Chapter 8
Limits: large N

Beside the limit $\hbar \to 0$, we consider the limit $N \to \infty$, where N could be the principal quantum number labeling orbits in atomic physics (as in Bohr's Correspondence Principle), or the number of particles or lattice sites, or the number of identical experiments in a long run measuring the relative frequencies of possible outcomes.

The case of large quantum numbers will be dealt with first: as our toy model of an classical orbit we take a *coadjoint orbit* in the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of a compact connected Lie group G, see §5.9; for $G = SU(2)$ or $SO(3)$ these are simply two-spheres S^2_r. The corresponding quantum theories are indexed by their spin $j = \frac{1}{2}n$, where $n \in \mathbb{N}$, which we send to infinity in order to recover the classical orbit. This can be done more generally by rescaling the highest weight λ of some fixed irreducible representation of G to $n\lambda$ and again letting $n \to \infty$.

The second case, where the limit $N \to \infty$ is typically the thermodynamic limit (namely if the density N/V is kept fixed, where V is the volume of the system sent to infinity, too), has been rigorously studied using operator algebras since the 1960s. In such work the system constructed *at* the limit $N = \infty$ is typically quantum statistical mechanics in infinite volume, whose existence (followed by the establishment of e.g. phase transitions) was a major achievement of mathematical physics.

However, our goal in taking the limit $N \to \infty$ is quite different, in that—in the spirit of Bohrification—our limiting system will be *classical*; from the traditional point of view we look at the macroscopic rather than the quasi-local observables. Nonetheless, for each finite value of $N \in \mathbb{N}$ our (quantum) system will be the same as in the usual theory! Like the first case, in which increasingly large matrix algebras converge to an algebra of continuous functions on some compact space, this apparent miracle is described by the theory of continuous bundles of C*-algebras, as outlined in §C.19. As in the case $\hbar \to 0$ studied in the previous chapter, this theory provides a convenient mathematical machinery for studying the limit $N \to \infty$ also.

We then apply the the limit $N \to \infty$ to N repeated experiments, and, applying the doctrine of classical concepts, rederive the Born rule (avoiding the conceptual and mathematical pitfalls of various previous attempts to do so).

Bridging the gap to the next two chapters, we close with an introduction to quantum spin systems (as a later playing ground for spontaneous symmetry breaking).

K. Landsman, *Foundations of Quantum Theory*,
Fundamental Theories of Physics 188, DOI 10.1007/978-3-319-51777-3_8

8.1 Large quantum numbers

As in §5.9, let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$ and dual $\mathfrak{g}^*$, and let $T \subset G$ be a maximal torus with Lie algebra $\mathfrak{t}$ and dual $\mathfrak{t}^*$. Let $\mathscr{O}_\lambda$ be a ***regular integral coadjoint orbit*** in $\mathfrak{g}^*$, labeled by a dominant weight $\lambda \in \Lambda_d$. This means that there is a point $\theta \in \mathscr{O}_\lambda$ whose stabilizer G_θ is T, and $\lambda = \theta_{|\mathfrak{t}}$; conversely, $\lambda \in \mathfrak{t}^*$ determines $\theta \in \mathfrak{g}^*$, which vanishes on each generator E_α of $\mathfrak{g}_\mathbb{C}$ ($\alpha \in \Delta$).

Following Theorems 5.49 and 5.51, we associate a unitary irreducible representation $u_\lambda : G \to U(H_\lambda)$ to $\mathscr{O}_\lambda$ (or rather to λ), whose underlying Hilbert space H_λ contains a unique highest weight vector υ_λ. We then have (5.228). We abbreviate

$$d_\lambda = \dim(H_\lambda). \tag{8.1}$$

For $SU(2)$ we have $\lambda \in \mathbb{N}_0/2 = \{0, \frac{1}{2}, 1, \ldots\}$, usually called j, and the (regular) coadjoint orbits in $\mathfrak{g}^* \cong \mathbb{R}^3$ are the spheres S^2_j with radius j (with $j \neq 0$). The corresponding highest weight representation u_j is carried by H_j with $d_j = 2j+1$, whose highest weight vector υ_j is an eigenvector of $L_3 = iu'(S_3)$ with eigenvalue j.

We are going to define a continuous bundle of C*-algebras over the base space

$$I = (1/\mathbb{N}) \cup \{0\} \equiv 1/\dot{\mathbb{N}}, \tag{8.2}$$

where $\mathbb{N} = \{1, 2, \ldots\}$ and $\dot{N} = \mathbb{N} \cup \{\infty\}$; as required, I contains 0 as an accumulation point. One may think of elements of I as "quantized" values of Planck's constant $\hbar = 1/N$, upon which the limit $N \to \infty$ is formally the same as the limit $\hbar \to 0$.

If $\lambda \in \Lambda_d$, then $n\lambda \in \Lambda_d$ for all $n \in \mathbb{N}$. We may therefore define the C*-algebras

$$A_0 = C(\mathscr{O}_\lambda); \tag{8.3}$$

$$A_{1/n} = B(H_{n\lambda}). \tag{8.4}$$

For each $f \in C(\mathscr{O}_\lambda)$ we define $f_\lambda = \pi^* f$ under the canonical projection $\pi : G \to G/G_\theta \cong \mathscr{O}_\lambda$ (i.e., $f_\lambda(x) = f(\pi(x))$), which enables us to define the operators

$$Q_{1/n}(f) = d_{n\lambda} \int_G dx\, f_\lambda(x) |u_{n\lambda}(x)\upsilon_{n\lambda}\rangle\langle u_{n\lambda}(x)\upsilon_{n\lambda}| \in A_{1/n}. \tag{8.5}$$

In fact, the entire integrand in (8.5) is a function on $\mathscr{O}_\lambda$, because for $z \in T$ we have

$$u_{n\lambda}(xz)\upsilon_{n\lambda} = u_{n\lambda}(x)u_{n\lambda}(z)\upsilon_{n\lambda} = \chi_{n\lambda}(z)u_{n\lambda}(x)\upsilon_{n\lambda},$$

and $\chi_{n\lambda}(z) \in \mathbb{T}$ cancels the factor $\overline{\chi_{n\lambda}(z)}$ from the last term in (8.5). Note that

$$Q_{1/n}(1_{\mathscr{O}_\lambda}) = 1_{H_{n\lambda}}, \tag{8.6}$$

as follows by taking $\psi_2 = \psi_3 = \upsilon_{n\lambda}$ in Schur's well-known orthogonality relations

$$d_{n\lambda} \int_G dx\, \langle \psi_1, u_{n\lambda}(x)\psi_2\rangle \langle u_{n\lambda}(x)\psi_3, \psi_4\rangle = \langle \psi_1, \psi_4\rangle \langle \psi_3, \psi_2\rangle \;\; (\psi_i \in H_{n\lambda}). \tag{8.7}$$

Other properties of the maps $Q_{1/n} : C(\mathscr{O}_\lambda) \to B(H_{n\lambda})$ (between C*-algebras) are:

- ***Self-adjointness***, i.e., $Q_{1/n}(f)^* = Q_{1/n}(f^*)$.
- ***Positivity***, i.e., $Q_{1/n}(f) \geq 0$ whenever $f \geq 0$.
- ***Equivariance***, i.e., writing $L_y f(x) = f(y^{-1}x)$ as usual, for any $y \in G$ we have

$$Q_{1/n}(L_y f) = u_{n\lambda}(y) Q_{1/n}(f) u_{n\lambda}(y)^*. \tag{8.8}$$

Positivity does not follows from self-adjointness, as $Q_{1/n}$ is not a homomorphism.

Theorem 8.1. *There exists a continuous bundle of C*-algebras A over I as defined in* (8.2)*, with fibers* (8.3) - (8.4)*, whose continuous sections are given by all sequences* $(a_{1/n})_{n\in\dot{\mathbb{N}}} \in \prod_{n\in\dot{\mathbb{N}}} A_{1/n}$ *for which* $a_0 \in C(\mathscr{O}_\lambda)$ *and* $a_{1/n} \in B(H_{n\lambda})$*, and the sequence* $(a_{1/n})_{n\in\mathbb{N}}$ *is asymptotically equivalent to* $(Q_{1/n}(a_0))_{n\in\mathbb{N}}$*, in the sense that*

$$\lim_{n\to\infty} \|a_{1/n} - Q_{1/n}(a_0)\| = 0. \tag{8.9}$$

In particular, if $f \in C(\mathscr{O}_\lambda)$, then the cross-section of $\prod_{n\in\dot{\mathbb{N}}} A_{1/n}$ defined by

$$a_0 = f; \tag{8.10}$$
$$a_{1/n} = Q_{1/n}(f), \tag{8.11}$$

is continuous. In fact, we have a deformation quantization of $\mathscr{O}_\lambda$ in the sense of Definition 7.1, where the Poisson structure of $\mathscr{O}_\lambda$ is inherited from (minus) the canonical one on the Poisson manifold $\mathfrak{g}^*$, but we shall merely prove the claim of the theorem.

Proof. This will follow from Proposition C.124, in whose notation $\tilde{A}$ (which will actually coincide with A) consists of all $\tilde{a} = (\tilde{a}_\hbar)_{\hbar\in I}$ where f runs through $C(\mathscr{O}_\lambda)$ in

$$\tilde{a}_0 = f; \tag{8.12}$$
$$\tilde{a}_{1/n} = Q_{1/n}(f). \tag{8.13}$$

To verify the conditions for Proposition C.124 we start with the property that the set $\{\tilde{a}_\hbar \mid \tilde{a} \in \tilde{A}\}$ be dense in $A_\hbar$; we will show that it even coincides with $A_\hbar$. At $\hbar = 0$ this is true by construction. At $\hbar = 1/n$, the required property

$$Q_{1/n}(C(\mathscr{O}_\lambda)) = B(H_{n\lambda}) \tag{8.14}$$

can be proved in two steps. For simplicity we set $n = 1$; the proof is the same for any $n \in \mathbb{N}$. The first step is to define a function L_a on G for each $a \in B(H_\lambda)$ by

$$L_a(x) = \mathrm{Tr}\,(a|u_\lambda(x)\upsilon_\lambda\rangle\langle u_\lambda(x)\upsilon_\lambda|) = \langle \upsilon_\lambda, u_\lambda(x)^* a u_\lambda(x)\upsilon_\lambda\rangle. \tag{8.15}$$

This function is continuous and is right-invariant under T, so that L_a is really an element of $C(\mathscr{O}_\lambda)$. Thus we have a map $L : B(H_\lambda) \to C(\mathscr{O}_\lambda)$, $a \mapsto L_a$. Furthermore,

$$\langle a, Q_1(f)\rangle_{HS} = \langle L_a, f\rangle_2, \tag{8.16}$$

where the Hilbert–Schmidt inner product on left-hand side is $\langle a,b\rangle_{HS} = \mathrm{Tr}\,(a^*b)$, cf. (B.495)—which is well defined since H_λ is finite-dimensional—and the right-hand side is the inner product on $L^2(\mathscr{O}_\lambda)$ with respect to the measure induced by the subspace of $L^2(G, d_\lambda \cdot dx)$ consisting of T-invariant functions. Now $Q_{1/n}(C(\mathscr{O}_\lambda))$ is a (necessarily closed) linear subspace of $B(H_\lambda)$, which coincides with $B(H_\lambda)$ iff its orthogonal complement in the Hilbert–Schmidt inner product is zero.

Hence (8.14) is equivalent to the implication: $a \in (Q_{1/n}(C(\mathscr{O}_\lambda)))^\perp \Rightarrow a = 0$. By (8.16), the antecedent holds iff $\langle L_a, f\rangle_2 = 0$ for each $f \in C(\mathscr{O}_\lambda)$, which, because $C(\mathscr{O}_\lambda)$ is dense in $L^2(\mathscr{O}_\lambda)$, holds iff $L_a = 0$. Hence the the above implication is equivalent to: $L_a = 0 \Rightarrow a = 0$, i.e., $\ker L = \{0\}$. We must therefore prove the latter.

If $L_a(x) = 0$ for all $x \in G$, then, taking $x = \exp(t_1A_1)\cdots\exp(t_nA_n)$, where each $A_i \in \mathfrak{g}$, and applying (5.156) for each t_i to the right-hand side of (8.15), we obtain

$$\langle \upsilon_\lambda, [u'_\lambda(A_n), \cdots [u'_\lambda(A_2), [u'_\lambda(A_1), a]] \cdots]\upsilon_\lambda\rangle = 0. \tag{8.17}$$

This equality extends to $\mathfrak{g}_\mathbb{C}$, so we may take $A_i = E_{\alpha_i}$ for some positive root $\alpha_i \in \Delta^+$. Since $u'_\lambda(E_\alpha)\upsilon_\lambda = 0$ for $\alpha \in \Delta^+$, of each commutator $[u'_\lambda(E_{\alpha_i}), a]$ only the term $u'_\lambda(E_{\alpha_i})a$ contributes. Moving the $u'_\lambda(E_{\alpha_i})$ to act as $u'_\lambda(E_{\alpha_i})^* = u'_\lambda(E_{-\alpha_i})$ on the vector on the left in the inner product in (8.17) gives all other eigenvectors of $\mathfrak{t}$, so that (8.17) implies $\langle \psi, a\upsilon_\lambda\rangle = 0$ for each $\psi \in H_\lambda$, and hence $a\upsilon_\lambda = 0$. Now it is clear from (8.15) that $L_{u_\lambda(y)^*au_\lambda(y)}(x) = L_a(yx)$, so if $L_a(x) = 0$ for all $x \in G$, then also $L_{u_\lambda(y)^*au_\lambda(y)}(x) = 0$ for all $x \in G$. Hence we may replace a by $u_\lambda(y)^*au_\lambda(y)$ in the above argument, finding $u_\lambda(y)^*au_\lambda(y)\upsilon_\lambda = 0$ and hence $au_\lambda(y)\upsilon_\lambda = 0$ for each $y \in G$. Since u_λ is irreducible, this implies $a\psi = 0$ for any $\psi \in H_\lambda$, and hence $a = 0$.

This completes the proof of (8.14). Proposition C.124 furthermore requires

$$\lim_{n\to\infty} \|Q_{1/n}(f)\| = \|f\|_\infty, \tag{8.18}$$

This follows from the following key property (to be proved at the end):

$$\lim_{n\to\infty} \langle u_{n\lambda}(y)\upsilon_{n\lambda}, Q_{1/n}(f)u_{n\lambda}(y)\upsilon_{n\lambda}\rangle = f_\lambda(y), \tag{8.19}$$

for any $y \in G$ and $f \in C(\mathscr{O}_\lambda)$. Indeed, for any $y \in G$ we obviously have

$$\|Q_{1/n}(f)\| \geq \langle u_{n\lambda}(y)\upsilon_{n\lambda}, Q_{1/n}(f)u_{n\lambda}(y)\upsilon_{n\lambda}\rangle. \tag{8.20}$$

Since G and hence $\mathscr{O}_\lambda$ is compact, by Weierstrass's Theorem there is an $y \in G$ such that $|f_\lambda(y)| = \|f\|_\infty$. Using this y in (8.20) and (8.19), the two of these imply

$$\lim\inf_{n\to\infty} \|Q_{1/n}(f)\| \geq \|f\|_\infty. \tag{8.21}$$

Conversely, for any unit vector $\psi \in H_{n\lambda}$, eqs. (8.5) and (8.7) imply

$$\langle \psi, Q_{1/n}(f)\psi\rangle = |\langle \psi, Q_{1/n}(f)\psi\rangle| \leq \|f\|_\infty. \tag{8.22}$$

If f is real-valued, then $Q_{1/n}(f)^* = Q_{1/n}(f^*) = Q_{1/n}(f)$. In that case, (8.22) implies

$$\|Q_{1/n}(f)\| \le \|f\|_\infty. \tag{8.23}$$

By the C*-identity $\|a^*a\| = \|a\|^2$, this is true for any $f \in C(\mathscr{O}_\lambda)$. Therefore,

$$\limsup_{n\to\infty} \|Q_{1/n}(f)\| \le \|f\|_\infty. \tag{8.24}$$

Eqs. (8.21) and (8.24) yield (8.18). It remains to prove (8.19), i.e.,

$$\lim_{n\to\infty} d_{n\lambda} \int_G dx\, f_\lambda(x) |\langle u_{n\lambda}(y)\upsilon_{n\lambda}, u_{n\lambda}(x)\upsilon_{n\lambda}\rangle|^2 = f_\lambda(y). \tag{8.25}$$

The key to the proof is the fact that if λ and μ are dominant weights, with associated highest weight representations u_λ and u_μ, respectively, for any $x \in G$ one has

$$\langle \upsilon_\lambda, u_\lambda(x)\upsilon_\lambda\rangle \cdot \langle \upsilon_\mu, u_\mu(x)\upsilon_\mu\rangle = \langle \upsilon_{\lambda+\mu}, u_{\lambda+\mu}(x)\upsilon_{\lambda+\mu}\rangle. \tag{8.26}$$

Namely, because the exponential map is surjective for compact connected Lie groups, eq. (8.26) is equivalent to the property

$$\langle \upsilon_\lambda, u'_\lambda(A)\upsilon_\lambda\rangle + \langle \upsilon_\mu, u'_\mu(A)\upsilon_\mu\rangle = \langle \upsilon_{\lambda+\mu}, u'_{\lambda+\mu}(A)\upsilon_{\lambda+\mu}\rangle, \tag{8.27}$$

for any $A \in \mathfrak{g}$. For $A \in \mathfrak{t}$ this amounts to $\lambda + \mu = \lambda + \mu$, cf. (5.228), whereas for $A = E_\alpha$ for some root $\alpha \in \Delta$ we have $0 = 0$, so that (8.27) is true for all $A \in \mathfrak{g}$. This also proves (8.26), of which we need the special (and iterated) case

$$\langle \upsilon_{n\lambda}, u_{n\lambda}(x)\upsilon_{n\lambda}\rangle = \langle \upsilon_\lambda, u_\lambda(x)\upsilon_\lambda\rangle^n. \tag{8.28}$$

