{c}. \tag{12.184}$$

In particular, since $a \in W^*(a)$, we obtain a continuous function

$$\hat{a} : \Sigma^A_{\uparrow W^*(a)} \to \mathbb{R}. \tag{12.185}$$

Furthermore, we have an inclusion

$$\iota : \mathbb{R} \hookrightarrow \mathbb{IR}; \tag{12.186}$$

$$x \mapsto [x,x], \tag{12.187}$$

which is continuous, and hence induces a map $C(\Sigma^A, \mathbb{R}) \to C(\Sigma^A, \mathbb{IR})$, as well as maps $C(\Sigma^A_{\uparrow W^*(a)}, \mathbb{R}) \to C(\Sigma^A_{\uparrow W^*(a)}, \mathbb{IR})$. Then the following diagram commutes:

$$\begin{tikzcd} \Sigma^A_{\uparrow W^*(a)} \arrow[r, "\delta(a)"] \arrow[dr, "\hat{a}"'] & \mathbb{IR} \\ & \mathbb{R} \arrow[u, "\iota"'] \end{tikzcd} \tag{12.188}$$

In words, the restriction of the “Daseinisation” $\delta(a) : \Sigma^A \to \mathbb{IR}$ of a to the open subset $\Sigma^A_{\uparrow W^*(a)} \subset \Sigma^A$ takes values in $\mathbb{R} \subset \mathbb{IR}$, and as such coincides with the Gelfand transform $\hat{a}$ of a, seen as a map (12.185). Hence, as might be expected in quantum mechanics, any fuzziness of $\delta(a)$ is only noticeable outside its own context $W^*(a)$.

The “Daseinisation” construction enables one to interpret propositions $a \in (p,q)$ as open subsets of the “phase space” Σ^A, as in classical physics, where $a : X \to \mathbb{R}$ would be a continuous function on a phase space X, and one would say that

$$[[a \in (p,q)]]_{\mathrm{CM}} = a^{-1}(p,q) \in \mathcal{O}(X). \tag{12.189}$$

In quantum mechanics, one would interpret $a \in (p,q)$ as the spectral projection

$$[[a \in (p,q)]]_{\mathrm{QM}} = e^{(a)}_{(p,q)} \equiv 1_{(p,q)\cap\sigma(a)}(a), \tag{12.190}$$

or, equivalently, with the corresponding closed subset of the ambient Hilbert space. In our quantum toposophy setting, however, we may adapt (12.189) as

$$[[a \in (p,q)]]_{\mathrm{QT}} = \delta(a)^{-1}(U_{(p,q)}) \in \mathcal{O}(\Sigma^A). \tag{12.191}$$

Similarly, one may interpret $a \in (p,q)$ as an internal open subset of the internal Gelfand spectrum $\underline{\Sigma}(\underline{A})$, as follows. For any locale Y in a topos T, an internal open in $\mathcal{O}(Y)$ is defined as an arrow $\mathbf{1} \to \mathcal{O}(Y)$, where as usual $\mathbf{1}$ is the terminal object in T. In the case at hand we have $Y = \underline{\Sigma}(\underline{A})$, and use the composition

$$\underline{\mathbf{1}} \xrightarrow{\underline{(p,q)}} \mathcal{O}(\underline{\mathbb{IR}}) \xrightarrow{\underline{\delta}(a)^{-1}} \mathcal{O}(\underline{\Sigma}(\underline{A})), \tag{12.192}$$

where the natural transformation $\underline{(p,q)}$ has components

$$\underline{(p,q)}_C(*) = \uparrow C \times U_{(p,q)}, \tag{12.193}$$

cf. (12.170), and $\underline{\delta}(a)^{-1} : \mathcal{O}(\underline{\mathbb{IR}}) \to \mathcal{O}(\underline{\Sigma}(\underline{A}))$ is the frame version of the locale map (12.181), whose component at C, i.e.,

$$\underline{\delta}(a)_C^{-1} : \mathcal{O}((\uparrow C) \times \mathbb{IR}) \to \mathcal{O}(\Sigma^A_{\uparrow C}), \tag{12.194}$$

is given on basic opens in $(\uparrow C) \times \mathbb{IR}$, with $D \supseteq C$ and $p < q$, by

$$\underline{\delta}(a)_C^{-1}(\uparrow D \times U_{(p,q)}) = \delta(a)^{-1}(U_{(p,q)}) \cap \Sigma^A_{\uparrow D}. \tag{12.195}$$

