# Chapter 9
# Symmetry in algebraic quantum theory

In §3.9 we defined symmetries of classical physics as symmetries of either Poisson manifolds or Poisson algebras; these notions are equivalent. At the bare level of the underlying phase space $X$, merely seen as a locally compact space (rather than a Poisson manifold), the key result establishing this equivalence is this:

**Theorem 9.1.** *Let $X$ and $Y$ be locally compact Hausdorff spaces. Each isomorphism $\alpha : C_0(Y) \to C_0(X)$ is induced by a homeomorphism $\varphi : X \to Y$ via $\alpha = \varphi^*$ (and so each automorphism of $C_0(X)$ is induced by a homeomorphism of $X$).*

*More generally, if $A$ and $B$ are commutative C*-algebras, then each isomorphism $\alpha : A \to B$ is induced by a homeomorphism $\varphi : \Sigma(B) \to \Sigma(A)$ of the corresponding Gelfand spectra via $\alpha = G_B^{-1} \circ \varphi^* \circ G_A$, where $G_A : A \to C_0(\Sigma(A))$ is the Gelfand ismomorphism, cf.* (C.79)*, and similarly for $B$ (and so each automorphism of $A$ is induced by a homeomorphism of its Gelfand spectrum $\Sigma(A)$).*

This immediately follows from Theorems C.8 and C.45, and Corollary C.48.

In Chapter 5 we saw that even in elementary quantum mechanics, where $A = B(H)$ for some Hilbert space $H$, the concept of a symmetry is more diverse, as least apparently, since a non-commutative C*-algebra like $B(H)$ gives rise to numerous "quantum structures". The ones we looked at were listed after Proposition 5.3, viz.

1. The ***normal pure state space*** $\mathscr{P}_1(H)$, dressed with a transition probability (2.44).
2. The ***normal (total) state space*** $\mathscr{D}(H)$, seen as a convex set; see Theorem 2.8.
3. The ***self-adjoint operators*** $B(H)_{\mathrm{sa}}$ on $H$, seen as a Jordan algebra.
4. The ***effects*** $\mathscr{E}(H) = [0,1]_{B(H)}$ on $H$, seen as a convex poset.
5. The ***projections*** $\mathscr{P}(H)$ on $H$, seen as an orthocomplemented lattice.
6. The ***unital commutative C*-subalgebras*** $\mathscr{C}(B(H))$ ***of*** $B(H)$, seen as a poset.

Each structure comes with its own notion of a symmetry, see Definition 5.1. This raises two questions, which for $B(H)$ were completely answered in Chapter 5:

- The possible equivalence of the various notions of quantum symmetry;
- Unitary implementability of symmetries.

Indeed, it was found that if $\dim(H) > 2$, then all these notions of symmetry are equivalent, as well as unitarily implementable à la Wigner; see Theorem 5.4.



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## 9.1 Symmetries of C*-algebras and Hamhalter's Theorem

In this chapter we generalize this analysis from $A = B(H)$ to arbitrary C*-algebras $A$, which for simplicity we assume to have a unit $1_A$. See §C.25 for terminology.

**Definition 9.2.** *Let A be a unital C*-algebra.*

1. *The* **pure state space** $P(A) = \partial_e S(A)$ *is the extreme boundary of the state space* $S(A)$*, seen as a uniform space equipped with a transition probability*

$$\tau(\omega,\omega') = \inf\{\omega(a) \mid a \in A, 0 \leq a \leq 1_A, \omega'(a) = 1\}. \tag{9.1}$$

   *A* **Wigner symmetry** *of A is a uniformly continuous bijection* $\mathsf{W} : P(A) \to P(A)$ *with uniformly continuous inverse that preserves transition probabilities, i.e.,*

$$\tau(\mathsf{W}(\omega)\mathsf{W}(\omega')) = \tau(\omega,\omega'),\ \omega,\omega' \in P(A). \tag{9.2}$$

   If $A = B(H)$, Proposition C.177 guarantees that the above expression reproduces the standard quantum-mechanical transition probabilities (2.44), but compared to this special case, one novel aspect of $P(A)$ is that all pure states are now taken into account (as opposed to merely the normal ones, which notion is undefined for general C*-algebras anyway). Another is that in order to obtain the desired equivalence with other structures, the set $P(A)$ should carry a uniform structure, namely the $w^*$-uniformity inherited from $A^*$.
2. *The* **state space** $S(A)$ *is the set of all states on A, seen as a compact convex set in the* $w^*$*-topology inherited from the embedding* $S(A) \subset A^*$*. A* **Kadison symmetry** *of A is an affine homeomorphism* $\mathsf{K} : S(A) \to S(A)$*.*
   Compared to $A = B(H)$, firstly *all* states are now taken into account (instead of all *normal* states), and secondly we have added a continuity condition on $\mathsf{K}$.
3. *Any C*-algebra A defines an associated* **Jordan algebra** *(more precisely, a JB-algebra), namely* $A_{\mathrm{sa}}$ *equipped with the commutative product* $a \circ b = \frac{1}{2}(ab+ba)$*. A* **Jordan symmetry** $\mathsf{J}$ *of A is a Jordan isomorphism of* $(A_{\mathrm{sa}},\circ)$ *(or, equivalently, an invertible unital linear isometry of* $(A_{\mathrm{sa}},\|\cdot\|)$*, which in turn is the same as a unital linear order isomorphism of* $(A_{\mathrm{sa}},\leq)$*, cf. Lemma C.173). A* **weak Jordan symmetry** *of A is an invertible map* $\mathsf{J} : A_{\mathrm{sa}} \to A_{\mathrm{sa}}$ *whose restriction to each subspace* $C_{\mathrm{sa}}$ *of* $A_{\mathrm{sa}}$*, where* $C \in \mathscr{C}(A)$*, is linear and preserves the Jordan product.*
4. *The* **effects** *in A comprise the order unit interval* $\mathscr{E}(A) = [0,1_A]$*, i.e., the set of all* $a \in A_{\mathrm{sa}}$ *such that* $0 \leq a \leq 1_A$*, seen as a convex poset in the obvious way. A* **Ludwig symmetry** *of A is an affine order isomorphism* $\mathsf{L} : \mathscr{E}(A) \to \mathscr{E}(A)$*.*
5. *The projections* $\mathscr{P}(A)$ *in A form an orthomodular poset (cf. Definition D.1) with* $e \leq f$ *iff* $ef = e$ *and* $e^{\perp} = 1_A - e$*; if A is a von Neumann algebra (cf. Proposition C.136), or more generally an AW*-algebra or a Rickart C*-algebra (see §C.24),* $\mathscr{P}(A)$ *is even an orthomodular lattice. A* **von Neumann symmetry** *of A is an isomorphism* $\mathsf{N} : \mathscr{P}(A) \to \mathscr{P}(A)$ *of orthomodular posets.*
6. *The poset* $\mathscr{C}(A)$ *(lying at the heart of exact Bohrification) consists of all* commutative *C*-subalgebras of A that contain the unit* $1_A$*, partially ordered by inclusion. A* **Bohr symmetry** *of A, then, is an order isomorphism* $\mathsf{B} : \mathscr{C}(A) \to \mathscr{C}(A)$*.*

The structures 1, 2, 3 (with Jordan symmetries), and 4 are equivalent; see Theorem C.179 for $1 \leftrightarrow 2$ and Theorem C.172 for $2 \leftrightarrow 3$; the equivalence $3 \leftrightarrow 4$ is proved in exactly the same way as in Proposition 5.21, with Lemma 5.20 for the special case $A = B(H)$ replaced by Lemma C.173 (which has the same proof). From 1–4 we pick the Jordan algebra structure of $A$, since it gives the most straightforward results.

Henceforth, $A$ and $B$ are unital C*-algebras, and we define a ***weak Jordan isomorphism*** of $A$ and $B$ as an invertible map $\mathsf{J} : A_{\mathrm{sa}} \to B_{\mathrm{sa}}$ whose restriction to each subspace $C_{\mathrm{sa}}$ of $A_{\mathrm{sa}}$, where $C \in \mathscr{C}(A)$, is linear and preserves the Jordan product $\circ$ (so that a Jordan symmetry of $A$ alone is a weak Jordan automorphism of of $A$). Such a map complexifies to a map $\mathsf{J}_{\mathbb{C}} : A \to B$ in the usual way, i.e. writing $a \in A$ as $a = b + ic$, with $b^* = b$ and $c^* = c$, cf. (C.9), and put $\mathsf{J}_{\mathbb{C}}(a) = \mathsf{J}(b) + i\mathsf{J}(c)$). If no confusion arises, we just write $\mathsf{J}$ for $\mathsf{J}_{\mathbb{C}}$. We first turn to Bohr symmetries.

**Proposition 9.3.** *Given a weak Jordan isomorphism* $\mathsf{J} : A_{\mathrm{sa}} \to B_{\mathrm{sa}}$*, the ensuing map* $\mathsf{B} : \mathscr{C}(A) \to \mathscr{C}(B)$ *defined by* $\mathsf{B}(C) = \mathsf{J}_{\mathbb{C}}(C) \equiv \mathsf{J}(C)$ *is an order isomorphism.*

Note that as an argument of $\mathsf{B}$ the symbol $C$ is a point in the poset $\mathscr{C}(A)$, whereas as an argument of $\mathsf{J}_{\mathbb{C}}$ it is a subset of $A$, so that $\mathsf{J}_{\mathbb{C}}(C)$ stands for $\{\mathsf{J}_{\mathbb{C}}(c) \mid c \in C\}$.

*Proof.* The restriction $\mathsf{J}_{|C} : C \to B$ is a homomorphism of C*-algebras on each *commutative* C*-algebra $C \subset A$ (although $\mathsf{J} : A \to B$ may not be). Since $\mathsf{J}_{|C}$ is injective on $C_{\mathrm{sa}}$ (where it coincides with $\mathsf{J}$), it is also injective on $C$. Hence $\mathsf{J}_{|C}$ is isometric by Theorem C.62.3, so that its range is closed and therefore $\mathsf{J}(C)$ is a commutative C*-algebra in $B$, which is unital if $C$ is. Trivially, if $C \subseteq D$ in $A$ (so that $C \leq D$ in $\mathscr{C}(A)$), then $\mathsf{J}(C) \subseteq \mathsf{J}(D)$ in $B$ (so that $\mathsf{J}(C) \leq \mathsf{J}(D)$ in $\mathscr{C}(B)$). $\square$

The converse, however, is a deep result, which we call ***Hamhalter's Theorem***:

**Theorem 9.4.** *Let $A$ and $B$ be unital C*-algebras and let* $\mathsf{B} : \mathscr{C}(A) \to \mathscr{C}(B)$ *be an order isomorphism. Then there is a weak Jordan isomorphism* $\mathsf{J} : A_{\mathrm{sa}} \to B_{\mathrm{sa}}$ *such that* $\mathsf{B} = \mathsf{J}_{\mathbb{C}}$*. Moreover, if $A$ is isomorphic to neither* $\mathbb{C}^2$ *nor* $M_2(\mathbb{C})$*, then* $\mathsf{J}$ *is uniquely determined by* $\mathsf{B}$*, so in that case there is a bijective correspondence* $\mathsf{J} \leftrightarrow \mathsf{B}$ *between weak Jordan symmetries* $\mathsf{J}$ *of $A$ and Bohr symmetries* $\mathsf{B}$ *of $A$.*

Before proving this, let us explain why $\mathbb{C}^2$ and $M_2(\mathbb{C})$ are exceptional. In the first case, $\mathscr{C}(\mathbb{C}^2) \cong \{0, 1\}$ (with $0 \equiv \mathbb{C} \cdot 1_2$ and $1 \equiv \mathbb{C}^2$), which admits just one order isomorphism (viz. the identity map), which is induced by both the map $(a, b) \mapsto (b, a)$ and by the identity map on $\mathbb{C}^2$ (each of which is a weak Jordan automorphism).

In the second case, the poset $\mathscr{C}(M_2(\mathbb{C}))$ has a bottom element $0 \equiv \mathbb{C} \cdot 1_2$, as before, but no top element; each element $C \neq \mathbb{C} \cdot 1_2$ of $\mathscr{C}(M_2(\mathbb{C})$ is a unitary conjugate of the diagonal subalgebra $D_2(\mathbb{C})$, with $0 \leq C$ but no other orderings. Furthermore, $C \cap C' = \mathbb{C} \cdot 1_2$ whenever $C \neq C'$. Hence any order isomorphism of $\mathscr{C}(M_2(\mathbb{C}))$ maps $\mathbb{C} \cdot 1_2$ to itself and permutes the $C$'s. Thus each map $\mathsf{J} : M_2(\mathbb{C})_{\mathrm{sa}} \to M_2(\mathbb{C})_{\mathrm{sa}}$ whose complexification $\mathsf{J}_{\mathbb{C}} : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ shuffles the $C$'s isomorphically (as C*-algebras) gives a weak Jordan automorphism. For example, take $(a, b) \mapsto (b, a)$ on $D_2(\mathbb{C})$ and the identity on each $C \neq D_2(\mathbb{C})$); this induces the identity map on $\mathscr{C}(M_2(\mathbb{C})$. It follows that there are vastly more weak Jordan automorphisms of $M_2(\mathbb{C})$ than there are order isomorphisms of $\mathscr{C}(M_2(\mathbb{C}))$.

*Proof.* The key to the proof lies in the commutative case, which can be reduced to topology. If $A = C(X)$, any $C \in \mathscr{C}(A)$ induces an equivalence relation $\sim_C$ on $X$ by

$$x \sim_C y \text{ iff } f(x) = f(y) \, \forall f \in C. \tag{9.3}$$

This, in turn, defines a partition $X = \bigsqcup_\lambda K_\lambda$ of $X$ (henceforth called $\pi$), whose blocks $K_\lambda \subset X$ are the equivalence classes of $\sim_C$. To study a possible inverse of this procedure, for any closed subset $K \subset X$ we define the ideal

$$I_K = C(X;K) = \{f \in C(X) \mid f(x) = 0 \, \forall x \in K\}, \tag{9.4}$$

in $C(X)$, and its unitization $\dot{I}_K = I_K \oplus \mathbb{C} \cdot 1_X$, which evidently consists of all continuous functions on $X$ that are constant on $K$. If $X$ is finite (and discrete), each partition $\pi$ of $X$ defines some unital C*-algebra $C \subseteq C(X)$ through

$$C = \bigcap_{K_\lambda \in \pi} \dot{I}_{K_\lambda}, \tag{9.5}$$

which consists of all $f \in C(X)$ that are constant on each block $K_\lambda$ of the given partition $\pi$. In that case, the correspondence $C \leftrightarrow \pi$, where $\pi$ is defined by the equivalence relation $\sim_C$ in (9.3), gives a bijection between $\mathscr{C}(C(X))$ and the set $\mathfrak{P}(X)$ of all partitions of $X$. For example, the subalgebra $C = \dot{I}_K$ corresponds to the partition consisting of $K$ and all singletons not lying in $K$. Given the already defined partial order on $\mathscr{C}(C(X))$ (i.e., $C \leq D$ iff $C \subseteq D$), we may promote this bijection to an order isomorphism of posets if we define the partial order $\leq'$ on $\mathfrak{P}(X)$ to be the *opposite* of the natural one $\leq$ in which $\pi \leq \pi'$ (where $\pi$ and $\pi'$ consist of blocks $\{K_\lambda\}$ and $\{K'_{\lambda'}\}$, respectively) iff each $K_\lambda$ is contained in some $K'_{\lambda'}$ (i.e., $\pi$ is finer than $\pi'$). The partial ordering $\leq'$ makes $\mathfrak{P}(X)$ a complete lattice, whose top element consists of all singletons on $X$ and whose bottom element just consists of $X$ itself: the former corresponds to $C(X)$, which is the top element of $\mathscr{C}(C(X))$, whilst the latter corresponds to $\mathbb{C} \cdot 1_X$, which is the bottom element of $\mathscr{C}(C(X))$.

For general compact Hausdorff spaces $X$, since $C(X)$ is sensitive to the topology of $X$ the equivalence relation (9.3) does not induce arbitrary partitions of $X$. It turns out that each $C \in \mathscr{C}(C(X))$ induces an ***upper semicontinuous partition*** (abbreviated by *u.s.c. decomposition*) of $X$, i.e.,

- Each block $K_\lambda$ of the partition $\pi$ is closed;
- For each block $K_\lambda$ of $\pi$, if $K_\lambda \subseteq U$ for some open $U \in \mathscr{O}(X)$, then there is $V \in \mathscr{O}(X)$ such that $K_\lambda \subseteq V \subseteq U$ and $V$ is a union of blocks of $\pi$ (in other words, if $K$ is such a block, then $V \cap K = \emptyset$ implies $K = \emptyset$).

This can be seen as follows. Firstly, if we equip $\pi$ with the quotient topology with respect to the the natural map $q : X \to \pi$, $x \mapsto K_\lambda$ if $x \in K_\lambda$, then $\pi$ is compact, for $X$ is compact. Moreover, $\pi$ is Hausdorff. To see this, let $K_\lambda$ and $K_\mu$ be two distinct *points* in $\pi$. Recall that $x, y \in K_\lambda$ if and only if $f(x) = f(y)$ for each $f \in C$. Since $K_\lambda \neq K_\mu$, there is some $x \in K_\lambda$, some $y \in K_\mu$ and some $f \in C$ such that $f(x) \neq f(y)$, whence there are open disjoint $U, V \subseteq \mathbb{C}$ such that $f(x) \in U$ and $f(y) \in V$.

Define $\hat{f} : \pi \to \mathbb{C}$ by $\hat{f}(K_\lambda) = f(x)$ for some $x \in K_\lambda$. By definition of $K_\lambda$, this is independent of the choice of $x \in K_\lambda$, hence $\hat{f}$ is well defined. Again by definition, we have $f = \hat{f} \circ q$, hence $q^{-1}(\hat{f}^{-1})[U] = f^{-1}[U]$, which is open in $X$ since $f$ is continuous. Since $\pi$ is equipped with the quotient topology, it follows that $\hat{f}^{-1}[U]$ is open in $\pi$, and similarly $\hat{f}^{-1}[V]$ is open. Moreover, we have $\hat{f}(K_\lambda) = f(x)$ and $f(x) \in U$, hence $K_\lambda \in \hat{f}^{-1}[U]$, and similarly, $K_\mu \in \hat{f}^{-1}[V]$. We conclude that $\pi$ is also Hausdorff. Since $q$ is a continuous map between compact Hausdorff spaces, it follows that $q$ is closed. It is a standard result in topology that $q$ is closed iff $\pi$ is a u.s.c. decomposition, so we have now proved the latter.