This motivates us to introduce a sequence (μ_n) of probability measures on G by

$$d\mu_n(x) = d_{n\lambda} \cdot dx\, |\langle \upsilon_\lambda, u_\lambda(x)\upsilon_\lambda\rangle|^{2n}, \tag{8.29}$$

so that, after a change $x \mapsto yx$ of the integration variable, eq. (8.25) reads

$$\lim_{n\to\infty} d_{n\lambda} \int_G d\mu_n(x)\, f_\lambda(yx) = f_\lambda(y), \tag{8.30}$$

for any $f \in C(\mathscr{O}_\lambda)$. Now $F(x) = |\langle \upsilon_\lambda, u_\lambda(x)\upsilon_\lambda\rangle|$ takes values in $(0,1]$ and hence the measure (8.29) is $d\mu_n(x) \sim \exp(-nS(x))$ for $S(x) = -\ln(F(x))$, with $S \ge 0$ and $S(x) = 0$ iff $x \in G_{\theta_\lambda} = T$ (using regularity of the orbit). In that case, i.e., if $z \in T$, then $f_\lambda(yz) = f(\pi(yz)) = f(\pi(y)) = f_\lambda(y)$. The method of steepest descent shows that any part of G (of positive Haar measure) where $S(x) > 0$ makes no contribution as $n \to \infty$, so that we may replace $f_\lambda(yx)$ in (8.30) by $f_\lambda(y)$, obtaining

$$\lim_{n\to\infty} \int_G d\mu_n(x)\, f_\lambda(yx) = f_\lambda(y) \lim_{n\to\infty} \int_G d\mu_n(x) = f_\lambda(y) \lim_{n\to\infty} 1 = f_\lambda(y). \tag{8.31}$$

We have now verified conditions 1 and 2 in Proposition C.124, and no. 3 is trivially satisfied since in condition 1 we have equality with $A_\hbar$, as shown above. □

8.2 Large systems

We now move from large quantum numbers within a single system to large quantum systems that consist of N identical sites, where we eventually study what happens as $N \to \infty$ (as is customary in quantum statistical mechanics we change notation from $n \in \mathbb{N}$ to $N \in \mathbb{N}$). This limit gives rise to two different continuous bundles $A^{(q)}$ and $A^{(c)}$ of C*-algebras over I as given by (8.2), which have exactly the same fibers at $1/N$ but, amazingly, differ dramatically at $N = \infty$, i.e., $1/N = 0$. This difference reflects two choices one may make for the N-particle observables that have a limit as $N \to \infty$, namely *local* ones, giving rise to a highly *non-commutative* limit algebra $A_0^{(q)}$ (which is the one usually studied in quantum statistical mechanics of infinite systems), and *macroscopic* ones, which generate a *commutative* algebra $A^{(c)}$ of observables of an infinite quantum system (describing classical thermodynamics as a limit of quantum statistical mechanics). It is the latter that we need for Bohrification.

Let B be a fixed *unital* C*-algebra, describing a single quantum system. The case of a two-level system, where $B = M_2(\mathbb{C})$, is already fascinating, and many other interesting examples are described by finite-dimensional C*-algebras. Though irrelevant in finite dimension, we note that the constructions below are generally valid if (for technical reasons to be found in Proposition C.97) we use the *projective* tensor product $\hat{\otimes}_{\max}$ between C*-algebras; see §C.13. For any $N \in \mathbb{N}$ we put

$$A_{1/N}^{(c)} = A_{1/N}^{(q)} = B^N, \tag{8.32}$$

i.e., the N-fold (projective) tensor product $\hat{\otimes}_{\max}^N B$ of B with itself. Furthermore,

$$A_0^{(c)} = C(S(B)); \tag{8.33}$$

$$A_0^{(q)} = B^\infty, \tag{8.34}$$

where $S(B)$ is the state space of B, seen as a compact convex set in the weak*-topology, as usual, and B^∞ is the infinite (projective) tensor product of B with itself as described in §C.14; see especially (C.318) with $C_i = B$ for each i. For example, the state space of $B = M_2(\mathbb{C})$ is affinely homeomorphic to the unit ball in $\mathbb{R}^3$, whose boundary is the familiar Bloch sphere of qubits; see Proposition 2.9.

We now explain how (8.32) and (8.33) - (8.34) give rise to continuous bundles $A^{(c)}$ and $A^{(q)}$ of C*-algebras, starting with the former. First, for each $N \in \mathbb{N}$, let $\mathfrak{S}_N$ be the permutation group (i.e. symmetric group) on N objects, acting on B^N in the obvious way, i.e., by linear and continuous extension of

$$\alpha_p^{(N)}(b_1 \otimes \cdots \otimes b_N) = b_{p(1)} \otimes \cdots \otimes b_{p(N)}, \tag{8.35}$$

where $b_i \in B$. This yields a ***symmetrization operator*** $S_N : B^N \to B^N$ defined by

$$S_N = \frac{1}{N!} \sum_{p \in \mathfrak{S}_N} \alpha_p^{(N)}. \tag{8.36}$$

If B is infinite-dimensional, these maps can be extended by continuity to the completion $B^{\infty} = \hat{\otimes}_{\max}^{\infty} B$ of the *algebraic* tensor product $\otimes^{\infty} B$; indeed, passing to any faithful representation of B it is easy to see that S^N is even continuous with respect to the minimal cross-norm (cf. §C.13). For $N \geq M$ we then define

$$S_{M,N} : B^M \to B^N \tag{8.37}$$

by linear (and if necessary continuous) extension of

$$S_{M,N}(a_{1/M}) = S_N(a_{1/M} \otimes 1_B \otimes \cdots \otimes 1_B) \;\; (a_{1/M} \in B^M), \tag{8.38}$$

with $N - M$ copies of the unit $1_B \in B$ so as to obtain an element of B^N. Clearly, $S_{N,N} = S_N$. In particular, $S_{1,N} : B \to B^N$ gives the average of b over N copies of B:

$$S_{1,N}(b) = \frac{1}{N} \sum_{k=1}^{N} 1_B \otimes \cdots \otimes b_{(k)} \otimes 1_B \cdots \otimes 1_B,. \tag{8.39}$$

For example, take $B = M_n(\mathbb{C})$ for simplicity, and pick some $a = a^* \in B$ and $\lambda \in \sigma(a)$, with associated spectral projection e_λ. Putting $b = e_\lambda$ in (8.39) gives

$$f_N^{(\lambda)} = S_{1,N}(e_\lambda). \tag{8.40}$$

This is a ***frequency operator***: applied to states of the kind $\upsilon_1 \otimes \cdots \otimes \upsilon_N \in (\mathbb{C}^n)^N$, where each υ_i is an eigenstate of a, so that $a\upsilon_i = \lambda_i \upsilon_i$ for some $\lambda_i \in \sigma(a)$, the corresponding operator counts the relative frequency of λ in the list $(\lambda_1, \ldots, \lambda_N)$. The commutative case $B = C(X)$ provides a classical analogue. Eq. (C.271) gives

$$B^N = C(X)^N \cong C(X^N), \tag{8.41}$$

so that, identifying elements of B^N with functions on X^N, for $f \in C(X)$ we have

$$S_{1,N}(f)(x_1, \ldots, x_N) = \frac{1}{N} \sum_{k=1}^{N} (f(x_1) + \cdots f(x_N)). \tag{8.42}$$

We return to the construction of a continuous bundle of C*-algebras with fibers (8.32) and (8.33). As in §8.1, we construct this bundle by specifying a preliminary family of continuous cross-sections and then using Proposition C.124 to finish.

Definition 8.2. *We say that a sequence $(a_{1/N})_{N \in \mathbb{N}}$, with $a_{1/N} \in B^N$, is* **symmetric** *when there exist $M \in \mathbb{N}$ and $a_{1/M} \in B^M$ such that for each $N \geq M$ one has*

$$a_{1/N} = S_{M,N}(a_{1/M}). \tag{8.43}$$

This implies $a_{1/M} = S_M(a_{1/M})$. Symmetric sequences can start in any finite way they like, but their infinite tails consist of averaged observables. Hence *symmetric sequences asymptotically commute*: if $(a_{1/N})$ and $(b_{1/N})$ are symmetric, then

$$\lim_{N\to\infty} \|a_{1/N}b_{1/N} - b_{1/N}a_{1/N}\|_{B^N} = 0, \tag{8.44}$$

simply because the commutators of single-site operators are nonvanishing only at finitely many positions, upon which the factor $1/N$ in (8.39) guarantees (8.44).

For example, if $B = M_2(\mathbb{C})$, and (σ_i) are the Pauli matrices, we have

$$[S_{1,N}(\tfrac{1}{2}\hbar\sigma_1), S_{1,N}(\tfrac{1}{2}\hbar\sigma_2)] = i\frac{\hbar}{N}S_{1,N}(\tfrac{1}{2}\hbar\sigma_3), \tag{8.45}$$

et cetera, showing that the averaged spin-$\frac{1}{2}$ operators effectively rescale $\hbar$ by $\hbar/N$.

In view of this, it is reasonable to expect that we may be able to assemble the algebra B^N into a continuous bundle whose limit algebra at $N = \infty$ is commutative.

For each symmetric sequence $(a_{1/N})$ we define a function $a_0 : S(B) \to \mathbb{C}$ by

$$a_0(\omega) = \lim_{N\to\infty} \omega^N(a_{1/N}), \tag{8.46}$$

where $\omega \in S(B)$, and $\omega^N \in S(B^N)$ is defined by linear (and continuous) extension of

$$\omega^N(b_1 \otimes \cdots \otimes b_N) = \omega(b_1)\cdots\omega(b_N); \tag{8.47}$$

continuity of ω^N on the algebraic tensor product $\otimes^N B$ (and hence extendibility to $A_{1/N}$) is guaranteed by Proposition C.98, although this is not really needed here because a_0 only requires the values of ω^N on $\otimes^N B$ itself. In any case, the limit exists by definition of a symmetric sequence, from which we also see that $a_0 \in C(S(B))$, because it is a finite sum of finite products of the type $\omega(b_1)\cdots\omega(b_M)$, each of which is continuous in ω by definition of the w^*-topology on $S(B)$.

For example, the frequency operators (8.40) define a symmetric sequence $(f_N^\lambda)_{N\in\mathbb{N}}$, whose the limit function $f_0^\lambda : S(B) \to \mathbb{C}$ in the sense of (C.560) or (8.46) is

$$f_0^\lambda(\omega) = \omega(e_\lambda). \tag{8.48}$$

Thus (8.46) gives the Born probability for the outcome $a = \lambda$ in the state ω; see §8.4. Classically, identifying elements of $S(C(X))$ with probability measures μ on X, the limit of the sequence $a_{1/N} = S_{1,N}(f)$ for fixed $f \in C(X)$, cf. (8.42), is

$$a_0(\mu) = \int_X d\mu\, f. \tag{8.49}$$

This convergence is an example of the strong law of large numbers, see §8.3.

We return to the general case.

Definition 8.3. *A sequence $(a_{1/N})_{N\in\mathbb{N}}$ as above is* **quasi-symmetric** *if for each $N \in \mathbb{N}$ one has $a_{1/N} = S_N(a_{1/N})$ and for any $\varepsilon > 0$ there is a symmetric sequence $(\tilde{a}_{1/N})$ and some $M \in \mathbb{N}$ such that $\|a_{1/N} - \tilde{a}_{1/N}\| < \varepsilon$ for all $N > M$.*

For example, if $\lim_{N\to\infty} \|a_{1/N} - \tilde{a}_{1/N}\| = 0$ for some fixed symmetric sequence $(\tilde{a}_{1/N})$, then $(a_{1/N})_{N\in\mathbb{N}}$ is obviously quasi-symmetric.

Theorem 8.4. *For any unital C*-algebra B, the C*-algebras* (8.32) *and* (8.33)*, i.e.,*

$$A_0^{(c)} = C(S(B)); \tag{8.50}$$

$$A_{1/N}^{(c)} = B^N, \tag{8.51}$$

where B^N *is N-fold projective tensor power* $\hat{\otimes}_{\max}^N B$*, are the fibers of a continuous bundle* $A^{(c)}$ *of C*-algebras over* $I = (1/\mathbb{N}) \cup \{0\} \equiv 1/\dot{\mathbb{N}}$ *whose continuous cross-sections are the quasi-symmetric sequences* $(a_{1/N})$ *with limit* a_0 *given by* (8.46)*.*

As in Theorem 8.1, also here we have a deformation quantization of $S(B)$ in the sense of Definition 7.1, where the Poisson bracket on $S(B)$ may be defined by specifying its value on linear function $\hat{b} \in C(S(B))$, where $b \in B$ and $\hat{b}(\omega) = \omega(b)$, by

$$\{\hat{a}, \hat{b}\} = i\widehat{[a,b]}. \tag{8.52}$$

Unfortunately, this involves the theory of infinite-dimensional Poisson manifolds, which we prefer to omit. Thus we shall only prove Theorem 8.4 as stated.

The proof relies on Størmer's ***quantum De Finetti Theorem*** 8.6 below.

Definition 8.5. *Let B be a unital C*-algebra. A state* ρ *on* B^N *is called:*

- **permutation-invariant** *if* $\rho \circ \alpha_p^{(N)} = \rho$ *for any* $p \in \mathfrak{S}_N$.
- K**-exchangeable** *(*$K \in \mathbb{N}$*) if it is permutation-invariant and in addition* ρ *is the restriction to* B^N *of some permutation-invariant state on* B^{N+K}.
- **Infinitely exchangeable** *if it is K-exchangeable for all* $K \in \mathbb{N}$.

The set of all permutation-invariant states / K-exchangeable states / infinitely exchangeable states on B^N *is denoted by* $S^{\mathfrak{S}_N}(B^N)$ / $S_K^{\mathfrak{S}_N}(B^N)$ / $S_\infty^{\mathfrak{S}_N}(B^N)$.

Theorem 8.6. *Let B be a unital C*-algebra. For any* $N \in \mathbb{N}$ *the correspondence* $\omega^N \leftrightarrow \omega$*, where* $\omega \in S(B)$ *and* $\omega^N \in S(B^N)$*, cf.* (8.47)*, gives a bijection*

$$\partial_e S_\infty^{\mathfrak{S}_N}(B^N) \cong S(B). \tag{8.53}$$

This theorem was originally stated (in the language of infinite tensor products) as Theorem 8.9 in §8.3, where it (and hence Theorem 8.6) will also be proved.

We also need a formula for the norm of any self-adjoint element a of any C*-algebra A in terms of the state space A and the pure state space $P(A)$, viz.

$$\|a\| = \sup\{|\omega(a)| : \omega \in S(A)\} = \sup\{|\omega(a)|, \omega \in P(A)\}. \tag{8.54}$$

This follows from Proposition C.15, the spectral radius formula (B.254), and compactness of $\sigma(a)$, implying that the supremum in (B.254) is reached on $\sigma(a)$.

Proof. The proof of Theorem 8.4 is quite similar to the proof of Theorem 8.1, in that we once again rely on Proposition C.124, where the symmetric sequences are going to play the role of $\tilde{A}$. To apply Proposition C.124, we should prove that:

1. The set $\tilde{A}_0$ (consisting of all $\tilde{a}_0 \in A_0 = C(S(B))$ as defined by (8.46), where $(\tilde{a}_{1/N})$ runs through all symmetric sequences) is a *-algebra which is dense in A_0.
2. For any symmetric sequence $(\tilde{a}_{1/N})$ with limit $\tilde{a}_0$ as given by (8.46), one has

$$\lim_{N\to\infty} \|\tilde{a}_{1/N}\| = \|\tilde{a}_0\|_\infty. \tag{8.55}$$

To prove the first claim, we first note that $\tilde{a}_0$ is the linear span of all finite products $\omega(b_1)\cdot\omega(b_N)$, where $N \in \mathbb{N}$ and $b_1,\ldots,b_N \in B$. Since $\overline{\omega(b)} = \omega(b^*)$ this is obviously a *-algebra. The monomials $\hat{b}(\omega) = \omega(b)$ already separate points of $S(B) \subset B^*$, since if $\omega' \neq \omega$ then clearly is there some $b \in B$ for which $(\omega - \omega')(b) \neq 0$. Hence claim no. 1 follows from the Stone–Weierstrass Theorem B.51.

For the second, let $(\tilde{a}_{1/N})$ be a symmetric sequence. Since there are $M \in \mathbb{N}$ and $\tilde{a}_{1/M} \in B^M$ with $\tilde{a}_{1/M} = S_M(\tilde{a}_{1/M})$ and $\tilde{a}_{1/(M+K)} = S_{M,M+K}(\tilde{a}_{1/M})$ for all $K \in \mathbb{N}$,

$$\begin{aligned}
\|\tilde{a}_{1/M}\| &= \sup\{|\rho(\tilde{a}_{1/M})| : \rho \in S(B^M)\} = \sup\{|\rho(\tilde{a}_{1/M})| : \rho \in S^{\mathfrak{S}_M}(B^M)\};\\
\|\tilde{a}_{1/(M+K)}\| &= \sup\{|\rho(S_{M,M+K}(\tilde{a}_M))| : \rho \in S^{\mathfrak{S}_{M+K}}(B^{M+K})\}\\
&= \sup\{|\rho(\tilde{a}_{1/M})| : \rho \in S_K^{\mathfrak{S}_M}(B^M)\},
\end{aligned}$$

where we used (8.54) and (8.43). Theorem 8.6 and (8.46) then yield (8.55):

$$\begin{aligned}
\lim_{N\to\infty} \|\tilde{a}_{1/N}\| &= \lim_{K\to\infty} \|\tilde{a}_{1/(M+K)}\|\\
&= \sup\{|\rho(\tilde{a}_{1/M})| : \rho \in S_\infty^{\mathfrak{S}_M}(B^M)\}\\
&= \sup\{|\rho(\tilde{a}_{1/M})| : \rho \in \partial_e S_\infty^{\mathfrak{S}_M}(B^M)\} = \sup\{|\omega^M(\tilde{a}_{1/M})| : \omega \in S(B)\}\\
&= \sup\{|\lim_{N\to\infty} \omega^N(\tilde{a}_{1/N})| : \omega \in S(B)\} = \sup\{|\tilde{a}_0(\omega)| : \omega \in S(B)\}\\
&= \|\tilde{a}_0\|_\infty
\end{aligned}$$

The proof that the sequences $(a_{1/N})$ for which condition (C.552) in Proposition C.124 holds are precisely the approximately symmetric sequences is the same as the proof of the equivalence of the two conditions in Lemma C.125, taking $\hbar_0 = 0$.