We therefore obtain the quantum-toposophical interpretation of $a \in (p,q)$ as:

$$[[\underline{a \in (p,q)}]]_{\mathrm{QT}} : \underline{\mathbf{1}} \to \mathcal{O}(\underline{\Sigma}(\underline{A})); \tag{12.196}$$

$$[[\underline{a \in (p,q)}]]_{\mathrm{QT}} = \underline{\delta}(a)^{-1} \circ \underline{(p,q)}. \tag{12.197}$$

We are now going to combine this expression with a construction relating states $\omega \in S(A)$ to arrows from $\mathcal{O}(\underline{\Sigma}(\underline{A}))$ to the truth object $\underline{\Omega}$ in $\mathsf{T}(A)$. This construction generalizes the fundamental bijective correspondence between states on commutative (unital) C*-algebras A and probability measures on its Gelfand spectrum $\Sigma(A)$ (cf. Theorem B.24) to the non-commutative case.

To this end, we first need to replace probability measures on *spaces* by probability measures on *locales*. This, in turn, requires the ***lower real numbers*** $\mathbb{R}_l$, which may be identified with proper subsets $x_l \subset \mathbb{Q}$ with the following two properties:

1. If $p \in x_l$, then there exists $q \in x_l$ with $p < q$.
2. If $p < q \in x_l$, then $p \in x_l$ (i.e., x_l is a lower subset of $\mathbb{Q}$).

In Sets, the lower reals may be identified with $\mathbb{R}$ (in Hilbert's definition) by identifying x_l with its supremum $x = \sup x_l$, but in arbitrary toposes (that admit internal natural and hence rational numbers) they drift apart. Similarly, one defines the ***upper real numbers*** $\mathbb{R}_u$ as proper upper subsets $x_u \subset \mathbb{Q}$ such that $p \in x_u$ implies that there exists $q \in x_u$ with $p > q$; once again, in Sets, $\mathbb{R}_u$ may be identified with Hilbert's $\mathbb{R}$ by taking $x = \inf x_u$. The ***Dedekind real numbers*** $\mathbb{R}_d$, then, are pairs (x_l, x_u) where $x_l \in \mathbb{R}_l$ and $x_u \in \mathbb{R}_u$ are such that $x_l \cap x_u = \emptyset$ and for each $p, q \in \mathbb{Q}$ with $p < q$, either $p \in x_l$ or $q \in x_u$. In Sets these may be identified with $\sup x_l = \inf x_u = x$, so that $\mathbb{R}_d \cong \mathbb{R}$, but in many toposes $\mathbb{R}_l$, $\mathbb{R}_u$, and $\mathbb{R}_d$ are all different. For example, we have already seen that in sheaf toposes $\mathsf{Sh}(X)$, the Dedekind reals are given by the sheaf (E.150), but the lower reals turn out to be defined by

$$(\mathbb{R}_l)_0 : U \mapsto L(U, \mathbb{R}), \tag{12.198}$$

where $U \in \mathscr{O}(X)$ and $L(U, \mathbb{R})$ is the set of all lower semicontinuous functions from U to $\mathbb{R}$ that are locally bounded from above (and similarly for $\mathbb{R}_u$, *mutatis mutandis*). In particular, in $\mathsf{T}(A)$ we have the functor

$$(\underline{\mathbb{R}}_l)_0 : C \mapsto L(\uparrow C, \mathbb{R}), \tag{12.199}$$

which is quite different from (12.3) (and similarly for $\underline{\mathbb{R}}_u$).