Consequently, by the same maps (9.3) and (9.5), the poset $\mathscr{C}(C(X))$ is anti-isomorphic to the poset $\mathfrak{F}(X)$ of all u.s.c. decompositions of $X$ in the natural ordering $\leq$ (which proves that $\mathfrak{F}(X)$ is a complete lattice, since $\mathscr{C}(C(X))$ is). This is still a complicated poset; assuming $X$ to be larger than a singleton, the next step is to identify the simpler poset $\mathscr{F}_2(X)$ of all closed subsets of $X$ containing at least two elements within $\mathfrak{F}(X)$, where (as above) we identify a closed $K \subseteq X$ with the (u.s.c.) partition $\pi_K$ of $X$ whose blocks are $K$ and all singletons not lying in $K$ (note that the poset $\mathscr{F}(X)$ of all closed subsets of $X$ is less useful, since any singleton in $\mathscr{F}(X)$ gives rise to the bottom element of $\mathfrak{F}(X)$). To do so, we first recall that $\beta$ is said to *cover* $\alpha$ in some poset if $\alpha < \beta$, and $\alpha \leq \gamma < \beta$ implies $\alpha = \gamma$. If the poset has a bottom element, then its covers are precisely its *atoms*. Furthermore, note that since the bottom element 0 of $\mathfrak{F}(X)$ consists of singletons, the atoms in $\mathfrak{F}(X)$ are the partitions of the form $\pi_{\{x_1,x_2\}}$ (where $x_1 \neq x_2$). It follows that some partition $\pi \in \mathfrak{F}(X)$ lies in $\mathscr{F}_2(X) \subset \mathfrak{F}(X)$ iff exactly one of the following conditions holds:

- $\pi$ is an atom in $\mathfrak{F}(X)$, i.e., $\pi = \pi_{\{x_1,x_2\}}$ for some $x_1, x_2 \in X$, $x_1 \neq x_2$;
- $\pi$ covers three (distinct) atoms in $\mathfrak{F}(X)$, in which case $\pi = \pi_{\{x_1,x_2,x_3\}}$ where all $x_i$ are different, which covers the atoms $\pi_{\{x_1,x_2\}}$, $\pi_{\{x_1,x_3\}}$, and $\pi_{\{x_2,x_3\}}$;
- If $\alpha \neq \beta$ are atoms in $\mathfrak{F}(X)$ such that $\alpha \leq \pi$ and $\beta \leq \pi$, there is an atom $\gamma \leq \pi$ such that there are three (distinct) atoms covered by $\alpha \vee \gamma$ and three (distinct) atoms covered by $\beta \vee \gamma$. In that case, $\pi = \pi_K$ where $K$ has more than three elements: if $\alpha = \pi_{\{x_1,x_2\}}$ and $\beta = \pi_{\{x_3,x_4\}}$, then due to the assumption $\alpha \neq \beta$, the set $\{x_1,x_2,x_3,x_4\}$ (which lies in $K$) has at least three distinct elements, say $\{x_1,x_2,x_3\}$. Hence we may take $\gamma = \pi_{\{x_2,x_3\}}$, in which case $\alpha \vee \gamma = \pi_{\{x_1,x_2,x_3\}}$, which covers the atoms $\alpha$, $\gamma$, and $\pi_{\{x_1,x_3\}}$. Likewise, we have $\beta \vee \gamma = \pi_{\{x_2,x_3,x_4\}}$, which covers three atoms $\beta$, $\gamma$, and $\pi_{\{x_2,x_4\}}$.

In order to see that $\pi$ satisfying the third condition must be of the form $\pi_K$, assume the converse. So $\pi$ contains two blocks $K_\lambda$ and $K_\mu$ consisting of two or more elements. Say $\{x_1,x_2\} \subseteq K_\lambda$ and $\{x_3,x_4\} \subseteq K_\mu$. Then $\alpha = \pi_{\{x_1,x_2\}}$ and $\beta_{\{x_3,x_4\}}$ are atoms such that $\alpha, \beta < \pi$, and there is an atom $\gamma = \pi_{\{x_5,x_6\}} \leq \pi$ such that there are three atoms covered by $\alpha \vee \gamma$, and there are three atoms covered by $\beta \vee \gamma$. It follows from the second condition that $\alpha \vee \gamma = \pi_L$ with $L$ a three-point set. This implies that $\{x_1,x_2\} \cap \{x_5,x_6\}$ is not empty, from which it follows that $\alpha \vee \gamma = \pi_{\{x_1,x_2,x_5,x_6\}}$. Similarly, we find $\beta \vee \gamma = \pi_{\{x_3,x_4,x_5,x_6\}}$. Since $\{x_1,x_2,x_5,x_6\}$ and $\{x_3,x_4,x_5,x_6\}$ overlap, we obtain $\alpha \vee \beta \vee \gamma = \pi_{\{x_1,x_2,x_3,x_4,x_5,x_6\}}$. Moreover, $\alpha, \beta, \gamma \leq \pi$, so $\alpha \vee \beta \vee \gamma \leq \pi$. However, since $x_1, x_2 \in K_\lambda$, we must have $\{x_1,x_2,x_3,x_4,x_5,x_6\} \subseteq K_\lambda$ by definition of the order on $\mathfrak{F}(X)$. But since $x_3, x_4 \in K_\mu$, we must also have $\{x_1, x_2, x_3, x_4, x_5, x_6\} \subseteq K_\mu$, which is not possible, since $K_\lambda$ and $K_\mu$ are distinct blocks, hence disjoint. We conclude that $\pi$ can have only one block $K$ of two or more elements, hence $\pi = \pi_K$.

Thus $\mathscr{F}_2(X) \subset \mathfrak{F}(X)$ has been characterized order-theoretically. Moreover,

$$\pi = \vee_{x \in X} \pi_{K(x)}, \tag{9.6}$$

where $K(x)$ is the unique block of $X$ that contains $x$. Hence $\mathscr{F}_2(X)$ determines $\mathfrak{F}(X)$.

Let $X$ and $Y$ be compact Hausdorff spaces of cardinality at least two (so that the empty set and singletons are excluded). By the previous analysis, an order isomorphism $\mathsf{B} : \mathscr{C}(C(X)) \to \mathscr{C}(C(Y))$ is equivalent to an order isomorphism $\mathfrak{F}(X) \to \mathfrak{F}(Y)$, which in turn restricts to an order isomorphism $\mathscr{F}_2(X) \to \mathscr{F}_2(Y)$.

**Lemma 9.5.** *If $X$ and $Y$ are compact Hausdorff spaces of cardinality at least two, then any order isomorphism $\mathsf{F} : \mathscr{F}_2(X) \to \mathscr{F}_2(Y)$ is induced by a homeomorphism $\varphi : X \to Y$ via $\mathsf{F}(F) = \varphi(F)$, i.e., $\mathsf{F}(F) = \cup_{x \in F}\{\varphi(x)\}$. Moreover, if $X$ and $Y$ have cardinality at least three, then $\varphi$ is uniquely determined by $\mathsf{F}$.*

To see the idea, we first prove this for finite $X$, where $\mathscr{F}_2(X)$ simply consists of all subsets of $X$ having at least two elements, etc. It is easy to see that $X$ and $Y$ must have the same cardinality $|X| = |Y| = n$. If $n = 2$, then $\mathscr{F}_2(X) = X$ etc., so there is only one map $\mathsf{F}$, which is induced by each of the two possible maps $\varphi : X \to Y$, so that $\varphi$ exists but fails to be unique. If $n > 2$, then $\mathsf{F}$ must map each subset of $X$ with $n-1$ elements to some subset of $Y$ with $n-1$ elements, so that taking complements we obtain a unique bijection $\varphi : X \to Y$. To show that $\varphi$ induces $\mathsf{F}$, note that the meet $\wedge$ in $\mathscr{F}_2(X)$ is simply intersection $\cap$, and also that for any $F \in \mathscr{F}_2(X)$,

$$F = \cup_{x \in F}\{x\} = \cap_{x \notin F}\{x\}^c = (\cup_{x \notin F}\{x\})^c, \tag{9.7}$$

where $A^c = X \backslash A$. Since $\mathsf{F}$ is an order isomorphism, it preserves $\wedge = \cap$, so that

$$\mathsf{F}(F) = \cap_{x \notin F}\mathsf{F}(\{x\}^c) = \cap_{x \notin F} X \backslash \{\varphi(x)\} = (\cup_{x \notin F}\{\varphi(x)\})^c = \cup_{x \in F}\{\varphi(x)\}. \tag{9.8}$$

Now assume that $X$ is infinite. Let $x \in X$. If $x$ is not isolated, we define $\varphi(x)$ as follows. Let $\mathscr{O}(x)$ denote the set of all open neighborhoods of $x$. Since $x$ is not isolated, each $O \in \mathscr{O}(x)$ contains at least another element, so $\overline{O} \in \mathscr{F}_2(X)$. Moreover, finite intersections of elements of $\{\overline{O} : O \in \mathscr{O}(x)\}$ are still in $\mathscr{F}_2(X)$. Indeed, if $O_1, \ldots, O_n \in \mathscr{O}(x)$, then $O_1 \cap \ldots \cap O_n$ is an open set containing $x$, and since $\overline{O_1 \cap \ldots \cap O_n} \subseteq \overline{O_1} \cap \ldots \cap \overline{O_n}$, it follows that $\overline{O_1} \cap \ldots \cap \overline{O_n} \in \mathscr{F}_2(X)$. Since $\mathsf{F}$ is an order isomorphism, we find that finite intersections of $\{\mathsf{F}(\overline{O}) : O \in \mathscr{O}(x)\}$ are contained in $\mathscr{F}_2(Y)$. This implies that $\{\mathsf{F}(\overline{O}) : O \in \mathscr{O}(x)\}$ satisfies the finite intersection property. As $Y$ is compact, it follows that $I_x = \bigcap_{O \in \mathscr{O}(x)} \mathsf{F}(\overline{O})$ is nonempty. We can say more: it turns out that $I_x$ contains exactly one element. Indeed, assume that there are two different points $y_1, y_2 \in I_x$. Then $\{y_1, y_2\} \in \mathscr{F}_2(Y)$, so $\mathsf{F}^{-1}(\{y_1, y_2\}) \in \mathscr{F}_2(X)$. Since $\{y_1, y_2\} \in \mathsf{F}(\overline{O})$ for each $O \in \mathscr{O}(x)$, we also find that $\mathsf{F}^{-1}(\{y_1, y_2\}) \subseteq \overline{O}$ for each $O \in \mathscr{O}(x)$. This implies that

$$\mathsf{F}^{-1}(\{y_1,y_2\}) \subseteq \bigcap_{O\in\mathscr{O}(x)} \overline{O} = \{x\}, \tag{9.9}$$

where the last equality holds by normality of $X$. But this is a contradiction with $\mathsf{F} : \mathscr{F}_2(X) \to \mathscr{F}_2(Y)$ being a bijection. So $I_x$ contains exactly one point. We define $\varphi(x)$ such that $\{\varphi(x)\} = I_x$. Notice that $\varphi(x)$ cannot be isolated in $Y$, since if we assume otherwise, then $Y \setminus \{\varphi(x)\}$ must be a co-atom in $\mathscr{F}_2(Y)$, whence $\mathsf{F}^{-1}(Y \setminus \{\varphi(x)\})$ is a co-atom in $\mathscr{F}_2(X)$, which must be of the form $X \setminus \{z\}$ for some isolated $z \in X$. Since $x$ is not isolated, we cannot have $x = z$, so $X \setminus \{z\}$ is an open neighborhood of $x$, which is even clopen since $z$ is isolated. By definition of $\varphi(x)$, we must have $\varphi(x) \in \mathsf{F}(X \setminus \{z\})$, but $\mathsf{F}(X \setminus \{z\}) = Y \setminus \{\varphi(x)\}$. We found a contradiction, hence $\varphi(x)$ cannot be isolated. Now assume that $x$ is an isolated point. Then $X \setminus \{x\}$ is a co-atom in $\mathscr{F}_2(X)$, so $\mathsf{F}(X \setminus \{x\})$ is a co-atom in $\mathscr{F}_2(Y)$, too. Clearly this implies that $\mathsf{F}(X \setminus \{x\}) = Y \setminus \{y\}$ for some unique $y \in Y$, which must be isolated, since $Y \setminus \{y\}$ is closed. We define $\varphi(x) = y$.

In an analogous way, $\mathsf{F}^{-1}$ induces a map $\psi : Y \to X$. We shall show that $\varphi$ and $\psi$ are each other's inverses. Let $x \in X$ be isolated. We have seen that $\varphi(x)$ must be isolated as well, and that $\varphi(x)$ is defined by the equation $\mathsf{F}(X \setminus \{x\}) = Y \setminus \{\varphi(x)\}$. Since $\mathsf{F}$ is an order isomorphism, we have $X \setminus \{x\} = \mathsf{F}^{-1}(Y \setminus \{\varphi(x)\})$. Since $\varphi(x)$ is isolated, we find by definition of $\psi$ that $\psi(\varphi(x)) = x$. In a similar way we find that $\varphi(\psi(y)) = y$ for each isolated $y \in Y$. Now assume that $x$ is not isolated and let $F \in \mathscr{F}_2(X)$ such that $x \in F$. Then

$$\begin{aligned}\{\varphi(x)\} &= \bigcap_{O\in\mathscr{O}(x)} \mathsf{F}(\overline{O}) \subseteq \bigcap\{\mathsf{F}(\overline{O}) : O \text{ open}, F \subseteq O\} \\ &= \mathsf{F}\left(\bigcap\{\overline{O} : O \text{ open}, F \subseteq O\}\right) = \mathsf{F}(F),\end{aligned} \tag{9.10}$$

where the last equality follows by completely regularity of $X$. The penultimate equality follows from the following facts. Firstly, the set $\bigcap\{\overline{O} : O \text{ open}, F \subseteq O\}$ is closed since it is the intersection of closed sets. Moreover, the intersection contains more than one point, since $F$ contains two or more points and $F \subseteq \overline{O}$ for each $O$. Hence $\bigcap\{\overline{O} : O \text{ open}, F \subseteq O\} \in \mathscr{F}_2(X)$, and since $\mathsf{F}$ is an order isomorphism, it preserves infima, which justifies the penultimate equality. Hence $\varphi(x) \in \mathsf{F}(F)$ for each $F \in \mathscr{F}_2(X)$ containing $x$. Since $x$ is not isolated, $\varphi(x)$ is not isolated either. Hence in a similar way, we find that $\psi(\varphi(x)) \in \mathsf{F}^{-1}(G)$ for each $G \in \mathscr{F}_2(Y)$ containing $\varphi(x)$. Let $z = \psi(\varphi(x))$. Combining both statements, we find that $z \in F$ for each $F \in \mathscr{F}_2(X)$ such that $x \in F$. In other words, $z \in \bigcap\{F \in \mathscr{F}_2(X) : x \in F\}$. Since $x$ is not isolated, we each $O \in \mathscr{O}(x)$ contains at least two points. Hence

$$\bigcap\{F \in \mathscr{F}_2(X) : x \in F\} \subseteq \bigcap\{\overline{O} : O \in \mathscr{O}(x)\} = \{x\}, \tag{9.11}$$

where we used complete regularity of $X$ in the last equality. We conclude that $z = x$, so $\psi(\varphi(x)) = x$. In a similar way, we find that $\varphi(\psi(y)) = y$ for each non-isolated $y \in Y$. We conclude that $\varphi$ is a bijection with inverse $\varphi^{-1} = \psi$.

Continuing the proof of Lemma 9.5, we have to show that if $F \in \mathscr{F}_2(X)$, then $\varphi[F] = \mathsf{F}(F)$. Let $x \in F$. In the proof that $\varphi$ is a bijection we already noticed that $\varphi(x) \in \mathsf{F}(F)$ if $x$ is not isolated. If $x$ is isolated in $X$, then we first assume that $F$ has at least three points. Since $\{x\}$ is open, $G = F \setminus \{x\}$ is closed. Since $F$ contains at least three points, $G \in \mathscr{F}_2(X)$. So $G$ is covered by $F$ in $\mathscr{F}_2(X)$, so $\mathsf{F}(F)$ covers $\mathsf{F}(G)$. It follows that there must be an element $y_G \in Y \setminus \mathsf{F}(G)$ such that

$$\mathsf{F}(F) = \mathsf{F}(G \cup \{x\}) = \mathsf{F}(G) \cup \{y_G\}. \tag{9.12}$$

Both $G \cup \{x\}$ and $X \setminus \{x\}$ are elements of $\mathscr{F}_2(X)$, so

$$\begin{aligned}\mathsf{F}(G) &= \mathsf{F}(G \cup \{x\} \cap X \setminus \{x\}) = \mathsf{F}(G \cup \{x\}) \cap \mathsf{F}(X \setminus \{x\}) \\ &= (\mathsf{F}(G) \cup \{y_G\}) \cap (Y \setminus \{\varphi(x)\}),\end{aligned} \tag{9.13}$$

where $\mathsf{F}(X \setminus \{x\}) = Y \setminus \{\varphi(x)\}$ by definition of values of $\varphi$ at isolated points. Since $x \notin G$ and $\mathsf{F}$ preserves inclusions, this latter equation also implies $\mathsf{F}(G) \subseteq Y \setminus \{\varphi(x)\}$. Hence we find

$$\mathsf{F}(G) = (\mathsf{F}(G) \cup \{y_G\}) \cap (Y \setminus \{\varphi(x)\}) = \mathsf{F}(G) \cup (\{y_G\} \cap Y \setminus \{\varphi(x)\}). \tag{9.14}$$

Thus we obtain $\{y_G\} \cap Y \setminus \{\varphi(x)\} \subseteq \mathsf{F}(G)$, but since $y_G \notin \mathsf{F}(G)$, we must have $\varphi(x) = y_G$. As a consequence, we obtain $\mathsf{F}(F) = \mathsf{F}(G) \cup \{\varphi(x)\}$, so $\varphi(x) \in \mathsf{F}(F)$.