Finally, it is easy to show that the limit (8.46) exists also for quasi-symmetric observables a: take $\varepsilon > 0$ and find $\tilde{a}$ and M as in Definition 8.3. For this $\tilde{a}$, let M_0 be such that (8.43) holds (with $M \rightsquigarrow M_0$). For all N, N' greater than both M and M_0,

$$\begin{aligned}
|\omega^N(a_{1/N}) - \omega^{N'}(a_{1/N'})| &\leq |\omega^N(a_{1/N} - \tilde{a}_{1/N}) - \omega^{N'}(a_{1/N'} - \tilde{a}_{1/N'})|\\
&\quad + |\omega^N(\tilde{a}_{1/N}) - \omega^{N'}(\tilde{a}_{1/N'})|\\
&\leq \|a_{1/N} - \tilde{a}_{1/N}\| + \|a_{1/N'} - \tilde{a}_{1/N'})\| + 0\\
&< 2\varepsilon,
\end{aligned} \tag{8.56}$$

since $\|\omega^N\| = 1$. Hence $(\omega^N(a_{1/N}))$ is a Cauchy sequence (in $\mathbb{C}$). □

Our second continuous bundle of C*-algebras of interest is described by the following changes in Definitions 8.2 and 8.3.

Definition 8.7. *Let B be a unital C*-algebra and let $a_{1/N} \in B^N$ for each $N \in \mathbb{N}$.*

- *A sequence $(a_{1/N})_{N\in\mathbb{N}}$ is called* **local** *when there exist $M \in \mathbb{N}$ and $a_{1/M} \in B^M$ such that for each $N \geq M$ one has*

$$a_{1/N} = a_{1/M} \otimes 1_B \otimes \cdots \otimes 1_B, \tag{8.57}$$

with $N-M$ copies of the unit $1_B \in B$ (so that indeed $a_{1/N} \in B^N$).
- *A sequence $(a_{1/N})_{N\in\mathbb{N}}$ is* **quasi-local** *if for any $\varepsilon > 0$ there is a local sequence $(\tilde{a}_{1/N})$ and some $M \in \mathbb{N}$ such that $\|a_{1/N} - \tilde{a}_{1/N}\| < \varepsilon$ for all $N > M$.*

For the right analogue of Theorem 8.4 we recall the description of the infinite tensor product B^∞; cf. §C.14, especially the explanation preceding (C.315). Accordingly, a dense subspace of B^∞ is given by equivalence classes of local sequences $(a_{1/N})_{N\in\mathbb{N}}$ under the equivalence relation $a \sim a'$ iff $\lim_{N\to\infty} \|a_{1/N} - a'_{1/N}\| = 0$; the C*-algebraic operations in B^∞ are inherited from the B^N, and if we denote the equivalence class of $(a_{1/N})_N$ by $[a_{1/N}]_N$, the norm in B^∞ is given by

$$\|[a_{1/N}]_N\| = \lim_{N\to\infty} \|a_{1/N}\|. \tag{8.58}$$

By construction, this number is independent of the representative $(a_{1/N})_N$ in the class $[a_{1/N}]_N$. By definition, B^∞ is the completion of the space of these equivalence classes in the norm (8.58). As explained after (C.315), for each $M \in \mathbb{N}$ we have an injective (and hence isometric) homomorphism $\varphi_M : B^M \to B^\infty$ that maps $a_{1/M} \in B^M$ to the equivalence class $[a_{1/N}]_N$ of the sequence $(a_{1/N})_N$ defined by

$$a_{1/N} = 0, \ (N < M); \tag{8.59}$$

$$a_{1/N} = a_{1/M}, \ (N = M); \tag{8.60}$$

$$a_{1/(M+K)} = a_{1/M} \otimes 1_B \otimes \cdots \otimes 1_B, \ (K > 0), \tag{8.61}$$

with K copies of 1_B. It is easy to verify that one might as well have started from quasi-local sequences and their equivalence classes, for which the limit (8.58) exists by an argument similar to (8.56). In that case the ensuing C*-algebra is already complete, which leads to a direct description of the elements of B^∞ as equivalence classes of quasi-local sequences. This fact also follows from the following analogue of Theorem 8.4, which may be proved in the same way, i.e., from Proposition C.124, where this time the elements of $\tilde{A}$ are local sequences rather than symmetric ones (in fact, the proof is much easier, since this time we obtain (C.552) for free):

Theorem 8.8. *For any unital C*-algebra B, the C*-algebras* (8.32) *and* (8.34)*, i.e.,*

$$A_0^{(q)} = B^\infty; \tag{8.62}$$

$$A_{1/N}^{(q)} = B^N, \tag{8.63}$$

are the fibers of a continuous bundle $A^{(q)}$ of C-algebras over $I = 1/\dot{\mathbb{N}}$ whose continuous cross-sections are the quasi-local sequences $(a_{1/N})$ with limit $a_0 = [a_{1/N}]_N$.*

8.3 Quantum de Finetti Theorem

As an initial step in exploring the connection between the bundles $A^{(c)}$ and $A^{(q)}$ we prove Theorem 8.6, which we first restate in an equivalent form. Let $\mathfrak{S}_\infty$ be the group of bijections of $\mathbb{N}$ that differ from the identity only on a finite set. Each such finite permutation $p \in \mathfrak{S}_\infty$ defines a map $\alpha_p : B^\infty \to B^\infty$, as follows. Let $S \subset \mathbb{N}$ be the finite subset of $\mathbb{N}$ on which p acts nontrivially (if $S = \emptyset$ we have $p = \mathrm{id}_{\mathbb{N}}$, in which case also $\alpha_p = \mathrm{id}_{B^\infty}$, see below). Take a local sequence $(a_{1/N})_N$, so that (8.57) holds, in which we may assume $M \geq \max S$; we also redefine $a_{1/N} = 0$ for each $N < M$. For each $N \geq M \geq \max S$, the map p may be regarded as an element p_N of $\mathfrak{S}_N$ by restriction to $\{1,\ldots,N\} \subset \mathbb{N}$ and hence p acts on B^N by permuting the entries in elementary tensor products of operators, cf. (8.35). For each $p \in \mathfrak{S}_\infty$, define a map

$$\alpha_p : B^\infty \to B^\infty; \tag{8.64}$$

$$\alpha_p([a_{1/N}]_N) = [\alpha_p^{(N)}(a_{1/N})]_N. \tag{8.65}$$

This uses a specific representative of the equivalence class $[a_{1/N}]_N \in B^\infty$, but nonetheless the map α_p is well defined. Furthermore, since each $\alpha_p^{(N)} : B^N \to B^N$ is an automorphism (i.e., an invertible homomorphism), it is an isometry, so that also α_p is an isometry on its domain and hence extends to an automorphism of B^∞. The ensuing map $p \mapsto \alpha_p$ from $\mathfrak{S}_\infty$ to the group $\mathrm{Aut}(B^\infty)$ of all automorphisms of B^∞ is a homomorphism of groups, and we say that $\mathfrak{S}_\infty$ is an *automorphism group* of B^∞.

Writing $S^{\mathfrak{S}_\infty}(B^\infty)$ for the set of all $\mathfrak{S}_\infty$-invariant states on B^∞, i.e., $\rho \in S^{\mathfrak{S}_\infty}(B^\infty)$ iff $\rho \circ \alpha_p = \rho$ for each $p \in \mathfrak{S}_\infty$, we may now rephrase Theorem 8.6 as follows:

Theorem 8.9. *Let B be a unital C^*-algebra. There is a bijection*

$$\partial_e S^{\mathfrak{S}_\infty}(B^\infty) \cong S(B), \tag{8.66}$$

given by $\omega^\infty \leftrightarrow \omega$, where $\omega \in S(B)$, and $\omega^\infty \in S(B^\infty)$ is defined by, cf. (8.47),

$$\omega^\infty([a_{1/N}]_N) = \lim_{N\to\infty} \omega^N(a_{1/N}). \tag{8.67}$$

This is essentially the same as Theorem 8.6: for any $M \in \mathbb{N}$, a state on B^M is infinitely exchangeable iff it is the restriction of an element of $S^{\mathfrak{S}_\infty}(B^\infty)$ to $B^M \subset B^\infty$, where the inclusion is given by the map φ_M defined below (8.58).

Proof. Let $S(B) \subset S^{\mathfrak{S}_\infty}(B^\infty)$ under the map $\omega \mapsto \omega^\infty$. We first show the inclusion

$$\partial_e S^{\mathfrak{S}_\infty}(B^\infty) \subseteq S(B) \tag{8.68}$$

contrapositively, i.e., if $\rho \in S^{\mathfrak{S}_\infty}(B^\infty)$ does not lie in $S(B)$, then ρ has a nontrivial convex decomposition in $S^{\mathfrak{S}_\infty}(B^\infty)$. We identify B^N with $\varphi_N(B^N) \subset B^\infty$ and denote the restriction of ρ to B^N by ρ_N. If $\rho = \omega^\infty$ for some $\omega \in S(B)$, then

$$\rho_{M+K}(a'_{1/M} \otimes a'_{1/K}) = \rho_M(a'_{1/M})\rho_K(a'_{1/K}), \tag{8.69}$$

for each $a'_{1/M} \in B^M$ and $a'_{1/K} \in B^K$. If (8.69) holds whenever $0 \leq a'_{1/M} \leq 1_{B^M}$, then by Lemma C.53 and (C.8) it always holds. Adding suitable multiples of the unit and rescaling, it follows that if (8.69) holds whenever

$$\tfrac{1}{3} \cdot 1_{B^M} \leq a'_{1/M} \leq \tfrac{2}{3} \cdot 1_{B^M}; \tag{8.70}$$

then it always holds. Therefore, if (8.69) fails, then it fails for some $a'_{1/M}$ satisfying (8.70) and some and $a'_{1/K}$, in which case $\frac{1}{3} \leq \rho_M(a'_{1/M}) \leq \frac{2}{3}$. However, such a failure implies the existence of a nontrivial convex decomposition

$$\rho = t\rho' + (1-t)\rho'', \tag{8.71}$$

with $t = \rho_M(a'_{1/M})$, and the functionals ρ' and ρ'' on B^∞ are defined by

$$\rho'([a_{1/N}]_N) = \lim_{N\to\infty} \rho_{M+N}(a'_{1/M} \otimes a_{1/N})/\rho_M(a'_{1/M}); \tag{8.72}$$

$$\rho''([a_{1/N}]_N) = \lim_{N\to\infty} \rho_{M+N}((1_{B^N} - a'_{1/M}) \otimes a_{1/N})/\rho_M(1_{B^M} - a'_{1/M}). \tag{8.73}$$

These limits exist on symmetric sequences (where they stabilize), and hence they exists in general. Furthermore, since $\rho_M(1_{B^M} - a'_{1/M}) = 1 - t$, the property (8.71) is obvious. Both ρ' and ρ'' belong to $S^{\mathfrak{S}_\infty}(B^\infty)$, since each functional ρ_{M+N} is an element of $S^{\mathfrak{S}_{M+N}}(B^{M_N})$. Finally, (8.71) is nontrivial, since if $\rho' = \rho''$, then $\rho'_K = \rho''_K$, and hence (8.69) would hold (whose violation we assumed). This proves (8.68).

Though it is always true, for simplicity we prove the converse inclusion

$$S(B) \subseteq \partial_e S^{\mathfrak{S}_\infty}(B^\infty) \tag{8.74}$$

just for the case where B is generated by projections, as in the case $B = M_n(\mathbb{C})$, $B = B(H)$, or B a von Neumann algebra, or more generally an AW*-algebra (see §C.24). In that case also each B^N is generated by its projections.

For each $\rho \in S^{\mathfrak{S}_\infty}(B^\infty)$, each $N \in \mathbb{N}$, and each projection $e \in B^N$, we have

$$\rho_N(e)^2 \leq \rho_{2N}(e \otimes e), \tag{8.75}$$

see below. Assuming (8.75), suppose $\omega \in S(B)$ and $\omega^\infty = t\rho' + (1-t)\rho''$ for some $t \in (0,1)$ and $\rho', \rho'' \in S^{\mathfrak{S}_\infty}(B^\infty)$. Since $\omega^\infty_N = \omega^N$, we then have

$$\begin{aligned}
\omega^N(e)^2 &= (t\rho'_N(e) + (1-t)\rho''_N(e))^2 = \left\langle \begin{pmatrix} \sqrt{t} \\ \sqrt{1-t} \end{pmatrix}, \begin{pmatrix} \rho'_N(e)\sqrt{t} \\ \rho''_N(e)\sqrt{1-t} \end{pmatrix} \right\rangle^2 \\
&\leq \left\langle \begin{pmatrix} \sqrt{t} \\ \sqrt{1-t} \end{pmatrix}, \begin{pmatrix} \sqrt{t} \\ \sqrt{1-t} \end{pmatrix} \right\rangle \cdot \left\langle \begin{pmatrix} \rho'_N(e)\sqrt{t} \\ \rho''_N(e)\sqrt{1-t} \end{pmatrix}, \begin{pmatrix} \rho'_N(e)\sqrt{t} \\ \rho''_N(e)\sqrt{1-t} \end{pmatrix} \right\rangle \\
&= t\rho'_N(e)^2 + (1-t)\rho''_N(e)^2 \\
&\leq t\rho'_{2N}(e \otimes e) + (1-t)\rho''_{2N}(e \otimes e) \\
&= \omega^{2N}(e \otimes e) = \omega^N(e)^2,
\end{aligned}$$

where the inner product in the first line is the usual one in $\mathbb{R}^2$, and, noting it is positive, we have used the Cauchy–Schwarz inequality for this inner product, as well as (8.75). Hence both inequalities must be equalities, and for the first one this implies $\rho'_N(e) = \rho''_N(e)$. Since this is true for all N and all projections in B^N, this implies $\rho' = \rho'' = \omega^\infty$, so that $\omega^\infty \in \partial_e S^{\mathfrak{S}_\infty}(B^\infty)$, and (8.74) has been established, up to the proof of (8.75). To this effect, note for each $M \in \mathbb{N}$ and $t \in \mathbb{R}$ we have

$$\rho_{MN}((1_{B^N} \otimes \cdots \otimes 1_{B^N} \otimes e + \cdots + e \otimes 1_{B^N} \otimes \cdots \otimes 1_{B^N} + t \cdot 1_{B^{MN}})^2) \quad (8.76)$$
$$= M(M-1)\rho_{2N}(e \otimes e) + M\rho_N(e) + 2tM\rho_N(e) + t^2, \quad (8.77)$$

with $M-1$ copies of 1_{B^N} and e moving from right to left in the first line, leaving M terms before the final one $t \cdot 1_{B^{MN}}$ in (8.76). In working out the square in (8.76) and moving to the second line we used $e^2 = e$ as wel as permutation invariance of the state ρ_{MN}. The point is that (8.76) is positive, so that (8.77) must be positive, too, for all $M \in \mathbb{N}$ and $t \in \mathbb{R}$. Now a function $f(t) = t^2 + 2bt + c = (t+b)^2 - b^2 + c$ obviously satisfies $f(t) \geq 0$ for each t iff $b^2 \leq c$, so that (8.76) is positive for all t iff

$$M^2\rho_N(e)^2 \leq M(M-1)\rho_{2N}(e \otimes e) + M\rho_N(e).$$

Letting $M \to \infty$ gives (8.75). □

Taking $B = C(X)$ for some compact Hausdorff space X, in view of (8.41) the situation may be transferred to the Cartesian product X^N, equipped with the product topology (which is generated by products $A_1 \times \cdots \times A_N \subset X^N$ with each $A_i \subset X$ open) and the ensuing Borel σ-algebra (generated by the above products with each A_i Borel). If $\mu_1, \ldots, \mu_N$ are (probability) measures on X (in which case we write $\mu_i \in \Pr(X)$), then there is a unique (probability) measure $\mu_1 \times \cdots \times \mu_N$ whose value on a product as above is equal to $\mu_1(A_1) \cdots \mu_N(A_N)$. In particular, any probability measure $\mu \in \Pr(X)$ on X defines a probability measure μ^N on X^N.

The symmetric group $\mathfrak{S}_N$ acts on X^N in the obvious way, and hence its acts on the power set $\mathscr{P}(X^N)$. We call the latter action $\sigma^{(N)}$, so that for $p \in \mathfrak{S}_N$ we have

$$\sigma_p^{(N)}(A_1 \times \cdots \times A_N) = A_{p(1)} \times \cdots \times A_{p(N)}. \quad (8.78)$$

The Cartesian product $X^\infty \equiv X^{\mathbb{N}}$ is well defined both topologically and measure-theoretically (the topology is generated by all products $\prod_i A_i$ with finitely many A_i open and different from X, and likewise for the Borel structure), and the infinite symmetric group $\mathfrak{S}_\infty = \cup_N \mathfrak{S}_N$ acts on it in the obvious way, in that $p \in \mathfrak{S}_N \subset \mathfrak{S}_\infty$ permutes the first N coordinates. Specializing Definition 8.5 to $B = C(X)$, we obtain:

Definition 8.10. *A probability measure ν_N on X^N is called:*

- **permutation-invariant** *if $\nu_N \circ \sigma_p^{(N)} = \nu_N$ for any $p \in \mathfrak{S}_N$.*
- ***K*-exchangeable** *($K \in \mathbb{N}$) if it is permutation-invariant and in addition ν_N is the restriction to B^N of some permutation-invariant probability measure on X^{N+K}.*
- **exchangeable** *if it is K-exchangeable for all $K \in \mathbb{N}$.*

A probability measure ν_∞ on X^∞ is called **permutation-invariant** *if $\nu_\infty \circ \sigma_p^{(N)} = \nu_\infty$ for any $p \in \mathfrak{S}_N$ and $N \in \mathbb{N}$, where $\sigma_p^{(N)}$ acts on $\prod_i A_i$ by* (8.78) *on the first N factors $A_1, \ldots, A_N$ whilst acting trivally on all remaining A_i's.*

The connection between the two parts of this definition is that ν_N is exchangeable iff it is the restriction to X^N of some permutation-invariant measure ν_∞ on X^∞.

