# Chapter 7
# Limits: Small $\hbar$

Limits are essential to the asymptotic Bohrification program. It was recognized at an early stage in the development of quantum mechanics that the limit $\hbar \to 0$ of Planck's constant going to zero should play a role in the derivation of classical physics from quantum theory, and later on also the thermodynamic limit (which often means "$\lim_{N\to\infty}$", where $N$ is the number of particles in the system) became a subject of interest in quantum statistical mechanics. The conceptual status of these limits will be discussed in Chapter 10; in the present one we mainly explain the underlying mathematics. However, one question needs to be addressed immediately, since it is a source of much confusion. Varying $N$ seems a realistic thing to do in the lab or on paper, whereas $\hbar$ is a *constant*, so how can it be varied? The answer is that $\hbar$ is a *dimensionful* constant, from which one forms dimensionless combinations of $\hbar$ and other parameters; this combination then re-enters the theory as if it were a dimensionless version of $\hbar$ that can indeed be varied. The oldest example is Planck's radiation formula $E_\nu/N_\nu = h\nu/(e^{h\nu/kT} - 1)$, with temperature $T$ as the pertinent variable. Indeed, the observation of Einstein and Planck that in the limit $\hbar\nu/kT \to 0$ this formula converges to the classical equipartition law $E_\nu/N_\nu = kT$ may well be the first use of the $\hbar \to 0$ limit of quantum theory; note that Einstein put $\hbar\nu/kT \to 0$ by letting $\nu \to 0$ at fixed $T$ and $\hbar$, whereas Planck took $T \to \infty$ at fixed $\nu$ and $\hbar$!

Another example is the Hamiltonian $h = -\frac{\hbar^2}{2m}\Delta + V(x)$ in the Schrödinger equation of non-relativistic quantum mechanics, where $m$ is the mass of the pertinent particle. Here one may pass to dimensionless parameters by introducing an energy scale $\varepsilon$ typical of $H$, like $\varepsilon = \sup_x |V(x)|$, as well as a typical length scale $\ell$, such as $\ell = \varepsilon/\sup_x |\nabla V(x)|$ (if these quantities are finite). In terms of the dimensionless variable $\tilde{x} = x/\ell$, the rescaled Hamiltonian $\tilde{h} = h/\varepsilon$ is then dimensionless and equal to $\tilde{h} = -\tilde{\hbar}^2\tilde{\Delta} + \tilde{V}(\tilde{x})$, where $\tilde{\hbar} = \hbar/\ell\sqrt{2m\varepsilon}$, the operator $\tilde{\Delta}$ is the Laplacian for $\tilde{x}$, and $\tilde{V}(\tilde{x}) = V(\ell\tilde{x})/\varepsilon$. Here $\tilde{\hbar}$ is dimensionless, and one might study the regime where it is small. Similarly, it is often realistic to rescale the potential $V$ by a positive number $\lambda$, in which case $h_\lambda = -\frac{\hbar^2}{2m}\Delta + \lambda V(x)$ can be rescaled to $h_\lambda/\lambda = -\frac{\tilde{\hbar}^2}{2m}\Delta + V(x)$, with $\tilde{\hbar} = \hbar/\sqrt{\lambda}$, so that the "large $V$ limit" $\lambda \to \infty$ comes down to $\tilde{\hbar} \to 0$.



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In (older) textbooks on quantum mechanics the limit $\hbar \to 0$ is typically studied using the so-called WKB-approximation. This may be justified on historical grounds, but in fact this approximation is rarely applicable, and is extremely delicate even when it applies. Fortunately, a much more satisfactory and almost universally applicable framework has become available since the 1990s, namely ***(strict) deformation quantization***, where the word "strict" (which we will henceforth omit) refers to the fact that in this approach $\hbar$ is a real number that can "really" (!) be varied and hence can be made small (as opposed to *formal* deformation quantization, where $\hbar$ is a formal parameter having no actual value). Also, "strict" sometimes refers to the use of C*-algebras and the high mathematical standards this brings. In the formalism that follows, (deformation) quantization and the classical limit of quantum mechanics are seen as two sides of the same coin, as the axioms of quantization are predicated on recovering the correct classical limit, while conversely the classical limit only makes sense in the context of some correct notion of quantization.

The starting point of deformation quantization is a phase space $X$, mathematically described as a Poisson manifold, i.e., a manifold equipped with a Poisson bracket $\{\cdot,\cdot\}$ on its algebra of smooth functions $C^\infty(X)$, see §3.2. We recall that a Poisson bracket is a Lie bracket on $C^\infty(X)$ with the additional property that for each $h \in C^\infty(X)$, the map $\delta_h(f) = \{h, f\}$ is a derivation of $C^\infty(X)$ with respect to its structure as a commutative algebra under pointwise multiplication, i.e.,

$$\delta_h(fg) = f\delta_h(g) + \delta_h(f)g. \tag{7.1}$$

Furthermore, like pointwise multiplication, the Poisson bracket preserves real-valuedness, i.e., if $f \in C^\infty(X,\mathbb{R})$ and $g \in C^\infty(X,\mathbb{R})$, then also $\{f,g\} \in C^\infty(X,\mathbb{R})$.

As early as 1925, Dirac noted the formal analogy between Poisson brackets of functions on phase space and commutators of operators on Hilbert space (i.e., $[a,b] = ab - ba$). Indeed, if $A$ is any C*-algebra, the commutator is a Lie bracket on $A$, and if we use $[a,b]' = i[ab - ba]$, then also self-adjointness is preserved (in that $a^* = a$ and $b^* = b$ implies that also $[a,b]'$ is self-adjoint, which fails to be the case for the commutator itself unless it vanishes). Thus $[-,-]'$ is a Lie bracket on $A_{\mathrm{sa}}$. Moreover, if for fixed $a \in A$ we define $\delta_a(b) = [a,b]'$, then we have the product rule

$$\delta_a(bc) = \delta_a(b)c + b\delta_a(c), \tag{7.2}$$

which makes $\delta_a : A \to A$ a derivation. A problem arises if one wishes to restrict $\delta_a$ to $A_{\mathrm{sa}}$, since this subspace is not stable under multiplication. This may be remedied by passing to the Jordan product (5.14), i.e., $a \circ b = \frac{1}{2}(ab + ba)$, which is defined on $A_{\mathrm{sa}}$. If $a^* = a$, then $\delta_a : A_{\mathrm{sa}} \to A_{\mathrm{sa}}$ satisfies the rule (7.2) also with respect to $\circ$.

All this remains true if $[-,-]'$ is rescaled by a nonzero real number. Which number this should be was suggested by Schrödinger's construction of momentum and position operators on the Hilbert space $H = L^2(\mathbb{R})$ through the substitutions

$$p \rightsquigarrow \hat{p} = \frac{\hbar}{i}\frac{d}{dx}; \tag{7.3}$$

$$q \rightsquigarrow \hat{q} = x, \tag{7.4}$$

where "$x$" is the multiplication operator $m_{\mathrm{id}}$ (with $\mathrm{id}(x) = x$), i.e., $\hat{q}\psi(q) = x\psi(x)$; for the moment we will not be bothered by the fact that these operators are unbounded; let us say they are both defined on the domain $C_c^\infty(\mathbb{R}) \subset L^2(\mathbb{R})$.

This yields the ***canonical commutation relations*** (which formally hold on $C_c^\infty(\mathbb{R})$):

$$\frac{i}{\hbar}[\hat{p}, \hat{q}] = 1_H, \tag{7.5}$$

Noting the Poisson brackets (in which $p, q$ are the coordinate functions on $X = \mathbb{R}^2$)

$$\{p, q\} = 1_X, \tag{7.6}$$

it it clear that analogy should be between $\{-,-\}$ and $(i/\hbar)[-,-]$. Thus Dirac wrote:

> 'The strong analogy between the quantum P.B. defined by $[(i/\hbar)$ times the commutator] and the classical P.B. $(\ldots)$ leads us to make the assumption that the quantum P.B.'s, or at any rate the simpler ones of them, have the same values as the corresponding classical P.B.'s.'

Combined with Heisenberg's decisive idea that quantum mechanics should be an *Umdeutung* (i.e., reinterpretation) of classical mechanics, one is led to the idea that "quantization" should be given by a linear map

$$f \mapsto Q_\hbar(f), \tag{7.7}$$

where $f$ is some (smooth) function on phase space $X$ and $Q_\hbar(f)$ is some operator on some "corresponding" Hilbert space, whose identification or construction is a separate problem (but for $X = \mathbb{R}^2$ it should apparently be $L^2(\mathbb{R})$), such that

$$\frac{i}{\hbar}[Q_\hbar(f), Q_\hbar(g)] = Q_\hbar(\{f, g\}), \tag{7.8}$$

at least for functions $f, g \in C^\infty(X)$ with 'the simpler' Poisson brackets. If only to do justice to Schrödinger's example (7.3) - (7.4) with (7.5), one should also require

$$Q_\hbar(1_X) = 1_H. \tag{7.9}$$

The act of quantization should also preserve the adjoint, i.e., writing $f^*(x) = \overline{f(x)}$,

$$Q_\hbar(f^*) = Q_\hbar(f)^*. \tag{7.10}$$

Putting $\hbar$ on the right-hand side of eqs. (7.5) and (7.8), Dirac (and similarly the *Dreimännerarbeit* Born–Heisenberg–Jordan) concluded from these equations that:

> '*classical mechanics may be regarded as the limiting case of quantum mechanics when $\hbar$ tends to zero.*'

In the remainder of this chapter we try to do justice to this fabulous insight of Dirac's (and also of Born, Heisenberg, and Jordan, or even Planck, Einstein, and Bohr, none of whom seem to have quite appreciated the stupendous complexity of the claim).

## 7.1 Deformation quantization

Recall Definition C.121 of a continuous bundle of C*-algebras over some space $I$, which below is taken to be a subset of the unit interval $[0,1]$ that contains 0 as an accumulation point (so one may have e.g. $I = [0,1]$ itself, or $I = (1/\mathbb{N}) \cup \{0\}$).