Summarizing, if $F$ has at least three points, then $\varphi(x) \in \mathsf{F}(F)$ for $x \in F$, regardless whether $x$ is isolated or not. So $\varphi[F] \subseteq \mathsf{F}(F)$ for each $F \in \mathscr{F}_2(X)$ such that $F$ has at least three points. Let $F \in \mathscr{F}_2(X)$ have exactly two points. Then there are $F_1, F_2 \in \mathscr{F}_2(X)$ with exactly three points such that $F = F_1 \cap F_2$. Then since $\varphi$ is a bijection and $\mathsf{F}$ as an order isomorphism both preserve intersections in $\mathscr{F}_2(X)$, we find

$$\varphi[F] = \varphi[F_1 \cap F_2] = \varphi[F_1] \cap \varphi[F_2] \subseteq \mathsf{F}(F_1) \cap \mathsf{F}(F_2) = \mathsf{F}(F_1 \cap F_2) = \mathsf{F}(F). \tag{9.15}$$

So $\varphi[F] \subseteq \mathsf{F}(F)$ for each $F \in \mathscr{F}_2(X)$. In a similar way, we find $\varphi^{-1}[G] \subseteq \mathsf{F}^{-1}[G]$ for each $G \in \mathscr{F}_2(Y)$. So if we substitute $G = \mathsf{F}(F)$, we obtain $\varphi^{-1}[\mathsf{F}(F)] \subseteq F$. Since $\varphi$ is a bijection, it follows that $\mathsf{F}(F) = \varphi[F]$ for each $F \in \mathscr{F}_2(X)$. As a consequence, $\varphi$ induces a one-one correspondence between closed subsets of $X$ and closed subsets of $Y$. Hence $\varphi$ is a homeomorphism. This proves Lemma 9.5. □

The special case of Theorem 9.4 where $A$ and $B$ are commutative now follows if we combine all steps so far:

1. The Gelfand isomorphism allows us to assume $A = C(X)$ and $B = C(Y)$, as above.
2. The order isomorphism $\mathsf{B} : \mathscr{C}(A) \to \mathscr{C}(B)$ determines an order isomorphism $F : \mathfrak{F}(X) \to \mathfrak{F}(Y)$ of the underlying lattices of u.s.c. decompositions, and *vice versa*.
3. Because of (9.6), the order isomorphism $F$ in turn determines and is determined by an order isomorphism $\mathsf{F} : \mathscr{F}_2(X) \to \mathscr{F}_2(Y)$.
4. Lemma 9.5 yields a homeomorphism $\varphi : X \to Y$ inducing $\mathsf{F} : \mathscr{F}_2(X) \to \mathscr{F}_2(Y)$.
5. The inverse pullback $(\varphi^{-1})^* : C(X) \to C(Y)$ is an isomorphism of C*-algebras, which (running backwards) reproduces the initial map $\mathsf{B} : \mathscr{C}(C(X)) \to \mathscr{C}(C(Y))$.

Therefore, in the commutative case we apparently obtain rather more than a weak Jordan isomorphism $\mathsf{J} : A_{\mathrm{sa}} \to B_{\mathrm{sa}}$; we even found an isomorphism $\mathsf{J} : A \to B$ of C*-algebras. However, if $A$ and $B$ are commutative, the condition of linearity on each commutative C*-subalgebra $C$ of $A$ includes $C = A$, so that (after complexification) weak Jordan isomorphisms are the same as isomorphisms of C*-algebras.

We now turn to the general case, in which $A$ and $B$ are both noncommutative (the case where one, say $A$, is commutative but the other is not, cannot occur, since $\mathscr{C}(A)$ would be a complete lattice but $\mathscr{C}(B)$ would not). Let $D$ and $E$ be maximal abelian C*-subalgebras of $A$, so that the corresponding elements of $\mathscr{C}(A)$ are maximal in the order-theoretic sense. Given an order isomorphism $\mathsf{B} : \mathscr{C}(A) \to \mathscr{C}(B)$, we restrict the map $\mathsf{B}$ to the down-set $\downarrow D = \mathscr{C}(D)$ in $\mathscr{C}(A)$ so as to obtain an order homomorphism $\mathsf{B}_{|D} : \mathscr{C}(D) \to \mathscr{C}(B)$. The image of $\mathscr{C}(D)$ under $\mathsf{B}$ must have a maximal element (since $\mathsf{B}$ is an order isomorphism), and so there is a maximal commutative C*-subalgebra $\tilde{D}$ of $B$ such that $\mathsf{B}_{|D} : \mathscr{C}(D) \to \mathscr{C}(\tilde{D})$ is an order isomorphism. Applying the previous result, we obtain an isomorphism $\mathsf{J}_D : D \to \tilde{D}$ of commutative C*-algebras that induces $\mathsf{B}_{|D}$. The same applies to $E$, so we also have an isomorphism $\mathsf{J}_E : E \to \tilde{E}$ of commutative C*-algebras that induces $\mathsf{B}_{|E}$. Let $C = D \cap E$, which lies in $\mathscr{C}(A)$. We now show that $\mathsf{J}_D$ and $\mathsf{J}_E$ coincide on $C$. There are three cases.

1. $\dim(C) = 1$. In that case $C = \mathbb{C} \cdot 1_A$ is the bottom element of $\mathscr{C}(A)$, so it must be sent to the bottom element $\tilde{C} = \mathbb{C} \cdot 1_B$ of $\mathscr{C}(B)$, whence the claim.
2. $\dim(C) = 2$. This the hard case dealt with below.
3. $\dim(C) > 2$. This case is settled by the uniqueness claim in Lemma 9.5.

So assume $\dim(C) = 2$. In that case, $C = C^*(e)$ for some proper projection $e \in \mathscr{P}(A)$, which is equivalent to $C$ being an atom in $\mathscr{C}(A)$. Recall that all our C*-algebras are unital, and that by assumption C*-subalgebras $C$ share the unit of the ambient C*-algebra $A$, hence $C^*(e)$ contains the unit of $A$. Hence $\tilde{C} \equiv \mathsf{B}(C) = \mathsf{B}_{|D}(C) = \mathsf{B}_{|E}(C)$ is an atom in $\mathscr{C}(B)$, which implies that $\tilde{C} = C^*(\tilde{e})$ for some projection $\tilde{e} \in \mathscr{P}(B)$. If $\mathsf{J}_D(e) = \mathsf{J}_E(e)$ we are ready, so we must exclude the case $\mathsf{J}_D(e) = \tilde{e}$, $\mathsf{J}_E(e) = 1_B - \tilde{e}$. This exclusion again requires a case distinction:

$$\dim(eAe) = \dim(e^{\perp}Ae^{\perp}) = 1; \tag{9.16}$$
$$\dim(eAe) = 1, \ \dim(e^{\perp}Ae^{\perp}) > 1; \tag{9.17}$$
$$\dim(eAe) > 1, \ \dim(e^{\perp}Ae^{\perp}) > 1, \tag{9.18}$$

where $e^{\perp} = 1_A - e$. Each of these cases is nontrivial, and we need another lemma.

**Lemma 9.6.** *Let $C \in \mathscr{C}(A)$ be maximal (i.e., $C \subset A$ is maximal abelian).*

*1. For each projection $e \in \mathscr{P}(C)$ we have $\dim(eCe) = 1$ iff $\dim(eAe) = 1$.*
*2. We have $\dim(C) = 2$ iff either $A \cong \mathbb{C}^2$ or $A \cong M_2(\mathbb{C})$.*

*Proof.* For the first claim $\dim(eAe) = 1$ clearly implies $\dim(eCe) = 1$. For the converse implication, assume *ad absurdum* that $\dim(eAe) > 1$, so that there is an $a \in A$ for which $eae \neq \lambda \cdot e$ for any $\lambda \in \mathbb{C}$. If also $\dim(eCe) = 1$, then any $c \in C$ takes the form $c = \mu \cdot e + e^{\perp}ce^{\perp}$ for some $\mu \in \mathbb{C}$. Indeed, since $c, e, e^{\perp}$ commute within $C$,

$$c = ce + ce^{\perp} = ce^2 + c(e^{\perp})^2 = ece + e^{\perp}ce^{\perp} = \mu e + e^{\perp}ce^{\perp}, \tag{9.19}$$

where the last equality follows since $ece \in eCe$, which is spanned by $e$. This implies that $eae \in C'$ (where $C'$ is the commutant of $C$ within $A$), and since $C$ is maximal abelian, we have $C = C'$, whence $eae \in C$. Now $eae = e(eae)e$, hence $eae \in eCe$, whence $eae = \lambda \cdot e$ for some $\lambda \in \mathbb{C}$. Contradiction. According to Theorem C.169.1, the assumption $\dim(C) = 2$ implies that $A$ is finite-dimensional, upon which Theorem C.163 and (C.641) yield the second claim. □

Having proved Lemma 9.6, we move on the analyze the cases (9.16) - (9.18).

- Eq. (9.16) implies that $C$ is maximal, as follows. Any element $a \in A$ is a sum of $eae$, $e^{\perp}ae^{\perp}$, $eae^{\perp}$, and $e^{\perp}ae$; nonzero elements of $C' = \{e\}'$ can only be of the first two types. If (9.16) holds, then $\dim(C') = 2$, but since $C$ is abelian we have $C \subseteq C'$ and since $\dim(C) = 2$ we obtain $C' = C$. Lemma 9.6.2 then implies that either $A \cong \mathbb{C}^2$ or $A \cong M_2(\mathbb{C})$. These C*-algebras have been analyzed after the statement of Theorem 9.4, and since those two $A$'s conversely imply (9.16), we may exclude them in dealing with (9.17) - (9.18). By Lemma 9.6.2 (applied to $D$ and $E$ instead of $C$), in what follows we may assume that $\dim(D) > 2$ and $\dim(E) > 2$ (as $D$ and $E$ are maximal).
- Eq. (9.17) implies $\dim(eD) = 1$. Assuming $\mathsf{J}_D(e) = \tilde{e}$, this implies $\dim(\tilde{e}\tilde{D}) = 1$ (since $\mathsf{J}_D$ is an isomorphism). Applying Lemma 9.6.1 to $B$ gives $\dim(\tilde{e}B\tilde{e}) = 1$ (since $\tilde{D}$ is maximal). If also $\dim((1_B - \tilde{e})B(1_B - \tilde{e})) = 1$, then $\dim(\tilde{D}) = 2$, whence $\dim(D) = 2$, which we excluded. Hence
$$\dim((1_B - \tilde{e})B(1_B - \tilde{e})) > 1. \tag{9.20}$$
Applied to $\mathsf{J}_E$ this gives $\mathsf{J}_E(e) = \tilde{e}$, and hence $\mathsf{J}_D$ and $\mathsf{J}_E$ coincide on $C = C^*(e)$.
- Eq. (9.18) implies that $\dim(eDe) > 1$ as well as $\dim(e^{\perp}Ee^{\perp}) > 1$ (apply Lemma 9.6.1 to $D$ and $E$). Since $\dim(eDe) > 1$, there is some $a \in D$ such that $e$ and $a' = eae \in D$ are linearly independent, and similarly there is some $b \in E$ such that $b' = e^{\perp}be^{\perp}$ is linearly independent of $e^{\perp}$. Then $a', b', e$ commute (in fact, $a'b' = b'a' = 0$), so that we may form the abelian C*-algebras $C_1 = C^*(e, a') \subseteq D$ and $C_2 = C^*(e, b') \subseteq E$, which (also containing the unit $1_A$) both have dimension at least three. We also form $C_3 = C^*(e, a', b')$, which contains $C_1$ and $C_2$ and hence is at least three-dimensional, too. Because $D$ and $E$ are maximal abelian, $C_3$ must lie in both $D$ and $E$. Applying the abelian case of the theorem already proved to $D$ and $E$, as before, but replacing $C$ used so far by $C_3$, we find that $\mathsf{J}_D$ and $\mathsf{J}_E$ coincide on $C_3$ (as its dimension is $> 2$). In particular, $\mathsf{J}_D(e) = \mathsf{J}_E(e)$.

To finish the proof, we first note that Theorem 9.4 holds for $A = B = \mathbb{C}$ by inspection, whereas the cases $A \cong B \cong \mathbb{C}^2$ or $\cong M_2(\mathbb{C})$ have already been discussed.

In all other cases we define $\mathsf{J} : A_{\mathrm{sa}} \to B_{\mathrm{sa}}$ by putting $\mathsf{J}(a) = \mathsf{J}_D(a)$ for any maximal abelian unital C*-subalgebra $D$ containing $C = C^*(a)$ and hence $a$; as we just saw, this is independent of the choice of $D$. Since each $\mathsf{J}_D$ is an isomorphism of commutative C*-algebras, $\mathsf{J}$ is a weak Jordan isomorphism. Finally, uniqueness of $\mathsf{J}$ (under the stated restriction on $A$) follows from Lemma 9.5. □

Theorem 9.4 begs the question if we can strengthen weak Jordan isomorphisms to Jordan isomorphism (i.e. invertible linear maps that preserve the Jordan product, cf. Appendix C.25). This hinges on the extendibility of weak Jordan isomorphisms to linear maps (which of course continue to preserve the Jordan product and hence are automatically Jordan isomorphisms). A general result in this direction is:

**Theorem 9.7.** *Let A and B be unital AW*-algebras, where A contains no summand of type* $\mathrm{I}_2$. *Then there is a bijective correspondence between order isomorphisms* $\mathsf{B} : \mathscr{C}(A) \to \mathscr{C}(B)$ *and Jordan isomorphisms* $\mathsf{J} : A_{\mathrm{sa}} \to B_{\mathrm{sa}}$.

This follows from Gleason's Theorem for AW*-algebras, which we will neither state nor prove. If $A = B = B(H)$, then the ordinary Gleason Theorem suffices to yield the crucial lemma for Wigner's Theorem for Bohr symmetries (i.e. Theorem 5.4.6):

**Lemma 9.8.** *Let H be a Hilbert space of dimension greater than two. Then any Bohr symmetry of* $\mathscr{C}(B(H))$ *is induced by a Jordan symmetry of* $B(H)_{\mathrm{sa}}$.

*Proof.* This follows from Theorem 9.4 and Corollary 5.22, which for the case at hand turns weak Jordan isomorphisms into Jordan isomorphisms. □

We finally turn to symmetries of projection lattices. Theorem C.174 shows that for von Neumann algebras (and more generally for AW*-algebras) $A$ (without summand of type $\mathrm{I}_2$) and $B$, any isomorphism $\mathsf{N} : \mathscr{P}(A) \to \mathscr{P}(B)$ of the corresponding orthocomplemented projection lattices (which automatically preserves arbitrary suprema) is the restriction of a unique Jordan isomorphism $\mathsf{J} : A_{\mathrm{sa}} \to B_{\mathrm{sa}}$.

This completes the argument to the effect that for many C*-algebras of observables $A$ (including $B(H)$ for $\dim(H) > 1$ as far as nos. 1–4 are concerned, and having $\dim(H) > 2$ if we also include nos. 5–6) our six seemingly different notions of symmetry of a quantum system described by a C*-algebra are equivalent. In particular, they are equivalent to Jordan isomorphisms, which are also the easiest ones to use, as they involve a readily identifiable part $A_{\mathrm{sa}}$ of $A$, and (by complexification, as explained above) may even be defined on $A$ itself (namely as those complex-linear isomorphisms that preserve the involution $*$ as well as the Jordan product $\circ$).

Putting $B = A$ and assuming (without loss of generality) that $A \subseteq B(H)$, Theorem C.175 then yields a separation of Jordan automorphisms into three disjoint classes:

**Corollary 9.9.** *If* $\mathsf{J}$ *is a Jordan symmetry of a unital C*-algebra* $A \subseteq B(H)$, *then there are three mutually orthogonal projections* $e_1$, $e_2$, $e_3$ *in* $A' \cap A''$ *such that:*

1. $e_1 + e_2 + e_3 = 1_H$;
2. *The map* $a \mapsto \mathsf{J}(a)e_1$ *from A to* $B(e_1H)$ *is a homomorphism (of C*-algebras);*
3. *The map* $a \mapsto \mathsf{J}(a)e_2$ *from A to* $B(e_2H)$ *is an anti-homomorphism (*ibid.*);*
4. *The map* $a \mapsto \mathsf{J}(a)e_3$ *from A to* $B(e_3H)$ *is both a homomorphism and an anti-homomorphism of C*-algebras (so that the "corner"* $\mathsf{J}(A)e_3$ *is commutative).*

*If in addition* $a \mapsto \mathsf{J}(a)e_1$ *is not an anti-homomorphism and* $a \mapsto \mathsf{J}(a)e_2$ *is not a homomorphism, then* $e_1$, $e_2$, *and* $e_3$ *are uniquely determined by these conditions.*

As we shall now see, if the symmetries form a (Lie) group, then this result often justifies restricting our attention simply to homomorphisms of C*-algebras.

## 9.2 Unitary implementability of symmetries

There are good reasons for the dichotomy (or even trichotomy) between homomorphisms and anti-homomorphisms of C*-algebras left by Corollary 9.9, since in physics certain discrete symmetries of quantum theory indeed give rise to anti-homomorphisms: the best-known examples are time inversion $T$ and charge conjugation $C$ combined with space inversion (i.e. parity) $P$, giving $CP$ (there are also other examples in condensed matter physics, like quantum spin flip). However, for the kind of problems mainly addressed in this book it is sufficient to restrict our attention to homomorphisms. One reason is that even if we use discrete symmetries (where the simplest non-trivial group $\mathbb{Z}_2$ often suffices to make our point), the models we treat simply realize these symmetries as homomorphisms. Another reason is that if symmetries join to form a *connected* topological group $G$ (typically a Lie group) and the maps $x \mapsto \mathsf{J}_x$ sending $x \in G$ to some Jordan symmetry $\mathsf{J}_x$ of the given C*-algebra $A$ of observables form a (strongly) continuous homomorphism (see below), then the identity $e \in G$ must be mapped to the identity $\mathrm{id}_A$, which of course is a homomorphism of $A$. Continuity then implies that all $\mathsf{J}_x$ must be homomorphisms.