From Theorems 8.6 and 8.3 we obtain the ***Hewitt–Savage Theorem***:

Corollary 8.11. *Let X be a compact Hausdorff space. For any $N \in \mathbb{N}$, any infinitely exchangeable probability measure ν_N on X^N takes the form*

$$\nu_N = \int_{\mathrm{Pr}(X)} dP(\mu)\, \mu^N \tag{8.79}$$

for some probability measure P on $\mathrm{Pr}(X)$ *that is uniquely determined by ν_N, and similarly for $N = \infty$, where ν_∞ is a permutation-invariant probability measure.*

The two claims in the theorem are equivalent by the remark after Definition 8.10.

The probability measure $P \in \mathrm{Pr}(\mathrm{Pr}(X))$ has the following interpretation. For $N \in \mathbb{N}$ and $(x_1, \ldots, x_N) \in X^N$, define the so-called ***empirical measure*** $E_N^{(x_1,\ldots,x_N)}$ on X as

$$E_N^{(x_1,\ldots,x_N)} = \frac{1}{N} \sum_{i=1}^N \delta_{x_i}, \tag{8.80}$$

where δ_x is the Dirac measure on X. Seen as a map on $C(X)$, this is the same as

$$\int_X dE_N^{(x_1,\ldots,x_N)} f = \frac{1}{N} \sum_{i=1}^N f(x_i). \tag{8.81}$$

Given a probability measure ν_N on X^N, these formulae give a random probability measure on X depending on a drawing from X^N, i.e., a map

$$E_N : X^N \to \mathrm{Pr}(X); \tag{8.82}$$

$$(x_1, \ldots, x_N) \mapsto E_N^{(x_1,\ldots,x_N)}. \tag{8.83}$$

Proposition 8.12. *The probability measure P in Corollary 8.11 is given by*

$$\lim_{N\to\infty} \int_{\mathrm{Pr}(X)} dP_N\, F = \int_{\mathrm{Pr}(X)} dP\, F, \tag{8.84}$$

for each $F \in C(\mathrm{Pr}(X))$ (that is, $P = \lim_{N\to\infty} P_N$ weakly), where $P_N \in \mathrm{Pr}(\mathrm{Pr}(X))$ is the probability measure on $\mathrm{Pr}(X)$ *defined by $\nu_N \in \mathrm{Pr}(X^N)$ and* (8.82) - (8.83)*, i.e.,*

$$P_N(A) = \nu_N(E_N^{-1}(A)) \;\; (A \subset \mathrm{Pr}(X)). \tag{8.85}$$

Proof. By the Stone–Weierstrass Theorem it suffices to prove (8.84) for linear combinations of monomials like $F(\mu) = \mu(f_1) \cdots \mu(f_K)$, where $f_1, \ldots, f_K \in C(X)$ are arbitrary and $\mu(f) = \int_X d\mu\, f$. This is a simple computation: using (8.85), we have

$$\begin{aligned}\int_{\Pr(X)} dP_N F &= \int_{X^N} d\nu_N(x_1,\ldots,x_N)\, F(E_N^{(x_1,\ldots,x_N)})\\ &= \int_{X^N} d\nu_N(x_1,\ldots,x_N) \prod_{j=1}^{K} \left(\frac{1}{N}\sum_{i=1}^{N} f_j(x_i)\right)\\ &= \int_{\Pr(X)} dP(\mu) \int_{X^N} d\mu^N(x_1,\ldots,x_N) \prod_{j=1}^{K} \left(\frac{1}{N}\sum_{i=1}^{N} f_j(x_i)\right),\end{aligned}$$

where in the third step we used (8.79). The result follows, since clearly

$$\lim_{N\to\infty} \int_{\Pr(X)} dP(\mu) \int_{X^N} d\mu^N(x_1,\ldots,x_N) \prod_{j=1}^{K} \left(\frac{1}{N}\sum_{i=1}^{N} f_j(x_i)\right) =$$
$$\int_{\Pr(X)} dP(\mu) \int_X d\mu(x_1)\, f_1(x_1)\cdots \int_X d\mu(x_K)\, f_k(x_K) = \int_{\Pr(X)} dP F. \quad \square$$

We can also say more about the limit of the sum (8.81), So far, we have been dealing with the Borel σ-algebras $\mathscr{B}_N \subset \mathscr{P}(X^N)$ and $\mathscr{B}_\infty \subset \mathscr{P}(X^\infty)$ generated by the topology (i.e., by the open sets). On top of this, consider $\mathscr{S}_N \subset \mathscr{B}_N$, defined as the σ-algebra generated by the permutation-invariant Borel subsets of X^N, or, equivalently, as the smallest σ-algebra for which the permutation-invariant Borel measurable functions on X^N are measurable. Likewise, $\mathscr{S}_\infty \subset \mathscr{B}_\infty$; regarding $A \subset X^N$ as a subset $A \times \prod_{K>N} X$ of X^∞, we have $\mathscr{S}_\infty = \cap_{N\in\mathbb{N}} \mathscr{S}_N$. For any permutation-invariant probability measure ν_N on X^N, the Hilbert space $L^2(X, \mathscr{S}_N, \nu_N)$ is a closed subspace of $L^2(X^N, \mathscr{B}_N, \nu_N)$, and the associated conditional expectation

$$E_{(\mathscr{S}_N,\nu_N)} : L^2(X^N, \mathscr{B}_N, \nu_N) \to L^2(X, \mathscr{S}_N, \nu_N) \tag{8.86}$$

is defined as the corresponding orthogonal projection. Since $C(X^N) \subset L^2(X^N)$, this map restricts to $C(X^N)$. Similarly for $N = \infty$. For each $N \in \mathbb{N}$, and also for $N = \infty$, we may regard $f \in C(X)$ as a function f_K on X^N through

$$f_K(x_1,\ldots,x_N) = f(x_K) \;\; K = 1,\ldots,N. \tag{8.87}$$

Proposition 8.13. *Let ν_∞ be a permutation-invariant probability measure on X^∞, with restriction ν_N to X^N. Recall* (8.42). *For any $f \in C(X)$ we have* pointwise*:*

$$S_{1,N}(f) = E_{(\mathscr{S}_N,\nu_N)}(f_1), \;\; \nu_N\text{-almost surely}; \tag{8.88}$$
$$\lim_{N\to\infty} S_{1,N}(f) = E_{(\mathscr{S}_\infty,\nu_\infty)}(f_1), \;\; \nu_\infty\text{-almost surely}, \tag{8.89}$$

where the left-hand sides of (8.88) *and* (8.89) *are functions on X^N and X^∞, respectively. Furthermore, if $\nu_\infty = \mu^\infty$ for some $\mu \in$* Pr(X)*, then pointwise on X^∞,*

$$\lim_{N\to\infty} S_{1,N}(f) = \int_X d\mu\, f, \;\; \mu^\infty\text{-almost surely } (f \in C(X)). \tag{8.90}$$

Equivalently, if $L_\mu \subset X^\infty$ is the set of infinite sequences $(x_1, x_2, \ldots)$ in X^∞ for which the limit in (8.90) exists for each $f \in C(X)$ and equals $\int_X d\mu\, f$, then

$$\mu^\infty(L_\mu) = 1. \tag{8.91}$$

Proof. Eq. (8.88) is almost trivial, since $S_{1,N}(f)$ is permutation invariant and hence already lies in $L^2(X, \mathscr{S}_N, \nu_N)$, so that the equality just expresses the projection property $E^2_{(\mathscr{S}_N,\nu_N)} = E_{(\mathscr{S}_N,\nu_N)}$. Eq. (8.89) follows from the ergodic theorem, applied to the probability space $(X^\infty, \mathscr{B}_\infty, \nu_\infty)$, the ***unilateral shift***

$$T : (x_1, x_2, \ldots) \mapsto (x_2, x_3, \ldots),$$

and the random variable f_1 defined by $f \in C(X)$ via (8.87). Since ν_∞ is permutation invariant, it is also T-invariant (in the sense that $\nu_\infty(T^{-1}(A)) = \nu_\infty(A)$ for any $A \subset B_\infty$). This follows either directly, where one has to realize firstly that

$$T^{-1}(A_1 \times A_2 \times \cdots A_n \times \cdots) = X \times A_1 \times A_2 \times \cdots \times \cdots A_n \times \cdots,$$

and secondly that $\mathscr{B}_\infty$ is generated by products $\prod_i A_i$ with finitely many A_i different from X, or, more easily, from Corollary 8.11. The (pointwise) ergodic theorem gives

$$\lim_{N\to\infty} S_{1,N}(f) = E_{(\mathscr{B}_T,\nu_\infty)}(f_1), \;\; \nu_\infty\text{-almost surely } (f \in C(X)), \tag{8.92}$$

where $\mathscr{B}_T$ is the σ-algebra within $\mathscr{B}_\infty$ by the T-invariant sets, and $f_1 \in C(X^\infty)$ is still defined by (8.87). Since $\mathscr{S}_\infty \subset \mathscr{B}_S$ and the left-hand side of (8.89) is $\mathscr{S}_\infty$-measurable (provided it exists, as we have just shown), eq. (8.89) follows from (8.92).

If $\nu_\infty = \mu^\infty$, then the unilateral shift on X^∞ is ergodic by Kolmogorov's 0–1 law, and hence the ergodic theorem gives (8.90). Alternatively, if $\nu_\infty = \mu^\infty$, then the random variables (f_N), defined by (8.87) with $N = \infty$, are i.i.d. (i.e., independent and identically distributed) and (8.90) follows from the strong law of large numbers (which, coherently, in turn may be derived from the ergodic theorem!). □

Note that (8.92) has been proved for $f \in C(X)$, but it holds for many other functions, including $f = 1_A$, where $A \in \mathscr{B}$. This gives ***Borel's law of large numbers***

$$\lim_{N\to\infty} S_{1,N}(1_A) = \mu(A), \;\; \mu^\infty\text{-almost surely.} \tag{8.93}$$

For example, take $X = \{0, 1\}$ (e.g., a coin toss with outcomes $1 =$ heads and $0 =$ tails). With $f(x) = x$ in (8.90) or $A = \{1\}$ in (8.93), writing $p = \mu(\{1\})$, we obtain

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} x_i = p, \;\; \mu^\infty\text{-almost surely on } \underline{2}^{\mathbb{N}}. \tag{8.94}$$

Equivalently, if $L_p \subset \underline{2}^{\mathbb{N}}$ is the set of infinite binary sequences $x_1 x_2 \cdots$ for which the limit in (8.94) exists and equals p, then $\mu^\infty(L_p) = 1$, cf. (8.91).

8.4 Frequency interpretation of probability and Born rule

Results like (8.90), (8.93), and (8.94) give a relationship between the single-case probabilities $\mu(A)$ or p and the limits of long series of trials on samples drawn according to μ or p. Despite the seemingly comforting appearance of $N < \infty$ on the left-hand side, this relationship depends in an essential way on the infinite idealization X^∞, which is strictly necessary in order to be able to say that the limit (8.94) holds almost surely relative to the measure μ^∞. This violates Earman's Principle (cf. the Introduction), which is the reason why we prefer the limit (8.49) over (8.93).

Although these results are mathematically equivalent, both formalizing the idea that if $(x_1,\ldots,x_N)$ are sampled from X according to some probability measure μ, then $(1/N)\sum_{i=1}^N f(x_i)$ converges to $\int_X d\mu\, f$ as $N\to\infty$, in (8.49) we never need to work with the "actual infinity" $N=\infty$ and (8.49) holds everywhere on $\mathrm{Pr}(X)$ rather than almost everywhere on X^∞. One reason for this is that in (8.93) etc. the choice of the sampling measure μ has to be made at the beginning, whereas in (8.49) it only comes in at the very end. But it has to made either way, and similarly for any other serious effort to relate probability to frequencies in long runs of measurements.

The extreme delicacy of such efforts is clear from the fact that limiting results like (8.90), (8.93), and (8.94) are insensitive to any finite part of the sum, whereas any practical use of probability only involves finite trials. As Lord Keynes once said:

> 'In the long run we are all dead.'

The founder of the mathematical theory of probability expressed himself likewise:

> 'The frequency concept based on the notion of limiting frequency as the number of trials increased to infinity, does not contribute anything to substantiate the applicability of the results of probability theory to real practical problems where we have always to deal with a finite number of trials.' (Kolmogorov).

Moreover, a *definition* of probability based on e.g. (8.93) is well known to be circular: although superficially the "almost sure" terminology in the statement of the result might instill confidence in the reader, in fact it is an exceptionally strong constraint on the sequences $(x_n)\in X^\infty$ in question that the limit should exist *and* has the right value $\mu(A)$, i.e., that $(x)\in L_\mu$, cf. (8.91), and we see that this constraint can only be formulated if the single-case probability μ was already defined in the first place. This shows that the link between probability and frequencies of outcomes of long runs of trials only exists and makes sense if single-case probabilities are prior.

On the other hand, if single-case probabilities are "objective", as those provided by the Born measure in quantum mechanics ought to be at least in remotely realistic interpretations of the theory (as opposed to "personal" or "subjective" probabilities construed as "degrees of belief" or "rationality constraints" or whatever other decision-theoretic concept in human psychology), then it is hard to say what they really mean, since it is precisely about single cases that they do not seem to say anything. This brings us to what we propose to call the ***Paradox of Probability***:

Although single-case probabilities must be logically prior to probabilities construed as frequencies, the numerical values of the former have no bearing on single trials and can only be validated through their predictions about (finite) frequencies.

This paradox imposes the following consistency requirement (which philosophers may want to compare with Lewis's "Principal Principle" that regulates credences):

The assumption that a single-case probability measure be μ must imply that the probabilities for the various outcomes of long runs of repetitions of identical experiments (provided these are possible) are distributed according to μ.

This describes the relationship between theoretical and experimental physics quite well, but still leaves us in the dark as to the meaning of single-case probabilities!

We are now ready to revisit the Born rule, which we already discussed from a purely mathematical point of view in §§§2.1, 2.5, and 4.1. To repeat the main point, if $a = a^* \in B(H)$ is a bounded self-adjoint operator on a Hilbert space, with spectrum $\sigma(a)$, then any state ω on $B(H)$ defines a unique probability measure μ_ω on $\sigma(a) \subset \mathbb{R}$, called the ***Born measure***, such that

$$\omega(f(a)) = \int_{\sigma(a)} d\mu_\omega f, \;\; f \in C(\sigma(a)), \tag{8.95}$$

where $f(a) \in C^*(a) \subset B(H)$ is defined through the continuous functional calculus (Theorem 4.3). For example, for $f = \mathrm{id}_{\sigma(a)}$, i.e., the function $x \mapsto x$, eq. (8.95) yields

$$\omega(a) = \int_{\sigma(a)} d\mu_\omega(\lambda)\,\lambda. \tag{8.96}$$

The point of this construction of the Born measure is that it is obtained by simply restricting the state ω, initially defined on $B(H)$, to its commutative C*-subalgebra $C^*(a)$. If, in the spirit of (exact) Bohrification, such commutative algebras are identified with corners of classical physics within quantum theory, one may argue that Heisenberg gave the right picture of the origin of probability in quantum mechanics:

> 'One may call these uncertainties objective, in that they are simply a consequence of the fact that we describe the experiment in terms of classical physics; they do not depend in detail on the observer. One may call them subjective, in that they reflect our incomplete knowledge of the world.' (Heisenberg, 1958, pp. 53–54)

See, however, §11.1. In any case, there are extensions of this construction to unbounded self-adjoint operators as well as to families of commuting self-adjoint operators, to which the following discussion applies, too, *mutatis mutandis*.

The ***Born rule*** relates the Born measure for a to measurements of a and as such is responsible for most predictions of quantum physics, especially in quantum field theory, where the connection between theory and experiment mainly involves the measurement of cross-sections computed from the Born measure via Feynman rules. The Born rule and the Heisenberg uncertainty relations are often seen as a turning point where indeterminism entered fundamental physics. Nonetheless, it is hard to say what this Born rule actually states! We made a first attempt in §4.1:

If an observable a is measured in a state ω, then the probability $P_\omega(a \in A)$ that the outcome lies in some measurable subset $A \subseteq \sigma(a) \subset \mathbb{R}$ is given by

$$P_\omega(a \in A) = \mu_\omega(A). \tag{8.97}$$

Two questions immediately arise:

1. What is meant by a "measurement" of a (and by its "outcome")?
2. What does the "probability" $P_\omega(a \in A)$ mean?

Perhaps these are even the main questions in the foundations of quantum mechanics. The first will be taken up in Chapter 11; for now, we simply assume that measurements of quantum-mechanical observables a are defined and have outcomes in $\sigma(a)$. The second has just been answered (or some might say evaded): through the Born measure, the formalism of quantum mechanics provides numerical values of $\mu_\omega(A)$, whose *mathematical* meaning seems unquestionable, and whose *operational* meaning is given by the predictions they give for outcomes of long runs of repetitions of identical experiments. Therefore, all that remains to be done is derive these predictions by analogy with the results in §8.3 for the commutative C*-algebra $C(X)$.