**Definition 7.1.** *A* **deformation quantization** *of a Poisson manifold $X$ consists of a continuous bundle of C*-algebras $(A, \{\varphi_\hbar : A \to A_\hbar\}_{\hbar \in I})$ over $I$, along with maps*

$$Q_\hbar : \tilde{A}_0 \to A_\hbar \ (\hbar \in I), \tag{7.11}$$

*where $\tilde{A}_0$ is a dense subspace of $A_0 = C_0(X)$, such that:*

1. *$Q_0$ is the inclusion map $\tilde{A}_0 \hookrightarrow A_0$;*
2. *Each map $Q_\hbar$ is linear and satisfies* (7.10)*;*
3. *For each $f \in \tilde{A}_0$ the following map is a continuous section of the bundle:*

$$0 \mapsto f; \tag{7.12}$$

$$\hbar \mapsto Q_\hbar(f) \ (\hbar > 0); \tag{7.13}$$

4. *For all $f, g \in \tilde{A}_0$ one has the* **Dirac–Groenewold–Rieffel condition**

$$\lim_{\hbar \to 0} \left\| \frac{i}{\hbar}[Q_\hbar(f), Q_\hbar(g)] - Q_\hbar(\{f,g\}) \right\|_\hbar = 0. \tag{7.14}$$

It follows from the definition of a continuous bundle that continuity properties like

$$\lim_{\hbar \to 0} \|Q_\hbar(f)\| = \|f\|_\infty; \tag{7.15}$$

$$\lim_{\hbar \to 0} \|Q_\hbar(f)Q_\hbar(g) - Q_\hbar(fg)\| = 0, \tag{7.16}$$

are automatically satisfied. Let us note that condition (7.9) is absent from this definition, because $1_X \notin C_0(X)$ whenever $X$ is not compact, in which case typically also the C*-algebras $A_\hbar$ have no unit (see below). However, the given conditions turn out to be sufficiently powerful to produce the "right" examples. We give one of the main such examples without proof (the underlying analysis is quite forbidding). We put

$$A_0 = C_0(T^*\mathbb{R}^n); \tag{7.17}$$

$$A_\hbar = B_0(L^2(\mathbb{R}^n)) \ (\hbar > 0), \tag{7.18}$$

where $T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$ carries the canonical Poisson structure (3.34), and $A_\hbar$ is the C*-algebra of compact operators on the familiar Hilbert space $L^2(\mathbb{R}^n)$ of wave-functions on $\mathbb{R}^n$. For the sake of completeness we also mention that

$$A = C^*_r((\mathbb{R}^n \times \mathbb{R}^n)^T) \tag{7.19}$$

is the (reduced) C*-algebra of the tangent groupoid $(\mathbb{R}^n \times \mathbb{R}^n)^T$ to the pair groupoid $\mathbb{R}^n \times \mathbb{R}^n$ on $\mathbb{R}^n$, see §§C.16,C.19, where one may also find the maps $\varphi_\hbar$.

Let us summarize the situation. Continuity of the limit $\hbar \to 0$ is hard to envisage if one merely has the classical phase space $X = T^*\mathbb{R}^n$ and the quantum Hilbert space $L^2(\mathbb{R}^n)$ in mind. However, the move to either: the underlying Lie groupoids $T\mathbb{R}^n$ and $\mathbb{R}^n \times \mathbb{R}^n$, which jointly comprise the smooth tangent groupoid $\mathbb{R}^n \times \mathbb{R}^n)^T$, or: the corresponding canonically defined C*-algebras $C_0(T^*\mathbb{R}^n)$ and $B_0(L^2(\mathbb{R}^n))$, which are glued together as a continuous bundle (7.17) - (7.19), does give rise to a satisfactory structure that makes the limit $\hbar \to 0$ "continuous".

In this example, various possibilities for the quantization maps $Q_\hbar$ arise. As explained in §C.19, the groupoid structure underlying (7.17) - (7.18) suggests Weyl's prescription (C.549), which for convenience we reproduce:

$$Q^W_\hbar(f)\psi(x) = \int_{T^*\mathbb{R}^n} \frac{d^n p d^n y}{(2\pi\hbar)^n}\, e^{ip(x-y)/\hbar}\psi(y) f(\tfrac{1}{2}(x+y), p), \tag{7.20}$$

where $f$ lies in the image of $C_c^\infty(T\mathbb{R}^n)$ under the fiberwise Fourier transform (C.547). This image, then, is the space $\tilde{A}_0$ in Definition 7.1. We may rewrite (7.20) as

$$Q^W_\hbar(f) = \int_{T^*\mathbb{R}^n} \frac{d^n p d^n q}{(2\pi\hbar)^n}\, f(q,p)\Omega^W_\hbar(q,p), \tag{7.21}$$

where the operators in the integrand are given by

$$\Omega^W_\hbar(q,p)\psi(x) = 2^n e^{2ip(x-q)/\hbar}\psi(2q-x). \tag{7.22}$$

The purpose of (7.21) is that for each $\psi \in L^2(\mathbb{R}^n)$ we then obviously have

$$\langle\psi, Q^W_\hbar(f)\psi\rangle = \int_{T^*\mathbb{R}^n} \frac{d^n p d^n q}{(2\pi\hbar)^n}\, f(q,p) W^\psi_\hbar(p,q), \tag{7.23}$$

where $W^\psi_\hbar : T^*\mathbb{R}^n \to \mathbb{R}$ is the ***Wigner function***, given by

$$W^\psi_\hbar(p,q) = \hbar^{-n}\langle\psi, \Omega^W_\hbar(q,p)\psi\rangle \tag{7.24}$$

$$= \int_{\mathbb{R}^n} d^n v e^{ipv}\overline{\psi(q+\tfrac{1}{2}\hbar v)}\psi(q-\tfrac{1}{2}\hbar v). \tag{7.25}$$

If $\|\psi\| = 1$, then $W^\psi_\hbar$ gives a "phase space portrait" of the corresponding pure state $e_\psi$ on $B_0(L^2(\mathbb{R}))$. However, this portrait cannot be interpreted as a probability density on $T^*\mathbb{R}^n$, since the Wigner function is not necessarily positive. This reflects a problem with Weyl's quantization map $Q^W_\hbar$ itself (at fixed $\hbar > 0$). We say that $Q_\hbar$ as introduced in (7.11) is ***positive*** if, for each $f \in \tilde{A}_0 \subset A_0$ (seen as a C*-algebra),

$$f \geq 0 \Rightarrow Q_\hbar(f) \geq 0, \tag{7.26}$$

where positivity of $Q_\hbar(f)$ is defined in the C*-algebra $A_\hbar$ (which in the case at hand is $B_0(L^2(\mathbb{R}^n))$). This is not the case for $Q^W_\hbar$. Moreover, $Q^W_\hbar$ fails to be continuous, and for this reason it cannot be extended to $A_0$ (at least not in the obvious way, viz. by continuity). Fortunately, both problems can be resolved by a change in $Q_\hbar$.

A strict deformation quantization of $\mathbb{R}^2$ that *is* positive exists under the name of ***Berezin quantization***, denoted by $Q_\hbar^B$. However, the fundamental idea of the underlying coherent states goes back to Schrödinger. For each $(p,q) \in \mathbb{R}^2$ and $\hbar > 0$, define a unit vector $\phi_\hbar^{(p,q)} \in L^2(\mathbb{R})$, called a ***coherent state***, by

$$\phi_\hbar^{(p,q)}(x) = (\pi\hbar)^{-n/4} e^{-ipq/2\hbar} e^{ipx/\hbar} e^{-(x-q)^2/2\hbar}. \tag{7.27}$$

Writing $z = p + iq$, the transition probability between two coherent states is

$$|\langle \phi_\hbar^{(z)}, \phi_\hbar^{(z')} \rangle|^2 = e^{-|z-z'|^2/2\hbar}. \tag{7.28}$$

In terms of these coherent states, we define $Q_\hbar^B : C_0(T^*\mathbb{R}^n) \to B_0(L^2(\mathbb{R}^n))$ by

$$Q_\hbar^B(f) = \int_{T^*\mathbb{R}^n} \frac{d^n p d^n q}{2\pi\hbar} f(p,q) |\phi_\hbar^{(p,q)}\rangle\langle\phi_\hbar^{(p,q)}|, \tag{7.29}$$

where the integral is meant in the sense that for each $\psi, \varphi \in L^2(\mathbb{R}^n)$ we have

$$\langle \varphi, Q_\hbar(f)\psi \rangle = \int_{\mathbb{R}^{2n}} \frac{d^n p d^n q}{2\pi\hbar} f(p,q) \langle \varphi, \phi_\hbar^{(p,q)} \rangle \langle \phi_\hbar^{(p,q)}, \psi \rangle. \tag{7.30}$$

In particular, for each unit vector $\psi \in L^2(\mathbb{R}^n)$ we may write

$$\langle \psi, Q_\hbar(f)\psi \rangle = \int_{T^*\mathbb{R}^n} d\mu_\psi f, \tag{7.31}$$

where $\mu_\psi$ is the probability measure on $T^*\mathbb{R}^n$ with density

$$B_\hbar^\psi(p,q) = |\langle \phi_\hbar^{(p,q)}, \psi \rangle|^2, \tag{7.32}$$

called the ***Husimi function*** of $\psi \in L^2(\mathbb{R}^n)$; in other words, $\mu_\psi$ is given by

$$d\mu_\psi(p,q) = \frac{d^n p d^n q}{2\pi\hbar} B_\hbar^\psi(p,q). \tag{7.33}$$

Weyl and Berezin quantization are related in many ways, for example, by

$$Q_\hbar^B(f) = Q_\hbar^W(e^{\frac{\hbar}{4}\Delta_{2n}} f), \tag{7.34}$$

where $\Delta_{2n} = \sum_{j=1}^n (\partial^2/\partial p_j^2 + \partial^2/\partial(q^j)^2)$, from which it follows that Weyl and Berezin quantization are *asymptotically equal* in the sense that for any $f \in \tilde{A}_0$,

$$\lim_{\hbar\to 0} \|Q_\hbar^B(f) - Q_\hbar^W(f)\| = 0. \tag{7.35}$$

Indeed, this provides one way (among various others) of proving that $Q_\hbar^B$ satisfies Definition 7.1, where we note that even though $Q_\hbar^B$ is defined on all of $C_0(T^*\mathbb{R}^n)$, eq. (7.14) only holds on a suitable dense subspace thereof, such as $C_0^\infty(T^*\mathbb{R}^n)$.

## 7.2 Quantization and internal symmetry

In the presence of symmetries, Dirac's condition (7.8) can often be met by suitable functions $f$ and $g$ related to the symmetries in question, though such functions may be unbounded. This blasts the C*-algebraic framework, but it does so in a controlled way. We start with internal symmetries, like spin, which will be coupled to motion in the next step. Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$, to which we associate:

- The "classical" ***Lie–Poisson manifold*** $\mathfrak{g}^*$, see (3.98), whose Poisson bracket we now preface with a minus sign, so that instead of (3.98) and (3.99) we now have

$$\{f,g\}_-(\theta) = -C^c_{ab}\theta_c \frac{\partial f(\theta)}{\partial \theta_a}\frac{\partial g(\theta)}{\partial \theta_b}; \tag{7.36}$$

$$\{\hat{A},\hat{B}\}_- = -\widehat{[A,B]}. \tag{7.37}$$

  We write $\mathfrak{g}^*_-$ for this Poisson manifold.
- The "quantum-mechanical" reduced ***group(oid) C*-algebra*** $C^*_r(G)$, cf. §C.18, defined as the norm-closure of $\pi(C^\infty_c(G))$ within $B(L^2(G))$, where

$$\pi(\check{f})\psi = \check{f}*\psi; \tag{7.38}$$

$$\check{f}*\psi(x) = \int_G dy\,\check{f}(xy)\psi(y^{-1}), \tag{7.39}$$

  where $\check{f}\in C^\infty_c(G)$ and $\psi\in L^2(G)$, cf. (C.481), and $dy$ is Haar measure on $G$ (which also provides the measure defining the Hilbert space $L^2(G)$).