Definition 12.25. *A* **probability measure on a locale** X *is a monotone map*

$$\mu : \mathscr{O}(X) \to [0,1]_l, \tag{12.200}$$

where $[0,1]_l$ *is the collection of lower reals between 0 and 1 (defined by replacing* $\mathbb{Q}$ *in the definition of* $\mathbb{R}_l$ *by the set of all rationals* $0 \leq q \leq 1$*), that satisfies*

$$\mu(\top) = 1; \tag{12.201}$$

$$\mu(U) + \mu(V) = \mu(U \wedge V) + \mu(U \vee V); \tag{12.202}$$

$$\mu\left(\bigvee\nolimits_\lambda U_\lambda\right) = \bigvee\nolimits_\lambda \mu(U_\lambda), \tag{12.203}$$

for any directed family (U_λ) *in* $\mathscr{O}(X)$.

Compared with (probability) measures on σ-algebras, we see that (probability) measures on locales are merely defined on *open* sets (as opposed to *measurable* sets, which include opens), but this weakening is compensated for by the much stronger (i.e. uncountable) additivity axiom (12.203). Indeed, in Sets, if X is a compact Hausdorff space, one even has a bijective correspondence between *regular* probability measures μ' on X as a space and probability measures μ on X as a locale.

This definition makes sense in constructive mathematics, and hence it may be internalized to $\mathsf{T}(A)$. Doing so, probability measures on the internal Gelfand spectrum $\underline{\Sigma}(\underline{A})$ turn out to correspond to the following notion (cf. Definition 2.26).

Definition 12.26. *A* **quasi-state** *on a unital C*-algebra A is a map $\omega : A \to \mathbb{C}$ that is positive and normalized ($\omega(1_A) = 1$), satisfies $\omega(b+ic) = \omega(b) + i\omega(c)$ for $b^* = b$ and $c^* = c$, and is linear on each commutative unital C*-algebra in A.*

Theorem 12.27. *There is a bijective correspondence between quasi-states ω on A and probability measures $\underline{\mu}_\omega$ on the internal Gelfand spectrum $\underline{\Sigma}(\underline{A})$.*

The proof uses the fact that given the (Alexandrov) topology on $\mathscr{C}(A)$, a function $\uparrow C \to [0,1]$ is lower semicontinuous iff it is order-preserving (i.e., monotone); since $[0,1]$ is bounded, the condition of local boundedness is trivially satisfied and hence $L(\uparrow C, [0,1])$ consists of all order-preserving functions from $\uparrow C \subset \mathscr{C}(A)$ to $[0,1]$.

Proof. Any probability measure on $\underline{\Sigma}(\underline{A})$ is a natural transformation

$$\underline{\mu} : \underline{\Sigma}(\underline{A}) \to \underline{[0,1]}_l, \tag{12.204}$$

whose component at $C \in \mathscr{C}(A)$, according to (12.138) and (12.199), is a map

$$\underline{\mu}_C : \mathscr{O}(\Sigma^A_{\uparrow C}) \to L(\uparrow C, [0,1]), \tag{12.205}$$

satisfying properties dictated by Definition 12.25. In particular, if C is maximal abelian in A, then by the comment preceding the proof, $\underline{\mu}_C$ is simply a function $\mathscr{O}(\Sigma(C)) \to [0,1]$ that satisfies (12.201) - (12.203) and hence is a (regular) probability measure μ_C on $\Sigma(C)$. Thus by Riesz–Markov one obtains a state ω_C on each maximal abelian C. From the topology on Σ^A and (12.137) we see that if D is not maximal, $\underline{\mu}_D$ is determined by $\underline{\mu}_C$ for any $C \supset D$, so that we also obtain a probability measure μ_D on $\Sigma(D)$, or, equivalently, a state ω_D, by restriction of ω_C to D. One might fear that μ_D and ω_D could depend on the chosen embedding $D \subset C$, but naturality of $\underline{\mu}$ implies that if $D \subset C$ as well as $D \subset C'$, where both C and C' are maximal, then the ensuing measures μ_D are the same. This implies the same property for the corresponding states ω_D, which in turn shows that the collection of all μ_D and μ_C thus obtained organizes itself into a single quasi-state ω on A.