One such attempt is—in its strengths and its weaknesses—quite analogous to the Borel's law of large numbers (8.93). Although we will soon move to $B = B(H)$, the following result is valid for any unital C*-algebra B, with infinite tensor product B^∞ as defined in §C.14 and recalled at the end of §8.2, including the map $\varphi_M : B^M \to B^\infty$.

Proposition 8.14. *If $\omega \in S(B)$, there is a unique state ω^∞ on B^∞ such that*

$$\omega^\infty(\varphi_M(b_1 \otimes \cdots \otimes b_M)) = \prod_{n=1}^{M} \omega(b_n), \;\; M \in \mathbb{N}, b_1, \ldots, b_M \in B. \tag{8.98}$$

Moreover, ω^∞ is pure iff ω is pure.

This is a special case of Proposition C.105, with $C_i = B$ and $\omega_i = \omega$ for all $i \in \mathbb{N}$.

We now take $B = B(H)$ for some separable Hilbert space H, some observable $a = a^* \in B(H)$ with spectrum $\sigma(a) \subset \mathbb{R}$, and some unit vector $\upsilon \in H$, with associated (normal) pure state ω_υ in $B(H)$ defined by $\omega_\upsilon(b) = \langle \upsilon, b\upsilon \rangle$, and Born measure $\mu_{\omega_\upsilon} \equiv \mu_\upsilon$ on $\sigma(a)$. Now take the corresponding pure state ω_υ^∞ on $B(H)^\infty$ and construct the associated GNS-representation $\pi_{\omega_\upsilon^\infty}(B(H)^\infty)$. The Hilbert space $H_{\omega_\upsilon^\infty}$ carrying this representation is an example of an ***infinite tensor product of Hilbert spaces*** in the sense of von Neumann, which may also be defined directly, as follows.

Take sequences $(\psi_n) \equiv (\psi_1, \psi_2, \ldots)$ with $\psi_n \in H$ satisfying the condition

$$\sum_n |\|\psi_n\| - 1| < \infty; \tag{8.99}$$

the rationale behind this condition is that for any sequence (z_n) of complex numbers, the product $\prod_n z_n$ converges *and* has a nonzero limit iff $\sum_n |z_n - 1| < \infty$, so (8.99) is equivalent to the requirement that $\prod_n \|\psi_n\|$ converges to some nonzero value. Following von Neumann, we now introduce the convention that if, for some sequence (z_n) of complex numbers, $\prod_n |z_n|$ converges but $\prod_n z_n$ does not, we define the latter to be zero. On this convention, linear and continuous extension of the expression

$$\langle (\psi_n), (\psi'_n) \rangle = \prod_n \langle \psi_n, \psi'_n \rangle_H, \tag{8.100}$$

defines an inner product on the finite linear span H_0^∞ of all sequences (ψ_n) satisfying (8.99); the ***complete tensor product*** H^∞ is defined as the closure of H_0^∞ in the ensuing norm. However, this is not the Hilbert space of interest, since it is far too large (e.g., it is not separable even if H is). To define interesting separable subspaces of H^∞, we call sequences (ψ_n) and (ψ'_n) that both satisfy (8.99) ***equivalent*** if

$$\sum_n |\langle \psi_n, \psi'_n \rangle - 1| < \infty; \tag{8.101}$$

this turns out to be a *bona fide* equivalence relation. In particular, if (ψ_n) and (ψ'_n) are *in*equivalent, then $\langle (\psi_n), (\psi'_n)\rangle = 0$. For any unit vector $\upsilon \in H$, we now define the ***incomplete tensor product*** H_υ^∞ as the closure of the linear span of all sequences (ψ_n) that satisfy (8.99) and are equivalent to υ^∞ (i.e., the sequence (ψ'_n) with $\psi'_n = \upsilon$ for each n), with inner product borrowed from H^∞ (note that von Neumann's terminology "incomplete" is somewhat confusing, since H_υ^∞ is complete as a normed vector space and in particular it is a Hilbert space). By construction, $\upsilon^\infty \in H_\upsilon^\infty$, and it is easy to show that H_υ^∞ is the closed linear span of all sequences (ψ_n) that differ from $\upsilon \in H$ in at most finitely many places. We often write $\otimes_n \psi_n$ or $\psi_1 \otimes \psi_2 \otimes \cdots$ for (ψ_n). Furthermore, for any $M \in \mathbb{N}$, any $b \in B(H)$ defines a bounded operator $b_\upsilon^{(M)}$ on H_υ^∞ by continuous linear extension of

$$b_\upsilon^{(M)}(\psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_M \otimes \cdots) = \psi_1 \otimes \psi_2 \otimes \cdots \otimes b\psi_M \otimes \cdots . \tag{8.102}$$

This extends to a representation π_υ^∞ of B^∞ on H_υ^∞, as follows. Define $b^{(M)} \in B^\infty$ by

$$b^{(M)} = \varphi_M(1_H \otimes \cdots \otimes 1_H \otimes b), \tag{8.103}$$

in which $1_H \otimes \cdots \otimes 1_H \otimes b \in B^M$, and $\varphi_M : B^M \to B^\infty$ was defined after (8.58). In other words, for $b \in B(H)$, the operator $b^{(M)}$ is the element of B^∞ given by the equivalence class $[a_{1/N}]_N$ of the sequence $(a_{1/N})_N$ with 1_B in every place except $a_{1/M} = b$. We then define $\pi_\upsilon^\infty(B^\infty)$ by linear and continuous extension of

$$\pi_\upsilon^\infty(b_1^{(M_1)} \cdots b_N^{(M_N)}) = b_{1\upsilon}^{(M_1)} \cdots b_{N\upsilon}^{(M_N)}. \tag{8.104}$$

Proposition 8.15. *For any unit vector $\upsilon \in H$, the* GNS*-representation $\pi_{\omega_\upsilon^\infty}(B^\infty)$ on $H_{\omega_\upsilon^\infty}$ is unitarily equivalent with $\pi_\upsilon^\infty(B^\infty)$ on H_υ^∞, under which equivalence the cyclic vector $\Omega_{\omega_\upsilon^\infty} \in H_{\omega_\upsilon^\infty}$ corresponds with $\upsilon^\infty \in H_\upsilon^\infty$.*

Proof. This is a simple consequence of Proposition C.91 and the equality

$$\omega_\upsilon^\infty(a) = \langle \upsilon^\infty, \pi_\upsilon^\infty(a)\upsilon^\infty \rangle_{H_\upsilon^\infty}, \tag{8.105}$$

initially for $a = b^{(M)}$, subsequently for $a = b_1^{(M_1)} \cdots b_N^{(M_N)}$, and finally, by linearity and continuity, for any $a \in B^\infty$. □

In view of this, we will henceforth identify the two Hilbert spaces etc., so that:

$$H_{\omega_\upsilon^\infty} = H_\upsilon^\infty; \tag{8.106}$$

$$\pi_{\omega_\upsilon^\infty}(b^{(M)}) = b_\upsilon^{(M)}; \tag{8.107}$$

$$\Omega_{\omega_\upsilon^\infty} = \upsilon^\infty. \tag{8.108}$$

Recall that $\mathscr{P}(H)$ is the set of all projections on H, seen as a lattice ordered by $e \leq f$ iff $ef = e$, which is equivalent to $eH \subseteq fH$, and coincides with the order in $B(H)_{\mathrm{sa}}$, cf. Proposition C.170. Also, $\mathscr{B}$ is the Boolean lattice of Borel subsets of $\sigma(a)$, ordered by inclusion. For each Borel set $A \subset \sigma(A)$ we have an associated spectral projection $e_A \in \mathscr{P}(H)$, and the map $A \mapsto e_A$ defined by the Borel functional calculus, i.e., Theorem B.102, is a lattice homomorphism from $\mathscr{B}$ to $\mathscr{P}(H)$. This follows because from the perspective of the Borel functional calculus the map $A \mapsto e_A$ is really the map $1_A \mapsto e_A$, which is the restriction of a homomorphism between C*-algebras and hence preserves positivity. Let $\mathscr{B}^\infty$ be the Boolean lattice of Borel sets $\mathscr{B}^\infty$ in $\sigma(a)^\infty$. As above, take some unit vector $\upsilon \in H$, with corresponding vector state ω_υ on $B(H)$ and associated state ω_υ^∞ on $B(H)^\infty$ as defined in Proposition 8.14, which in turn defines the GNS-representation $\pi_{\omega_\upsilon^\infty}$ of $B(H)^\infty$ on the Hilbert space $H_{\omega_\upsilon^\infty}$. The lattice homomorphism $A \mapsto e_A$ then extends to a homomorphism

$$e^\infty : \mathscr{B}^\infty \to \mathscr{P}(H_{\omega_\upsilon^\infty}); \tag{8.109}$$

$$A_1 \times \cdots \times A_M \times \prod_{M+1}^{\infty} \sigma(a) \mapsto \pi_{\omega_\upsilon^\infty}(e_{A_1}^{(1)} \cdots e_{A_M}^{(M)}); \tag{8.110}$$

this defines e^∞ on the basis Borel sets in $\sigma(a)^\infty$ and extends to all of $\mathscr{B}^\infty$. Realizing $H_{\omega_\upsilon^\infty}$ as the infinite tensor product H_υ^∞, cf. (8.106) - (8.108), we rewrite this as

$$e^\infty\left(A_1 \times \cdots \times A_M \times \prod_{M+1}^{\infty} \sigma(a)\right) = e_{A_1\upsilon}^{(1)} \cdots e_{A_M\upsilon}^{(M)}. \tag{8.111}$$

Theorem 8.16. *Let $a = a^* \in B(H)$, let μ_υ be the Born measure on $\sigma(a)$ defined by some unit vector $\upsilon \in H$, and define e^∞ by* (8.111)*. Let $\sigma(a)_\upsilon^\infty$ be the set of all points in $\sigma(a)^\infty$ for which* (8.92)*, or, equivalently,* (8.93) *holds (with $\mu \rightsquigarrow \mu_\upsilon$). Then*

$$e^\infty(\sigma(a)_\upsilon^\infty) = 1_{H_{\omega_\upsilon^\infty}}. \tag{8.112}$$

Furthermore, if $A \subseteq \sigma(a)$ is Borel measurable, then, using the notation (8.39)*,*

$$\lim_{N\to\infty} S_{1,N}(e_A) = \mu_\upsilon(A) \cdot 1_{H_{\omega_\upsilon^\infty}}, \tag{8.113}$$

in the strong operator topology (i.e., applied to each fixed vector in $H_{\omega_\upsilon^\infty}$).

This is the ***quantum-mechanical law of strong numbers***, plus its Borel version. In comparison, the strong law of large numbers or Borel's law of large numbers gives

$$\mu_\upsilon^\infty(\sigma(a)_\upsilon^\infty) = 1. \tag{8.114}$$

Proof. For any probability measure μ on any σ-finite compact space X, the corresponding probability measure μ^∞ on X^∞ is characterized by the property

$$\mu^\infty\left(A_1\times\cdots\times A_M\times A\times\prod_{M+2}^{\infty}\sigma(a)\right)=\mu(A)\mu^\infty\left(A_1\times\cdots\times A_M\times\prod_{M+1}^{\infty}\sigma(a)\right),$$

for any $M\in\mathbb{N}$ and Borel sets $A_i\subseteq X$. The measure ν on $\sigma(a)^\infty$ defined by

$$\nu\left(A_1\times\cdots\times A_M\times\prod_{M+1}^{\infty}\sigma(a)\right)=\omega_\upsilon^\infty\left(e_{A_1}^{(1)}\cdots e_{A_M}^{(M)}\right) \tag{8.115}$$

satisfies the above property for $\mu=\mu_\upsilon$ and hence coincides with μ_υ. In view of this, eqs. (C.196) and (8.114) give

$$\langle\Omega_{\omega_\upsilon^\infty}, e^\infty(\sigma(a)_\upsilon^\infty)\Omega_{\omega_\upsilon^\infty}\rangle=1. \tag{8.116}$$

For any projection e' and any unit vector $\psi'\in H'$ in any Hilbert space H', the properties $\langle\psi',e'\psi'\rangle=1$, $\|e'\psi'\|=1$, and $e'\psi'=\psi'$ are equivalent. Therefore,

$$e^\infty(\sigma(a)_\upsilon^\infty)\Omega_{\omega_\upsilon^\infty}=\Omega_{\omega_\upsilon^\infty}. \tag{8.117}$$

Consider a vector $\otimes_n\psi_n\in H_\upsilon^\infty$, where only $\psi_1,\ldots,\psi_K$ possibly differ from υ ($K<\infty$). Noting that by (8.106) - (8.107) the right-hand side of (8.115) may be written as

$$\begin{aligned}\omega_\upsilon^\infty\left(e_{A_1}^{(1)}\cdots e_{A_M}^{(M)}\right)&=\langle\Omega_{\omega_\upsilon^\infty},\pi_{\omega_\upsilon^\infty}\left(e_{A_1}^{(1)}\cdots e_{A_M}^{(M)}\right)\Omega_{\omega_\upsilon^\infty}\rangle\\&=\langle\upsilon^\infty,(e_{A_1\upsilon}^{(1)}\otimes\cdots\otimes e_{A_M\upsilon}^{(M)})\upsilon^\infty\rangle,\end{aligned} \tag{8.118}$$

we modify (8.115) so as to define a new measure ν' on $\sigma(a)^\infty$ by

$$\nu'\left(A_1\times\cdots\times A_M\times\prod_{M+1}^{\infty}\sigma(a)\right)=\langle\otimes_n\psi_n,(e_{A_1\upsilon}^{(1)}\otimes\cdots\otimes e_{A_M\upsilon}^{(M)})\otimes_n\psi_n\rangle.$$

Generalizing the above case of μ^∞, the measure $\nu''=\mu_{\psi_1}\times\cdots\times\mu_{\psi_K}\times\prod_{K+1}^\infty\mu_\upsilon$ on σ^∞ is characterized by the following two properties:

$$\nu''\left(A_1\times\cdots\times A_K\times\prod_{K+1}^{\infty}\sigma(a)\right)=\mu_{\psi_1}(A_1)\cdots\mu_{\psi_K}(A_K); \tag{8.119}$$

$$\nu''\left(A_1\times\cdots\times A_M\times A\times\prod_{M+2}^{\infty}\sigma(a)\right)=\mu_\upsilon(A)\nu''\left(A_1\times\cdots\times A_M\times\prod_{M+1}^{\infty}\sigma(a)\right),\quad (M>K), \tag{8.120}$$

and hence $\nu'=\nu''$. Therefore, even though $\nu'\neq\mu_\upsilon^\infty$, we have $\nu'(\sigma(a)_\upsilon^\infty)=1$, since membership of $\sigma(a)_\upsilon^\infty$ is entirely defined by the tail of the event. Hence we obtain

$$e^{\infty}(\sigma(a)^{\infty}_{\upsilon}) \otimes_n \psi_n = \otimes_n \psi_n, \tag{8.121}$$

by the same reasoning as for $\upsilon^{\infty} \equiv \Omega_{\omega^{\infty}_{\upsilon}}$. Since the linear span of such vectors is dense in $H^{\infty}_{\upsilon} \equiv H_{\omega^{\infty}_{\upsilon}}$ and the projection $e^{\infty}(\sigma(a)^{\infty}_{\upsilon})$ is bounded, we obtain (8.112).

To derive (8.113), we use the definition of the Born measure μ_{υ} to find

$$\|(S_{1,N}(e_A) - \mu_{\upsilon}(A))\upsilon^{\infty}\| = \frac{1}{N}(\mu_{\upsilon}(A) - 2\mu_{\upsilon}(A)^2), \tag{8.122}$$

which vanishes as $N \to \infty$, so that (8.113) holds on υ^{∞}. A similar computation proves (8.113) on vectors $\otimes_n \psi_n$ as above, since the initial K terms where possibly $\psi_n \neq \upsilon$ drop out in the limit $N \to \infty$. Thus we have (8.113) on a dense subspace of $H_{\omega^{\infty}_{\upsilon}}$. Since the strong limit operator $\mu_{\upsilon}(A) \cdot 1_{H_{\omega^{\infty}_{\upsilon}}}$ is bounded, this proves (8.113). □

An alternative argument shows the mere existence of the limit on the left-hand side of (8.113) on the same dense set, upon which the limit operator is seen to commute with all local and hence (by norm-continuity) with all quasi-local operators. Since ω_{υ} is pure, so is $\omega^{\infty}_{\upsilon}$, and hence $\pi_{\omega^{\infty}_{\upsilon}}$ is irreducible. Thus the limit is a multiple of the unit, and the coefficient $\mu_{\upsilon}(A)$ then follows from the computation

$$\lim_{N\to\infty} \langle \upsilon^{\infty}, S_{1,N}(e_A)\upsilon^{\infty}\rangle = \mu_{\upsilon}(A). \tag{8.123}$$

To reduce the level of abstraction and since it is an important case, we now specialize Theorem 8.16 to a two-level system, i.e., $B = M_2(\mathbb{C})$. In other words, we take $H = \mathbb{C}^2$, and pick a simple observable $a = \mathrm{diag}(1,0)$ with non-degenerate spectrum $\sigma(a) = \underline{2} = \{0,1\}$, so that measurements outcomes are just strings of zero's and one's. Furthermore, we take a unit vector $\upsilon = c_0|0\rangle + c_1|1\rangle$, where $|0\rangle = (1,0)$ and $|1\rangle = (0,1)$ form the standard basis of $\mathbb{C}^2$, and $|c_0|^2 + |c_1|^2 = 1$. We write $p = |c_1|^2$. The Born measure μ_{υ} on $\sigma(a) = \{0,1\}$ is then given by $\mu_{\upsilon}(\{1\}) = p$ and $\mu_{\upsilon}(\{0\}) = 1-p$; cf. (2.10) - (2.11). Taking $A = \{1\}$, we have $e_A = |1\rangle\langle 1|$. The Hilbert space $(\mathbb{C}^2)^{\infty}_{\upsilon}$ is the closure of the finite linear span of vectors of the kind $\psi_1 \otimes \psi_2 \cdots$ with $\psi_n \in \mathbb{C}^2$ and only finitely many ψ_n possibly different from υ. For $M \in \mathbb{N}$, the operator$|1\rangle\langle 1|^{(M)})$ sends such a vector to $\psi_1 \otimes \psi_2 \cdots \otimes (|1\rangle\langle 1|\psi_M) \otimes \cdots$, with all ψ_n unaffected except for $n = M$. Eqs. (8.112) - (8.113) then simply read

$$e^{\infty}(\underline{2}^{\infty}_p) = 1_{(\mathbb{C}^2)^{\infty}_{\upsilon}}; \tag{8.124}$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{M=1}^{N} (|1\rangle\langle 1|^{(M)}) = p \cdot 1_{(\mathbb{C}^2)^{\infty}_{\upsilon}}, \tag{8.125}$$

where $\underline{2}^{\infty}_p$ denotes the set of all infinite binary strings $x_1x_2\cdots$ for which $x_i \in \underline{2}$ and

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} x_1 = p, \tag{8.126}$$

and once again the limit in (8.125) is meant strongly, i.e., the expression on the left-hand side must be applied to a fixed vector in $(\mathbb{C}^2)^{\infty}_{\upsilon}$.