We then obtain a continuous bundle of C*-algebras, with fibers and total C*-algebra

$$A_0 = C^*_r(\mathfrak{g}); \tag{7.40}$$

$$A_\hbar = C^*_r(G)\ (\hbar > 0); \tag{7.41}$$

$$A = C^*_r(G^T), \tag{7.42}$$

where $\mathfrak{g}$ is seen as an abelian Lie group under addition, cf. Theorem C.123. We have

$$C^*_r(\mathfrak{g}) \cong C_0(\mathfrak{g}^*_-), \tag{7.43}$$

which isomorphism (i.e. of C*-algebras) is given by the Fourier transform

$$f(\theta) = \int_{\mathfrak{g}} d^nA\, e^{-i\theta(A)}\check{f}(A); \tag{7.44}$$

$$\check{f}(A) = \int_{\mathfrak{g}^*} \frac{d^n\theta}{(2\pi)^n}\, e^{i\theta(A)}\, f(\theta), \tag{7.45}$$

where initially $\check{f}\in C^\infty_c(G)$, and the map $\check{f}\mapsto f$ is subsequently extended to $C^*_r(G)$ by continuity. Here the normalization of Lebesgue measure $d^nA$ on $\mathfrak{g}$ is arbitrary, but the normalization of $d^n\theta$ is thereby fixed. In what follows, we take a (left-invariant)

Haar measure $dx$ on $G$ and fix the normalization of $d^nA$ by the condition

$$J(0) = 1 \tag{7.46}$$

in the definition of the Jacobian under the exponential map $\exp : \mathfrak{g} \to G$, i.e.,

$$J(A) = \frac{d(\exp(A))}{d^nA}. \tag{7.47}$$

With $\tilde{A}_0 = C_c^\infty(\mathfrak{g})$, the quantization map $Q_\hbar : C_c^\infty(\mathfrak{g}) \to C_r^*(G)$ is then given by

$$Q_\hbar(\check{f})(e^A) = \hbar^{-n}\check{f}(A/\hbar), \tag{7.48}$$

where $n = \dim(G)$ and we assume that $\hbar > 0$ is small enough that $\hbar$ times the support of $\check{f} \in C_c^\infty(\mathfrak{g})$ is contained in an open neighbourhood $U$ of $0 \in \mathfrak{g}$ where the exponential map is a diffeomorphism onto some open neighbourhood $U'$ of $e \in G$; otherwise a cutoff function should be included. Equivalently, defining $\tilde{A}_0 \subset C_0(\mathfrak{g}^*_-)$ as the image of $C_c^\infty(\mathfrak{g})$ under the Fourier transform $\check{f} \mapsto f$ (which consists of the so-called Paley–Wiener functions on $\mathfrak{g}^*$), the map $Q_\hbar : \tilde{A}_0 \to C_r^*(G)$ is given by

$$Q_\hbar(f)(e^A) = \int_{\mathfrak{g}^*} \frac{d^n\theta}{(2\pi\hbar)^n}\, e^{i\theta(A)/\hbar}\, f(\theta). \tag{7.49}$$

Although these maps satisfy (7.14), if $G$ is non-abelian there are no natural functions on $\mathfrak{g}^*$ whose quantizations satisfy the exact Dirac condition (7.8). This is a limitation of the C*-algebraic framework, since candidate functions like

$$\hat{A} : \mathfrak{g}^* \to \mathbb{R}; \tag{7.50}$$

$$\hat{A}(\theta) = \theta(A), \tag{7.51}$$

whose Poisson brackets (3.99) are promising, are unbounded. However, this is easily remedied by regarding $C_r^*(G)$ as an algebra of bounded operators on the Hilbert space $L^2(G)$—which indeed is the way it was originally defined—rather than abstractly. This "spatial" context allows the passage to the Lie algebra, as reviewed in §5.6, see especially (5.156) - (5.161). First note that (7.38) - (7.39) is a special case of (5.172), where $H = L^2(G)$ and $u = u_L$, i.e., the ***left-regular representation***

$$u_L(y)\psi(x) = \psi(y^{-1}x). \tag{7.52}$$

In this representation, the construction (5.156) then realizes $\mathfrak{g}$ as right-invariant differential operators on the Gårding domain$D_G \subset C^\infty(G)$. By definition of $C_r^*(G)$, seen as an operator on $L^2(G)$ the function $Q_\hbar(f)$ is given in coordinates by

$$Q_\hbar(f) = \int_{\mathfrak{g}} d^nX\, J(X) \int_{\mathfrak{g}^*} \frac{d^n\theta}{(2\pi\hbar)^n}\, e^{i\theta(X)/\hbar}\, f(\theta) u_L\left(\exp\left(\sum_j X_jT_j\right)\right). \tag{7.53}$$

Here $(X_1,\ldots,X_n)$ in (7.53) are coordinates on $\mathfrak{g}$ defined by a basis choice $(T_1,\ldots,T_n)$, i.e., $A=\sum_i X_i T_i$. The function $\hat{T}_j$ on $\mathfrak{g}^*$ is then simply given by the coordinate function $\hat{T}_j(\theta)=\theta_j$. Now take $A\in\mathfrak{g}$ and assume that $f=\hat{A}$. This function is unbounded, but the following formal calculation is rigorously correct on the Gårding domain and may be justified by some distribution theory. For simplicity we assume that $G$ is unimodular, in which case $J(X)=1+O(X^2)$ as $X\to 0$, so that all first derivatives of $J$ vanish at $X=0$. Taking $f=\hat{T}_j$ in (7.53) then gives

$$
\begin{aligned}
Q_\hbar(\hat{T}_j) &= \int_{\mathfrak{g}} d^nX\, J(X) \int_{\mathfrak{g}^*} \frac{d^n\theta}{(2\pi\hbar)^n}\, e^{i\theta(X)/\hbar}\, \theta_j u_L\left(\exp\left(\sum_j X_j T_j\right)\right) \\
&= -i\int_{\mathfrak{g}} d^nX\, J(\hbar X) u_L\left(\exp\left(\hbar\sum_j X_j T_j\right)\right)\frac{\partial}{\partial X_j}\delta(X) \\
&= i\hbar u_L'(X_j),
\end{aligned}
\tag{7.54}
$$

from which we obtain

$$
Q_\hbar(\hat{A}) = i\hbar u_L'(A) = \pi_L(A). \tag{7.55}
$$

This explains the need for *minus* the Lie–Poisson bracket, since instead of (3.99) we now have (7.37), so that (5.160) gives the exact result (7.8) for $f=\hat{A}$ and $g=\hat{B}$:

$$
\frac{i}{\hbar}[Q_\hbar(\hat{A}),Q_\hbar(\hat{B})] = Q_\hbar(\{\hat{A},\hat{B}\}_-). \tag{7.56}
$$

The minus sign in the Lie–Poisson bracket could have been avoided by writing $\check{f}(-A/\hbar)$ in (7.48), whose minus sign would have propagated into (5.159) and hence in the commutation relations (5.160), but the latter are so engrained in the physics literature that we see the minus sign on the bracket in (7.56) as the lesser evil.

Any continuous unitary representation $u_\lambda$ of $G$ (where $\lambda$ is some label) induces a representation $u_\lambda^{\int}$ of $C_c^\infty(G)$ by (5.173), which may be extended to a representation of $C^*(G)$ by continuity (the same is true for $C_r^*(G)$ provided $u_\lambda$ is weakly contained in $L^2(G)$, cf. §C.18). This gives operators $u^{\int}(Q_\hbar(f))$ which, by the same formal computation as for the case $u=u_L$ above, for $A\in\mathfrak{g}$ rigorously give rise to operators

$$
\pi_\lambda(A) = i\hbar u_\lambda'(A), \tag{7.57}
$$

satisfying the like of (5.160) for fixed values of $\hbar$ (but without control over the limit $\hbar\to 0$). Many commutation relations in quantum mechanics take this form, where both irreducible and reducible representations $u$ give rise to interesting examples. The reducible case typically comes from group actions and is best studied using the formalism of action groupoids reviewed in the next section, where we will see that further operators start playing a role. The irreducible case, on the other hand, gives rise to intriguing new examples of continuous bundles of C*-algebras, where $\hbar$ (now related the label $\lambda$) takes values in a discrete set and may be sent to zero, cf. §8.1.

## 7.3 Quantization and external symmetry

We now generalize the setting of the preceding section from groups taken by themselves to group actions. Let a Lie group $G$ act smoothly on some manifold $Q$; for example, we may have $Q = \mathbb{R}^3$ with either $G = SO(3)$ acting by rotations, or $G = \mathbb{R}^3$ action by translations. We now take $X = \mathfrak{g}^* \times Q$. Recalling the notation (3.71) and writing $\delta_a \equiv \delta_{T_a}$, we define the ***action Poisson bracket***

$$\{f,g\} = -C^c_{ab}\theta_c \frac{\partial f}{\partial \theta_a}\frac{\partial g}{\partial \theta_b} + \xi_a f \frac{\partial g}{\partial \theta_a} - \frac{\partial f}{\partial \theta_a}\xi_a g. \tag{7.58}$$

Interesting special cases arise if we take $A \in \mathfrak{g}$ and define $\hat{A} \in C^\infty(\mathfrak{g}^*)$ as before, i.e., $\hat{A}(\theta) = \theta(A)$, now regarded as a function on $\mathfrak{g}^* \times Q$ (ignoring the *second* argument $q$). Similarly, if $\tilde{f} \in C^\infty(Q)$ we write $\hat{f}$ for the corresponding function on $\mathfrak{g}^* \times Q$ (ignoring the *first* argument $\theta$). This gives the coordinate-independent expressions

$$\{\hat{A},\hat{B}\} = -\widehat{[A,B]}; \tag{7.59}$$
$$\{\hat{A},\hat{f}\} = -\delta_A f; \tag{7.60}$$
$$\{\hat{f},\hat{g}\} = 0. \tag{7.61}$$

Clearly, if $Q$ is a point (with trivial $G$-action) we recover (minus) the Lie–Poisson structure on $\mathfrak{g}^*$. If, on the other hand, $Q = \mathbb{R}^3$ and $G = \mathbb{R}^3$ acts on $Q$ by translation, i.e., $\mathbf{a}\cdot\mathbf{x} = \mathbf{x}+\mathbf{a}$, we recover the canonical Poisson bracket (3.34), where the momenta $p_a$ ($a = 1,\ldots,n$) are identified with the coordinates $\theta_a$ on the dual of the Lie algebra of $\mathbb{R}^3$, which is just $\mathbb{R}^3$ itself (with the usual basis $(e_1,e_2,e_3)$). Therefore, the Poisson bracket (3.34) on $\mathbb{R}^{2n}$ may be generalized in two ways:

1. By passing to arbitrary cotangent bundles $T^*M$, whose canonical Poisson bracket is still given in local coordinates by (3.34), which emphasizes the role of momenta as fiber coordinates on $T^*M$.
2. By passing to the setting discussed here, which emphasizes the role of momenta as generators of global translations of the base space $\mathbb{R}^3$ (a property that breaks the $p$-$q$ symmetry and cannot be generalized to arbitrary cotangent bundles).