In what follows we therefore assume that $G$ is a (topological) group and that we are given a (continuous) homomorphism $x \mapsto \alpha_x$from $G$ into the group $\mathrm{Aut}(A)$ of all automorphisms of $A$; note that, given our restriction to homomorphisms, we switch notation from $\mathsf{J}$ to the customary symbol $\alpha$. Continuity here always means ***strong continuity***, in that for each $a \in A$ the map $x \mapsto \alpha_x(a)$ from $G$ to $A$ is continuous (so that the map $G \times A \to A$ given by $(x,a) \mapsto \alpha_x(a)$ is continuous, as usually required for group actions in a topological setting, cf. Proposition 5.35).

It follows from Theorem 5.4 (technically, from part 4 of that theorem, but "morally" from all of it, including the equivalences between all kinds of symmetries) that if $A = B(H)$, then a homomorphism $\alpha : G \to \mathrm{Aut}(B(H))$ is always implemented by a family $u(x)$ of unitary operators on $H$, in that

$$\alpha_x(a) = u(x)au(x)^* \quad (x \in G). \tag{9.21}$$

The group representation property $\alpha_x\alpha_y = \alpha_{xy}$ does not enforce $u(x)u(y) = u_{xy}$: indeed, as we saw in detail in §5.10 one may have a projective unitary representation $g \mapsto u(x)$ of $G$ on $H$. However, by Theorem 5.62 one may usually pass to a central extension $\check{G}$ of $G$ for which this problem does not arise (e.g., $\check{SO}(3) = SU(2)$). In Corollary 9.12 below (unbroken symmetry), even such a passage is not necessary.

For general C*-algebras $A$—especially those modeling either classical systems (in which case $A$ is commutative) or infinite quantum systems (where $A$ is typically an infinite tensor product), one rarely has $\alpha(a) = uau^*$ for some $u \in A$ even for single automorphisms $\alpha$, let alone for a whole group of them. Instead, we settle for a weaker notion of unitary implementability, where the unitary $u$ need not be in $A$.

**Definition 9.10.** *Let $\pi : A \to B(H)$ be a representation of $A$. An automorphism $\alpha \in$* $\mathrm{Aut}(A)$ *is* implemented *in $H$ if there exists a unitary operator $u : H \to H$ such that*

$$\pi(\alpha(a)) = u\pi(a)u^* \quad (a \in A). \tag{9.22}$$

The fundamental criterion for implementability uses the pullback $\alpha^* : S(A) \to S(A)$ of $\alpha : A \to A$ to the state space $S(A)$, defined by $\alpha^*\omega = \omega \circ \alpha^{-1}$; cf. §C.25.

**Theorem 9.11.** *An automorphism $\alpha : A \to A$ can be implemented in the* GNS*-representation $\pi_\omega$ defined by a state $\omega$ on $A$ iff $\pi_{\alpha^*\omega}$ and $\pi_\omega$ are unitarily equivalent.*

*Proof.* Whether or not $\pi_{\alpha^*\omega}$ and $\pi_\omega$ are unitarily equivalent, we may define

$$w : H_\omega \to H_{\alpha^*\omega}; \tag{9.23}$$

$$w\pi_\omega(a)\Omega_\omega = \pi_{\alpha^*\omega}(\alpha(a))\Omega_{\alpha^*\omega}. \tag{9.24}$$

This operator is well defined and unitary, and satisfies $w\Omega_\omega = \Omega_{\alpha^*\omega}$ as well as $w\pi_\omega(a)w^* = \pi_{\alpha^*\omega}(\alpha(a))$; these properties even characterize $w$. If $\pi_{\alpha^*\omega} \cong \pi_\omega$, there exists a unitary $v : H_\omega \to H_{\alpha^*\omega}$ satisfying $v\pi_\omega(a)v^* = \pi_{\alpha^*\omega}(a)$, $a \in A$. Then $u = v^*w$ satisfies (9.22) for $\pi = \pi_\omega$. The converse is similar. □

An important special case arise if $\omega$ is invariant under $\alpha$.

**Corollary 9.12.** *If $\alpha^*\omega = \omega$ (that is, $\omega(\alpha(a)) = \omega(a)$ for all $a \in A$), then $\alpha$ is implemented by a unitary operator $u_\omega : H_\omega \to H_\omega$ satisfying $u_\omega\Omega_\omega = \Omega_\omega$. In particular, given a continuous homomorphism $\alpha : G \to \mathrm{Aut}(A)$ such that $\alpha_x^*\omega = \omega$ for each $x \in G$, one has a family of unitaries $u_\omega(x) : H_\omega \to H_\omega$ that for all $x \in G$ satisfy*

$$u_\omega(x)\Omega_\omega = \Omega_\omega; \tag{9.25}$$

$$\pi_\omega(\alpha_x(a)) = u_\omega(x)\pi_\omega(a)u_\omega(x)^*, \tag{9.26}$$

*and form a continuous unitary representation of $G$ on $H_\omega$.*

*Proof.* One easily checks that the following operators do the job:

$$u_\omega(x)\pi_\omega(a)\Omega_\omega = \pi_\omega(\alpha_x(a))\Omega_\omega. \qquad \square$$

Given some $\alpha \in \mathrm{Aut}(A)$, a weak form of ***spontaneous symmetry breaking*** (SSB) is that some state $\omega$—it is always a *state* that breaks a symmetry—satisfies $\alpha^*\omega \neq \omega$; a stronger one states that the two equivalent conditions in Theorem 9.11 are violated, i.e., that $\alpha$ cannot be implemented in the GNS-representation $\pi_\omega(A)$ (cf. Definition 9.10). In order to be physically relevant, the weaker notion has to be supplemented with additional structure, which also guarantees that generically the weak form implies the strong one. Part of this structure involves the identification of suitable classes of states within which we define SSB; these classes are predicated on a time-evolution on $A$. We also need a symmetry *group* instead of a single automorphism $\alpha$ (which implicitly uses the group $\mathbb{Z}_p = \mathbb{Z}/p\cdot\mathbb{Z}$, where $p$ is the smallest integer such that $\alpha^p = \mathrm{id}_A$; if no such $p$ exists the group is just $\mathbb{Z}$). Thus we need:

- A C*-algebra $A$ with ***time-evolution***, i.e., a homomorphism $\alpha : \mathbb{R} \to \mathrm{Aut}(A)$;
- A preferred class of states defines via $\alpha$, viz. ***ground states*** or ***equilibrium states***;
- A ***symmetry group*** $G$ acting on $A$ via a homomorphism$\gamma : G \to \mathrm{Aut}(A)$ satisfying

$$\alpha_t\gamma_g = \gamma_g\alpha_t \quad (t \in \mathbb{R}, g \in G). \tag{9.27}$$

## 9.3 Motion in space and in time

The C*-algebras $A$ we are going to use are the ***quasi-local*** ones introduced in §8.5 for quantum spin systems; especially recall (8.130). Also, the C*-algebra $A = B^{\infty}$ in §8.2 is a case in point, but this would require some changes in what follows. The last expression in (8.130) is convenient for introducing ***spatial translation symmetry***

$$\tau : \mathbb{Z}^d \to \mathrm{Aut}(A) \tag{9.28}$$

of $\mathbb{Z}^d$, as follows: for $x \in \mathbb{Z}^d$, define $\tau_x : A_\Lambda \to A_{x+\Lambda}$ initially by

$$\tau_x(b(y)) = b(x+y), \tag{9.29}$$

where, for given $b \in B(H)$ and $y \in \Lambda$, the operator $b(y) \in A_\Lambda$ is the element $\otimes_{z\in\Lambda} a_z$ with $a_y = b$ and $a_z = 1_H$ whenever $z \neq y$. Since arbitrary elements of $A_\Lambda$ are (norm-limits of) finite linear combinations of products of such operators $b(y)$, the automorphic (and hence isometric) property of $\tau_x$ defines its action on all of $A_\Lambda$ (if necessary by continuous extension). Note that for $a \in A_\Lambda$ the operator $\tau_x(a)$ thus defined is independent of the (typically non-unique) realization of $a$ in terms of the $b(y)$, because $\tau_x$ is an isometry. The group homomorphism property of the map (9.28) thus constructed is guaranteed by (9.29), whilst continuity is no issue since $\mathbb{Z}^d$ is discrete.

Since $A_\Lambda = \otimes_{y\in\Lambda} A_y$ with $A_y = B(H)$, an equivalent way to define $\tau_x$ is to use identifications $\mathrm{id}_{yz} : A_y \to A_z$ (since $A_y = A_z = B(H)$), which, taking tensor products, yield isomorphisms $\mathrm{id}_{\Lambda,\Lambda'} : A_\Lambda \to A_{\Lambda'}$ whenever some bijection $\Lambda \cong \Lambda'$ is given. In terms of those, we simply have $(\tau_x)_{|A_\Lambda} = \mathrm{id}_{\Lambda,x+\Lambda}$. Either way, the maps $(\tau_x)_{|A_\Lambda}$ extend to $\tau_x : A \to A$ by continuity. The following property then holds:

**Proposition 9.13.** *An automorphic action $\tau$ of $\mathbb{Z}^d$ on a quasi-local C*-algebra $A$ is* **asymptotically abelian** *in the sense that* $\lim_{x\to\infty}[a, \tau_x(b)] = 0$ *for all $a, b \in A$.*

Here $x \to \infty$ means that any sequence $(x_n)$ with $|x_n| \to \infty$ with respect to the Euclidean norm on $\mathbb{Z}^d$ has a subsequence $(x'_n)$ for which the stated result holds.

*Proof.* For $a$ and $b$ local, i.e., $a \in A_{\Lambda(1)}$ and $b \in A_{\Lambda(2)}$ this follows from Einstein locality. The general case follows by approximating $a$ and $b$ by local elements. □

Thus quasi-local C*-algebras $A$ satisfy the assumptions in the following theorem, which will be important in linking the various notions of SSB discussed earlier.

**Theorem 9.14.** *Let $A$ be a C*-algebra $A$ equipped with an asymptotically abelian action $\tau$ of $\mathbb{Z}^d$, and let $\omega$ be a translation-invariant primary state on $A$ (i.e., $\tau_x^*\omega = \omega$ for all $x \in \mathbb{Z}^d$). Then $\Omega_\omega$ is the only translation-invariant vector in $H_\omega$. Moreover,*

$$\lim_{x\to\infty} \omega(a\tau_x(b)) = \omega(a)\omega(b) \quad (a, b \in A); \tag{9.30}$$

$$\lim_{x\to\infty} \pi_\omega(\tau_x(b)) = \omega(b)\cdot 1_{H_\omega} \quad (b \in A); \tag{9.31}$$

$$\lim_{\Lambda\uparrow\mathbb{Z}^d} |\Lambda|^{-1} \sum_{x\in\Lambda} \pi_\omega(\tau_x(b)) = \omega(b)\cdot 1_{H_\omega} \quad (b \in A). \tag{9.32}$$

Here (9.31) and (9.32) hold in the weak operator topology on $B(H_\omega)$, and the limit $\Lambda \uparrow \mathbb{Z}^d$ in is taken along the hypercubes $\Lambda_N$ in (8.153) as $N \to \infty$.

*Proof.* If $\omega$ is primary, Theorem 8.23 (or its proof) yields

$$\lim_{x\to\infty} |\omega(a\tau_x(b)) - \omega(a)\omega(\tau_x(b))| = 0. \tag{9.33}$$

Translation-invariance of $\omega$ then yields (9.30), which also is a lemma for (9.31) - (9.32). Towards (9.31) we compute $\omega(a\tau_x(b))$ in terms of the projection

$$e_0 = \lim_{\Lambda\uparrow\mathbb{Z}^d} |\Lambda|^{-1} \sum_{x\in\Lambda} u(x) \tag{9.34}$$

onto the translation-invariant subspace of $H_\omega$, where $u$ is the unitary representation of $\mathbb{Z}^d$ on $H_\omega$ from Corollary 9.12 (with $G = \mathbb{Z}^d$), and the limit is taken in the strong operator topology. Eq. (9.34) is a special case of von Neumann's $L^2$ ergodic theorem (which generalizes the Peter–Weyl–Schur relation $e_0 = \int_G dx\, u(x)$ for compact groups $G$ to amenable groups like $\mathbb{Z}^d$ or $\mathbb{R}^d$). Since $e_0\Omega_\omega = \Omega_\omega$, we have

$$\begin{aligned}\omega(a\tau_x(b)) &= \langle \Omega_\omega, \pi_\omega(a)\pi_\omega(\tau_x(b))\Omega_\omega\rangle \qquad (9.35)\\ &= \langle \Omega_\omega, \pi_\omega(a)([\pi_\omega(\tau_x(b)), e_0] + e_0\pi_\omega(b))\Omega_\omega\rangle. \qquad (9.36)\end{aligned}$$

We now let $x \to \infty$. The commutator then vanishes, because the weak limit of $\pi_\omega(\tau_x(b))$ lies in the center of $\pi_\omega(A)''$, which is trivial since $\omega$ is primary. The remaining term matches with (9.30) iff $e_0$ is one-dimensional, so that $\Omega_\omega$ is the only translation-invariant vector in $H_\omega$, and $e_0 = |\Omega_\omega\rangle\langle\Omega_\omega|$. A similar trick then yields

$$\pi_\omega(\tau_x(b))\pi_\omega(a)\Omega_\omega = ([\pi_\omega(\tau_x(b)), \pi_\omega(a)] + \pi_\omega(a)([\pi_\omega(\tau_x(b)), e_0] + \omega(b)))\Omega_\omega.$$

Both commutators vanish (weakly) as $x \to \infty$, proving (9.31). Similarly, write

$$\pi_\omega(\tau_x(b))\pi_\omega(a)\Omega_\omega = ([\pi_\omega(\tau_x(b)), \pi_\omega(a)] + \pi_\omega(a)u(x)\pi_\omega(b))\Omega_\omega, \tag{9.37}$$

and use (9.34) and the previous formula for $e_0$ to prove (9.32). □

In the C*-algebraic formalism, ***dynamics*** is described by a continuous homomorphism $\alpha : \mathbb{R} \to \mathrm{Aut}(A)$, $t \mapsto \alpha_t$. For $A = B(H')$, where $H'$ is some Hilbert space (not to be confused with our earlier $H$ in the quasi-local setting), Theorem 5.4 yields

$$\alpha_t(a) = u_t a u_t^* \tag{9.38}$$

for some family of unitaries $u_t \equiv u(t)$, $t \in \mathbb{R}$. Eq. (5.268) and Proposition 5.53 then imply that the family $u_t$ may be redefined so as to make the map $t \mapsto u_t$ a continuous unitary representation of $\mathbb{R}$ on $H'$. Stone's Theorem 5.73 finally gives the familiar expression for time evolution in the so-called Heisenberg picture in terms of the ***Hamiltonian*** $h$, which is a (possibly unbounded) self-adjoint operator on $H'$, i.e.,

$$\alpha_t(a) = e^{ith} a e^{-ith}. \tag{9.39}$$

For arbitrary (unital) C*-algebras $A$ one has no counterpart of Theorem 5.4, and one cannot rely on Theorem 9.11 either because there are no preferred states to begin with; such states typically require a time-evolution for their definition (see below). For quantum spin systems (still with $H = \mathbb{C}^n$ and hence $B(H) \cong M_n(\mathbb{C})$), one tries to construct the map $t \mapsto \alpha_t$ from local approximations: with $A_\Lambda$ given by (8.129) with (8.128), we pick local Hamiltonians $h_\Lambda \in B(H_\Lambda)$ and define maps $t \mapsto \mathrm{Aut}(A_\Lambda)$ by

$$\alpha_t^\Lambda(a) = e^{ith_\Lambda} a e^{-ith_\Lambda}, \tag{9.40}$$

where $a \in A_\Lambda$. Letting $\Lambda \nearrow \mathbb{Z}^d$, we would then like to assemble the family $\alpha^\Lambda$ into a single automorphism group $\alpha : \mathbb{R} \to \mathrm{Aut}(A)$, which describes the dynamics of the corresponding infinite quantum system. Towards this aim, we start from a ***potential*** (also called an ***interaction***) $\Phi(X) \in B(H_X)$, which is defined for any finite sublattice $X$ of $\mathbb{Z}^d$, in terms of which the local Hamiltonians $h_\Lambda$ take the form

$$h_\Lambda = \sum_{X \subseteq \Lambda} \Phi(X), \tag{9.41}$$

where the sum is over all sublattices $X$ of $\Lambda$. For ***nearest-neighbour interactions***, $\Phi(X)$ is nonzero iff $X = \{x, y\}$ is a pair of neighbours, and in the presence of an external magnetic field one also has terms proportional to $\Phi(\{x\})$. For example, the ***quantum Ising model*** is defined by $H = \mathbb{C}^2$ and $\Phi(\{x,y\}) = -J\sigma_3(x)\sigma_3(y)$ for nearest neighbours and $\Phi(\{x\}) = -B\sigma_1(x)$ for all $x$, where $J > 0$ and $B \in \mathbb{R}$. The local Hamiltonians are therefore given by

$$h_\Lambda = -J \sum_{\langle xy \rangle \in \Lambda} \sigma_3(x)\sigma_3(y) - B \sum_{x \in \Lambda} \sigma_1(x), \tag{9.42}$$

where the sum over $\langle xy \rangle \in \Lambda$ denotes summing over nearest neighbours in $\Lambda$. The expression (9.42) implicitly has so-called ***free boundary conditions***, in that only neighbours inside $\Lambda$ take part in $h_\Lambda$. Alternatively, one could use ***periodic boundary conditions***, which in $d = 1$ define the ***quantum Ising chain***

$$h_N = -J \left( \sum_{x=1}^{N-1} (\sigma_3(x)\sigma_3(x+1) + \sigma_3(N)\sigma_3(1) \right) - B \sum_{x=1}^{N} \sigma_1(x). \tag{9.43}$$

In (9.42) - (9.43) the operators $\sigma_i(x)$ in $A_\Lambda$ is defined as explained after (9.29). We are going to study the quantum Ising chain in detail in connection with SSB; for the moment, we just mention another popular spin model, namely the ***Heisenberg model*** for magnetism. This also has $H = \mathbb{C}^2$, but the local Hamiltonians are

$$h_\Lambda = J \sum_{\langle xy \in \Lambda \rangle} \sum_{i=1}^{3} \sigma_i(y)\sigma_i(y), \tag{9.44}$$

with free boundary conditions, where $J < 0$ ( $J > 0$) yields (anti) ferromagnetism.