Theorem 8.16 forms the (mathematical) culmination of attempts that started in 1960s to derive the Born rule from other postulates of quantum mechanics, notably the so-called ***eigenvalue-eigenvector link***, according to which a quantum-mechanical observable has a definite value if and only if the current quantum state is an eigenvector of the associated operator. This link is applied to the state υ^∞ (or to any other state with approximately the same tail) and the operators $e^\infty(\sigma(a)_\upsilon^\infty)$ and $\lim_{N\to\infty} S_{1,N}(e_A)$. The idea, then, is that according to (8.112), the property expressed by the projection $e^\infty(\sigma(a)_\upsilon^\infty)$ is certain in the state υ^∞ (for qubits this means that any possible infinite string of binary measurement outcomes has average value p). This is reinforced by (8.113), which states that the frequency operator for the outcome A has a sharp limit equal to $\mu(A)$ (for qubits, with $A = \{1\}$ this limit is p).

However, although the mathematics is suggestive, apart from the fact that the eigenvalue-eigenvector link itself falls prey to Earman's Principle (in that sharp eigenvalues and eigenvectors are an idealization in a world full of continuous spectra), this particular application of the link makes sense only *at* $N = \infty$. In this respect, eq. (8.124) has the same drawback as the strong law of large numbers (on which its derivation indeed relies), including the fact that attempts to define probabilities through (8.113) or its special case (8.125) are inherently circular. Moreover, υ^∞ fails to be an eigenvector of any finite-N approximant to (8.125), and by the same token, the limit operator defined by (8.125) can only be measured via its individual contributions $|1\rangle\langle 1|^{(M)}$, none of which has υ^∞ as an eigenvector; in fact, it can be shown that any joint eigenvector of all projections $|1\rangle\langle 1|^{(M)}$ is orthogonal to the entire space $(\mathbb{C}^2)_\upsilon^\infty$ with the complete infinite tensor product $(\mathbb{C}^2)^\infty$.

Problems with Earman's Principle are avoided if we use Theorem 8.4 (applied to $B = B(H)$) rather than Theorem 8.16: the sequence of operators $S_{1,N}(e_A)$ forms a continuous section of the continuous bundle of C*-algebras with fibers (8.50) - (8.51), whose limit at $N = \infty$, in the sense of (8.46) or (C.560), is given by

$$S_{1,\infty}(e_A) : \omega \mapsto \omega(A); \tag{8.127}$$

recall that $S_{1,\infty}(e_A) \in C(S(B(H)))$. In particular, for pure states $\omega = \omega_\upsilon$ we obtain the Born probability $\mu_\upsilon(A)$. As we have also seen in the commutative case, this limit avoids infinite idealizations and other problems with the law of large numbers.

From the point of view of (asymptotic) Bohrification, $C(S(B(H)))$ provides a classical description of a long run of identical experiments, which becomes increasingly accurate as $N \to \infty$; this is the whole point of the limits (8.46) and (C.560). In particular, the unsound eigenvalue-eigenvector link has been replaced by the role of points $\omega \in S(B(H))$ as truthmakers, which is uncontroversial in classical physics. If the quantum state in each identical experiment on the given (single) system is ω, then the above derivation shows that in the limit $N \to \infty$, this state acquires a classical meaning (which according to Bohr would even be the *only* meaning it has), namely as the point in the "classical phase space" $S(B(H))$ that gives the relative frequencies of outcomes of the given long runs of identical experiments. Short of deriving the Born rule, this at least provides the reasoning that links the Born measure (which is canonically given by the theory) to experiment.

8.5 Quantum spin systems: Quasi-local C*-algebras

Beside the Born rule, our second application of the previous formalism is to ***quantum spin systems***, especially to spontaneous symmetry breaking (SSB), see Chapter 10. Postponing a conceptual discussion of infinite systems in their role of idealizations of finite systems to the preamble of that chapter, for the moment we just describe infinite quantum spin systems mathematically. As in §C.14, we take a Hilbert space H, here assumed *finite-dimensional*, i.e., $H \cong \mathbb{C}^n$, and use the standard lattice $\mathbb{Z}^d \subset \mathbb{R}^d$ in dimension d. For any *finite* subset $\Lambda \subset \mathbb{Z}^d$, i.e., $\Lambda \in \mathscr{P}_f(\mathbb{Z}^d)$, we put

$$H_\Lambda = \otimes_{x\in\Lambda} H_x; \tag{8.128}$$

$$A_\Lambda = B(H_\Lambda) \cong \otimes_{x\in\Lambda} B(H_x), \tag{8.129}$$

where $H_x = H$ for each $x \in \Lambda$, cf. (C.297) and (C.303). The symbolic notations

$$A = \otimes_{x\in\mathbb{Z}^d} B(H) = \underline{\lim}_\Lambda A_\Lambda = \overline{\bigcup_{\Lambda\in\mathscr{P}_f(\mathbb{Z}^d)} A_\Lambda}^{\|\cdot\|}, \tag{8.130}$$

all come down to the same thing—see §C.14, notably (C.323) and (C.317)—and define a ***quasi-local C*-algebra***. Elements of each $A_\Lambda \subset A$ are called ***local observables***, those in the closure of their union are referred to as ***quasi-local observables***.

Eq. (8.129) defines a map $\Lambda \mapsto A_\Lambda$, which has three important properties:

$$A_{\Lambda^{(1)}} \subseteq A_{\Lambda^{(2)}} \text{ if } \Lambda^{(1)} \subseteq \Lambda^{(2)} \textbf{ (Isotony)}; \tag{8.131}$$

$$[A_{\Lambda^{(1)}}, A_{\Lambda^{(2)}}] = 0 \text{ if } \Lambda^{(1)} \cap \Lambda^{(2)} = \emptyset \textbf{ (Einstein locality)}; \tag{8.132}$$

$$A'_\Lambda = A_{\Lambda'} \textbf{ (Haag duality)}, \tag{8.133}$$

where A'_Λ in (8.133) is the commutant of A_Λ within A, and, in cute notation, we put $\Lambda' = \mathbb{Z}^d \backslash \Lambda$ (which is infinite), so that the right-hand side of (8.133) denotes

$$A_{\Lambda'} = \otimes_{x\in\Lambda'} B(H) = \overline{\bigcup_{\Lambda^{(1)}\in\mathscr{P}_f(\mathbb{Z}^d\backslash\Lambda)} A_{\Lambda^{(1)}}}^{\|\cdot\|}, \tag{8.134}$$

which is a C*-subalgebra of A. Since $\Lambda^{(2)} \subset \mathbb{Z}^d \backslash \Lambda^{(1)}$ whenever $\Lambda^{(1)} \cap \Lambda^{(2)} = \emptyset$, Haag duality implies Einstein locality (and sharpens it), but it is still worth mentioning these properties separately: although in quantum spin systems (8.133)—and hence (8.132)—holds, Einstein locality is a more fundamental property (e.g. it is also valid in algebraic quantum field theory, where Haag duality may well fail).

We now discuss some C*-algebraic concepts that will be needed for the analysis of SSB. Through the associated GNS-representation $\pi_\omega : A \to B(H_\omega)$, any state ω on A defines two interesting subalgebras of $B(H_\omega)$, which *a priori* may be different:

- The ***center*** $A^c_\omega = \pi_\omega(A)'' \cap \pi_\omega(A)'$;
- The ***algebra at infinity*** $A^\infty_\omega = \bigcap_{\Lambda\in\mathscr{P}_f(\mathbb{Z}^d)} \pi_\omega(A_{\Lambda'})''$.

Recall that the center of a von Neumann algebra $M \subset B(H)$ is $M \cap M'$, and that M is called a factor if $M \cap M = \mathbb{C} \cdot 1$ (cf. §C.21), so A_ω^c is the center of the von Neumann algebra $\pi_\omega(A)''$. It is easy to show from Einstein locality that $A_\omega^\infty \subseteq A_\omega^c$. If each local algebra A_Λ is simple, Haag duality yields the opposite inclusion, so in that case,

$$A_\omega^\infty = A_\omega^c. \tag{8.135}$$

Given (8.129), this applies as long as $\dim(H) < \infty$, in which case also A is simple.

The algebra at infinity provides a new perspective on the macroscopic observables in §8.2. Averages like $|\Lambda|^{-1} \sum_{x\in\Lambda} b(x)$, where $b \in B(H)$, do not have a limit in A as $\Lambda \uparrow \mathbb{Z}^d$, but (depending on ω) their representatives $|\Lambda|^{-1} \sum_{x\in\Lambda} \pi_\omega(b(x))$ may have a weak limit in $B(H_\omega)$. If they do, Einstein locality implies that the limit operator lies in algebra at infinity A_ω^∞ (and hence, assuming (8.135), in A_ω^c). If the algebra of infinity is trivial (i.e. $\mathbb{C} \cdot 1_{H_\omega}$), macroscopic observables are therefore "c-numbers", i.e., multiples of the unit operator. In particular, they do not fluctuate, which is among the defining properties of *pure* thermodynamic phases. Formally, this idea is captured by the following generalization of the notion of a pure state:

Definition 8.17. *A representation $\pi(A)$ is* **primary** *if $\pi(A)'' \cap \pi(A)'$ is trivial.*
A state $\omega \in S(A)$ is **primary** *if the* GNS*-representation π_ω is primary.*

For compact groups G (or rather their group C*-algebras $C^*(G)$), all representations are completely reducible, and a representation is primary iff it is a (possibly infinite) multiple of some irreducible representation. However, this is not the right picture for general groups or C*-algebras, which requires some discussion. In preparation, we call some representation $\pi'(A)$ on a Hilbert space $H' \subset H$ a ***subrepresentation*** of a representation $\pi(A)$ on H, written $\pi' \subset \pi$, if $\pi' = \pi_{|H'}$. Subrepresentations π' of π correspond to projections $e \in \pi(A)'$, such that $\pi'(a) = e\pi(a)$. It follows that $\pi_1(A)$ and $\pi_2(A)$ have equivalent subrepresentations iff there exists a nonzero partial isometry $w : H_1 \to H_2$ such that $w\pi_1(a) = \pi_2(a)w$ for all $a \in A$.

Definition 8.18. *Two representations π_1 and π_2 of a C*-algebra A are called:*

1. **equivalent** *if there is a unitary $u : H_1 \to H_2$ such that $u\pi_1(a)u^* = \pi_2(a)$ $(a \in A)$;*
2. **quasi-equivalent** *if every subrepresentation of π_1 has a subrepresentation that is equivalent to some subrepresentation of π_2, and* vice versa*;*
3. **disjoint** *if they do not have any equivalent subrepresentations.*

We say that two states ω_1 and ω_2 on A equivalent, disjoint, or quasi-equivalent if the corresponding GNS*-representations π_{ω_1} and π_{ω_2} have the said property.*

In other words, π_1 and π_2 are quasi-equivalent iff π_1 has no subrepresentations disjoint from π_2, and *vice versa*. This, in turn, is equivalent to the property that the set of ***π_i-normal states*** on A, i.e. states of the form $a \mapsto \mathrm{Tr}\,(\rho\pi_i(a))$ with $\rho \in \mathscr{D}(H_i)$, is the same for $i = 1$ as it is for $i = 2$. Contrapositively, π_1 and π_2 are disjoint iff no state exists that is both π_1-normal and π_2-normal. For example, taking $A = C(X)$, in which case states are probability measures μ on X, equivalence and disjointness of states recovers the usual notions of equivalence and disjointness of measures, respectively (i.e., having the same null sets and having disjoint supports).

Proposition 8.19. *For any state ω, if $\omega = t\omega_1 + (1-t)\omega_2$ for some $t \in (0,1)$, then ω_1 and ω_2 are disjoint iff there is a projection $e \in A^c_\omega = \pi_\omega(A)' \cap \pi_\omega(A)''$ such that*

$$\pi_\omega(A)_{|eH_\omega} \cong \pi_{\omega_1}(A); \tag{8.136}$$
$$\pi_\omega(A)_{|e^\perp H_\omega} \cong \pi_{\omega_2}(A). \tag{8.137}$$

Since subrepresentations of $\pi_\omega(A)$ always correspond to projections $e \in \pi_\omega(A)'$; the key assumption being made here is that e also lies in the weak closure $\pi_\omega(A)''$.

Proof. One direction is easy: if (8.136) - (8.137) hold, then (arguing by contradiction) equivalent subrepresentations $\pi_1(A)$ of $\pi_{\omega_1}(A)$ and $\pi_2(A)$ of $\pi_{\omega_2}(A)$ are given by projections $e_1 \le e$ and $e_2 \le e^\perp = 1_{H_\omega} - e$, respectively, through

$$\pi_i(a) = \pi_\omega(a)_{|e_iH_\omega}, \ \ (i = 1,2, a \in A), \tag{8.138}$$

and the partial isometry w on H_ω whose restriction to e_1H_ω implements a (unitary) equivalence between $\pi_1(A)$ and $\pi_2(A)$ by definition satisfies $w^*w = e_1$, $ww^* = e_2$. Moreover, $e_1 \le e$ implies $we = w$ and $e_2 \le e^\perp$ implies $e^\perp w = w$, which together give $e^\perp we = w$. Furthermore, again by definition, $w \in \pi_\omega(A)'$. If now $e \in \pi_\omega(A)''$, then $we = ew$. Combining these equalities gives $w = 0$, which is the desired contradiction.

Lemma 8.20. *For any functional $\omega' \in A^*$ such that $0 \le \omega' \le \omega$, where $\omega \in S(A)$, there is an operator $c \in \pi_\omega(A)'$ on H_ω such that $0 \le c \le 1_H$ and*

$$\omega'(a) = \langle \Omega_\omega, c\pi_\omega(a)\Omega_\omega\rangle \ \ (a \in A). \tag{8.139}$$

In particular, there is a vector $\xi \in H_\omega$ such that

$$\omega'(a) = \langle \xi, \pi_\omega(a)\xi\rangle_{H_\omega}. \tag{8.140}$$

Proof. Cauchy–Schwarz for the positive semidefinite form $\langle a,b\rangle' = \omega'(a^*b)$ gives

$$|\omega'(a^*b)|^2 \le \omega'(a^*a)\omega'(b^*b) \le \omega(a^*a)\omega(b^*b) = \|\pi_{\omega_i}(a)\Omega_{\omega_i}\|^2\|\pi_{\omega_i}(b)\Omega_{\omega_i}\|^2.$$

Hence we obtain a well-defined positive quadratic form B on H_ω, initially defined on the dense domain $\pi_\omega(A)\Omega_\omega \times \pi_\omega(A)\Omega_\omega$ by the formula

$$B(\pi_\omega(a)\Omega_\omega, \pi_\omega(b)\Omega_\omega) = \omega'(a^*b), \tag{8.141}$$

and extended to $H_\omega \times H_\omega$ by continuity; the above inequality immediately gives $|B(\varphi,\psi)| \le \|\varphi\|\|\psi\|$, and hence Proposition B.79 yields an operator $0 \le c \le 1_H$ such that $B(\varphi,\psi) = \langle \varphi, c\psi\rangle$. With (8.141), this gives (8.139). We now compute

$$\begin{aligned}\omega'(a^*b^*d) &= B(\pi_\omega(ba)\Omega_\omega, \pi_\omega(d)\Omega_\omega) = \langle \pi_\omega(a)\Omega_\omega, \pi_\omega(b^*)c\pi_\omega(d)\Omega_\omega\rangle \\ &= B(\pi_\omega(a)\Omega_\omega, \pi_\omega(b^*d)\Omega_\omega = \langle \pi_\omega(a)\Omega_\omega, c\pi_\omega(b^*)\pi_\omega(d)\Omega_\omega\rangle,\end{aligned}$$

so that $[c, \pi_\omega(b^*)] = 0$ for each $b \in A$, i.e., $c \in \pi_\omega(A)'$. Writing $c = c_1^2$ with $c_1^* = c_1$, and then $\xi = c_1\Omega_\omega$, completes the proof. □