A richer structure emerges if we keep $Q = \mathbb{R}^3$ but now take $G = E(3)$, i.e.,

$$E(3) = SO(3) \ltimes \mathbb{R}^3, \tag{7.62}$$

known as the ***Euclidean group***. To explain its group structure, let some group $L$ act on a vectors space $V$, seen as an abelian group under addition. Then the operations

$$(\lambda,v)\cdot(\lambda',v') = (\lambda\lambda', v+\lambda\cdot v'); \tag{7.63}$$
$$(\lambda,v)^{-1} = (\lambda^{-1}, -\lambda^{-1}\cdot v), \tag{7.64}$$

turn $G = L \ltimes V$ into a group, called the ***semi-direct product*** of $L$ and $V$.

Then $E(3)$ acts on $\mathbb{R}^3$ in the obvious way, giving rise to the Poisson manifold $\mathfrak{g}^* \times Q = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ (since $\mathfrak{so}(3) \cong \mathbb{R}^3$). We now also have generators $(J_1, J_2, J_3)$ of the Lie algebra of $SO(3)$, with corresponding functions $\hat{J}_i$, as well as standard coordinate functions $(q_1, q_2, q_3)$ on $Q = R^3$, giving rise to the Poisson brackets

$$\{\hat{J}_i, \hat{J}_j\} = -\varepsilon_{ijk}\hat{J}_k;\ \ \{\hat{J}_i, p_j\} = -\varepsilon_{ijk}p_k;\ \ \{p_i, p_j\} = 0; \tag{7.65}$$
$$\{\hat{J}_i, q_j\} = -\varepsilon_{ijk}q_k;\ \ \{p_i, q_j\} = \delta_{ij};\ \ \{q_i, q_j\} = 0. \tag{7.66}$$

The appropriate target C*-algebra $C^*_r(G, Q)$ for quantization is a generalization of $C^*_r(G)$, constructed in a similar way, as explained in §C.18. For the moment it is enough to know that $C^*_r(G, Q)$ is the completion of the function space $C^\infty_c(G \times Q)$, seen as a $^*$-algebra in the operations (C.526) - (C.527), in a suitable norm, namely

$$\|f\|_r = \|\tilde{\rho}(f)\|, \tag{7.67}$$

where the representation $\tilde{\rho} : C^\infty_c(G \times Q) \to B(L^2(G \times Q))$ is given by (C.530). In case that $Q$ has a $G$-invariant measure $\nu$ (still with support $Q$), the operator

$$w : L^2(G \times Q) \to L^2(G \times Q); \tag{7.68}$$
$$w\psi(x, q) = \psi(x, x^{-1}q), \tag{7.69}$$

is unitary, and in terms of the notation

$$\tilde{u}(y) = wu(y)w^*,\ \tilde{\pi}(\tilde{f}) = w\pi(\tilde{f})w^*,\ \tilde{\rho}(f) = w\rho(f)w^*, \tag{7.70}$$

the formulae (C.528) - (C.530) take the slightly more appealing form

$$\tilde{u}(y)\psi(x, q) = \psi(y^{-1}x, y^{-1}q); \tag{7.71}$$
$$\tilde{\pi}(\tilde{f})\psi(x, q) = \tilde{f}(q)\psi(x, q); \tag{7.72}$$
$$\tilde{\rho}(f)\psi(x, q) = \int_G dy\, f(y, q)\psi(y^{-1}x, y^{-1}q). \tag{7.73}$$

The simplification thus gained especially concerns the position functions (7.72).

Analogously to (7.49), the quanitzation maps are given by

$$Q_\hbar\ :\ C_0(\mathfrak{g}^* \times Q) \to C^*_r(G, Q); \tag{7.74}$$
$$Q_\hbar(f)(e^A, q) = \int_{\mathfrak{g}^*} \frac{d^n\theta}{(2\pi\hbar)^n}\, e^{i\theta(A)/\hbar}\, f(\theta, e^{-\frac{1}{2}A} \cdot q), \tag{7.75}$$

where, as in the pure group case, strictly speaking $f$ must lie in the dense subspace of $C_0(\mathfrak{g}^* \times Q)$ consisting of Paley–Wiener functions (in $A$) that are the Fourier transform (in the first argument) of functions that lie in $C^\infty_c(\mathfrak{g} \times Q)$.

Computations similar to (7.54) then establish, for $A \in \mathfrak{g}$ and $\tilde{f} \in C^\infty(Q)$ as before,

$$Q_\hbar(\hat{A}) = i\hbar\tilde{u}'(A); \tag{7.76}$$
$$Q_\hbar(\hat{f}) = \tilde{\pi}(\tilde{f}). \tag{7.77}$$

Form these formulae and (7.59) - (7.60), it is easy to verify that Dirac's exact condition (7.8) holds in the following special cases:

$$\frac{i}{\hbar}[Q_{\hbar}(\hat{A}), Q_{\hbar}(\hat{B})] = Q_{\hbar}(\{\hat{A}, \hat{B}\}); \tag{7.78}$$
$$\frac{i}{\hbar}[Q_{\hbar}(\hat{A}), Q_{\hbar}(\hat{f})] = Q_{\hbar}(\{\hat{A}, \hat{f}\}); \tag{7.79}$$
$$\frac{i}{\hbar}[Q_{\hbar}(\hat{f}), Q_{\hbar}(\hat{g})] = Q_{\hbar}(\{\hat{f}, \hat{g}\}) = 0. \tag{7.80}$$

These might be regarded as infinitesimal versions of the covariance condition (C.514), specialized to the case at hand. We formalize this special case as follows.

**Definition 7.2.** *Let $G$ be a locally compact group and let $Q$ be a space equipped with some continuous $G$-action. A* **system of imprimitivity** $(u(G), \pi(C_0(Q)))$ *for the given group action $G \circlearrowright Q$ is a combination of a strongly continuous unitary representation $u$ of $G$ and a nondegenerate representation $\pi$ of $C_0(Q)$, both defined on the same Hilbert space, that for each $x \in G$ and $\tilde{f} \in C_0(Q)$ satisfies*

$$u(x)\pi(\tilde{f})u(x)^* = \pi(\widetilde{L_x f}). \tag{7.81}$$

Here $\widetilde{L_x f}(q) = \tilde{f}(x^{-1}q)$, as usual. We recall from §C.18 that such systems of imprimitivity bijectively correspond to degenerate representations $\rho \equiv \pi \rtimes u^{\int}$ of $C^*(G, Q)$ through (C.515), which in the special case (C.524) - (C.525) comes down to

$$\rho(f) = \int_G dx\, \pi(f(x, \cdot))u(x). \tag{7.82}$$

The formulae (7.71) - (7.73) define such a system of imprimitivity on the Hilbert space $H = L^2(G \times Q)$. However, this cannot be the end result of quantization, since this space is typically reducible under the pair $(u(G), \pi(C_0(Q)))$, or, equivalently, under $\rho(C^*(G, Q))$. For example, this is the case for $G = \mathbb{R}^3$ or $G = E(3)$ acting on $Q = \mathbb{R}^3$ in the natural way discussed above, for which we obtain $H = L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ or even $H = L^2(E(3) \times \mathbb{R}^3)$. In the former case we do obtain the correct position operators $q^i$, but for the momentum operators we find the curious expression $-i\hbar(\partial/\partial x^i + \partial/\partial q^i)$—to their credit, these do satisfy the canonical commutation relations (7.5), since these follow from (7.78) - (7.80), which in turn follow from the covariance condition (7.81) defining a system of imprimitivity.

Instead, we would prefer the Hilbert space $H = L^2(\mathbb{R}^3)$ expected from elementary quantum mechanics (without spin), equipped with the system of imprimitivity

$$u(y)\psi(q) = \psi(y^{-1}q); \tag{7.83}$$
$$\pi(\tilde{f})\psi(q) = \tilde{f}(q)\psi(q). \tag{7.84}$$

The answer lies in the search for *irreducible* systems of imprimitivity $(u(G), \pi(C_0(Q)))$, or, equivalently, *irreducible* representations of $\rho(C^*(G, Q))$; see §7.5.

## 7.4 Intermezzo: The Big Picture

First, however, we summarize and generalize the results in this chapter so far into what we call ***The Big Picture***. This arose in the 1990s from efforts to relate Mackey's quantization theory based on systems of imprimitivity (which Mackey himself saw as the natural implementation of what he called ***Weyl's Program***, i.e. the construction of the basic operators of quantum mechanics from group-theoretical considerations) to deformation quantization (and hence to the tradition started by Dirac, as continued by Groenewold, Moyal, Berezin, Flato, Rieffel, and others).

The Big Picture is technically based on the theory of ***Lie groupoids*** (already alluded to in the preceding sections) and ***Lie algebroids***. For a precise definition of the former we refer to Definition C.115; briefly, a ***groupoid*** $G$ is an object like a group, where however *multiplication* is defined only partially (although the *inverse* is defined for each element). To see which elements can be multiplied, one has maps $s,t : G_1 \to G_0$ from the *total space* $G_1$ of the groupoid to its *base space* $G_0$, such that the product $xy \in G_1$ of $x,y \in G_1$ is defined whenever $s(x) = t(y)$, and satisfies $s(xy) = s(y)$, $t(xy) = t(x)$, and $s(x^{-1}) = t(x)$. Four relevant examples are:

- ***Spaces***, where $G_1 = G_0 = Q$ for some set $Q$, with $s(x) = t(x) = x$ for all $x \in G_1$, and hence $xy$ is defined iff $y = x$, with result $xx = x$; furthermore, $x^{-1} = x$.
- ***Groups***, where $G_1 = G$ and $G_0 = \{e\}$, with $s(x) = t(x) = e$ for all $x$, so that all elements can be multiplied and the notion of a groupoid reduces to a group.
- ***Pair groupoids*** over a set $Q$ have base space $G_0 = Q$, total space $G_1 = Q \times Q$, and projections $s(q,q') = q'$ and $t(q,q') = q$, so that $(q,q')(r,r')$ is defined iff $q' = r$, resulting in $(q,q')(q',r') = (q,r')$. The inverse is given by $(q,q')^{-1} = (q',q)$.
- ***Action groupoids*** (also called ***semi-direct product groupoids***) are important in what follows. These originate in some group action we denote by $G \circlearrowright Q$, where $G$ is a group and $Q$ is a set. The ensuing groupoid is called $\Gamma = G \ltimes Q$, where

$$\Gamma_1 = G \times Q,\ \ \Gamma_0 = Q,\ \ s(x,q) = x^{-1}q,\ \ t(x,q) = q, \tag{7.85}$$

so that products $(x,q)(y,q')$ are defined iff $q' = x^{-1}q$, with result

$$(x,q)(y,x^{-1}q) = (xy,q). \tag{7.86}$$

Finally, the inverse is (necessarily) given by

$$(x,q)^{-1} = (x^{-1},x^{-1}q). \tag{7.87}$$

A ***Lie groupoid*** is a groupoid $G$ where $G_1$ and $G_0$ are manifolds and all operations are smooth. In all examples just given this requires $Q$ to be a manifold, and in the last one $G$ should be a Lie group, and the given action $G \times Q \to Q$ must be smooth.