Although we do not have (9.38) for any $u_t \in A$, we may construct $\alpha_t$ as follows.

**Theorem 9.15.** *Let $\Phi$ be a* **short-range potential** *in that there is $r \in \mathbb{N}$ such that $\Phi(X) \neq 0$ only if $|x-y| \leq r$ for all $x, y \in X$, and define local Hamiltonians $h_\Lambda$ by* (9.41). *For fixed finite $\Lambda \subset \mathbb{Z}^d$ and $a \in A_\Lambda$, the following (norm) limit exists and defines an automorphism $\alpha_t$ of $\cup_{\Lambda \subset \mathbb{Z}^d} A_\Lambda$ and hence by continuity also of $A$:*

$$\alpha_t(a) = \lim_{N \to \infty} e^{ith_{\Lambda_N}} a e^{-ith_{\Lambda_N}}, \tag{9.45}$$

*Proof.* Note that for large enough $N$, the hypercube $\Lambda_N$ contains any $\Lambda \in \mathscr{P}_f(\mathbb{Z}^d)$. Take $a \in A_\Lambda$, take $\Lambda_{N_2} \supset \Lambda_{N_1} \supset \Lambda$, and use (9.40) and (9.41) to compute

$$\begin{aligned}
&\|\alpha_t^{(\Lambda_{N_2})}(a) - \alpha_t^{(\Lambda_{N_1})}(a)\| = \|\int_0^t ds\, \frac{d}{ds}(\alpha_s^{(\Lambda_{N_2})} \circ \alpha_{t-s}^{(\Lambda_{N_1})}(a))\| \\
&= \left\| \int_0^t ds \left( [h_{\Lambda_{N_2}}, \alpha_s^{(\Lambda_{N_1})} \circ \alpha_{t-s}^{(\Lambda_{N_1})}(a)] - \alpha_s^{(\Lambda_{N_2})}([h_{\Lambda_{N_1}}, \alpha_{t-s}^{(\Lambda_{N_1})}(a)]) \right) \right\| \\
&= \left\| \int_0^t ds\, \alpha_s^{(\Lambda_{N_2})}([h_{\Lambda_{N_2}} - h_{\Lambda_{N_1}}, \alpha_{t-s}^{(\Lambda_{N_1})}(a)]) \right\| \\
&\leq \int_0^t ds\, \|\alpha_s^{(\Lambda_{N_2})}([h_{\Lambda_{N_2}} - h_{\Lambda_{N_1}}, \alpha_{t-s}^{(\Lambda_{N_1})}(a)])\| \\
&\leq \int_0^t ds\, \|[h_{\Lambda_{N_2}} - h_{\Lambda_{N_1}}, \alpha_{t-s}^{(\Lambda_{N_1})}(a)]\| \\
&= \int_0^t ds \left\| \sum_{x \in \Lambda_{N_2} \backslash \Lambda_{N_1}} \sum_{X \ni x} [\Phi(X), \alpha_{t-s}^{(\Lambda_{N_1})}(a)] \right\| \\
&\leq \sum_{x \in \Lambda_{N_2} \backslash \Lambda_{N_1}} \sum_{X \ni x} \int_0^t ds\, \|[\Phi(X), \alpha_{t-s}^{(\Lambda_{N_1})}(a)]\|.
\end{aligned} \tag{9.46}$$

We now show that the left-hand side of the first line is a Cauchy sequence. Since

$$\alpha_{t-s}^{(\Lambda_{N_1})}(a) = e^{i(t-s)\sum_{Y \subseteq \Lambda_{N_1}} \Phi(Y)} a e^{-i(t-s)\sum_{Y \subseteq \Lambda_{N_1}} \Phi(Y)} \in B(H_{\Lambda_{N_1}}), \tag{9.47}$$

which is finite-dimensional (as $\Lambda_{N_1}$ is finite), we have a norm-convergent expansion

$$\alpha_t^{(\Lambda_{N_1})}(a) = a + it \sum_{Y_1 \subseteq \Lambda_{N_1}} [\Phi(Y_1), a] + \frac{(it)^2}{2!} \sum_{Y_1, Y_2 \subseteq \Lambda_{N_1}} [\Phi(Y_2), [\Phi(Y_1), a]] + \cdots \tag{9.48}$$

Let $\Lambda(r)$ consist of all $y \in \mathbb{Z}^d$ for which there is some $x \in \Lambda$ for which $|x-y| \leq r$. Then the zeroth term $a$ in (9.48) is in $A_\Lambda$, the first is in $A_{\Lambda(r)}$, ..., the $n$'th is in $A_{\Lambda(nr)}$. Therefore, we can find $n = n(N_1, N_2, 3)$ such that the only terms in (9.48) that contribute to the commutator in (9.46) are the $n$'th and beyond. Taking $\Lambda_{N_1}$ and $\Lambda_{N_2}$ large enough, this tail can be made arbitrarily small, so that $(\alpha_t^{(\Lambda_N)}(a))_N$ is a Cauchy sequence in $A$. This gives convergence of (9.45) for $a \in A_\Lambda$, where $\Lambda$ is arbitrary (but finite), yielding an automorphism $\alpha_t$ in $\cup_\Lambda A_\Lambda$. Being an automorphism, $\alpha_t$ is isometric, so that it extends to $A$ by continuity. $\square$

## 9.4 Ground states of quantum systems

A ground state of a finite system $A_\Lambda = B(H_\Lambda)$ is an eigenstate of the local Hamiltonian $h_\Lambda$ with the lowest eigenvalue; because $\dim(H_\Lambda) < \infty$, the spectrum of $h_\Lambda$ is discrete and hence local ground states exist. For infinite systems, no Hamiltonian is yet defined, so we need to define ground states in terms of the dynamics $\alpha_t$.

**Definition 9.16.** *Let A be a C*-algebra with time evolution, i.e., a continuous homomorphism* $\alpha : \mathbb{R} \to \mathrm{Aut}(A)$ *(which gives the dynamics of the underlying physical system). A* **ground state** *of* $(A, \alpha)$ *is a state* $\omega$ *on A such that:*

1. $\omega$ *is time-independent, i.e.* $\alpha_t^* \omega = \omega$ *(or* $\omega(\alpha_t(a)) = \omega(a)$ *for all* $a \in A$*)* $\forall t \in \mathbb{R}$*;*
2. *The generator* $h_\omega$ *of the ensuing continuous unitary representation*

$$t \mapsto u_t = e^{ith_\omega} \tag{9.49}$$

*of* $\mathbb{R}$ *on* $H_\omega$ *has positive spectrum, i.e.,* $\sigma(h_\omega) \subseteq \mathbb{R}^+$*, or, equivalently,*

$$\langle \psi, h_\omega \psi \rangle \geq 0 \;\; (\psi \in D(h_\omega)). \tag{9.50}$$

Note that the existence of the operator $h_\omega$ is guaranteed by Corollary 9.12 and the arguments after (9.38). Since Corollary 9.12 yields

$$h_\omega \Omega_\omega = 0; \tag{9.51}$$

$$\pi_\omega(\alpha_t(a)) = e^{ith_\omega} \pi_\omega(a) e^{-ith_\omega}, \tag{9.52}$$

it follows that $h_\omega$ is a Hamiltonian in the usual sense, implementing the Heisenberg-picture time evolution (albeit in the representation $\pi_\omega(A)$ rather than in $A$ itself). Moreover, in view of (9.51) and the assumed positivity of $\sigma(h_\omega)$, the unit vector $\Omega_\omega$ of the GNS-representation $\pi_\omega$ induced by a ground state $\omega$ is a ground state for the Hamiltonian $h_\omega$ in the usual sense. If $\omega$ is pure (see below for a discussion of this desirable possibility), then obviously $\exp(ith_\omega) \in \pi_\omega(A)''$, since the latter equals $B(H_\omega)$. A deep result states that this is always the case (***Borchers Theorem***):

**Theorem 9.17.** *If* $\omega$ *is a ground state on A, then* $\exp(ith_\omega) \in \pi_\omega(A)''$ *for all* $t \in \mathbb{R}$.

As we shall see, this contrasts with equilibrium states. The Heisenberg equation of motion for operators $a(t)$ has a counterpart in the C*-algebraic formalism, which requires a concept already encountered in §3.1, but repeated here for convenience:

**Definition 9.18.** *A* **derivation** *on a C*-algebra A is a linear map* $\delta : A \to A$ *with*

$$\delta(ab) = \delta(a)b + a\delta(b), \;\; (a, b \in A) \;\; \textbf{(Leibniz rule)}. \tag{9.53}$$

*An* **unbounded derivation** *is a linear map* $\delta : \mathrm{Dom}(\delta) \to A$*, where the domain* $\mathrm{Dom}(\delta) \subset A$ *of* $\delta$ *is a dense linear subspace of A, that satisfies the Leibniz rule.*

*An (unbounded) derivation* $\delta$ *is* **symmetric** *when* $\delta(a^*) = \delta(a)^*$ *for all a (in* $\mathrm{Dom}(\delta)$*, which must be self-adjoint in that* $a \in \mathrm{Dom}(\delta)$ *iff* $a^* \in \mathrm{Dom}(\delta)$*).*

Bounded derivations are rare in classical physics; nonzero derivations of $A = C_0(\mathbb{R}^d)$ do not even exist, but it has plenty of *unbounded* derivations, viz. $\delta(f) = \xi f$ for some vector field $\xi$ on $\mathbb{R}^d$. In quantum mechanics, $A = B(H')$ does have derivations, all given by $\delta(a) = i[h,a]$ for some bounded (self-adjoint) operator $h$ on $H'$.

**Proposition 9.19.** *Any continuous homomorphism $\alpha : \mathbb{R} \to \mathrm{Aut}(A)$ on any $C^*$-algebra A defines an unbounded symmetric derivation $\delta$ on A by the norm limit*

$$\delta(a) = \frac{d}{dt}\alpha_t(a)_{|t=0} \equiv \lim_{t\to 0}\frac{\alpha_t(a)-a}{t}, \tag{9.54}$$

*where* $\mathrm{Dom}(\delta)$ *consists of all $a \in A$ for which this limit exists. Moreover, this domain is stable under $\alpha_t$ in that if $a \in \mathrm{Dom}(\delta)$, then $\alpha_t(a) \in \mathrm{Dom}(\delta)$ ($t \in \mathbb{R}$).*

The proof is an elementary verification (cf. Theorem 5.73). On $H_\omega$ we then have

$$\pi_\omega(\delta(a)) = i[h_\omega, \pi_\omega(a)], \tag{9.55}$$

which, then, is "Heisenberg's equation of motion revisited." One may also reformulate Definition 9.16 in terms of the derivation $\delta$ associated to $\alpha$ by (9.54):

**Proposition 9.20.** *A state $\omega \in S(A)$ is a ground state for given dynamics $\alpha$ iff*

$$-i\omega(a^*\delta(a)) \geq 0 \ \ (a \in \mathrm{Dom}(\delta)). \tag{9.56}$$

*Proof.* If $\omega$ is a ground state according to Definition 9.16, we may use (9.55), (C.196), (9.51), and finally (9.50) to compute

$$\begin{aligned} -i\omega(a^*\delta(a)) &= -i\langle\Omega_\omega, \pi_\omega(a^*\delta(a))\Omega_\omega\rangle = \langle\Omega_\omega, \pi_\omega(a)^*[h_\omega, \pi_\omega(a)]\Omega_\omega\rangle \\ &= \langle\pi_\omega(a)\Omega_\omega, h_\omega\pi_\omega(a)\Omega_\omega\rangle \geq 0. \end{aligned} \tag{9.57}$$

Conversely, we first show that if $\omega$ satisfies (9.56), then it is $\alpha_t$-invariant. We initially assume $a = a^*$, so that $\delta(a)^* = \delta(a^*) = \delta(a)$, as $\delta$ is symmetric by construction. Since $\omega$ is a state, one has $\omega(b^*) = \overline{\omega(b)}$ for any $b \in A$, so taking $b = \delta(a)a$, using (9.56) just in that $\omega(a^*\delta(a)) \in i\mathbb{R}$, we obtain $\omega(\delta(a)a) = -\omega(a\delta(a))$. Hence

$$\omega(\delta(a^2)) = 0, \tag{9.58}$$

by (9.53), so also $\omega(\delta(\alpha_s(a)^2)) = 0$, $s \in \mathbb{R}$. With (9.54), we find

$$\begin{aligned} 0 &= \int_0^u ds\,\omega(\delta(\alpha_s(a)^2)) = \int_0^u ds\,\omega\left(\frac{d}{dt}\alpha_t(\alpha_s(a)^2)_{|t=0}\right) \\ &= \int_0^u ds\,\frac{d}{dt}\omega(\alpha_{t+s}(a)^2))_{|t=0} = \int_0^u ds\,\frac{d}{ds}\omega(\alpha_s(a)^2)) = \omega(\alpha_u(a^2)) - \omega(a^2). \end{aligned}$$

Hence $\omega(\alpha_u(a^2)) = \omega(a^2)$ for each $u > 0$ (and analogously for each $u < 0$), whenever $a^* = a$, i.e., $\omega(\alpha_u(b) = \omega(b)$ for each $b \geq 0$. But any $b \in A$ may be written as a sum of at most four positive elements, so $\omega \circ \alpha_u = \omega$ for all $u \in \mathbb{R}$. We therefore have a Hamiltonian $h_\omega$, whose positivity follows from (9.57), ran backwards. □

## 9.5 Ground states and equilibrium states of classical spin systems

Thermal equilibrium states are arguably physically more relevant than ground states, as the latter rely on the idealization of temperature zero. Since in statistical mechanics infinite systems are used to approximate very large ones, it will be of particular interest to define equilibrium states in infinite volume. If only to highlight contrasts with quantum theory, we take a long run and start with the classical case.

Classical spin systems on a lattice are defined by a single-site configuration space $\underline{n} \cong \{0,1,\ldots,n\}$, where $m \in \underline{n}$ may either be interpreted as some spin-like degree of freedom (as in the Ising model, where $n=2$) or as the number of (structureless) particles occupying a given site (in which case one has a ***lattice gas***). As in (C.310), for any finite sublattice $\Lambda \subset \mathbb{Z}^d$, the local algebra of observables is given by

$$A_{\Lambda}^{(c)} = C(\underline{n}^{\Lambda}), \tag{9.59}$$

where $\underline{n}^{\Lambda} = C(\Lambda, \underline{n})$ consists of all functions $s : \Lambda \to \underline{n}$. For finite $\Lambda$ this is a finite set (of cardinality $n^{|\Lambda|}$), so that all functions in question are continuous and hence $C(\underline{n}^{\Lambda})$ just stands for the commutative C*-algebra of *all* functions from $\underline{n}^{\Lambda}$ to $\mathbb{C}$. If $\Lambda_1 \subseteq \Lambda^{(2)}$, we have maps $\iota_{\Lambda_1\Lambda^{(2)}}^{(c)} : A_{\Lambda_1}^{(c)} \hookrightarrow A_{\Lambda^{(2)}}^{(c)}$, written $f_1 \mapsto f_2$, which are given by

$$f_2(s) = f_1(s_{|\Lambda_1}), \tag{9.60}$$

where $s : \Lambda^{(2)} \to \underline{n}$. As these maps are injective, the ensuing inductive limit is simply

$$A^{(c)} = \cup_{\Lambda \subset \mathbb{Z}^d} A_{\Lambda}^{(c)} \cong C\left(\underline{n}^{\mathbb{Z}^d}\right), \tag{9.61}$$

where $\underline{n}^{\mathbb{Z}^d} = \prod_{x \in \mathbb{Z}^d} \underline{n}$ is endowed with the product topology and hence (by Tychonoff's theorem) is compact (for $n=2, d=1$ this is a model of the Cantor set).

As in the quantum case, local Hamiltonians are defined via an ***interaction*** $\boldsymbol{\Phi}$, which now is an assignment $X \mapsto \Phi(X)$, where $X \subset \mathbb{Z}^d$ is finite and $\Phi(X) \in A_X^{(c)}$. If $X \subset Y$, we regard $\Phi(X)$ an an element in $A_Y^{(c)}$ through the inclusion $A_X^{(c)} \subset A_Y^{(c)}$, indicating this explicitly by writing $\Phi(X)_Y \in A_Y^{(c)}$. We then define $h_{\Lambda} \in A_{\Lambda}^{(c)}$ by

$$h_{\Lambda} = \sum_{X \subset \Lambda} \Phi(X)_{\Lambda}, \tag{9.62}$$

where the the sum is over all subsets $X$ of $\Lambda$. For example, the Ising Hamiltonian

$$h_{\Lambda}(s) = -J \sum_{\langle ij \rangle_{\Lambda}} s_i s_j - B \sum_{i \in \Lambda} s_i, \tag{9.63}$$

where the sum is over nearest neighbours in $\Lambda$, and we assume $\underline{2} = \{-1,1\}$ (rather than the usual c-bit $\{0,1\}$), comes from the following potential:

- $\Phi(X) = 0$ if either $|X| > 2$ or, if $|X| = 2$, its elements are not nearest neighbours;
- $\Phi(\{i\}) : s \mapsto -Bs_i$, and $\Phi(\{i,j\}) : s \mapsto -Js_i s_j$ if $i$ and $j$ are nearest neighbours.