The converse follows by running this argument backwards. □

Combining (12.196) with Theorem 12.27, we obtain a state-proposition pairing that is no longer probabilistic, as in ordinary quantum mechanics, but defines a proposition in the internal language of $\mathsf{T}(A)$ and as such may or may not be true at each stage $C \in \mathscr{C}(A)$. The final ingredient for this is an arrow

$$\underline{1} : \underline{\Sigma}(\underline{A}) \to \underline{[0,1]}_l, \tag{12.206}$$

defined by its components $\underline{1}_C : \mathscr{O}(\Sigma^A_{\uparrow C}) \to L(\uparrow C, [0,1])$ that map each open subset of $\Sigma^A_{\uparrow C}$ to the constant function on $\uparrow C$ taking the value $1 \in [0,1]$. The internal language of $\mathsf{T}(A)$ (cf. §E.5) turns this into a formula $\underline{\mu}_\omega = \underline{1}$ with the following interpretation:

$$[[\underline{\mu}_{\omega} = \underline{1}]] : \underline{\Sigma}(\underline{A}) \to \underline{\Omega}. \tag{12.207}$$

We combine this with (12.196) so as to obtain an internal state-proposition pairing

$$[[\underline{\mu}_{\omega}(\underline{a \in (p,q)}) = \underline{1}]]_{\mathrm{QT}} : \underline{\mathbf{1}} \to \underline{\Omega}, \tag{12.208}$$

where we have abbreviated

$$[[\underline{\mu}_{\omega}(\underline{a \in (p,q)}) = \underline{1}]]_{\mathrm{QT}} = [[\underline{\mu}_{\omega} = \underline{1}]] \circ [[\underline{a \in (p,q)}]]_{\mathrm{QT}}. \tag{12.209}$$

The truth of the proposition (12.208) at stage C may be determined from Kripke–Joyal semantics; a straightforward computation for $A = B(H)$ shows that

$$C \Vdash \underline{\mu}_{\omega}(\underline{a \in (p,q)}) = \underline{1} \tag{12.210}$$

iff there exists a projection $e \in \mathscr{P}(C)$ with $e \leq e^{(a)}_{(p,q)}$ and $\omega(e) = 1$. Assuming ω is a vector state $\omega(a) = \langle \psi, a\psi \rangle$ for some unit vector $\psi \in H$, this means that (12.210) holds iff $\psi \in eH \subseteq e^{(a)}_{(p,q)}H$ for some $e \in \mathscr{P}(C)$, i.e., if the proposition $a \in (p,q)$ has (Born) probability one in state ψ *and* there is a yes-no measurement in context C verifying this probability. In comparison, in classical mechanics a pure state $x \in X$ makes $a \in (p,q)$ true iff $a(x) \in (p,q)$, where $a \in C(X,\mathbb{R})$ as before.

We close this chapter with a topos-theoretical (or, one might say, topological) reinterpretation of the Kochen–Specker Theorem, which to some extent explains why the previous construction had to use the fuzzy interval domain $\mathbb{IR}$ rather than the sharp reals $\mathbb{R}$. To this end, we first generalize the notion of a quasi-linear non-contextual hidden variable (cf. Definitions 6.1 and 6.3) to any (unital) C*-algebra:

Definition 12.28. *1. A* **valuation** *on a unital C*-algebra A is a unital map*

$$V : A_{\mathrm{sa}} \to \mathbb{R} \tag{12.211}$$

that is dispersion-free (i.e. multiplicative) and linear on commuting operators.
2. A **point** *in a frame $\mathscr{O}(X)$ in some topos T is defined as a frame homomorphism*

$$p : \mathscr{O}(X) \to \Omega, \tag{12.212}$$

where Ω is the truth object in T.