Generalizing the construction of a Lie algebra $\mathfrak{g}$ from a given Lie group $G$, a Lie groupoid comes with an associated linearized (or "infinitesimal") structure, called a *Lie algebroid*. As in the group case, this differential-geometric notion can also be defined independently of its origin in the theory of Lie groupoids, as follows:

**Definition 7.3.** *A* **Lie algebroid** *$E$ over a manifold $Q$ is a vector bundle $E \xrightarrow{\pi} Q$ with a vector bundle map $E \xrightarrow{\alpha} TQ$ (called the* **anchor***), as well as with a Lie bracket $[\,,\,]$ on the space $C^\infty(Q,E)$ of smooth cross-sections of $E$, satisfying the Leibniz rule*

$$[\sigma_1, f\cdot\sigma_2] = f\cdot[\sigma_1,\sigma_2] + (\alpha\circ\sigma_1 f)\cdot\sigma_2 \tag{7.88}$$

*for all $\sigma_1,\sigma_2 \in C^\infty(Q,E)$ and $f\in C^\infty(Q)$ (here $\alpha\circ\sigma_1$ is a vector field on $Q$).*

It follows that the map $\sigma \mapsto \alpha\circ\sigma : C^\infty(Q,E) \to C^\infty(Q,TQ)$ induced by the anchor is a homomorphism of Lie algebras, where the latter is equipped with the usual commutator of vector fields (this homomorphism property used to be part of the definition of a Lie algebroid, but in fact it follows from the stated definition).

Lie algebroids generalize (finite-dimensional) Lie algebras as well as tangent bundles, and the (infinite-dimensional) Lie algebra $C^\infty(Q,E)$ could be said to be of geometric origin in the sense that it derives from an underlying finite-dimensional geometrical object. Similar to the above list of examples of Lie groupoids, one has the following basic classes of Lie algebroids.

- ***Manifolds***, where $E = Q$, seen as the zero-dimensional vector bundle over $Q$, evidently with identically vanishing Lie bracket and anchor.
- ***Lie algebras***, where $E = \mathfrak{g}$ and $Q$ is a point (which may be identified with the identity element of any Lie group with Lie algebra $g$) and anchor $\alpha = 0$.
- ***Tangent bundles*** over a manifold $Q$, where $E = TQ$ and $\alpha = \mathrm{id} : TQ \to TQ$, with the Lie bracket given by the usual commutator of vector fields (or derivations).
- ***Action algebroids*** (or ***semi-direct product algebroids***) are defined by a $\mathfrak{g}$-action on a manifold $Q$, i.e. a Lie algebra homomorphism $\mathfrak{g} \to C^\infty(Q,TQ)$, $A \mapsto \delta_A$, where we identify vector fields on $Q$ with derivations on $C^\infty(Q)$—these are often, but not necessarily, obtained from a $G$-action on $Q$ via see (3.71). We write $E = \mathfrak{g} \ltimes Q$, which is $E = \mathfrak{g}\times Q$ as a trivial bundle (with $\pi$ the projection on the second space), and $\alpha(A,q) = -\delta_A(q) \in T_qQ$, where $A\in\mathfrak{g}$. The Lie bracket is given by

$$[\sigma_1,\sigma_2](q) = [\sigma_1(q),\sigma_2(q)]_{\mathfrak{g}} + \delta_{\sigma_2}\sigma_1(q) - \delta_{\sigma_1}\sigma_2(q). \tag{7.89}$$

These examples may also be recovered as special cases of the following construction that canonically associates a Lie algebroid $\mathrm{Lie}(G)$ to a Lie groupoid $G$: as a vector bundle, $\mathrm{Lie}(G)$ is the restriction of $\ker(t_*)$ to $G_0$ (where $t_* : TG_1 \to TG_0$ is the derivative map of the source projection $t : G_1 \to G_0$), and the anchor is $\alpha = s_*$ (one may alternatively define $\mathrm{Lie}(G)$ as the normal bundle to the object inclusion map $i : G_0 \hookrightarrow G_1$, cf. Definition C.115, but this makes the definition of the anchor a bit more complicated). As in the Lie group case, one may identify sections of $\mathrm{Lie}(G)$ with left-invariant vector fields on $G$, and under this identification the Lie bracket on $C^\infty(G_0,\mathrm{Lie}(G))$ is by definition given by the commutator of vector fields.

Conversely, one may ask whether a given Lie algebroid $E$ is ***integrable***, in that $E \cong \mathrm{Lie}(G)$ for some Lie groupoid $G$ (where the isomorphism sign $\cong$ means that a pertinent vector bundle isomorphism $E \cong \ker(t_*)_{|G_0}$ should preserve all relevant structure). Unlike the special case of Lie groups (where Lie's Third Theorem 5.41 settles this in the positive), this is not necessarily the case, but that is of no concern.

We now state a crucial connection between Lie algebroids and Poisson geometry.

**Proposition 7.4.** *The dual vector bundle $E^*$ of a Lie algebroid $E$ is a Poisson manifold, whose Poisson bracket on $C^\infty(E^*)$ is defined by the following special cases:*

$$\{f,g\} = 0 \;\; (f,g \in C^\infty(Q)); \tag{7.90}$$

$$\{\tilde{\sigma}, f\} = -\alpha \circ \sigma f \;\; (\sigma \in C^\infty(Q,E), f \in C^\infty(Q)); \tag{7.91}$$

$$\{\tilde{\sigma}_1, \tilde{\sigma}_2\} = -\widetilde{[\sigma_1, \sigma_2]}, \tag{7.92}$$

*where $\tilde{\sigma} \in C^\infty(E^*)$ is defined by a given section $\sigma$ of $E$ through the obvious pairing.*

*Conversely, if the dual $F^*$ to a given vector bundle $F \to Q$ is a Poisson manifold such that the Poisson bracket of two linear functions is linear, then $F \cong E$ for some Lie algebroid $E$ over $Q$, with the above Poisson structure on $E^*$.*

Following our earlier lists, the main examples are:

- A manifold $Q$, seen as the dual to the zero-dimensional vector bundle $Q \to Q$, carries the zero Poisson structure.
- The dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ acquires (minus) the Lie–Poisson structure (3.98).
- A cotangent bundle $T^*Q$ acquires (minus) the Poisson structure defined by its standard symlectic structure, cf. (3.34).
- The dual $\mathfrak{g}^* \ltimes Q$ of an action algebroid acquires the Poisson bracket (7.58).

The following theorem displays a rich and physically relevant class of examples of Definition 7.1 of deformation quantization. The key point is that a Lie groupoid $G$ defines both classical and quantum data, namely the (reduced) Lie groupoid C*-algebra $C^*_{(r)}(G)$ (cf. §C.17) and the Poisson manifold $\mathrm{Lie}(G)^*$ (cf. Proposition 7.4), and these are continuously (even smoothly) related through the tangent groupoid $G^T$ (cf. Proposition C.117) and its associated Lie groupoid C*-algebra $C^*_{(r)}(G^T)$.

**Theorem 7.5.** *For any Lie groupoid $G$, the bundle of C*-algebras given by*

$$A_0 = C_0(\mathrm{Lie}(G)^*) \;\; (\hbar = 0); \tag{7.93}$$

$$A_\hbar = C^*(G) \qquad (0 < \hbar \leq 1); \tag{7.94}$$

$$A = C^*(G^T), \tag{7.95}$$

*defines a deformation quantization of the Poisson manifold* $\mathrm{Lie}(G)^*$ *over $I = [0,1]$. The same statement holds for the corresponding* reduced *groupoid C*-algebras.*

The key lemma for this theorem is Theorem C.123, which provides the continuity of the given bundle of C*-algebras. A lengthy computation shows that also the Dirac–Groenewold–Rieffel condition (7.14) is met. In this light, the quantization of the phase space $T^*\mathbb{R}^n$ in §7.1 then corresponds to the pair groupoid $G = \mathbb{R}^n \times \mathbb{R}^n$ on $\mathbb{R}^n$, the one in §7.2 follows from the special case where the Lie groupoid $G$ is "simply" a Lie group, and the case of §7.3, which puts Mackey's quantization theory in a deformation framework, is obviously given by the action groupoid $G \ltimes Q$. Finally, the space groupoid $G_0 = G_1 = Q$ gives a trivial continuous bundle of C*-algebras, where $A_\hbar = C_0(Q)$ for all $\hbar \in [0,1]$, and $Q$ carries the zero Poisson bracket.

## 7.5 Induced representations and the imprimitivity theorem

Returning to §7.3, we recall the bijective correspondence between systems of imprimitivity $(u(G), \pi(C_0(Q)))$ and non-degenerate representations of the C*-algebra $C^*(G,Q)$ of the action groupoid defined by the given action $G \circlearrowright Q$. This correspondence preserves irreducibility, and our task is to find irreducible representations.

It was recognized at least 50 years ago that this task can be carried out if the group action satisfies a certain regularity condition, and is hopeless otherwise. This is sometimes called the ***Mackey–Glimm dichotomy***. The condition in question may be stated in a number of equivalent ways (whose equivalence is not at all obvious).

First, we recall some terminology from topology. Let $X$ be a space. One calls $Y \subset Y' \subseteq X$ ***relatively open*** in $Y'$ if there is an open set $U \subset X$ such that $Y = Y' \cap U$. A subset $Y \subset X$ is ***locally closed*** if each $y \in Y$ has an open neighbourhood $U$ in $X$ such that $U \cap Y$ is closed, and finally "$X$ is $T_0$" if for any two distinct points there is an open set that contains exactly one of them. Furthermore, each $q \in Q$ defines a $G$-orbit through $q$ denoted by $G \cdot q$, as well as a stabilizer (or "little group")

$$G_q = \{x \in G \mid x \cdot q = q\}. \tag{7.96}$$

For any subgroup $H \subset G$, we denote the equivalence class of $x$ in $G/H$ by $[x]$.

**Definition 7.6.** *A smooth action of a Lie group G on a manifold Q is called* **regular** *if one and hence each of the following equivalent conditions is satisfied:*

1. *Each G-orbit in Q is relatively open in its closure;*
2. *Each G-orbit in Q is locally closed;*
3. *The quotient space $Q/G$ of G-orbits in Q is $T_0$;*
4. *Each map $[x] \mapsto xq$ is a homeomorphism from $G/G_q$ to the orbit $G \cdot q$ ($q \in Q$).*

Probably the simplest example of a non-regular action is the action $\mathbb{Z} \circlearrowright \mathbb{T}$ given by

$$n : z \mapsto e^{2\pi i n\theta} z, \tag{7.97}$$

where $\theta \in \mathbb{R}\backslash\mathbb{Q}$ (here $\mathbb{Z}$ may be seen as a zero-dimensional Lie group with infinitely many components—in fact, Definition 7.6 more generally applies to second countable locally compact groups and spaces that are "almost Hausdorff"). Indeed, each orbit is dense in $\mathbb{T}$ (but not open), and the orbit space $\mathbb{T}/\mathbb{Z}$ has no proper open sets.