As in (9.41), the prescription (9.62) has free boundary conditions, in that it only involves spins inside $\Lambda$. Another possibility is to fix a "boundary" spin configuration $b \in \underline{n}^{\mathbb{Z}^d}$, and define $h^b_\Lambda \in A^{(c)}_\Lambda$ by

$$h^b_\Lambda = \sum_{X\subset\mathbb{Z}^d, |X|<\infty, X\cap\Lambda\neq\emptyset} \Phi(X)^b_\Lambda. \tag{9.64}$$

This involves some new notation $\Phi(X)^b_\Lambda$, which means the following. In principle, $\Phi(X) \in A^{(c)}_X$ is a function on $\underline{n}^X$. We now turn $\Phi(X)$ into a function $\Phi(X)^b_\Lambda$ on $\underline{n}^\Lambda$ (so that $h^b_\Lambda$ is a function on $\underline{n}^\Lambda$ as required): for given $s : \Lambda \to \underline{n}$ and given $b : \mathbb{Z}^d \to \underline{n}$ we define $s' : X \to \underline{n}$ by putting $s' = s$ on $X \cap \Lambda$ and $s' = b$ on the remainder of $X$ (which is $X \cap \Lambda^c$, with $\Lambda^c = \mathbb{Z}^d \backslash \Lambda$). Then

$$\Phi(X)^b_\Lambda(s) = \Phi(X)(s'). \tag{9.65}$$

Physically, this simply means that those spins outside $\Lambda$ that interact with spins inside $\Lambda$ are set at a fixed value determined by the boundary condition $b$. For example, consider the Ising model in $d = 1$. If we take $\Lambda = \{2,3\}$, then from (9.62) we obtain $h_\Lambda = -Js_2s_3 - B(s_2 + s_3)$; spins outside $\Lambda$ do not contribute. From (9.64), on the other hand, we obtain $h^b_\Lambda = h_\Lambda - J(b_1s_2 + s_3b_4)$. Although the boundary condition $b$ is arbitrary, one may think of simple choices like $b_i = 1$ or $-1$ for each $i$.

We may actually rewrite (9.64) as a difference between Hamiltonians with free boundary conditions. To do so, for given finite $\Lambda$ we pick some finite $\Lambda' \supset \Lambda$ large enough that it contains all spins outside $\Lambda$ that interact with spins inside $\Lambda$ (provided this is possible). With the conventional notation $h_\Lambda(s|b) \equiv h^b_\Lambda(s)$, this yields

$$h_\Lambda(s|b) = h_{\Lambda'}(s,b) - h_{\Lambda'\backslash\Lambda}(b) = \sum_{X'\subset\Lambda'} \Phi(X')_{\Lambda'}(s,b) - \sum_{Y\subset\Lambda'\backslash\Lambda} \Phi(Y)_{\Lambda'\backslash\Lambda}(b).$$

Analogous to (9.65), the notation $\Phi(X')_{\Lambda'}(s,b)$ here means $\Phi(X')_{\Lambda'}(s')$, for the function $s' : \Lambda' \to \underline{n}$ that on $\Lambda \subset \Lambda'$ coincides with $s : \Lambda \to \underline{n}$, whilst on $(\Lambda'\backslash\Lambda) \subset \Lambda'$ it coincides with the restriction of $b$ to $\Lambda'\backslash\Lambda$. Thus we may also write

$$h_\Lambda(s|b) = \lim_{\Lambda'\uparrow\mathbb{Z}^d} (h_{\Lambda'}(s,b) - h_{\Lambda'\backslash\Lambda}(b)), \tag{9.66}$$

although neither $h_{\mathbb{Z}^d}(s,b)$ nor $h_{\mathbb{Z}^d\backslash\Lambda}(b)$ makes sense by itself. ***Periodic*** boundary conditions for local Hamiltonians may be defined for arbitrary interactions $\Phi$ and special lattices. For example, the Ising chain in $d = 1$ has local Hamiltonians

$$h^{pbc}_{\{1,2,\ldots,n\}}(s) = J\left(s_1s_n + \sum_{i=1}^{n-1} s_is_{i+1}\right) - B\sum_{i=1}^{n} s_i. \tag{9.67}$$

Naively, a ***ground state*** of a *finite* classical spin system, i.e., a system of the above kind defined on a *fixed* finite lattice $\Lambda \subset \mathbb{Z}^d$, is a spin configuration $s_0 \in \underline{n}^\Lambda$ that minimizes the local Hamiltonian $h_\Lambda$ (9.62), or its counterpart (9.64), that is,

$$h_\Lambda(s_0) \leq h_\Lambda(s), \tag{9.68}$$

for all $s \in \underline{n}^\Lambda$. For example, if $\Lambda$ is a hypercube $\Lambda_N$, then the Ising model (9.63) has a unique ground state for $B > 0$, namely $s_0(x) = 1$ for all $x \in \Lambda$, whereas it has two ground states $s_0^\pm$ for $B = 0$, given by $s_0^\pm(x) = \pm 1$ for all $x$. Ground states of finite classical systems always exist (since the space on which $h_\Lambda$ is finite), but they are not necessarily unique; we just gave a counterexample! The same is true for quantum theory, since for $B = 0$ also the quantum Ising model (9.42) has two degenerate symmetry-breaking ground states. Nonetheless, this case is special, since for nonzero small values of $B$ the ground state of the quantum Ising model is unique for finite $\Lambda$, whereas on the infinite lattice $\mathbb{Z}^d$ it is degenerate (cf. §10.7).

The definition of ground states of *infinite* classical spin systems is just slightly more involved: for local Hamiltonians $h_\Lambda$ with free boundary conditions defined by an interaction $\Phi$ à la (9.62), a ground state is a point $s_0 \in \underline{n}^{\mathbb{Z}^d}$ for which

$$h_\Lambda(s_{0|\Lambda}) \leq h_\Lambda(s_{|\Lambda}), \tag{9.69}$$

for any finite $\Lambda \subset \mathbb{Z}^d$ and any spin configuration $s \in \underline{n}^{\mathbb{Z}^d}$. Alternatively, one may ask

$$h_\Lambda^{s_0}(s_0) \leq h_\Lambda^{s_0}(s), \tag{9.70}$$

for all finite $\Lambda \subset \mathbb{Z}^d$ and all spin configurations $s \in \underline{n}^{\mathbb{Z}^d}$ *that coincide with* $s_0$ *outside* $\Lambda$, where $h_\Lambda^{s_0}$ stands for (9.64) with $b = s_0$. In other words, $s_0$ provides a boundary condition $b$, which is fixed for all $s$ that compete with $s_0$ in minimizing the local Hamiltonian $h_\Lambda^{b=s_0}$. Both definitions give the usual two ground states for the Ising model with $B = 0$ (in which all spins are either "up" or "down"), but the second one also opens the possibility of *domain walls*, where infinite chains of "spin up" alternate with infinite chains of "spin down", and similarly in higher $d$.

If different ground states in the above ("pure") sense exist, we may reinterpret such states $s_0$ as Dirac measures $\delta_{s_0}$ on the space $\underline{n}^\Lambda$ of all spin configurations on $\Lambda$, and may also allow convex combinations of ground states as ground states. This, as well as the analogy with Definition 9.16 (in which no purity condition is imposed) inspires a more liberal definition of a ground state, which is predicated on Boltzmann's idea that a state of a classical system of the kind we consider is a probability measure $\mu_\Lambda^0$ on $\underline{n}^\Lambda$, and likewise for $\underline{n}^{\mathbb{Z}^d}$. In the C*-algebraic formalism we use, this follows from (9.61) and the identification of states on $C(X)$ with completely regular probability measures on $X$ (assumed to be a compact Hausdorff space, cf. §B.5). A state $\mu$ on $C(\underline{n}^{\mathbb{Z}^d})$, i.e., a probability measure on $\underline{n}^{\mathbb{Z}^d}$, induces a state on each local algebra $C(\underline{n}^\Lambda)$, i.e., a probability measure $\mu_\Lambda$ on $\underline{n}^\Lambda$ simply by restriction, since

$$C(\underline{n}^\Lambda) \subset C(\underline{n}^{\mathbb{Z}^d}) \tag{9.71}$$

through the injection (9.60), according to which $f_\Lambda \in C(\underline{n}^\Lambda)$ has image $f \in C(\underline{n}^{\mathbb{Z}^d})$ defined by $f(s) = f_\Lambda(s_{|\Lambda})$. The measure $\mu_\Lambda$, then, is given in terms of $\mu$ by

$$\mu_\Lambda(f_\Lambda) = \mu(f); \tag{9.72}$$

the corresponding probability distribution $p_\Lambda$ (i.e., $p_\Lambda(s) = \mu_\Lambda(\{s\})$) is given by

$$p_\Lambda(s) = \mu\left(\{s' \in \underline{n}^{\mathbb{Z}^d} \mid s'_{|\Lambda} = s\}\right), \quad s \in \underline{n}^\Lambda. \tag{9.73}$$

The family of probability measures $(\mu_\Lambda)$ defined by $\mu$ is ***consistent*** in that if $\Lambda^{(1)} \subset \Lambda^{(2)}$ and $f_1 \in C(\underline{n}^{\Lambda^{(1)}})$ and $f_2 \in C(\underline{n}^{\Lambda^{(2)}})$ are related as in (9.60), then

$$\mu_{\Lambda^{(1)}}(f_1) = \mu_{\Lambda^{(2)}}(f_2). \tag{9.74}$$

Conversely, a consistent family of probability measures $(\mu_\Lambda)$ defines a unique probability measure $\mu$ on $\underline{n}^{\mathbb{Z}^d}$ which induces the given family through (9.72).

**Definition 9.21.** *For given finite $\Lambda \subset \mathbb{Z}^d$, a probability measure $\mu^0_\Lambda$ on $\underline{n}^\Lambda$ is a* **ground state** *of a local Hamiltonian $h_\Lambda$ (with free boundary conditions) if, in terms of the probabilities $p^0_\Lambda(s) = \mu^0_\Lambda(\{s\})$, for any probability measure $\mu_\Lambda$ on $\underline{n}^\Lambda$,*

$$\sum_{s \in \underline{n}^\Lambda} p^0_\Lambda(s)\, h_\Lambda \leq \sum_{s \in \underline{n}^\Lambda} p_\Lambda(s)\, h_\Lambda. \tag{9.75}$$

*A probability measure $\mu_0$ on $\underline{n}^{\mathbb{Z}^d}$ is a* **ground state** *for some interaction $\Phi$ if* (9.75) *holds for any probability measure $\mu$ on $\underline{n}^{\mathbb{Z}^d}$ and any finite subset $\Lambda \subset \mathbb{Z}^d$, where this time $p^0_\Lambda$ (and analogously $p_\Lambda$) is defined by* (9.73).

In particular, convex sums of pure ground states are ground states in this more general sense, so that, if all pure ground states break some symmetry (as is the case for the $\mathbb{Z}_2$-symmetry $s \mapsto -s$ of the Ising model at $B = 0$), symmetric convex sums will restore the symmetry. The set of all ground states of a given interaction $\Phi$ is a convex set, whose extreme points are the pure ground states (at least, under suitable hypotheses on $\Phi$). This leads to a discussion of SSB similar to the quantum case.

In the following discussion of equilibrium states, we use the notation

$$\Pr(X) \cong S(C(X)) \tag{9.76}$$

for the compact convex set of all completely regular probability measures on $X$, which as above will either be the finite set $\underline{n}^\Lambda$ (with discrete topology)—on which of course any probability measure is completely regular—or the compact space $\underline{n}^{\mathbb{Z}^d}$. In the first case we may as well use probability *distributions* $p_\Lambda$ (instead of probability measures) on $\underline{n}^\Lambda$. In the second, we could also use Baire measures.

Given an interaction $\Phi$ and the ensuing family (9.62) of local Hamiltonians $h_\Lambda$, we define the local ***energy*** for each finite $\Lambda \subset \mathbb{Z}^d$ as a function $\mathscr{E}_\Lambda : \Pr(\underline{n}^\Lambda) \to \mathbb{R}$ by

$$\mathscr{E}_\Lambda(p_\Lambda) = \sum_{s \in \underline{n}^\Lambda} p_\Lambda(s) h_\Lambda(s). \tag{9.77}$$

Of course, this is just the expectation value of the Hamiltonian in the state $p_\Lambda$. The local ***entropy*** $S_\Lambda : \Pr(\underline{n}^\Lambda) \to \mathbb{R}$ is a more subtle concept; rather than the expectation value of some (local) observable, it specifies a property of the probability distribution itself. With Boltzmann's constant $k_B$, we have

$$S_\Lambda(p_\Lambda) = -k_B \sum_{s \in \underline{n}^\Lambda} p_\Lambda(s) \ln(p_\Lambda(s)). \tag{9.78}$$

Note that $S_\Lambda(p_\Lambda) \geq 0$, with equality iff $p_\Lambda$ is a pure state (i.e., $p_\Lambda$ is supported at a single spin configuration). The local ***free energy*** $\mathscr{F}_\Lambda^\beta : \Pr(\underline{n}^\Lambda) \to \mathbb{R}$ is defined as

$$\mathscr{F}_\Lambda^\beta = \mathscr{E}_\Lambda - TS_\Lambda, \tag{9.79}$$

where $\beta = 1/k_B T$. A ***local equilibrium state***, then, is a probability distribution $p_\Lambda^\beta$ that minimizes the free energy (for fixed temperature $T$).

**Theorem 9.22.** *For each $T > 0$, there is a* unique *local equilibrium state, given by the* **Boltzmann distribution** *(and associated* **partition function***)*

$$p_\Lambda^\beta(s) = (Z_\Lambda^\beta)^{-1} e^{-\beta h_\Lambda(s)}; \tag{9.80}$$

$$Z_\Lambda^\beta = \sum_{s' \in \underline{n}^\Lambda} e^{-\beta h_\Lambda(s')}. \tag{9.81}$$

The associated ***free energy in equilibrium*** is then given by

$$F_\Lambda^\beta = \mathscr{F}_\Lambda^\beta(p_\Lambda^\beta) = -\beta^{-1} \ln Z_\Lambda^\beta. \tag{9.82}$$

*Proof.* The claim follows from the fact that any $p_\Lambda \in \Pr(\underline{n}^\Lambda)$ satisfies the inequality

$$\mathscr{F}_\Lambda^\beta(p_\Lambda) \geq -\beta^{-1} \ln Z_\Lambda^\beta, \tag{9.83}$$

with equality iff $p = p_\Lambda^\beta$, i.e., using (9.79), (9.77), and (9.78), we need to show that

$$\sum_{s \in E^\Lambda} p(s)(h_\Lambda(s) + \beta^{-1} \ln p(s)) + \beta^{-1} \ln Z_\Lambda^\beta \geq 0. \tag{9.84}$$

Using (9.80), for each $s \in E^\Lambda$ we obtain

$$-\beta h_\Lambda(s) = \ln Z_\Lambda^\beta + \ln p_\Lambda^\beta(s). \tag{9.85}$$

Substituting this in (9.84), using $\sum_s p(s) = 1$, omitting the ensuing prefactor $\beta^{-1}$, and noting that $p_\Lambda^\beta(s) > 0$ for all $s$, the inequality (9.84) to be proved becomes

$$\sum_{s \in E^\Lambda} p(s) \ln \left( \frac{p(s)}{p_\Lambda^\beta(s)} \right) \geq 0. \tag{9.86}$$

Hence we need to prove the inequality

$$\sum_{s\in E^{\Lambda}} p^{\beta}_{\Lambda}(s)\cdot\left(\frac{p(s)}{p^{\beta}_{\Lambda}(s)}\right)\ln\left(\frac{p(s)}{p^{\beta}_{\Lambda}(s)}\right)\geq 0, \tag{9.87}$$

with equality iff $p(s)=p^{\beta}_{\Lambda}(s)$ for all $s$. Let us note that the function $f(x)=x\ln x$ is strictly convex for all $x\geq 0$, that is, for any finite set of numbers $p'(s)\in(0,1)$ with $\sum_s p'(s)=1$ and any set of positive real numbers $(x_s)_s\geq 0$, we have

$$\sum_s p'(s)f(x_s)\geq f\left(\sum_s p'(s)x_s\right), \tag{9.88}$$

with equality iff all numbers $x_s$ are the same. Applying this with $p'(s)=p^{\beta}_{\Lambda}(s)$ and $x_s=p(s)/p^{\beta}_{\Lambda}(s)$, so that $p'(s)x_s=p(s)$ and hence $\sum_s p'(s)x_s=\sum_s p(s)=1$, which makes the right-hand side of (9.88) vanish since $\ln(1)=0$, finally leads to (9.87). Equality arises iff $p(s)/p^{\beta}_{\Lambda}(s)$ equals the same numer $c$ for all $s$; summing over all $s$ forces $c=1$, so that one has equality iff $p(s)=p^{\beta}_{\Lambda}(s)$ for all $s$, as desired. □

Neither the local Hamiltonians (9.62) nor the local partition functions (9.81) have a limit as $\Lambda\uparrow\mathbb{Z}^d$. A precise definition equilibrium states of infinite classical systems was given in 1968 by Dobrushin and by Lanford and Ruelle (DLR).

**Definition 9.23.** *For fixed inverse temperature $\beta\in(0,\infty)$ and fixed interaction $\Phi$, a* **Gibbs measure** *$\mu^{\beta}$ is a (Baire = regular Borel) probability measure on $\underline{n}^{\mathbb{Z}^d}$ such that for each finite $\Lambda\subset\mathbb{Z}^d$ and each pair $(s,b)$ of a spin configuration $s:\Lambda\to\underline{n}$ plus boundary condition $b:\Lambda^c\to\underline{n}$, the conditional probability $\mu^{\beta}(\mathsf{s}|\mathsf{b})$ for the events*

$$\mathsf{s}=\{s'\in\underline{n}^{\mathbb{Z}^d}\mid s'_{|\Lambda}=s\}\subset\underline{n}^{\mathbb{Z}^d}; \tag{9.89}$$

$$\mathsf{b}=\{s''\in\underline{n}^{\mathbb{Z}^d}\mid s''_{|\Lambda^c}=b\}\subset\underline{n}^{\mathbb{Z}^d}, \tag{9.90}$$

*is given in terms of the local Hamiltonian $h_{\Lambda}(s|b)$ as defined by* (9.66) *by*

$$\mu^{\beta}(\mathsf{s}|\mathsf{b})=(Z^{\beta}_{\Lambda}(b))^{-1}e^{-\beta h_{\Lambda}(s|b)}, \tag{9.91}$$

$$Z^{\beta}_{\Lambda}(b)=\sum_{s\in\underline{n}^{\Lambda}} e^{-\beta h_{\Lambda}(s|b)}. \tag{9.92}$$

Recall that $\mu^{\beta}(\mathsf{s}|\mathsf{b})=\mu^{\beta}(\mathsf{s}\cap\mathsf{b})/\mu^{\beta}(\mathsf{b})$, where $\mathsf{s}\cap\mathsf{b}=\{s_b\}$ consists of the single spin configuration $s_b:\mathbb{Z}^d\to\underline{n}$ that coincides with $s$ on $\Lambda$ and coincides with $b$ on $\Lambda^c$. Thus we may write $\mu^{\beta}(\mathsf{s}|\mathsf{b})=p^{\beta}(s_b)/\mu^{\beta}(\mathsf{b})$, where $p^{\beta}(s)=\mu^{\beta}(\{s\})$ as usual.