If A is commutative, the Gelfand spectrum $\Sigma(A)$ consists of the valuations on A. The second part generalizes the notion of a point of a frame in set theory (cf. §C.11).

Theorem 12.29. *For any unital C*-algebra A, there are canonical bijective correspondences between:*

- *Valuations on A.*
- *Points of $\underline{\Sigma}(\underline{A})$ in $\mathsf{Sh}(\mathscr{C}(A))$.*
- *Continuous cross-sections $\sigma : \mathscr{C}(A) \to \Sigma^A$ of the bundle $\pi : \Sigma^A \to \mathscr{C}(A)$.*

Proof. We first give the external description of points of a locale $\underline{Y}$ in a sheaf topos $\mathsf{Sh}(X)$ (cf. §E.4). The subobject classifier in $\mathsf{Sh}(X)$ is the sheaf $\underline{\Omega} : U \mapsto \mathscr{O}(U)$, in terms of which a point of $\underline{Y}$ is a frame map $\mathscr{O}(\underline{Y}) \to \underline{\Omega}$. Externally, the point-free space defined by the frame $\underline{\Omega}$ is given by the identity map $\mathrm{id}_X : X \to X$, so that a point of $\underline{Y}$ externally correspond to a continuous cross-section $\sigma : X \to Y$ of the bundle $\pi : Y \to X$ (i.e., $\pi \circ \sigma = \mathrm{id}_X$). In principle, π and σ are by definition frame maps in the opposite direction, but in the case at hand, namely $X = \mathscr{C}(A)$ and $Y = \Sigma^A$, the map $\sigma : \mathscr{C}(A) \to \Sigma^A$ may be interpreted as a continuous cross-section of the projection (12.134) in the usual sense. Being a cross-section simply means that $\sigma(C) \in \Sigma(C)$. As to continuity, by definition of the Alexandrov topology, σ is continuous iff the following condition is satisfied:

$$\text{For all } \mathscr{U} \in \mathscr{O}(\Sigma^A) \text{ and all } C \subseteq D, \text{ if } \sigma(C) \in \mathscr{U}, \text{ then } \sigma(D) \in \mathscr{U}.$$

Hence, given the definition of $\mathscr{O}(\Sigma^A)$, the following condition is sufficient for continuity: if $C \subseteq D$, then $\sigma(D)_{|C} = \sigma(C)$. However, this condition is also necessary. To explain this, let $\rho_{DC} : \Sigma(D) \to \Sigma(C)$ again be the restriction map. This map is continuous and open. Suppose $\rho_{DC}(\sigma(D)) \neq \sigma(C)$. Since $\Sigma(D)$ is Hausdorff, there is an open neighbourhood $\mathscr{U}_D$ of $\rho_{DC}^{-1}(\sigma(C))$ not containing $\sigma(D)$. Let $\mathscr{U}_C = \rho_{DC}(\mathscr{U}_D)$ and take any $\mathscr{U} \in \mathscr{O}(\Sigma^A)$ such that $\mathscr{U} \cap \mathscr{O}(\Sigma(C)) = \mathscr{U}_C$ and $\mathscr{U} \cap \mathscr{O}(\Sigma(D)) = \mathscr{U}_D$. This is possible, since $\mathscr{U}_C$ and $\mathscr{U}_D$ satisfy both conditions in the definition of $\mathscr{O}(\Sigma^A)$. By construction, $\sigma(C) \in \mathscr{U}$ but $\sigma(D) \notin \mathscr{U}$, so that σ is not continuous. Hence σ is a continuous cross-section of π iff

$$\sigma(D)_{|C} = \sigma(C) \text{ for all } C \subseteq D. \tag{12.213}$$

Now define a map $V : A_{\mathrm{sa}} \to \mathbb{C}$ by $V(a) = \sigma(C^*(a))(a)$, where $C^*(a)$ is the commutative unital C*-algebra generated by a. If $b^* = b$ and $[a,b] = 0$, then $V(a+b) = V(a) + V(b)$ by (12.213), applied to $C^*(a) \subset C^*(a,b)$ as well as to $C^*(b) \subset C^*(a,b)$. Furthermore, since $\sigma(C) \in \Sigma(C)$, the map V is dispersion-free.