We continue the proof of Proposition 8.19 in the converse direction. Assume

$$\omega = t\omega_1 + (1-t)\omega_2 = \omega_1' + \omega_2', \tag{8.142}$$

with $\omega_1' = t\omega_1$ and $\omega_2' = (1-t)\omega_2$, so that $0 \leq \omega_1' \leq \omega$ and $0 \leq \omega_2' \leq \omega$. It follows from the first claim in Lemma 8.20 that there is $c \in B(H_\omega)$ as stated such that

$$\omega_1'(a) = \langle \Omega_\omega, c\pi_\omega(a)\Omega_\omega \rangle; \tag{8.143}$$
$$\omega_2'(a) = \langle \Omega_\omega, (1_{H_\omega} - c)\pi_\omega(a)\Omega_\omega \rangle, \tag{8.144}$$

where (8.144) follows from (8.143), (C.196), and $\omega = \omega_1' + \omega_2'$. Define $\omega' \in A^*$ by

$$\omega'(a) = \langle \Omega_\omega, c(1_{H_\omega} - c)\pi_\omega(a)\Omega_\omega \rangle. \tag{8.145}$$

We have $0 \leq \omega' \leq \omega_1'$ (since $c(1_{H_\omega} - c) \leq c$) as well as $0 \leq \omega' \leq \omega_2'$ (since also $c(1_{H_\omega} - c) \leq 1_{H_\omega} - c$). Now assume that ω_1 and ω_2 are disjoint. Applying (8.140) with $\omega \rightsquigarrow \omega_i$ shows that ω' is π_1-normal as well as π_2-normal, so that it follows from the remarks following Definition 8.18 that $\omega' = 0$. Since Ω_ω is cyclic for $\pi_\omega(A)$ by the GNS-construction, this implies $c(1_{H_\omega} - c) = 0$, and hence $c^2 = c$. Since $c \geq 0$, which implies $c^* = c$, it follows that c is a projection, henceforth called e. Therefore,

$$\omega_1(a) = \langle \Omega_\omega, e\pi_\omega(a)\Omega_\omega \rangle / \|e\Omega_\omega\|^2; \tag{8.146}$$
$$\omega_2(a) = \langle \Omega_\omega, e^\perp\pi_\omega(a)\Omega_\omega \rangle / \|e^\perp\Omega_\omega\|^2, \tag{8.147}$$

where $t = \|e\Omega_\omega\|^2$. We see from these formulae and Proposition C.91 that π_{ω_1} and π_{ω_2} are equivalent to the restrictions of π_ω to eH_ω and $e^\perp H_\omega$, respectively; under this equivalence, the cyclic vectors Ω_{ω_1} and Ω_{ω_2} correspond with $e\Omega_\omega/\|e\Omega_\omega\|$ and $e^\perp\Omega_\omega/\|e^\perp\Omega_\omega\|$, respectively. Since $e \in \pi_\omega(A)'$ by Lemma 8.20, it only remains to be shown that $e \in \pi_\omega(A)''$. To this effect, for any $b \in \pi_\omega(A)'$ and $\psi \in H_\omega$, define

$$\omega'' \in A^*;$$
$$\omega''(a) = \langle e^\perp b e\psi, \pi_\omega(a) e^\perp b e\psi \rangle. \tag{8.148}$$

Then ω'' is positive, as well as π_{ω_2}-normal, the latter because of the presence of the projection $e^\perp$ and (8.147). But for $a \in A^+$ we have the inequalities

$$0 \leq \omega''(a) \leq \|e^\perp b\|^2 \langle e\psi, \pi_\omega(a) e\psi \rangle, \tag{8.149}$$

so that $0 \leq \omega'' \leq \omega_1''$ for the state (assuming $e\psi$ is a unit vector)

$$\omega_1''(a) = \langle \psi, e\pi_\omega(a) e\psi \rangle. \tag{8.150}$$

Since $e\psi \in eH_\omega$, the latter state is π_{ω_1}-normal, so that ω_1'' is itself π_{ω_1}-normal by Lemma 8.20 (which argument by now should sound familiar). Again invoking disjointness of ω_1 and ω_2, it follows that $\omega'' = 0$, which, since ψ was arbitrary, in turn yields $e^\perp b e = 0$ for any $b \in \pi_\omega(A)'$. This forces $e \in \pi_\omega(A)''$. □

The first of the following corollaries to Proposition 8.19 is ***Hepp's Lemma:***

Lemma 8.21. *Let $\pi : A \to B(H)$ be a representation of A, and let ψ_1, ψ_2 be unit vectors in H. Then the vector states $\omega_i(a) = \langle \psi_i, \pi(a)\psi_i \rangle$ $(i = 1,2)$ are disjoint iff*

$$\langle \psi_1, \pi(a)\psi_2 \rangle = 0 \;\; (a \in A). \tag{8.151}$$

Proof. Take, for example, $\omega = \frac{1}{2}(\omega_1 + \omega_2)$ in Proposition 8.19. □

Corollary 8.22. *1. Two primary states are either disjoint or quasi-equivalent.*
2. A state is primary iff it has no convex decomposition into disjoint *states.*

Recall that a state is pure if it has no nontrivial convex decomposition *whatsoever*. The analogy between pure states and primary states may be completed as follows:

- ω pure $\leftrightarrow \pi_\omega(A)' = \mathbb{C} \cdot 1$ (cf. Theorem C.90);
- ω primary $\leftrightarrow \pi_\omega(A)' \cap \pi_\omega(A)'' = \mathbb{C} \cdot 1$ (cf. Definition 8.17).

A physical property of primary states is that the corresponding correlation functions have a clustering property of a kind that may even be experimentally accessible:

Theorem 8.23. *A state ω on a quasi-local C*-algebra A* (8.130) *has trivial algebra at infinity, i.e., $A_\omega^\infty = \mathbb{C} \cdot 1$, iff it is* **clustering**, *in the following sense: for each $a \in A$ and $\varepsilon > 0$ there is a finite $\Lambda \subset \mathbb{Z}^d$ such that for all $b \in A_{\Lambda'}$ with $\|b\| = 1$ one has*

$$|\omega(ab) - \omega(a)\omega(b)| \leq \varepsilon. \tag{8.152}$$

In particular, if ω is primary, then it is clustering and hence (8.152) *holds.*

Proof. The complete proof is quite technical, but the main idea is as follows. Choose finite regions Λ_n moving to infinity (i.e., eventually avoiding any given Λ), and pick elements $c_n \in A_{\Lambda_n}$), $\|c_n\| = 1$. The sequence $(\pi_\omega(c_n))$ in $B(H_\omega)$ has a weakly convergent subsequence with limit $c \in B(H_\omega)$. This follows from the Banach–Alaoglu Theorem B.48, applied to $B(H_\omega)$ seen as the dual space of $B_1(H_\omega)$): on the unit ball, the corresponding weak*-topology on $B(H_\omega)$ coincides with the weak operator topology, so that the unit ball in $B(H_\omega)$ is weakly compact and the theorem applies.

- By von Neumann's Bicommutant Theorem C.127 we have $c \in \pi_\omega(A)''$.
- By Einstein locality (8.132) and the delocalization of the Λ_n, also $c \in \pi_\omega(A)'$.

Hence $c \in A_\omega^c$, and by a more refined argument (which is unnecessary if if $A_\omega^\infty = A_\omega^c$), even $c \in A_\omega^\infty$. So if $A_\omega^\infty = \mathbb{C} \cdot 1$ we have $c = (\Omega_\omega, c\Omega_\omega) \cdot 1$. On the other hand,

$$\langle \Omega_\omega, c\Omega_\omega \rangle = \lim_n \langle \Omega_\omega, \pi_\omega(c_n)\Omega_\omega \rangle = \lim_n \omega(c_n),$$

so that we may compute:

$$\lim_n \omega(ac_n) = \lim_n \langle \Omega_\omega, \pi_\omega(a)\pi_\omega(c_n)\Omega_\omega \rangle = \langle \Omega_\omega, \pi_\omega(a)c\Omega_\omega \rangle = \omega(a) \lim_n \omega(c_n).$$

Thus for any $\varepsilon > 0$ there is an N such that $|\omega(ac_n) - \omega(a)\omega(c_n)| \leq \varepsilon$ for all $n > N$. To derive (8.152) from this, an easy *reductio ad absurdum* argument suffices.

The converse direction follows from Kaplansky's Density Theorem C.131. □

8.6 Quantum spin systems: Bundles of C*-algebras

In this section we reformulate the theory of quantum spin systems in the continuous C*-bundle language of §8.2. First, for each $N \in \mathbb{N}$ we define $\Lambda_N \in \mathscr{P}_f(\mathbb{Z}^d)$ by

$$\Lambda_N = \{x \in \mathbb{Z}^d \mid \|x\| \leq N\}. \tag{8.153}$$

We then have the following analogue of the continuous bundle of C*-algebras $A^{(q)}$ of C*-algebras of Theorem 8.8. The base space remains $I = 1/\dot{\mathbb{N}} \subset [0,1]$, where $\dot{\mathbb{N}} = \{1,2,\ldots,\infty\}$ (seen as possible values of $1/\hbar$), and the fibers are given by

$$A_0 = A = \varinjlim_N A_{\Lambda_N} = \overline{\bigcup_{N\in\mathbb{N}} A_{\Lambda_N}}^{\|\cdot\|}; \tag{8.154}$$

$$A_{1/N} = A_{\Lambda_N} = B(H_{\Lambda_N}) \;\; (N \in \mathbb{N}), \tag{8.155}$$

cf. (8.128) - (8.130), still assuming $\dim(H) < \infty$. As before, the topology of this bundle is defined through its continuous cross-sections $(a_{1/N})_{N\in\dot{\mathbb{N}}}$, which are the analogues of the quasi-local sequences of Definition 8.7. Given (8.154) - (8.155), each fiber algebra $A_{1/N}$ is a subalgebra of A_0, and some sequence $(a_{1/N})_{N\in\dot{\mathbb{N}}}$ simply defines a continuous cross-section of the bundle iff within A (i.e. in norm) we have

$$\lim_{N\to\infty} a_{1/N} = a_0. \tag{8.156}$$

In other words, a sequence $(a_{1/N})_{N\in\mathbb{N}}$ with $a_{1/N} \in A_{1/N} \subset A$ is quasi-local in the sense of Definition 8.7 iff it converges in A (i.e., iff it is Cauchy in the norm of A).

The continuous bundle of Theorem 8.4 makes equally good sense for quantum spin systems. First, with $B = B(H) \cong M_n(\mathbb{C})$, the fibers are obviously given by

$$A_0^{(c)} = C(S(B(H))); \tag{8.157}$$

$$A_{1/N}^{(c)} = B(H_{\Lambda_N}). \tag{8.158}$$

Second, the continuous sections are once again specified via symmetrization maps

$$S_{M,N} : B(H_{\Lambda_M}) \to B(H_{\Lambda_N}), \tag{8.159}$$

defined similarly to (8.39), namely via canonical symmetrizers

$$S_N : B(H_{\Lambda_N}) \to B(H_{\Lambda_N}) \tag{8.160}$$

that are defined à la (8.35) - (8.36), where this time the tensor product and ensuing permutation in (8.35) are over all sites $x \in \Lambda_N$. Regarding $a_{1/M} \in B(H_{\Lambda_M})$ as an element $a'_{1/M}$ of $B(H_{\Lambda_N})$ via the embedding $A_{\Lambda_M} \hookrightarrow A_{\Lambda_N}$, we finally define $S_{M,N}$ by

$$S_{M,N}(a_{1/M}) = S_N(a'_{1/M}). \tag{8.161}$$

Symmetric and quasi-symmetric sequences may then be defined exactly as in Definitions 8.2 and 8.3; each quasi-symmetric sequence $(a_{1/N})_{N\in\mathbb{N}}$ duly has a limit $a_0 \in A_0^{(c)}$ given by (8.46), where ω^N is defined as in (8.47), once again with a tensor product over all sites $x \in \Lambda_N$. By definition, the continuous sections of the bundle (8.157) - (8.158) are then given by the quasi-symmetric sequences.

Although the fibers A in (8.154) and $C(S(B(H)))$ in (8.157) are as wide apart as they could possibly be, they stunningly arise as limit algebras at $\hbar = 0$ (i.e., $N = \infty$ or $\Lambda = \mathbb{Z}^d$) for the same fiber algebras (8.155) and (8.158) at $\hbar > 0$ (i.e., $N < \infty$ or $\Lambda \in \mathscr{P}_f(\mathbb{Z}^d)$). As in §8.2, the difference lies in the choice of the topology on the bundle, defined via the continuous sections, which in the first case are the quasi-local sequences, and in the second are the quasi-symmetric (i.e., macroscopic) ones.

An interesting connection between these bundles can be obtained via the following concept, which in a way justifies the introduction of the bundles themselves.

Definition 8.24. *A* **continuous field of states** *on a continuous bundle of C*-algebras with fibers $(A_{1/N})_{N\in\dot{\mathbb{N}}}$ is a family $(\omega_{1/N})_{N\in\dot{\mathbb{N}}}$ where*

$$\omega_{1/N} \in S(A_{1/N}); \tag{8.162}$$

$$\lim_{N\to\infty} \omega_{1/N}(a_{1/N}) = \omega_0(a_0), \tag{8.163}$$

for each continuous cross-sections $(a_{1/N})$. In that case, we write

$$\omega_0 = \lim_{N\to\infty} \omega_{1/N}, \tag{8.164}$$

despite the fact that all states in question may be defined on different C-algebras.*

For example, any state ω on $A_0 = A$ as in (8.154) defines a continuous field:

Proposition 8.25. *For any state $\omega \in S(A)$, the set $(\omega_{1/N})_{N\in\dot{\mathbb{N}}}$ of states defined by*

$$\omega_0 = \omega; \tag{8.165}$$

$$\omega_{1/N} = \omega_{|A_{1/N}}, \tag{8.166}$$

is a continuous field of states on the bundle with fibers (8.154) - (8.155).

Proof. We use the notation of Definition 8.7. For local sequences (8.57) we have

$$\omega_{1/N}(a_{1/N}) = \omega(a_{1/N}) = \omega(a_{1/M}),$$

for all $N \geq M$. Since $a_0 = a_{1/M}$, this equals $\omega_0(a_0)$. For quasi-local sequences, a_0 is the limit of the sequence $(a_{1/N})$ in the norm of A, so that $\omega(a_{1/N}) \to \omega(a_0)$. □

Definition 8.26. *A state $\omega \in S(A)$ is* **macroscopic** *if* $\lim_{N\to\infty} \omega(a_{1/N})$ *exists for any (quasi-) symmetric sequence $(a_{1/N})$.*

It does not matter whether we put "symmetric" or "quasi-symmetric" here, since existence of the limit for symmetric sequences implies its existence on quasi-symmetric sequences. Indeed, using the fact that $\|\omega\| = 1$, we may estimate

$$|\omega(a_{1/N}) - \omega(a_{1/M})| \leq |\omega(\tilde{a}_{1/N}) - \omega(\tilde{a}_{1/N})| + |\|a_{1/N} - \tilde{a}_{1/N}\| + \|a_{1/M} - \tilde{a}_{1/M}\|, \tag{8.167}$$

for any sequence $(\tilde{a}_{1/M})$. Using Definition 8.3, and hence taking $(\tilde{a}_{1/M})$ symmetric, we see that if $(\omega(\tilde{a}_{1/N}))$ is a Cauchy sequence, then so is $(\omega(a_{1/N}))$.

Proposition 8.27. *A macroscopic state* ω *determines a state* $\omega_0^{(c)}$ *on* $C(S(B))$ *by*

$$\omega_0^{(c)}(a_0) = \lim_{N\to\infty} \omega(a_{1/N}), \tag{8.168}$$

where $(a_{1/N})$ *is any quasi-symmetric sequence with limit* $a_0 \in C(S(B))$*, cf.* (8.46).

Proof. First, note that $\omega_0^{(c)}$ is independent of the choice of the approximating sequence $(a_{1/N})$, since by the same argument as in the proof of Proposition C.126, if $a_{1/N} \to a_0$ as well as $a'_{1/N} \to a_0$, we have

$$\lim_{N\to\infty} \|a_{1/N} - a'_{1/N}\| = \|a_0 - a_0\| = 0, \tag{8.169}$$

and because $\|\omega\| = 1$ for any state ω, we also have

$$|\omega(a_{1/N} - a'_{1/N})| \leq \|a_{1/N} - a'_{1/N}\|. \tag{8.170}$$

Eqs. (8.169) - (8.170) obviously imply

$$\lim_{N\to\infty} \omega(a_{1/N}) = \lim_{N\to\infty} \omega(a'_{1/N}). \tag{8.171}$$

We next show that if $a_{1/N} \to a_0$ and $b_{1/N} \to b_0$ in the sense of (C.560), then

$$a_{1/N}b_{1/N} \to a_0b_0.$$

If $(a_{1/N})$ is a symmetric sequence à la (8.43), and likewise $(b_{1/N})$, where we may assume without loss of generality that M is the same for both, then

$$a_0(\rho) = \rho^M(a_{1/M}), \tag{8.172}$$

where $\rho \in S(B)$, and likewise for b_0. Using (8.38), we obtain

$$\lim_{N\to\infty} \rho^N(a_{1/N}b_{1/N}) = \rho^M(a_{1/M})\rho^M(b_{1/M}) = a_0(\rho)b_0(\rho) = (a_0b_0)(\rho). \tag{8.173}$$

In particular, if $a_{1/N} \to a_0$, then $a^*_{1/N}a_{1/N} \to a^*_0a_0$. Since ω is a state, it follows that $\omega_0^{(c)}(a^*_0a_0) \geq 0$, and since also $\omega_0^{(c)}(1_{S(B)}) = 1$ (because the sequence with $a_{1/N} = 1_{H_{\Lambda_N}}$ converges to $1_{S(B(H))}$), the claim follows for symmetric sequences. For quasi-symmetric sequences $(a_{1/N})$ the result follows by approximating $(a_{1/N})$ with symmetric sequences (cf. Definition 8.3). □

Each state $\omega_0^{(c)} \in S(A_0^{(c)})$ is represented by a probability measure μ on the state space $S(B(H))$ of $B(H)$. We compute this measure if $\omega \in S(A)$ is ***permutation-invariant*** in that each restriction $\omega_{1/N} = \omega_{|B(H_{\Lambda_N})}$ is invariant under the natural action of the permutation group $\mathfrak{S}_{|\Lambda_N|}$ on $B(H_{\Lambda_N}) \cong \otimes_{x\in\Lambda_N} B(H)$, where $N \in \mathbb{N}$ and $|\Lambda_N|$ is the number of points in Λ_N (as in the case of B^∞ in §8.2). It follows from the Quantum De Finetti Theorem 8.9 (and the fact that that the set $S^{\mathfrak{S}_\infty}(A)$ of permutation-invariant states on A is a so-called *Bauer simplex*) that each permutation-invariant state $\omega \in S^{\mathfrak{S}_\infty}(A)$ takes the form

$$\omega = \int_{S(B(H))} d\mu(\rho)\, \rho^\infty, \tag{8.174}$$

where μ is some probability measure on $S(B(H))$, and $\rho \in S(B(H))$; the associated state ρ^∞ on A is defined by its values on each $A_{\Lambda_N} \subset A$ via the isomorphism

$$A_{\Lambda_N} \cong \otimes_{x\in\Lambda_N} B(H). \tag{8.175}$$

Furthermore, the integral in (8.174) is defined weakly, i.e., for any $a \in A$ the number $\omega(a)$ is obtained by integrating the function $\rho \mapsto \rho^\infty(a)$ on $S(B(H))$ with respect to μ. In particular, $\omega \in \partial_e S^{\mathfrak{S}_\infty}(A)$ iff μ is a Dirac measure on $S(B(H))$.