It was initially unclear how to generalize this highly fruitful definition of equilibrium states in classical statistical mechanics to the quantum case, where conditional probabilities are not well defined (this was eventually resolved, however, through Definition 10.9 below). Thus a different (equally fruitful) approach to equilibrium states of (infinite) quantum systems was developed, to which we now turn.

## 9.6 Equilibrium (KMS) states of quantum systems

For *finite* quantum spin systems we have expressions for the energy $\hat{\mathscr{E}}_\Lambda^\beta$, the entropy $\hat{S}_\Lambda$, and the free energy $\hat{\mathscr{F}}_\Lambda$ that are analogous to their classical counterparts (9.77), (9.78), and (9.79). In particular, these quantities are functions on the state space $S(A_\Lambda)$. Since $A_\Lambda = B(H_\Lambda)$, where we assume that $H$ and hence $H_\Lambda$ is finite-dimensional, each state $\omega_\Lambda \in S(A_\Lambda)$ is given by a density operator $\rho_\Lambda$, so that

$$\hat{\mathscr{E}}_\Lambda(\omega_\Lambda) = \omega_\Lambda(h_\Lambda) = \mathrm{Tr}\,(\rho_\Lambda h_\Lambda); \tag{9.93}$$
$$\hat{S}_\Lambda(\omega_\Lambda) = -k_B \mathrm{Tr}\,(\rho_\Lambda \ln \rho_\Lambda); \tag{9.94}$$
$$\hat{\mathscr{F}}_\Lambda^\beta = \hat{\mathscr{E}}_\Lambda - T\hat{S}_\Lambda. \tag{9.95}$$

Defining a local equilibrium state as a density matrix $\rho_\Lambda^\beta$ that minimizes the free energy (for fixed $T$), we have the following quantum analogue of Theorem 9.22:

**Theorem 9.24.** *For each $T > 0$, there is a* unique *local equilibrium state $\omega_\Lambda^\beta$, viz.*

$$\omega_\Lambda^\beta(a) = \mathrm{Tr}\left(\rho_\Lambda^\beta a\right); \tag{9.96}$$
$$\rho_\Lambda^\beta = (\hat{Z}_\Lambda^\beta)^{-1} e^{-\beta h_\Lambda}; \tag{9.97}$$
$$\hat{Z}_\Lambda^\beta = \mathrm{Tr}\left(e^{-\beta h_\Lambda}\right). \tag{9.98}$$

Accordingly, the free energy $F_\Lambda^\beta$ in equilibrium is given by

$$F_\Lambda^\beta = \hat{\mathscr{F}}_\Lambda^\beta(\rho_\Lambda^\beta) = -\beta^{-1} \ln \hat{Z}_\Lambda^\beta. \tag{9.99}$$

*Proof.* One proof is analogous to the classical case, in that for all $\rho_\Lambda \in \mathscr{D}(B(H_\Lambda))$,

$$\hat{\mathscr{F}}_\Lambda^\beta(\rho_\Lambda) \geq -\beta^{-1} \ln \hat{Z}_\Lambda^\beta, \tag{9.100}$$

with equality iff $\rho_\Lambda = \rho_\Lambda^\beta$. This, in turn, follows from the inequality

$$\mathrm{Tr}\,(a(\ln b - \ln a)) \leq \mathrm{Tr}\,(b - a), \tag{9.101}$$

with equality iff $b = a$, which is valid for matrices $a, b$ for which $a \geq 0$ (in the usual sense that $\lambda \geq 0$ for each $\lambda \in \sigma(a)$) and $b > 0$ in that $\lambda > 0$ for each $\lambda \in \sigma(b)$. The case $a = \rho_\Lambda$ and $b = \rho_\Lambda^\beta$ immediately gives the claim. □

What remains to be done, however, is to define equilibrium states for infinite systems. This is achieved through the so-called KMS-***condition***, which is based on the observation that for any $a, b \in A_\Lambda$, in terms of (9.40) the state (9.96) satisfies

$$\omega_\Lambda^\beta(\alpha_t^{(\Lambda)}(a)b) = \omega_\Lambda^\beta(b\alpha_{t+i\beta}^{(\Lambda)}(a)) \quad (t \in \mathbb{R}). \tag{9.102}$$

Moreover, in finite systems this condition (even at $t = 0$) fully characterizes $\omega_\Lambda^\beta$:

**Proposition 9.25.** *Let $h$ be a self-adjoint operator on a finite-dimensional Hilbert space $H'$, with associated density operator $\rho$ and (complex) time-evolution given by*

$$\rho = \frac{e^{-h}}{\mathrm{Tr}\,(e^{-h})}; \tag{9.103}$$

$$\alpha_z(a) = e^{izh} a e^{-izh}, \; z \in \mathbb{C}, a \in B(H'), \tag{9.104}$$

*respectively (the exponentials being defined by a norm-convergent power series).*

*Then the associated two-point functions defined by $\omega(a) = \mathrm{Tr}\,(\rho a)$ satisfy*

$$\omega(ab) = \omega(b\alpha_i(a)) \;\; (a, b \in B(H)). \tag{9.105}$$

*Conversely, any state for which* (9.105) *holds for given $h$ and $\alpha_z$ is given by* (9.103).

*Proof.* Eq. (9.105) follows from (9.103) - (9.104) and cyclicity of the trace, i.e., (A.78). Similarly, given non-degeneracy of the Hilbert-Schmidt inner product (B.495) on $B(H)$, eq. (9.105) is equivalent to the condition

$$\rho a = e^{-h} a e^{h} \rho, \tag{9.106}$$

for each $a \in B(H')$. Multiplying with $\exp(h)$ shows that $\exp(h)\rho$ commutes with every $a \in B(H')$. Since $B(H')' = \mathbb{C} \cdot 1_{H'}$, we obatin $\exp(h)\rho = \lambda \cdot 1_H$. Since $\exp(h)$ is invertible with inverse $\exp(-h)$, we obtain $\rho = \lambda \cdot \exp(-h)$, upon which the normalization condition $\mathrm{Tr}\,(\rho) = 1$ yields (9.103). □

For arbitrary C*-algebras $A$ with time-evolution $t \mapsto \alpha_t$, expressions like $\alpha_{t+i\beta}(a)$ may not be defined, so one has to proceed more carefully, but the idea is the same.

**Definition 9.26.** *Let $A$ be a C*-algebra with an automorphism group $\mathbb{R}$. A* **KMS state** *at "inverse temperature" $\beta \in \mathbb{R}$ is a state $\omega$ on $A$ with the following property:*

1. *For any $a, b \in A$, the function $F_{a,b} : t \mapsto \omega(b\alpha_t(a))$ from $\mathbb{R}$ to $\mathbb{C}$ has an analytic continuation to the strip*

$$\mathscr{S}_\beta = \{z \in \mathbb{C} \mid 0 \leq \mathrm{Im}(z) \leq \beta\}, \tag{9.107}$$

*where it is holomorphic in the interior and continuous on the boundary*

$$\partial \mathscr{S}_\beta = \mathbb{R} \cup (\mathbb{R} + i\beta). \tag{9.108}$$

2. *The boundary values of $F_{a,b}$ are related, for all $t \in \mathbb{R}$, by*

$$F_{a,b}(t) = \omega(b\alpha_t(a)); \tag{9.109}$$

$$F_{a,b}(t + i\beta) = \omega(\alpha_t(a)b). \tag{9.110}$$

*If this is the case, $\omega$ satisfies the* **KMS-condition** *at (inverse temperature) $\beta$.*

It is easy to show that $A$ has a dense subset $A_\alpha$ such that for any $a \in A_\alpha$ the function $t \mapsto \alpha_t(a)$ from $\mathbb{R}$ to $A$ extends to an entire $A$-valued analytic function, written $z \mapsto \alpha_z(a)$ (i.e., for each $\varphi \in A^*$ the function $z \mapsto \varphi(\alpha_z(a))$ from $\mathbb{C}$ to $\mathbb{C}$ is entire analytic). Namely, for any $a \in A$ and $\varepsilon > 0$, define

$$a_\varepsilon = \int_{-\infty}^{\infty} \frac{dt}{\sqrt{2\pi\varepsilon}} e^{-t^2/2\varepsilon} \alpha_t(a), \tag{9.111}$$

which satisfies $a_\varepsilon \in A_\alpha$ and $\lim_{\varepsilon\downarrow 0} a_\varepsilon = a$. If $A = B(H')$ with $\dim(H') < \infty$, we even have $B(H')_\alpha = B(H')$, since (9.104) is entire analytic in $z$ for any $a \in B(H')$. For any $A$, the KMS-condition on $\omega$ is then equivalent to the simpler requirement

$$\omega(ab) = \omega(b\alpha_{i\beta}(a)) \ \ (a \in A_\alpha, b \in A). \tag{9.112}$$

**Corollary 9.27.** *If $A = B(H')$ with $\dim(H') < \infty$, then* KMS *states (at fixed $\beta$) are necessarily given by the equilibrium states of Theorem 9.24 and hence are unique.*

Although initially the characterization of equilibrium states of infinite systems by the KMS condition was tentative, in the 1970s and '80s it became clear that it was spot on, being equivalent to local and global thermodynamic stability (against perturbations of the dynamics), the (local) maximum entropy principle, etc. Also:

**Proposition 9.28.** *A* KMS *state at $\beta \in \mathbb{R}\backslash\{0\}$ is time-independent.*

*Proof.* We just sketch the proof if $A$ is unital. Taking $b = 1_A$, for fixed $a \in A_\alpha$ the function $F_{a,1_A} \equiv F$ defined by $F(z) = \omega(\alpha_z(a))$ is entire analytic on $\mathbb{C}$. Writing $z = t + is$ (with $s,t \in \mathbb{R}$), we have $\alpha_z = \alpha_t \circ \alpha_{is}$ and hence (since each $\alpha_t$ is an automorphism and hence an isometry), $|F(t+is)| \leq \|\alpha_{is}(a)\|$. Also, (9.112) yields $F(t+i(s+\beta)) = F(t+is)$. Hence $F(t+is)$ is bounded in $t$ and periodic in $s$; by the latter property its supremum on $\mathbb{C}$ may be computed by its supremum on the strip $\mathscr{S}_\beta$, and by the former property this supremum is finite. Therefore, $F$ is bounded, and so by Liouville's Theorem it must be constant, especially if $z = t \in \mathbb{R}$. Hence $\alpha_t^*\omega(a) = \omega(a)$ for each $a \in A_\alpha$, and since this is a dense set, $\alpha_t^*\omega = \omega$. □

By the argument for ground states following Definition 9.16, the automorphism group $t \mapsto \alpha_t$ is unitarily implemented in the GNS-representation $\pi_\omega$ induced by a KMS state $\omega$, such that (9.51) - (9.52) hold. However, the operator $h_\omega$ in this construction should not be confused with the Hamiltonian of the system. For example suppose $A = B(H')$ for some (not necessarily finite-dimensional) Hilbert space $H'$, so that (9.39) holds for some (not necessarily bounded) Hamiltonian $h$ with discrete spectrum, such that $\exp(-\beta h) \in B_1(H')$. If we now define the density operator

$$\rho = \frac{e^{-\beta h}}{\mathrm{Tr}\left(e^{-\beta h}\right)}, \tag{9.113}$$

then the corresponding state $\omega$ satisfies the KMS-condition at $\beta$. Generalizing the computations around (2.66) in §2.4, we then find (up to unitary equivalence):

$$H_\omega = B_2(H'); \tag{9.114}$$
$$\pi_\omega(a)b = ab; \tag{9.115}$$
$$\Omega_\omega = \rho^{1/2}; \tag{9.116}$$
$$e^{ith_\omega} = \pi_\omega\left(e^{ith}\right)\pi'_\omega\left(e^{-ith}\right), \tag{9.117}$$

where for any $a \in B(H')$, the operator $\pi'_\omega(a)$ on $B_2(H')$ is defined by

$$\pi'_\omega(a)b = ba. \tag{9.118}$$

Note that (9.115) is well defined, since $\rho \geq 0$ and $\rho \in B_1(H')$, whence $\rho^{1/2} \in B_2(H')$, and hence also $ab \in B_2(H')$ and $ba \in B_2(H')$, since $B_2(H')$ is a two-sided ideal in $B(H')$. If $h$ happens to be bounded, we may therefore write

$$h_\omega = \pi_\omega(h) - \pi'_\omega(h). \tag{9.119}$$

Note that the $\pi'_\omega$ term in (9.117) is not needed for (9.52), since $[\pi_\omega(a), \pi'_\omega(b)] = 0$ for any $a, b \in B(H')$, but it *is* necessary to secure (9.51). Another feature of this example is that the vector $\Omega_\omega$ is not only *cyclic* for $\pi_\omega(B(H'))$, which it has to be by virtue of the GNS-construction, but also *separating*, i.e., $\pi_\omega(a)\Omega_\omega = 0$ implies $\pi_\omega(a) = 0$. In other words, one has $\omega(a^*a) = 0$ iff $a = 0$ (which is by no means the case for ground states). If $\dim(H') < \infty$, this is obvious, because $\pi_\omega(a)\Omega_\omega = a\rho^{1/2}$ and $\rho^{1/2}$ is invertible. In general, for arbitrary C*-algebras $A$ we have:

**Proposition 9.29.** *Let $\omega$ be a* KMS *state on $A$ at $\beta \in \mathbb{R}$. Then $\Omega_\omega$ is both cyclic and separating for $\pi_\omega(A)$ and hence also for $\pi_\omega(A)''$ (as well as for $\pi_\omega(A)'$).*

*Proof.* Since $\omega(a^*a) = \|\pi_\omega(a)\Omega_\omega\|^2$, we have $\omega(a^*a) = 0$ iff $\pi_\omega(a)\Omega_\omega = 0$, so that

$$\omega(a^*\alpha_t(a)) = \langle \pi_\omega(a)\Omega_\omega, \pi_\omega(\alpha_t(a))\Omega_\omega \rangle = 0 \ \ (t \in \mathbb{R})$$

if $\omega(a^*a) = 0$, and hence $F_{a^*,a}(t) = 0$, cf. (9.109). The "edge of the wegde" theorem then gives $F_{a^*,a}(z) = 0$ for all $z \in \mathscr{S}_\beta$, upon which the KMS-condition gives

$$\omega(aa^*) = F_{a^*,a}(i\beta) = 0.$$

This means that $\omega(a^*a) = 0$ iff $\omega(aa^*) = 0$, or $\pi_\omega(a)\Omega_\omega = 0$ iff $\pi_\omega(a)^*\Omega_\omega = 0$, and hence $\pi_\omega(b^*)\pi_\omega(a)\Omega_\omega = 0$ iff $\pi_\omega(a^*)\pi_\omega(b)\Omega_\omega = 0$. Since $\Omega_\omega$ is cyclic for $\pi_\omega(A)$, the assumption $\pi_\omega(a)\Omega_\omega = 0$ therefore implies that the bounded operator $\pi_\omega(a^*)$ vanishes on a dense domain in $H_\omega$ and hence vanishes. Since $\pi_\omega(a) = (\pi_\omega(a^*))^*$, it follows that $\pi_\omega(a) = 0$. The extension to $\pi_\omega(A)''$ (and $\pi_\omega(A)'$) is obvious. □

**Corollary 9.30.** *If $\omega$ is a* KMS *state on a quasi-local algebra $A$, i.e., given by* (8.130) *with* $\dim(H) < \infty$, *then $\omega(a^*a) = 0$ iff $a = 0$ and hence the* GNS*-representation $\pi_\omega : A \to B(H_\omega)$ is injective.*

*Proof.* By the previous proof, the closed left-ideal (C.204) is actually a two-sided ideal, which must be zero, since $A$ is simple (as is easily shown from the simplicity of $B(H)$ for finite-dimensional $H$, cf. §8.5). □

Proposition 9.29 shows that the von Neumann algebra $\pi_\omega(A)''$ is in standard form (see Definition C.158), so that the KMS condition bring us into the realm of the Tomita–Takesaki theory. In particular, Theorem C.159 provides us with another time-evolution, namely the one given by the modular group. In the situation of Theorem C.159, we take $a \in M_\alpha$ and $b \in M$, and compute

$$\begin{aligned}\langle \Omega, b\alpha_{-i}(a)\Omega\rangle &= \langle \Omega, b\Delta a\Delta^{-1}\Omega\rangle = \langle \Omega, b\Delta a\Omega\rangle \\ &= \langle \Delta^{1/2}b^*\Omega, \Delta^{1/2}a\Omega\rangle = \langle J\Delta^{1/2}a\Omega, J\Delta^{1/2}b^*\Omega\rangle \\ &= \langle Sa\Omega, Sb^*\Omega\rangle = \langle a^*\Omega, b\Omega\rangle \qquad (9.120)\\ &= \langle \Omega, ab\Omega\rangle, \qquad (9.121)\end{aligned}$$

where we used the property $\Delta^{1/2}\Omega = \Omega$ as well as anti-unitarity of $J$, which implies $\langle J\psi, J\varphi\rangle = \langle \varphi, \psi\rangle$; these facts follow from the definitions of $\Delta$ and $J$ via $S$. Therefore, the state $\omega$ on $M$ defined by $\omega(a) = \langle \Omega, a\Omega\rangle$ ($a \in M$) satisfies the KMS-condition for the modular group at $\beta = -1$. If, on the other hand, we start with a $\beta$-KMS state $\omega$ on a C*-algebra $A$ with respect to some given time-evolution $\alpha_t$, and take $H = H_\omega$, $M = \pi_\omega(A)''$, and $\Omega = \Omega_\omega$, the normal extension of $\omega$ to $\pi_\omega(A)''$ given by $\langle \Omega_\omega, \cdot\Omega_\omega\rangle$ still satisfies the KMS condition with respect to the time-evolution on $\pi_\omega(A)''$ given by conjugation with $\exp(ith_\omega)$, as in (9.52). Comparing the latter with the time-evolution on $M$ defined by conjugation with $\Delta^{it}$ (cf. Theorem C.159) gives

$$e^{ith_\omega} = \Delta^{-it/\beta}, \qquad (9.122)$$

since both one-parameter groups of unitary operators satisfy the KMS-condition at $\beta$, and some time-evolution $\alpha_t$ that satisfies the KMS-condition relative to a given state $\omega$ and inverse temperature $\beta$ is unique. To see this (barring technicalities about unbounded operators that are easily dealt with), take $\beta = -1$ for simplicity, assume $\alpha_t$ is conjugation by $\Delta^{it} = \exp(ith)$ (i.e., $\Delta = \exp(h)$), and rewrite (9.112) as

$$\omega(ab) = \langle b^*\Omega, \Delta a\Omega\rangle. \qquad (9.123)$$

This determines $\langle \varphi, \Delta\psi\rangle$ between a dense set of vectors $\varphi, \psi$, and hence fixes $\Delta$.