Conversely, a valuation V defines a cross-section σ by complex linear extension of $\sigma(C)(a) = V(a)$, where $a \in C_{\mathrm{sa}}$. By the criterion (12.213) this cross-section is continuous, since the value $V(a)$ is independent of the choice of C containing a. □

Corollary 12.30. *The bundle $\pi : \Sigma^A \to \mathscr{C}(A)$ (cf. Corollary 12.22) admits no continuous cross-sections as soon as A has no valuations (e.g. if $A = M_n(\mathbb{C})$, $n > 2$).*

The contrast between the pointlessness of the internal spectrum $\underline{\Sigma}$ and the spatiality of the external spectrum Σ^A is striking, but easily explained: a point of Σ^A (in the usual sense, but also in the frame-theoretic sense if Σ^A is sober) necessarily lies in some $\Sigma(C) \subset \Sigma^A$, and hence is defined (and dispersion-free) only in the context C. For example, for $A = M_n(\mathbb{C})$, a point $V \in \Sigma(C)$ corresponds to a map

$$V^* : \mathscr{O}(\Sigma^A) \to \{0,1\},\ S \mapsto V(S(C)), \tag{12.214}$$

where $\mathscr{O}(\Sigma^A)$ is given by (12.95). Thus V^* is only sensitive to the value of S at C.

Notes

Previous advocates of intuitionistic logic for quantum mechanics include Popper (1968) and Coecke (2002). The earliest use of topos theory in quantum mechanics was probably by Adelman & Corbett (1995), but the founding papers of the topos approach to quantum mechanics as further developed in this chapter are Isham & Butterfield (1998), Butterfield & Isham (1999, 2002), and Hamilton, Isham & Butterfield (2000). This series of papers was predated by Isham (1997) and was followed by Döring & Isham (2008abcd, 2010); see also Flori (2013) for an introduction. Wolters (2013ab) gives a detailed comparison between the "contravariant" Butterfield–Döring–Isham approach and the "covariant" approach in this chapter.

The original motivation behind our approach to "quantum toposophy" was the ***Principle of General Tovariance*** (Heunen, Landsman, & Spitters, 2008), which was a pun on Einstein's *Principle of General Covariance* underlying General Relativity (Norton, 1993, 1995). Einstein based his theory of gravity and space-time on the mathematical postulate that all equations of physics be invariant under arbitrary coordinate transformation, and similarly we proposed that all physical theories should be invariant under so-called *geometric morphisms* between toposes and hence should be formulated in terms of what (confusingly) is called *geometric logic* (cf. Mac Lane & Moerdijk, 1992; Johnstone, 2002). Since in fact some of our constructions turned out be non-geometric in this sense, we subsequently dropped this principle and stopped even referring to the above paper. However, as Raynaud (2014) and, more generally, Henry (2015) show, our theory can actually be made geometric (in the topos-theoretical sense) provided one puts the entire theory of (internal) C*-algebras on a localic (i.e., pointfree) basis, as in Henry (2014ab). Other recent developments of the program (which are not discussed here) may be found in e.g. van den Berg & Heunen (2012, 2014), Spitters, Vickers, & Wolters (2014), Heunen (2014ab), and Heunen & Lindenhovius (2015).

§12.1. **C*-algebras in a topos**

C*-algebras in a topos, including a constructive version of Gelfand duality for commutative unital C*-algebras that is valid in arbitrary Grothendieck toposes, were first studied by Banaschewski & Mulvey (2000ab, 2006). The topos $\mathsf{T}(A)$ and the internal commutative C*-algebra $\underline{A}$ were introduced by Heunen, Landsman, & Spitters (2009). All these papers rely crucially on the theory of internal locales in toposes, which owes much to Johnstone (1982) and Joyal & Tierney (1984). See also Johnstone (1983) and Vickers (2007). It is possible to realize $\mathsf{T}(A)$ as the topos of sheaves on the locale $\mathrm{Idl}(\mathscr{C}(A))$, which is the ideal completion of the "mere" poset $\mathscr{C}(A)$, but we will not use this description (Raynaud, 2014).