**Theorem 7.7.** *Let a group action $G \circlearrowright Q$ be regular. Then the irreducible representations of the associated action groupoid C*-algebra $C^*(G,Q)$—and hence also the irreducible systems of imprimitivity $(u(G), \pi(C_0(Q)))$—are classified up to unitary equivalence by pairs $(\mathscr{O}, u_\chi)$, where $\mathscr{O}$ is a G-orbit in Q and $u_\chi$ is an irreducible representation of the stabilizer $G_q$ of an arbitrary point $q \in \mathscr{O}$, with an explicit construction of the corresponding representation $\rho_{(\mathscr{O},u_\chi)}(C^*(G,Q))$. Two such representations $\rho_{(\mathscr{O},u_\chi)}$ and $\rho_{(\mathscr{O}',u'_\chi)}$ are equivalent iff $\mathscr{O} = \mathscr{O}'$ and, given that $q' = xq$ and hence $G_{q'} = xG_qx^{-1}$ for some $x \in G$, $u'_\chi$ is unitarily equivalent to $u_\chi \circ \mathrm{Ad}(x)$. Finally, any irreducible representation $\rho$ is unitarily equivalent to some $\rho_{(\mathscr{O},u_\chi)}$.*

In the simplest case, $Q$ is equal to a point, so that $C^*(G,Q) = C^*(G)$, and we find that irreducible representations of $C^*(G)$ (which are necessarily non-degenerate) bijectively correspond to unitary irreducible representations of $G$. In the next easiest case, $G$ acts nontrivially but still transitively on $Q$, in which case the action is clearly regular and $Q \cong G/H$ through the $G$-equivariant map in no. 4 of the above definition (read in the opposite direction), i.e., we pick some $q_0 \in Q$, define $H = G_{q_0}$, and finally map $Q$ to $G/H$ by $q \mapsto [x]$, where $q = xq_0$ (this map is well defined); in that case, we might as well have assumed that $Q = G/H$ to begin with. The following important corollary of Theorem 7.7 is called the ***Imprimitivity Theorem***.

**Corollary 7.8.** *Up to unitary equivalence, irreducible representations of* $C^*(G,G/H)$ *(or, equivalently, of pairs* $(\pi(C_0(G/H)),u(G))$ *satisfying the covariance condition* (7.81)*) bijectively correspond to unitary irreducible representations of* $H$.

In preparation for the general case stated in Theorem 7.7, and also as a goal in itself, we first give an explicit construction of the irreducible representation $\rho^\chi$ of $C^*(G,G/H)$ corresponding to a given unitary irreducible representation $u_\chi(H)$, where we label the unitary irreducible representations of $H$ (up to unitary equivalence) by $\chi \in \hat{H}$ (where $\hat{H}$ is the set of unitary equivalence classes of unitary irreducible representations of $H$, cf. §C.15 for the abelian case), and let the corresponding representation $\rho^\chi(C^*(G,G/H))$—or the pair $\pi^\chi(C_0(G/H))$ and $u^\chi(G)$—inherit this label (in raised form, in order to prevent confusion between $u_\chi(H)$ and $u^\chi(G)_{|H}$).

The construction of $\rho^\chi(C^*(G,G/H))$—or, equivalently, of a system of imprimitivity $(\pi^\chi(C_0(G/H)),u^\chi(G))$—from $u_\chi(H)$ proceeds by the technique of ***induced representations*** (which physicists may be familiar with from the representation theory of the Poincaré group, see Theorem 7.9 below). We start from a specific realization of $u_\chi(H)$ on a Hilbert space $H_\chi$ (which is finite-dimensional if $H$ is compact or abelian). From this, we construct a new Hilbert space $H^\chi$, whose realization depends on the choice of a ***quasi-invariant measure*** $\nu$ on $G/H$, i.e., a (non-zero) measure whose null-sets are $G$-invariant in the sense that if $\nu(A) = 0$ for some (Borel) measurable $A \subset G/H$, then also $\nu(x \cdot A) = 0$ for each $x \in G$. This will surely be the case if $\nu$ is ***invariant***, i.e., if $\nu(x \cdot A) = \nu(A)$ for each measurable $A$, but invariant measures on $G/H$ may not exist, whereas quasi-invariant measures always do.

We now consider (measurable) functions $\psi : G \to H_\chi$ that satisfy

$$\psi(xh) = u_\chi(h^{-1})\psi(x), \tag{7.98}$$

for every $x \in G$ and $h \in H$; equivalently, we may say that

$$u_\chi(h) \circ R_h\psi = \psi, \tag{7.99}$$

for each $h \in H$, where $R_h\psi(x) = \psi(xh)$. Now if $\psi$ and $\varphi$ both satisfy (7.98), then, by unitarity of $u_\chi$, their inner product $\langle\varphi(x),\psi(x)\rangle_{H_\chi}$ in $H_\chi$ is $H$-invariant, in that

$$\langle\varphi(xh),\psi(xh)\rangle_{H_\chi} = \langle\varphi(x),\psi(x)\rangle_{H_\chi}. \tag{7.100}$$

Hence the function $x \mapsto \langle \varphi(x), \psi(x) \rangle_{H_\chi}$, *a priori* defined from $G$ to $\mathbb{C}$, induces a function $[x] \mapsto \langle \varphi(x), \psi(x) \rangle_{H_\chi}$ from $G/H$ to $\mathbb{C}$. We write the latter function as $\langle \varphi, \psi \rangle_{H_\chi}[x]$; in particular, taking $\varphi = \psi$, we write $\|\psi\|^2_{H_\chi}[x] = \langle \psi(x), \psi(x) \rangle_{H_\chi}$. We may then define a new Hilbert space $H^\chi$ that consists of all measurable functions $\psi : G \to H_\chi$ that for each $h \in H$ satisfy (7.98), and are square-integrable on $G/H$:

$$\int_{G/H} d\nu([x])\, \|\psi\|^2_{H_\chi}[x] < \infty. \tag{7.101}$$

This space turns out to be complete in the natural inner product

$$\langle \varphi, \psi \rangle = \int_{G/H} d\nu([x])\, \langle \varphi, \psi \rangle_{H_\chi}[x] \tag{7.102}$$

It also carries a system of imprimitivity: in case that $\nu$ is $G$-invariant we simply have

$$u^\chi(y)\psi(x) = \psi(y^{-1}x) \ \ (x, y \in G); \tag{7.103}$$

$$\pi^\chi(\tilde{f})\psi(x) = \tilde{f}([x])\psi(x) \ \ (\tilde{f} \in C_0(G/H)), \tag{7.104}$$

where we note that $u^\chi(y)\psi$ satisfies (7.98) if $\psi$ does. Unitarity of $u^\chi$ as well as the covariance condition (7.81) are easily checked. In general, we replace (7.103) by

$$u^\chi(y)\psi(x) = \sqrt{\frac{d\nu([y^{-1}x])}{d\nu([x])}}\, \psi(y^{-1}x), \tag{7.105}$$

where $d\nu([y^{-1}\cdot])/d\nu([\cdot])$ is the Radon–Nikodym derivative of the translated measure $L_y^*\nu$ with respect to $\nu$, cf. (B.137), which is well defined because by the assumption of quasi-invariance, $L_y^*\nu$ is absolutely continuous with respect to $\nu$ (indeed, on this assumption they are even equivalent). Here $L_y^*\nu(A) = \nu(L_y^{-1}(A))$, $A \subset G/H$.

Physicists do not like the Hilbert space $H^\chi$, preferring a different realization

$$\tilde{H}^\chi = L^2(G/H) \otimes H_\chi, \tag{7.106}$$

in which the wave-function $\psi$ is not constrained and one has a clean separation between the (typically) spatial degree of freedom $Q = G/H$ and the internal degree of freedom $H_\chi$. One half of the system of imprimitivity will then be given nicely by

$$\tilde{\pi}^\chi(\tilde{f})\tilde{\psi} = \tilde{f}\tilde{\psi} \ \ (\tilde{f} \in C_0(G/H)), \tag{7.107}$$

but this cleanliness comes at the cost of a more complicated formula for $\tilde{u}^\chi(y)$, as follows. Pick a (measurable) cross-section $s : G/H \to G$, i.e., a *right* inverse to the projection $p : G \to G/H$, $p(x) = [x]$, in other words, we have

$$p \circ s = \mathrm{id}_{G/H}. \tag{7.108}$$

It may not be possible to make $s$ continuous, and, crucially, $s$ is not a *left* inverse to $p$; instead, there exists a unique function $h_s : G \to H$ such that $s \circ p(x) = xh_s(x)$, i.e.,

$$h_s(x) = x^{-1}s([x]). \tag{7.109}$$

Such a cross-section $s$ gives rise to a unitary isomorphism

$$\begin{aligned} w_s : H^\chi &\to \tilde{H}^\chi; && (7.110)\\ w_s\psi(q) &= \psi(s(q)); && (7.111)\\ w_s^{-1}\tilde{\psi}(x) &= u_\chi(h_s(x))\tilde{\psi}([x]), && (7.112)\end{aligned}$$

which enables us to move the system of imprimitivity $(u^\chi, \pi^\chi)$ to $\tilde{H}^\chi$ by defining

$$\begin{aligned} \tilde{u}^\chi(y) &= w_s u^\chi(y) w_s^* \ \ (y \in G); && (7.113)\\ \tilde{\pi}^\chi(\tilde{f}) &= w_s \pi^\chi(\tilde{f}) w_s^* \ \ (\tilde{f} \in C_0(G/H)). && (7.114)\end{aligned}$$

This duly leads to (7.107), but instead of (7.105), we obtain the more cumbersome

$$\tilde{u}^\chi(y)\tilde{\psi}(q) = \sqrt{\frac{d\nu(y^{-1}q)}{d\nu(q)}}\, u_\chi(s(q)^{-1}ys(y^{-1}q))\tilde{\psi}(y^{-1}q), \tag{7.115}$$

where of course the square root may be omitted if $\nu$ is $G$-invariant, as in (7.103). The argument $h = s(q)^{-1}ys(y^{-1}q)$ of $u_\chi$ appearing here is called the ***Wigner cocycle*** (after the physicist who first introduced it in his classification of the irreducible representations of the Poincaré group). One may verify that $h \in H$ by applying $p$, which by construction is $G$-equivariant (i.e., $p(xy) = xp(y)$), which gives

$$p(h) = p(s(q)^{-1}ys(y^{-1}q)) = s(q)^{-1}yp(s(y^{-1}q)) = s(q)^{-1}yy^{-1}q = s(q)^{-1}q,$$

where in the third step we used (7.108). For any $x \in G$ we have $x^{-1}[x] = [x^{-1}x] = [e]$, so taking $x = s(q)$ in this computation we find $p(h) = [e]$, which is true iff $h \in H$.