The operators $J$ and $\Delta$ from the Tomita–Takesaki theory can explicitly be computed in the example (9.113); the antilinear operator $J : B_2(H') \to B_2(H')$ reads

$$Jb = b^*, \qquad (9.124)$$

so that the isomorphism $a \mapsto JaJ$ between $\pi_\omega(A)'' = B(H')$ (where $B(H')$ acts on $B_2(H')$ by left multiplication) and its commutant $\pi_\omega(A)' = B(H')$ (which copy of $B(H')$ now acts on $B_2(H')$ by right multiplication) is given by $JaJb = ba$. Furthermore, the (generally unbounded) linear operator $\Delta : B_2(H') \to B_2(H')$ is given by

$$\Delta b = \rho b\rho^{-1}, \qquad (9.125)$$

which strictly speaking is defined as the closure of the expression (9.125) on the domain of all $b \in B_2(H')$ for which $b\rho^{-1/2} \in B(H')$.

**Theorem 9.31.** *For given unital C*-algebra A, dynamics* $\alpha : \mathbb{R} \to \mathrm{Aut}(\mathbb{R})$*, and inverse temperature* $\beta \in \mathbb{R}$*, let* $S_\beta(A)$ *be the compact convex set of* KMS *states. Then*

$$\partial_e S_\beta(A) = S_\beta(A) \cap S_p(A), \tag{9.126}$$

*where* $S_p(A)$ *is the set of primary states on A (cf. Definition 8.17). Consequently, extreme* KMS *states at fixed inverse temperature* $\beta$ *are either equal or disjoint.*

This suggests that extreme KMS states define ***pure thermodynamics phases***.

*Proof.* We enlarge $S_\beta(A)$ to the set $\hat{K}_\beta(A) \subset A^*$ of all continuous linear functionals on $A$ that satisfy the $\beta$-KMS condition (so that $S_\beta(A)$ consists of all positive elements in $\hat{K}_\beta(A)$ of unit norm). The key to the proof is a bijection between the set $S(\omega)$ of functionals $\rho \in \hat{K}_\beta(A)$ for which $0 \leq \rho \leq \omega$, where $\omega \in S_\beta(A)$ is fixed, and the set $T(\omega)$ of operators $c \in \pi_\omega(A)' \cap \pi_\omega(A)''$ such that $0 \leq c \leq 1_{H_\omega}$, given by

$$\rho(a) = \langle \Omega_\omega, c\pi_\omega(a)\Omega_\omega \rangle. \tag{9.127}$$

This implies the claim, since $\omega \in \partial_e S_\beta$ iff any $\rho \in S(\omega)$ takes the form $\rho = t\omega$ for some $t \in [0,1]$ (cf. Lemma C.17), which in turn is the case iff $c = t \cdot 1_{H_\omega}$.

First, for any state $\omega \in S(A)$ there is a bijection between the set of linear functionals $\rho \in A^*$ for which $0 \leq \rho \leq \omega$ and the set of operators $c \in \pi_\omega(A)'$ such that $0 \leq c \leq 1_{H_\omega}$, given by (9.127). Indeed, in one direction, given $a = b^*b \geq 0$, we have

$$(\omega - \rho)(a) = \langle \pi_\omega(b)\Omega_\omega, (1_{H_\omega} - c)\pi_\omega(b)\Omega_\omega \rangle \geq 0, \tag{9.128}$$

for if $0 \leq c \leq 1_{H_\omega}$, then $0 \leq (1_{H_\omega} - c) \leq 1_{H_\omega}$. Hence $\rho \leq \omega$, whilst from (9.127) we similarly find $\rho \geq 0$. Conversely, $\rho$ induces a quadratic form $R$ on $H_\omega$, defined initially on the dense domain $\pi_\omega(A)H_\omega$ by the formula

$$R(\pi_\omega(a)\Omega_\omega, \pi_\omega(b)\Omega_\omega) = \rho(a^*b), \tag{9.129}$$

which is easily seen to be well defined, positive, and bounded, and so Proposition B.79 supplies the operator $c$, which a simple computation shows to be in $\pi_\omega(A)'$.

For the bijection $S(\omega) \cong T(\omega)$, where $\omega$ is a $\beta$-KMS state as above, we therefore need the additional property $c \in \pi_\omega(A)''$. Putting $\beta = -1$ for convenience and using the notation of Theorem C.159, we first show that $\Delta^{-it} c \Delta^{it} = c$ for any $t \in \mathbb{R}$: indeed, since $\rho$ satisfies the KMS condition, it is time-translation invariant, so that

$$\begin{aligned}
\langle \pi_\omega(a^*)\Omega_\omega, \Delta^{-it} c \Delta^{it} \pi_\omega(b)\Omega_\omega \rangle &= \langle \Omega_\omega, c\Delta^{it}\pi_\omega(a)\Delta^{-it}\Delta^{-it}\pi_\omega(b)\Delta^{-it}\Omega_\omega \rangle \\
&= \langle \Omega_\omega, c\pi_\omega(\alpha_t(ab))\Omega_\omega \rangle \\
&= \rho(\alpha_t(ab)) = \rho(ab) \\
&= \langle \pi_\omega(a^*)\Omega_\omega, c\pi_\omega(b)\Omega_\omega \rangle,
\end{aligned}$$

so that $\Delta^{-it} c \Delta^{it} = c$ between a dense set of states, and hence this is valid as an operator equation. This also implies that $c$ commutes with any power of $\Delta$. Define $c' = JcJ$, which by Theorem C.159 is an element of $\pi_\omega(A)''$, and compute

$$\begin{aligned}\langle\Omega_\omega,\pi_\omega(a)c'\Omega_\omega\rangle &= \langle\Omega_\omega,\pi_\omega(a)Jc\Delta^{1/2}\Omega_\omega\rangle = \langle\Omega_\omega,\pi_\omega(a)J\Delta^{1/2}c\Omega_\omega\rangle\\ &= \langle\Omega_\omega,\pi_\omega(a)Sc\Omega_\omega\rangle = \langle\Omega_\omega,\pi_\omega(a)c^*\Omega_\omega\rangle\\ &= \langle\Omega_\omega,\pi_\omega(a)c\Omega_\omega\rangle\\ &= \rho(a), \end{aligned} \tag{9.130}$$

where we used the properties $J\Omega_\omega = \Omega_\omega$, $\Delta^{1/2}\Omega_\omega = \Omega_\omega$, $c\Delta^{1/2} = \Delta^{1/2}c$ as just mentioned, $S = J\Delta^{1/2}$, and $c^* = c$ (since $c \geq 0$). Finally, it follows from the KMS condition (applied to the normal extension of the state $\omega$ to $\pi_\omega(A)''$ given by $\langle\Omega_\omega,\cdot\Omega_\omega\rangle$ as well as to the normal extension of $\rho$ to $\pi_\omega(A)''$ given by $\langle\Omega_\omega,\cdot c'\Omega_\omega\rangle$ just computed) that $c' \in \pi_\omega(A)'$, since for arbitrary $a,b,d \in A_\alpha$ we have

$$\begin{aligned}\omega(ac'bd) &= \omega(\alpha_i(bd)ac') = \rho(\alpha_i(bd)a) = \rho(\alpha_i(b)\alpha_i(d)a)\\ &= \rho(\alpha_i(d)ab) = \omega(\alpha_i(d)abc') = \omega(abc'd).\end{aligned}$$

In other words, for any $a,b,d \in A$ we have

$$\langle\pi_\omega(a^*)\Omega_\omega, c'\pi_\omega(b)\pi_\omega(d)\Omega_\omega\rangle = \langle\pi_\omega(a^*)\Omega_\omega,\pi_\omega(b)\pi_\omega(d)c'\Omega_\omega\rangle, \tag{9.131}$$

so that $c'\pi_\omega(b) = \pi_\omega(b)c'$ between vectors in a dense domain, so that this is an operator equality. Hence $c' \in \pi_\omega(A)'$, and in view of this we may rewrite (9.130) as $\rho(a) = \langle\Omega_\omega, c'\pi_\omega(a)\Omega_\omega\rangle$. Since the operator $c' \in \pi_\omega(A)'$ in (9.127) is uniquely determined by $\rho$, this shows that $c' = c$. Since we already had $c' \in \pi_\omega(A)''$, it follows that $c \in \pi_\omega(A)' \cap \pi_\omega(A)''$. □

It can also be shown that $S_\beta(A)$ is a (Choquet) *simplex*, which is a property rather more typical of the state space of a *commutative* unital C*-algebra; this makes it especially remarkable for the set of $\beta$-KMS states on a highly *non-commutative* C*-algebra like the infinite tensor product of $B = M_n(\mathbb{C})$. In the physically relevant case where $S_\beta(A)$ is metrizable, this implies that for any given KMS state $\omega \in S_\beta(A)$ there is a *unique* probability measure $\mu$ on $\partial_e S_\beta(A)$, such that for each $a \in A$,

$$\omega(a) = \int_{\partial S_\beta(A)} d\mu(\omega')\,\omega'(a). \tag{9.132}$$

Conversely, any probability measure $\mu$ on $\partial_e S_\beta(A)$ *defines* a $\beta$-KMS state by reading this equality from right to left. Towards the next chapter, suppose for example that there is a $G$-action on $A$, i.e., a continuous homomorphism $\gamma : G \to \mathrm{Aut}(A)$ (where $G$ is a locally compact group). Then $G$ also acts on $S(A)$ via the dual maps $\gamma_g^*(\omega) = \omega \circ \gamma_{g^{-1}}$, and if $G$ is a symmetry of the dynamics in that $\alpha_t \circ \gamma_g = \gamma_g \circ \alpha_t$ for each $t \in \mathbb{R}$ and $g \in G$, then this dual action maps both $S_\beta(A)$ and $\partial_e S_\beta(A)$ into themselves. If $G$ is compact with normalized Haar measure $\mu$, then for any fixed extremal KMS state $\omega_0 \in \partial_e S_\beta(A)$, by (left) invariance of $\mu$ one obtains a $G$-invariant state by

$$\omega = \int_G d\mu(g)\,\gamma_g^*\omega_0. \tag{9.133}$$

## Notes

§9.1. **Symmetries of C*-algebras and Hamhalter's Theorem**

Theorem 9.4 is due to Hamhalter (2011). Our proof, taken almost *verbatim* from Landsman & Lindenhovius (2016) roughly follow his, but adds various details and also takes some different turns. The main differences with the original proof by Hamhalter are the following. Firstly, we give an order-theoretic characterization of u.s.c. decompositions of the form $\pi_K$ (and hence of the commutative algebras in $\mathscr{C}(C(X))$ that are the unitization of some ideal) by the three axioms stated in Lemma 3.1.1 in Firby (1973), whereas Hamhalter uses Proposition 7 in Mendivil (1999), which gives a different characterization of unitizations of ideals. Furthermore, Hamhalter only treats Lemma 9.5 in full generality, whereas in our opinion it is very instructive to take the case of finite sets first, where many of the key ideas already appear in a setting where they are not overshadowed by topological complications. Finally, our proof of Lemma 9.6.2 differs from Hamhalter's proof. The topology of partitions may be found in Willard (1970), especially Theorem 9.9.

Theorem 9.7 is due to Hamhalter (2015). Corollary 9.9 has a long history, starting with Jacobson & Rickart (1950) and ending with Thomsen (1982).

§9.2. **Unitary implementability of symmetries**

See Bratteli & Robinson (1987), §4.3.

§9.3. **Motion in space and in time**

For a far more detailed study of asymptotic abelianness see Bratteli & Robinson (1987), §4.3.2 and Bratteli & Robinson (1997), §5.4.1. Results like Theorem 9.14 may also be found in Sewell (2002). Theorem 9.14 is also valid for ***ergodic states*** with respect to the given $\mathbb{Z}^d$-action, where we say that a state on a C*-algebra $A$ with $G$-action is ergodic if it is an element of $\partial_e(S(A)^G)$, i.e., extreme in the convex set of $G$-invariant states on $A$. Also Theorem 9.15 holds (with a more complicated proof, of course) under weaker conditions on $\Phi$, typically exponential decay in $X$.

Theorem 9.15 is the simplest result in this direction; for similar results under weaker assumptions on the interaction $\Phi$, see Bratteli & Robinson (1997), §6.2.1.

§9.4. **Ground states of quantum systems**

The idea of a ground state of a quantum system may be attributed to Bohr (1913), who postulated that an atom has a state of lowest energy (which he called a "permanent state"). See e.g. Pais (1986), p. 199. In this section, which merely present some key points treated in far more detail in Bratteli & Robinson (1997), §5.3.3. and §6.2.7, we have just scratched the surface of the topic, which is basic to physics.

§9.5. **Ground states and equilibrium states of classical spin systems**

Basic references for the mathematical physics of classical spin systems on a lattice are Israel (1979), Simon (1993), van Enter, Fernandez, & Sokal (1993), and Georgii (2011). One may now define pure thermodynamics phases as extreme elements of the compact convex set of all Gibbs measures (or of the set of all translation-invariant Gibbs measures, as in Simon, 1993, §III.5), but there is no identification between pure thermodynamics phases with primary equilibrium states (as in the quantum case), because a state on a commutative C*-algebra like $C(\underline{n}^{\mathbb{Z}^d})$ is primary iff it is pure. Fortunately, the specific measure-theoretic setting of classical statistical mechanics provides its own resources. For any $\Lambda \subset \mathbb{Z}^d$, let $\Sigma_\Lambda$ be the smallest $\sigma$-algebra (within the Borel $\sigma$-algebra for $\underline{n}^{\mathbb{Z}^d}$) for which each $f \in C(\underline{n}^\Lambda)$ is measurable, and let

$$\Sigma_\infty = \bigcap_\Lambda \Sigma_\Lambda, \tag{9.134}$$

where each $\Lambda$ is *finite*, be the ***σ-algebra at infinity***, with associated commutative C*-algebra $\mathscr{B}_\infty(\underline{n}^{\mathbb{Z}^d})$ of all bounded measurable functions on $\underline{n}^{\mathbb{Z}^d}$ that are $\Sigma_\infty$-measurable. This is the home of the macroscopic observables, defined as averages analogously to the quantum case. The role of primary states (or rather of states whose algebra of observables is trivial at infinity, as in Theorem 8.23) is now played by ***states that are trivial at infinity***, that is, probability measures $\mu$ on $\underline{n}^{\mathbb{Z}^d}$ for which either $\mu(X) = 0$ or $\mu(X) = 1$ for $X \in \Sigma_\infty$ (cf. the Kolmogorov 0-1 law of probability theory). Indeed there is a classical version of Theorem 8.23, making exactly the same claim *mutatis mutandis*, see Theorem III.1.6 in Simon (1993). The main result (cf. Theorem 7.7 in Georgii, 2011), is that a state is extreme in the compact convex set of all Gibbs measures (at fixed temperature and potential, of course) iff it is a Gibbs measure that is trivial at infinity. It follows that two distinct extreme Gibbs measures are mutually singular on $\Sigma_\infty$ (which is the pertinent classical version of disjointness of primary states).

§9.6. **Equilibrium (KMS) states of quantum systems**

The KMS condition was introduced by Haag, Hugenholtz, and Winnink (1967), in the following equivalent form:

$$\int_{-\infty}^{\infty} dt\, f(t - i\beta)\omega(a\alpha_t(b)) = \int_{-\infty}^{\infty} dt\, f(t)\omega(\alpha_t(b)a), \tag{9.135}$$

for each $a, b \in A$ and each Schwartz function $f \in \mathscr{D}(\mathbb{R})$. The name KMS derives from the earlier observation (9.102) of Kubo (1957) and independently Martin & Schwinger (1957). See also Haag (1992), Simon (1993), Borchers (2000), Sewell (2002), Thirring (2002), Emch (2007), and perhaps also, at a heuristic level, Landsman & van Weert (1987), especially for applications of the KMS condition to quantum field theory at finite temperature and the quark-gluon plasma (this, incidentally, was the MSc thesis as well as the first major published paper by the author).

The KMS condition also plays a major role in operator algebras and noncommutative geometry; see Connes (1994) and Connes & Marcolli (2008).

For a proof of (9.101) see Bratteli & Robinson (1997, Lemma 6.2.21); this book is the bible about the KMS condition and its application to quantum spin systems.

The proof of Proposition 9.25 is taken from Simon (1993), Lemma IV.4.1 and Proposition IV.4.2. The terminology of pure thermodynamical phases for primary KMS states (introduced after Theorem 9.31) is not completely standard; also ergodic states are sometimes called ‘pure phases’.