§12.2. **The Gelfand spectrum in constructive mathematics**

This section is based on Coquand (2005) and Coquand & Spitters (2005, 2009), where also the missing details may be found. All necessary background on lattice theory is provided by Johnstone (1982), except the ingredients for the proof that the constructive Gelfand spectrum is compact and regular, which is due to Cederquist & Coquand (2000). Proposition 12.10 may be found in Aczel (2006).

§12.3. **Internal Gelfand spectrum and intuitionistic quantum logic**

This section is based on Caspers, Heunen, Landsman, & Spitters (2009), except for the final part on Kripke semantics, which is taken from Heunen, Landsman, & Spitters (2012). An interesting philosophical analysis of the intuitionistic logic emerging from this program may be found in Hermens (2016), to whom the interpretation elements of the frame $\mathcal{O}(\Sigma^A)$ as disjunctions is due.

§12.4. **Internal Gelfand spectrum for arbitrary C*-algebras**

This section is based on Caspers (2008), Caspers, Heunen, Landsman, & Spitters (2009), and Heunen, Landsman, & Spitters (2009). Complete proofs of Lemma 12.15 and Lemma 12.16 may be found in Caspers (2008), §5.2. For different proofs of these lemmas see Heunen, Landsman, & Spitters (2009) and Coquand (2005), respectively. A proof of Lemma 12.21 may be found in Wolters (2013b), Theorem 2.17, also available as `http://arxiv.org/pdf/1010.2031v2.pdf`.

§12.5. **"Daseinisation" and Kochen–Specker Theorem**

The spectral order was introduced by Olson (1971) and was rediscovered by De Groote (2011). For a devastating critique of Heidegger's philosophy see Philipse (1999). The first construction of a "Daseinisation" map was given by Döring & Isham (2008b). The version presented here is an improvement, due to Wolters (2013ab), of a previous adaptation of the Döring–Isham appraoch to the topos $\mathsf{T}(A)$ in Heunen, Landsman, & Spitters (2009). Similarly, Theorem 12.29, first published in Heunen, Landsman, Spitters, & Wolters (2012), is an improvement due to Wolters (2013a) of an earlier result in this direction in Heunen, Landsman, & Spitters (2009).

The work of Isham & Butterfield (1998), which, as already mentioned, started the entire quantum toposophy program, was actually motivated by an topos-theoretica reformulation of the Kochen–Specker Theorem. Isham and Butterfield started from the following observation. Let $\mathscr{C}(B(H))$ be the poset of commutative *von Neumann* subalgebras of $B(H)$, partially ordered by set-theoretic inclusion, seen as a category in the usual way. Consider the presheaf topos $[\mathscr{C}(H)^{\mathrm{op}}, \mathsf{Set}]$ of *contravariant* functors $\underline{F} : \mathscr{C}(H) \to \mathsf{Set}$, where Set is the category of sets. The ***spectral presheaf*** is the contravariant functor $\underline{\Sigma}$ defined on objects by $\underline{\Sigma}_0(C) = \Sigma(C)$, and by the natural map on arrows, that is, $\underline{\Sigma}_1(C \subset D)$ maps $\omega \in \Sigma(D)$ (which is a map $D \to \mathbb{C}$) to its restriction to C, i.e., to $\omega_{|C} \in \Sigma(C)$. A ***point*** of some object $\underline{F}$ in $[\mathscr{C}(B(H))^{\mathrm{op}}, \mathsf{Set}]$ is defined as a natural transformation $\mathbf{1} \to \underline{F}$, where $\mathbf{1}$ is the terminal object, i.e., the presheaf that maps everything into the singleton set $*$.

The Kochen–Specker Theorem à la Butterfield & Isham, then, states that if $\dim(H) > 2$ as usual, *the spectral presheaf has no points.*