Given an irreducible system of imprimitivity $(\tilde{u}^\chi, \tilde{\pi}^\chi)$, we obtain generalized momentum operators by passing to the associated representation of the Lie algebra $\mathfrak{g}$ of $G$ through (5.156) and (7.57), i.e.,

$$\tilde{\pi}^\chi(A) = i\hbar(\tilde{u}^\chi)'(A), \tag{7.116}$$

where $A \in \mathfrak{g}$, so that, cf. (7.78) - (7.80), we obtain from (5.160) and (7.81):

$$\begin{aligned} [\tilde{\pi}^\chi(A), \tilde{\pi}^\chi(B)] &= i\hbar\tilde{\pi}^\chi([A,B]); && (7.117)\\ [\tilde{\pi}^\chi(A), \tilde{\pi}^\chi(\tilde{f})] &= i\hbar\tilde{\pi}^\chi(\delta_A\tilde{f}); && (7.118)\\ [\tilde{\pi}^\chi(\tilde{f}), \tilde{\pi}^\chi(\tilde{g})] &= 0, && (7.119)\end{aligned}$$

where $A, B \in \mathfrak{g}$ and $\tilde{f}, \tilde{g} \in C_0(Q)$ (in fact, these formulae—defined on the right domain—work also for many unbounded functions on $Q$, see below), and $\delta_A$ is defined in (3.71). Let us take a look at a few illustrative special cases:

- If $H = G$, then $Q$ is a point, so that $C^*(G,Q) = G^*(G)$, and systems of imprimitivity are just irreducible representations of $G$. We have $H^\chi \cong H_\chi$ through the map $w: H^\chi \to H_\chi$ defined by $\psi \mapsto \psi(e) \equiv \psi' \in H_\chi$, with inverse $\psi(x) = u_\chi(x^{-1})\psi'$. This gives $wu^\chi(y)w^{-1} = u_\chi(y)$. Similarly, in (7.115) we take $s = e$, which gives $\tilde{u}^\chi(y) = u_\chi(y)$ on $\tilde{H}^\chi = H_\chi$.
- If $H = \{e\}$ we have $Q = G$ and $C^*(G,G) \cong B_0(L^2(G))$, which quantizes the underlying classical phase space $\mathfrak{g}^* \times G \cong T^*G$. We now have $H = L^2(G)$ carrying the left-regular representation of $G$.
- Let $G = E(3)$ act canonically on $Q = \mathbb{R}^3$. Taking $q_0 = 0$ gives $H = SO(3)$, so irreducible systems of imprimitivity are classified by $j = 0, 1, \ldots$, with corresponding irreducible representations $D_j(SO(3))$ on $H_j = \mathbb{C}^{2j+1}$, cf. §5.8. Hence

$$\tilde{H}^j = L^2(\mathbb{R}^3) \otimes H_j, \tag{7.120}$$

and using the cross-section $s(q) = (1_3, q)$ from $\mathbb{R}^3$ to $E(3)$ we obtain, from (7.115) with (7.63) - (7.64) and (7.107), the expressions

$$\tilde{u}^j(R,a))\tilde{\psi}(q) = D_j(R)\tilde{\psi}(R^{-1}(q-a)); \tag{7.121}$$
$$\tilde{\pi}^j(\tilde{f}))\tilde{\psi}(q) = \tilde{f}(q)\tilde{\psi}(q). \tag{7.122}$$

For $j = 0$ this gives the usual quantum theory of a spinless particle:

1. The Hilbert space is $\tilde{H}^0 = L^2(\mathbb{R}^3)$.
2. For the generators of $\mathbb{R}^3 \subset E(3)$ we duly obtain the momentum operators

$$P_i = -i\hbar\frac{\partial}{\partial q^i}, \tag{7.123}$$

   where $P_i = \tilde{\pi}^0(e_i)$ is defined in terms of the standard basis $(e_1, e_2, e_3)$ of $\mathbb{R}^3$, now seen as the Lie algebra of $\mathbb{R}^3$.
3. Using the basis (3.66) of the Lie algebra of $SO(3) \subset E(3)$, we obtain the orbital angular momentum operators (which pick up extra terms for $j > 0$):

$$\tilde{\pi}^0(J_1) = i\hbar\left(q^3\frac{\partial}{\partial q^2} - q^2\frac{\partial}{\partial q^3}\right); \tag{7.124}$$
$$\tilde{\pi}^0(J_2) = i\hbar\left(q^1\frac{\partial}{\partial q^3} - q^3\frac{\partial}{\partial q^1}\right); \tag{7.125}$$
$$\tilde{\pi}^0(J_3) = i\hbar\left(q^2\frac{\partial}{\partial q^1} - q^1\frac{\partial}{\partial q^2}\right). \tag{7.126}$$

4. The coordinate functions $\tilde{f}(q) = q^i$ yield the position operators $Q_i = \tilde{\pi}^0(q^i)$:

$$Q_i\tilde{\psi}(q) = q^i\tilde{\psi}(q). \tag{7.127}$$

5. Thus we obtain all the familiar commutation relations like $[Q_i, P_j] = i\hbar\delta_{ij}$, $[\tilde{\pi}^0(J_1), \tilde{\pi}^0(J_2)] = i\hbar\tilde{\pi}^0(J_3)$, etc., cf. (7.65) - (7.66).

- Let $G = \mathbb{R}$ act on $Q = \mathbb{T}$, which we parametrize by $z = \exp(2\pi iq)$, $q \in [0,1)$, by

$$a : \exp(2\pi iq) \mapsto \exp(2\pi i(q+a)), \tag{7.128}$$

so that $H = \mathbb{Z}$, with $\hat{H} = \mathbb{T}$ under $u_z(n) = z^n$, $z \in \mathbb{T}$, $n \in \mathbb{Z}$, cf. (C.349). We parametrize $\hat{H}$ by $z = \exp(i\theta)$, $\theta \in [0,2\pi)$, so that (with slight abuse of notation) $u_\theta(n) = e^{in\theta}$. In the second description (i.e. the one of the physicists) we have

$$\tilde{H}^\theta = L^2(\mathbb{T}) = L^2(0,1), \tag{7.129}$$

where topology of $Q$ is lost for the moment. Using the cross-section

$$s\left(e^{2\pi iq}\right) = q, \tag{7.130}$$

where $q \in [0,1)$, we obtain

$$\tilde{u}^\theta(a)\tilde{\psi}(q) = e^{in(a,q)\theta}\tilde{\psi}(q - a + n(a,q)), \tag{7.131}$$

where $n(a,q) \in \mathbb{Z}$ is the unique integer such that $q - a + n(a,q) \in [0,1)$. The corresponding momentum operator is formally given by the usual expression $P = -i\hbar\partial/\partial q$, cf. (7.123), which appears to be independent of $\theta$ (since for any $q \in (0,1)$ and $a$ small enough we have $n(a,q) = 0$), but in fact the $\theta$-dependence is in its domain, which can be shown to consist of the subspace of the Sobolev space $H^1(0,1)$—i.e. the closure of $C^\infty([0,1])$ in the inner product (5.318) adapted to $L^2(0,1)$, which implies $H^1(0,1) \subset C([0,1])$—whose elements satisfy

$$\psi(1) = e^{-i\theta}\psi(0). \tag{7.132}$$

To see this, we recall that

$$P\tilde{\psi} = i\hbar \lim_{\varepsilon\to 0}\left(\frac{\tilde{u}^\theta(\varepsilon)\tilde{\psi} - \tilde{\psi}}{\varepsilon}\right), \tag{7.133}$$

where the limit is taken in the $L^2$-norm, so that we need existence of

$$\lim_{\varepsilon\to 0}\varepsilon^{-2}\int_0^1 dq\,|e^{in(a,q)\theta}\tilde{\psi}(q - \varepsilon + n(\varepsilon,q)) - \tilde{\psi}(q)|^2.$$

For $0 < q < \varepsilon$ we have $n(\varepsilon,q) = 1$, whereas for $\varepsilon < q < 1$ we have $n(\varepsilon,q) = 0$, so it is convenient to split the integral as a sum of $\int_0^\varepsilon$ and $\int_\varepsilon^1$. The second term enforces the existence of derivatives in the $L^2$-sense (which in turn makes $\tilde{\psi}$ continuous on $[0,1]$) and is unproblematic, but the first requires the existence of

$$\lim_{\varepsilon\to 0}\varepsilon^{-2}\int_0^\varepsilon dq\,|e^{i\theta}\tilde{\psi}(q - \varepsilon + 1) - \tilde{\psi}(q)|^2.$$

This strange expression, then, enforces the boundary condition (7.132). In this case there is no single position operator, but the algebra $C(\mathbb{T})$ plays its role.

## 7.6 Representations of semi-direct products

The case $Q = G/H$ also provides the key for the general case, as long as the $G$-action on $Q$ is *regular*, cf. Theorem 7.7. In that case, the construction of the irreducible system of imprimitivity $(u(G), \pi(C_0(Q)))$ corresponding to a pair $(\mathscr{O}, u_\chi(H))$, where $\mathscr{O}$ is a $G$-orbit in $Q$, requires no new ideas: we have $\mathscr{O} \cong G/H$, and hence $u = u^\chi$ and $\pi = \pi^\chi$ as described in §7.5 (where the function $\tilde{f}$ in formulae like (7.104) or (7.114), which in these expression was defined on $G/H$, should be seen as the restriction of $\tilde{f} \in C_0(Q)$ to $\mathscr{O} \subset Q$). An important application of this construction is the representation theory of ***regular semi-direct products*** $L \ltimes V$ (cf. §7.3), where regularity means that the *dual* $L$-action on $V^*$ is regular; this action is given by

$$\lambda \cdot \theta(v) = \theta(\lambda^{-1} \cdot v) \; (\lambda \in L, \theta \in V^*, v \in V). \tag{7.134}$$

**Theorem 7.9.** *Up to unitary equivalence, the irreducible unitary representations of a regular semi-direct product $G = L \ltimes V$ are classified by pairs $(\mathscr{O}, \sigma)$, where $\mathscr{O}$ is an L-orbit in $V^*$ and $\sigma$ is an element of the unitary dual of the stabilizer $L_0 \subset L$ of an arbitrary point $\theta_0 \in \mathscr{O}$. The corresponding representation $\tilde{u}^{(\mathscr{O},\sigma)}(G)$ may be realized from an irreducible representation $u_\sigma$ of $L_0$ on a Hilbert space $H_\sigma$ combined with a cross-section $s : L/L_0 \to L$ of the canonical projection $p : L \to L/L_0$, namely through*

$$\tilde{H}^{(\mathscr{O},\sigma)} = L^2(L/L_0) \otimes H_\sigma; \tag{7.135}$$

$$\tilde{u}^{(\mathscr{O},\sigma)}(\lambda, v)\tilde{\psi}(\theta) = e^{i\theta(v)} u_\sigma(s(\theta)^{-1}\lambda s(\lambda^{-1}\theta))\tilde{\psi}(\lambda^{-1}\theta). \tag{7.136}$$

*Proof.* Let $u$ be a unitary representation of $G$. This implies

$$u(\lambda)u(v)u(\lambda^{-1}) = u(\lambda \cdot v), \tag{7.137}$$

in which $\lambda \equiv (\lambda, 0)$ and $v \equiv (e, v)$. Since $V \subset G$ is abelian, we have $C^*(V) \cong C_0(V^*)$ by the Fourier transform (cf. Theorem C.109 in §C.15), which here is given by (7.44) - (7.45), with $A \rightsquigarrow v$. Hence the representation $u^\int(C^*(V))$ defined by $u(V)$ via (5.172), seen as a representation of $C_0(V^*)$ via the Fourier transform, is given by

$$u^\int(f) = (2\pi)^{-n} \int_{V \times V^*} d^n v d^n \theta \, e^{i\theta(v)} f(\theta) u(v). \tag{7.138}$$

Using invariance of the measure $d^n v d^n \theta$ under the joint transformation $(v, \theta) \rightsquigarrow (\lambda \cdot v, \lambda \cdot \theta)$, from (7.137) we obtain, for $f \in C_0(V^*)$ in the image of $\check{f} \in C_c^\infty(V)$,

$$\begin{aligned} u(\lambda)u^\int(f)u(\lambda)^* &= (2\pi)^{-n} \int_{V \times V^*} d^n v d^n \theta \, e^{i\theta(v)} f(\theta) u(\lambda \cdot v) \\ &= (2\pi)^{-n} \int_{V \times V^*} d^n v d^n \theta \, e^{i(\lambda \cdot \theta)(\lambda \cdot v)} f(\lambda^{-1} \cdot \lambda \cdot \theta) u(\lambda \cdot v) \\ &= (2\pi)^{-n} \int_{V \times V^*} d^n v d^n \theta \, e^{i\theta(v)} f(\lambda^{-1} \cdot \theta) u(v) \\ &= u^\int(L_\lambda f). \end{aligned} \tag{7.139}$$

Consequently, a unitary representation $u(L \ltimes V)$ defines a system of imprimitivity $(u(L), u^{\int}(C_0(V^*)))$, and *vice versa*, since any pair of representations $(u(L), u(V))$ that satisfies (7.137) gives rise to a representation $u(G)$ by $u(\lambda, v) = u(v)u(\lambda)$.