Proposition 8.28. *Each permutation-invariant state $\omega \in S^{\mathfrak{S}_\infty}(A)$ is macroscopic (cf. Definition 8.26), and the probability measure μ on $S(B(H))$ defined by $\omega_0^{(c)}$ via* (8.168) *coincides with the one appearing in* (8.174).

Proof. Let $(a_{1/N})$ be a symmetric sequence (the quasi-symmetric case follows from this), so that $a_{1/N} = S_{M,N}(a_{1/M})$ for some M whenever $N > M$, cf. (8.43). The limit $a_0 \in C(S(B(H)))$ is given by (8.172), so that state $\omega_0^{(c)}$ on $C(S(B(H)))$ defined by

$$\omega_0^{(c)}(f) = \int_{S(B(H))} d\mu(\rho)\, f(\rho) \tag{8.176}$$

satisfies the required condition

$$\lim_{N\to\infty} \omega_{1/N}(a_{1/N}) = \omega_{1/M}(a_{1/M}) = \int_{S(B(H))} d\mu(\rho)\, \rho^M(a_{1/M}) = \omega_0^{(c)}(a_0). \qquad \square$$

To proceed we make the following technical assumption on $\omega \in S(A)$ (which is satisfied in typical physical models): if $\pi_\omega(a_{1/N}) \to 0$ weakly in $B(H_\omega)$, for some sequence $(a_{1/N})$ where $a_{1/N} \in A_{1/N}$, then $\pi_\omega(a_{1/N})\Omega_\omega \to 0$ in $B(H_\omega)$ (in norm).

Theorem 8.29. *Assume that the state ω in part 1 below (and likewise the states ω_1 and ω_2 in part 2) satisfies the above technical condition. Then:*

1. *If ω is a* primary *macroscopic state on A, then the corresponding state $\omega_0^{(c)}$ is* pure, *i.e., the probability measure μ on $S(B(H))$ is a Dirac measure.*
2. *If ω_1 and ω_2 are quasi-equivalent primary macroscopic state on A, then $\mu_1 = \mu_2$ (and hence if $\mu_1 \neq \mu_2$, then ω_1 and ω_2 are disjoint).*

The techniques in the proof below can be used to show that our additional assumption is equivalent to: if (8.178) below holds weakly in $B(H_\omega)$, then it also holds strongly. Thus we could have redefined a macroscopic state ω as one for which the strong limit $\lim_{N\to\infty}\pi_\omega(a_{1/N})$ exists in $B(H_\omega)$ (and some authors indeed do so).

Proof. We first show that if ω is a primary macroscopic state on A, and $(a_{1/N})$ is symmetric (from which the quasi-symmetric case duly follows) such that

$$\lim_{N\to\infty}\omega(a_{1/N})=\alpha, \tag{8.177}$$

then, in the weak operator topology on the GNS-representation space $B(H_\omega)$,

$$\lim_{N\to\infty}\pi_\omega(a_{1/N})=\alpha\cdot 1_{H_\omega}. \tag{8.178}$$

To this end, we first note that $\|a_{1/N}\|$ is uniformly bounded in N: if $(a_{1/N})$ is symmetric, as in (8.43), then obviously $\|a_{1/N}\|=\|a_{1/M}\|$ for all $N>M$, so that if $(a_{1/N})$ is merely quasi-symmetric we have $\|a_{1/N}\|\le\|a_{1/M}\|+\varepsilon$ for all $N>M$, where ε and M are the quantities appearing in Definition 8.3. Hence it is enough to establish the weak limit (8.178) between states in a dense set, viz. $\pi_\omega(b)\Omega_\omega$, where $b\in A$, or even in $\cup_N A_{1/N}$. Furthermore, using the polarization identity (A.5) and (C.8) - (C.9), it is enough to prove that for each $K\in\mathbb{N}$ and $b\in A_{1/K}$, we have

$$\lim_{N\to\infty}\omega(b^*a_{1/N}b)=\alpha\omega(b^*b), \tag{8.179}$$

since by the GNS-construction we obviously have

$$\langle\pi_\omega(b)\Omega_\omega,\pi_\omega(a_{1/N})\pi_\omega(b)\Omega_\omega\rangle=\omega(b^*a_{1/N}b). \tag{8.180}$$

Theorem 8.23 implies (or even states) that if ω is primary, for each $b\in A$ and $\varepsilon>0$ there is $M\in\mathbb{N}$ such that for all $a\in A'_{\Lambda_M}$ with $\|a\|=1$, we have

$$|\omega(b^*ba)-\omega(b^*b)\omega(a)|\le\varepsilon. \tag{8.181}$$

Assuming $b\in A_{1/K}$, we first note that $\lim_{N\to\infty}[a_{1/N},b]=0$ in norm (even though $\lim_{N\to\infty}a_{1/N}$ does not exist in norm), and secondly that, for any given $M\in\mathbb{N}$, if $\tilde{a}_{1/N}$ is the same as $a_{1/N}$ except that in any term $b_1\otimes\cdots\otimes b_{|\Lambda_N|}$ that contributes to $a_{1/N}$ we replace $b_i\rightsquigarrow 1_H$ whenever $b_i\in A_{1/M}$, then

$$\lim_{N\to\infty}\|\tilde{a}_{1/N}-a_{1/N}\|=0. \tag{8.182}$$

Given (8.177), these facts with (8.181) immediately give (8.179) and hence (8.178).

According to (8.177) and (8.178), the state $\omega_0^{(c)}\in S(C(S(B(H))))$ is given by

$$\omega_0^{(c)}(a_0)=\lim_{N\to\infty}\langle\Omega_\omega,\pi_\omega(a_{1/N})\Omega_\omega\rangle, \tag{8.183}$$

where $a_{1/N}$ is some symmetric sequence converging to $\to a_0$ in the sense of (C.560); as in the proof of Proposition 8.27, the left-hand side is independent of the particular choice of this sequence. The proof of Proposition 8.27 also showed that if $a_{1/N} \to a_0$ and $b_{1/N} \to b_0$, then $a_{1/N}b_{1/N} \to a_0 b_0$, so that

$$\begin{aligned}\omega_0^{(c)}(a_0 b_0) &= \lim_{N\to\infty} \langle \Omega_\omega, \pi_\omega(a_{1/N}b_{1/N})\Omega_\omega\rangle \\ &= \lim_{N\to\infty} \langle \Omega_\omega, \pi_\omega(a_{1/N}) - \alpha \cdot 1_{H_\omega})\pi_\omega(b_{1/N})\Omega_\omega\rangle + \alpha\beta,\end{aligned}$$

where α is defined by (8.177), and likewise β. At this point that we need our additional assumption, which, together with uniform boundedness of $\|\pi_\omega(a_{1/N})\|$ and hence of $\|\pi_\omega(a_{1/N})\Omega_\omega\|$ in N yields that the first term in the second line is zero. Therefore, $\omega_0^{(c)}$ is multiplicative and hence pure (cf. Proposition C.14).

To prove the second claim, first suppose ω_1 and ω_2 are quasi-equivalent. In that case, up to unitary equivalence, either π_{ω_1} is a subrepresentation of π_{ω_2}, or *vice versa*; assume the former. We then have a projection $e \in \pi_{\omega_2}(A)'$ such that

$$\pi_{\omega_1}(a) = e\pi_{\omega_2}(a), \tag{8.184}$$

for each $a \in A$, and since $e = 1_{H_{\omega_1}}$ by construction, eq. (8.178) gives

$$\lim_{N\to\infty} \pi_{\omega_1}(a_{1/N}) = \alpha_1 \cdot e; \tag{8.185}$$

$$\lim_{N\to\infty} \pi_{\omega_2}(a_{1/N}) = \alpha_2 \cdot 1_{H_{\omega_2}}. \tag{8.186}$$

Multiplying both sides of (8.186) with e gives $\alpha_1 = \alpha_2$. □

Corollary 8.30. *A permutation-invariant state $\omega \in S^{\mathfrak{S}_\infty}(A)$ is primary iff the corresponding measure μ in* (8.174) *is a Dirac measure, and it is pure iff the latter is supported by a pure state on $B(H)$.*

Proof. In the first claim, the inference from “primary“ to “Dirac” obviously follows from Theorem 8.29. The converse direction is a consequence of the commutation theorem (C.329) for von Neumann algebras, combined with the fact that each representation of $B(H)$ for finite-dimensional H is primary (which in turn follows from the fact, not proved in this book, that $B(H)$ has just one irreducible representation, up to equivalence). The second claim follows from Proposition C.105. □

Finally, *one* macroscopic state generates many others. A ***folium*** in the state space $S(A)$ of a C*-algebra A is a convex, norm-closed subspace $\mathscr{F}$ of $S(A)$ with the property that if $\omega \in \mathscr{F}$ and $b \in A$ such that $\omega(b^*b) > 0$, then the “reduced” state $\omega_b : a \mapsto \omega(b^*ab)/\omega(b^*b)$ must be in $\mathscr{F}$. For example, if π is a representation of A on a Hilbert space H, then the set of all density matrices on H (i.e. the π-normal states on A) comprises a folium $\mathscr{F}_\pi$. In particular, each state ω on A defines a folium $\mathscr{F}_\omega \equiv \mathscr{F}_{\pi_\omega}$ through its GNS-representation π_ω. It then follows from cyclicity of the GNS-representation that each state in the folium $\mathscr{F}_\omega$ of a macroscopic state $\omega \in S(A)$ is automatically macroscopic and even has the same limit state $\omega^{(c)}$ as ω.

Notes

§8.1. **Large quantum numbers**

Theorem 8.1 has been adapted from Landsman (1998b); the proof relies on Simon (1980), who, generalizing the case of $SU(2)$ treated by Lieb (1973), in turn uses the coherent states for Lie groups introduced by Perelomov (1972, 1986). Duffield (1999) gives the details of the method of steepest descent used in proving (8.30). Although this material was inspired by Bohr's Correspondence Principle, at the end of the day the relationship may seem remote.

§8.2. **Large systems**

The theory in this section, which elaborates on Landsman (2007), is a reformulation in terms of continuous bundles of C*-algebras of the formal parts of a series of papers on quantum mean-field systems by Raggio & Werner (1989, 1991), Duffield & Werner (1992a,b,c), and Duffield, Roos, & Werner (1992). These models have their origin in the treatment of the BCS theory of superconductivity due to Bogoliubov (1958) and Haag (1962); for further references see the notes to §10.8.

§8.3. **Quantum de Finetti Theorem**

Theorem 8.9 is due to Størmer (1969), whose proof was based on the fact that the $\mathfrak{S}_\infty$-action on B^∞ is ***asymptotically abelian***, in that for any $a, a' \in B^\infty$ one has

$$\inf\{\|[\alpha_p(a), a']\|, p \in \mathfrak{S}_\infty\} = 0.$$

This implies that $S^{\mathfrak{S}_\infty}(B^\infty)$ is a Choquet simplex, which quickly leads to (8.66). Our proof is taken from Hudson & Moody (1975). See also Caves, Fuchs, & Schack (2002a). Finite-size corrections to Theorem 8.9 are studied e.g. in König & Mitchison (2009). Corollary 8.11 is due to Hewitt & Savage (1955), who credit Jules Haag (rather than De Finetti) for the binary case (i.e., $X = \{0,1\}$). See Kallenberg (2005) for an exhaustive account of such results (in classical probability theory).

Proposition 8.12 is taken from Diaconis & Freedman (1980), who also give finite-size corrections to Corollary 8.11, as follows. Let a permutation-invariant probability measure ν_N on X^N be K-exchangeable, so that there is a permutation-invariant probability measure ν_{N+K} on X^{N+K} whose restriction to X^N is ν_N. Let P_{N+K} be the probability measure on $\mathrm{Pr}(X)$ defined by ν_{N+K} as in (8.85), i.e., $P_{N+K}(A) = \nu_{N+K}(E_{N+K}^{-1}(A))$, and finally define

$$\nu'_{N+K} = \int_{\mathrm{Pr}(X)} dP_{N+K}(\mu)\, \mu^{N+K},$$

as in (8.79). Then, in terms of the usual norm on the Banach dual $C(X^N)^*$,

$$\|\nu_N - \nu'_N\| \leq \frac{K(K-1)}{N}.$$

Proposition 8.13 is stated without proof in Kingman (1978). See Mackey (1974) or Gray (2009) for ergodic theory in connection with probability theory.

Of course, there are numerous results in probability theory that do not share the problems of the law of large numbers. For example, in the situation (8.94), for any $\varepsilon > 0$ one has the ***Chernoff–Hoeffding bound***

$$\mu^N \left(|\frac{1}{N}\sum_{i=1}^{N} x_i - p| \geq \varepsilon| \right) \leq e^{-2N\varepsilon^2},$$

which is superior to the weak law of large numbers, i.e., for every $\varepsilon > 0$,

$$\lim_{N\to\infty} \mu^N \left(|\frac{1}{N}\sum_{i=1}^{N} x_i - p| \geq \varepsilon| \right) = 0,$$

which from the point of view of Earman's Principle is already a marked conceptual improvement over the strong law (but which is mathematically weaker).

§8.4. **Frequency interpretation of probability and Born rule**

The Kolmogorov quote is from Fine (1973, p. 94), which even 40 years later is still to be recommended as one of the best (technical) book on the foundations of probability theory. See also Hájek & Hitchcock (2016) for a comprehensive recent survey of the philosophy of probability. The Keynes quote is from Hacking (2001, p. 149), which is a very elementary introduction to the foundations of probability At a more advanced level see also Gillies (2000), whilst Howson (1995) is a useful brief survey.

The original version of the *Principal Principle* (Lewis, 1980) equated probability (or chance) as subjective degree of belief (i.e. credence) with objective chance (though in the single case as opposed to relative frequency. Our own version in the main text is meant to clarify the relationship between singe-case probabilities and long run frequencies, both seen as objective.

Attempts to derive the Born rule started with Finkelstein (1965) and were continued e.g. by Hartle (1968), Farhi, Goldstone, & Gutmann (1989), Van Wesep (2006), Aguirre & Tegmark (2011), Moulay (2014), and others, partly based on indubitable mathematical arguments in the spirit of the strong law of large numbers supplied by e.g. Ochs (1977, 1980), Bugajski & Motyka (1981), Pulmannová & Stehlková (1986). Such attempts (typically presented as claims) provoked valid critiques of the kind mentioned in the main text from e.g. Cassinelli & Sánchez-Gómez (1996) and Caves & Schack (2005). For a balanced account see also Cassinelli & Lahti (1989). Infinite tensor products of Hilbert spaces were introduced by von Neumann (1938).

Our approach, which is sympathetic to both sides of the dispute, is a vast expansion of Landsman (2008). The existence of e_∞ as in (8.109) - (8.110) is based on the same extension argument that proves the Kolmogorov existence theorem for infinite product probabilities, see e.g. Dudley (1989), proof of Theorem 8.2.2, and Van Wesep (2006), who carries out the proof for $X = \{0,1\}$.

There is also a large (and inconclusive) literature on alleged derivations of the Born rule in the context of the Many-Worlds (i.e. Everettian) Interpretation of quantum mechanics, which may be traced back from Wallace (2012), who supports such derivations, and Dawid & Thébault (2015), who criticize them.

§8.5.**Quantum spin systems: Quasi-local C*-algebras**

Basic references are Ruelle (1969), Israel (1979), Bratteli & Robinson (1987, 1997), and Simon (1993); for macroscopic states see Hepp (1972) and Sewell (2002). Naaijkens (2013) is a useful brief introduction to quantum spin systems.

The proof that Haag duality holds for quantum spin systems is far from trivial: see Simon (1993), Prop. IV.1.6. In the proof of (8.135), simplicity of A given simplicity of each A_Λ is easily inferred from the fact that if $I \subset A$ is an ideal, then $I_\Lambda = I \cap A_\Lambda$ is an ideal in $A_\Lambda = B(H_\Lambda)$, which must be either zero or A_Λ, both of which contradict non-triviality of I. Theorem 8.23 is a famous result due to Lanford & Ruelle (1969), partly anticipated by Powers (1967). For a complete proof see also Simon (1993), Theorem IV.1.4.

§8.5.**Quantum spin systems: Bundles of C*-algebras**

This section was inspired by Landsman (2007), §6, and Gerisch (1993).

Folia of states (in the sense meant here) were introduced by Haag, Kadison, & Kastler (1970), but note that the name "folium" is poorly chosen, since $S(A)$ is by no means foliated by its folia (for example, a folium may contain subfolia).