Now apply Theorem 7.7 with $G \rightsquigarrow L$ and $Q \rightsquigarrow V^*$. All we need in order to obtain (7.135) - (7.136) from (7.106) and (7.107) - (7.115) is to find the representation $u(V)$ that induces the representation $u^{\int}(C_0(V^*))$ given by (7.107), namely

$$u(v)\tilde{\psi}(\theta) = e^{-i\theta(v)}\tilde{\psi}(\theta), \tag{7.140}$$

as is easily checked from (7.138). □

In view of this, we have a remarkable group–groupoid C*-algebra isomorphism

$$C^*(L \ltimes V) \cong C^*(L \ltimes V^*), \tag{7.141}$$

where the left-hand side is just the C*-algebra of the *group* $L \ltimes V$, whereas the right-hand side is the C*-algebra of the action group*oid* $L \ltimes V^*$ relative to (7.134). Also, a computation shows that the same formulae (7.135) - (7.136) are obtained if, given $\theta_0 \in V^*$ and hence given $L_0$ as its stabilizer, we define a subgroup $H \subset G$ by

$$H = L_0 \ltimes V, \tag{7.142}$$

and induce from the representation $u_{(\theta_0,\sigma)}$ of $H$ defined by

$$u_{(\theta_0,\sigma)}(\lambda, v) = e^{i\theta_0(v)}u_\sigma(\lambda). \tag{7.143}$$

We briefly discuss four basic examples from physics, each of which is easily seen to be regular. We write $a$ instead of $v$ in $(\lambda, v) \in G$ so as to emphasize the "spatial" character of $V$, whereas $V^*$ is labeled by a dual "momentum" variable $p$.

- $G = E(2) = SO(2) \ltimes \mathbb{R}^2$, defined like $E(3)$, i.e., with respect to the usual action of $SO(2)$ on $\mathbb{R}^2$ (this group will play a role in the representation theory of the Poincaré-group). We find the same action of $SO(2)$ on $(\mathbb{R}^2)^* = \mathbb{R}^2$, so that the orbits are $\mathscr{O}_0 = \{0\}$ with $G_0 = SO(2)$ and $\mathscr{O}_r = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = r^2\}$ for $r > 0$, with $G_r = \{e\}$. Thus the Hilbert spaces and representations are given by

$$\tilde{H}^{(0,n)} = \mathbb{C}; \tag{7.144}$$

$$\tilde{u}^{(0,n)}(\lambda, a) = e^{2\pi i n\lambda}; \tag{7.145}$$

$$\tilde{H}^r = L^2(0,1); \tag{7.146}$$

$$\tilde{u}^r(\lambda, a)\tilde{\psi}(p) = e^{ir(a_1\cos p' + a_2 \sin p')}\psi(p - \lambda|\mathrm{mod}\, 1), \tag{7.147}$$

where $n \in \mathbb{Z}$, $\lambda \in [0,1)$, $p \in (0,1)$, and $p' = 2\pi p$. In the first case $\mathbb{R}^2 \subset E(2)$ is represented trivially, whereas in the second the $r$-dependence of the representation lies entirely in $\mathbb{R}^2$ (since $\tilde{H}^r$ and $\tilde{u}^r(\lambda, 0)$ are evidently independent of $r$). The projective representations of $G$ are of considerable interest, too, cf. §5.10.

**Lemma 7.10.** *If $G = SO(p,q) \ltimes \mathbb{R}^{p+q}$ ($p > 0, q \geq 0$), then $H^2(\mathfrak{g}, \mathbb{R}) = 0$.*

Here $SO(p,q)$ is the subgroup of $SL_{p+q}(\mathbb{R}^{p+q})$ whose elements leave the form

$$x^2 = x_1^2 + \cdots + x_p^2 - (x_{p+1}^2 + \cdots + x_{p+q}^2)$$

invariant; the best-known example is the (proper) Lorentz group $SO(3,1)$, see below. This lemma may be proved by a straightforward but lengthy computation. By Theorem 5.59, the projective unitary representations of $G$ then correspond to the ordinary unitary representations of the universal covering

$$\tilde{G} = \mathbb{R} \ltimes \mathbb{R}^2, \tag{7.148}$$

where $\mathbb{R}$ acts on $\mathbb{R}^2$ through the covering projection $\tilde{\pi} : \mathbb{R} \to SO(2) = \mathbb{R}/\mathbb{Z}$, cf. Theorem 5.41 (with $D \rightsquigarrow \mathbb{Z}$). This changes the expressions (7.144) - (7.147) into

$$\tilde{H}^{(0,s)} = \mathbb{C}; \tag{7.149}$$
$$\tilde{u}^{(0,s)}(\lambda,a) = e^{is\lambda}; \tag{7.150}$$
$$\tilde{H}^{(r,\theta)} = L^2(0,1); \tag{7.151}$$
$$\tilde{u}^{(r,\theta)}(\lambda,a)\tilde{\psi}(p) = e^{ir(a_1\cos p' + a_2 \sin p')} e^{in(\lambda,p)\theta}\tilde{\psi}(p-\lambda+n(\lambda,p)), \tag{7.152}$$

where $\lambda \in \mathbb{R}$, $s \in \mathbb{R}$, $\theta \in [0,2\pi)$, $p \in (0,1)$, and $n(\lambda,p)$ is defined as in (7.131).

- $G = E(3) = SO(3) \ltimes \mathbb{R}^3$, as before with the defining action of $SO(3)$. The $SO(3)$-orbits in $(\mathbb{R}^3)^* = \mathbb{R}^3$ are spheres $S_r^2 \cong SO(3)/SO(2)$ with radius $r > 0$, as well as the origin ($r = 0$) with stabilizer $SO(3)$, so that for the Hilbert spaces we obtain

$$\tilde{H}^{(0,j)} = \mathbb{C}^{2j+1}; \tag{7.153}$$
$$\tilde{H}^{(r,n)} = L^2(S^2); \tag{7.154}$$

where $j = 0,1,\ldots$ labels the unitary irreducible representations of $SO(3)$ on $H_j = \mathbb{C}^{2j+1}$, whereas $n \in \mathbb{Z}$ labels the irreducible representations of $SO(2)$ on $\mathbb{C}$ (we write $S^2 \equiv S_1^2$). In the second case, the representation $u^{(r,n)}$ of $SO(3) \subset E(3)$ depends explicitly on $n$ through the Wigner cocycle; for $n = 0$ we simply obtain

$$\tilde{u}^{(r,0)}(R,a)\tilde{\psi}(p) = e^{irp\cdot a}\tilde{\psi}(R^{-1}p). \tag{7.155}$$

For $n \neq 0$ we just give a formula for $\tilde{u}^{(r,n)}(R,a)$ in case that $R$ is a rotation around the $z$-axis and $a = 0$; this is enough to make the point. To this end we parametrize $SO(3)$ by the well-known Euler angles, i.e., in terms of the matrices $J_i$, cf. (3.66),

$$R(\phi,\theta,\alpha) = e^{\phi J_3} e^{\theta J_2} e^{\alpha J_3}, \tag{7.156}$$

and write $q \in S^2$ as $q = (\phi,\theta) = R(\phi,\theta,0)e_3$ with $e_3 = (0,0,1)$ (the spherical coordinates of $q$ are $(\phi - \frac{1}{2}\pi, \theta)$). This also provides $S^2$ with an $SO(3)$-invariant measure $d\nu(\phi,\theta) = d\phi d\theta \sin\theta$. A convenient choice of $s : S^2 \to SO(3)$ is

$$s(\phi,\theta) = R(\phi,\theta,-\phi), \tag{7.157}$$

in which case we simply obtain, writing $R_z(\alpha) = R(\alpha,0,0)$,

$$\tilde{u}^{(r,n)}(R_z(\alpha),0)\tilde{\psi}(\phi,\theta) = e^{in\alpha}\tilde{\psi}(\phi-\alpha,\theta). \tag{7.158}$$

The universal covering group of $E(3)$ is

$$\widetilde{E(3)} = SU(2) \ltimes \mathbb{R}^3, \tag{7.159}$$

where $SU(2) = \widetilde{SO(3)}$ acts on $\mathbb{R}^3$ through its covering projection $\tilde{\pi}$ onto $SO(3)$, as in the previous case. By Theorem 5.59 and Lemma 7.10, the projective unitary irreducible representations of $E(3)$ are given by the unitary irreducible representations of $SU(2) \ltimes \mathbb{R}^3$. This obviously leads to additional half-integral values for $j$ in (7.153), since this number now labels the unitary irreducible representations of $SU(2)$. As to $n$ in (7.154), the subgroup $H \subset SU(2)$ that stabilizes $(0,0,r) \in S^2_r$ consists of all matrices $u_z = \mathrm{diag}(z,\overline{z})$, where $z \in \mathbb{T}$, so $H \cong \mathbb{T}$ and hence $\hat{H} = \mathbb{Z}$ under $u_z \mapsto z^m$, $m \in \mathbb{Z}$. We now recall from the proof of Proposition 5.5 that

$$u = \cos(\theta/2)\cdot 1_2 + i\sin(\theta/2)\mathbf{u}\cdot\sigma \in SU(2), \tag{7.160}$$

where $\mathbf{u}$ is a unit vector in $\mathbb{R}^3$, projects to $\tilde{\pi}(u) = R_\theta(\mathbf{u}) \in SO(3)$, i.e., the rotation around $\mathbf{u}$ by an angle $\theta$. Parametrizing $z = \cos(\alpha/2) + i\sin(\alpha/2)$, $\alpha \in [0,4\pi)$, therefore gives $\tilde{\pi}(u_z) = \exp(\alpha J_3)$. Besides (7.157), we now also need a cross-section $s : S^2_r \to SU(2)$, for which the above analysis suggests we take

$$s(\phi,\theta) = u^{(3)}(\phi)u^{(2)}(\theta)u^{(3)}(-\phi); \tag{7.161}$$

$$u^{(2)}(\theta) \equiv \cos(\tfrac{1}{2}\theta)\cdot 1_2 + i\sin(\tfrac{1}{2}\theta)\cdot\sigma_2; \tag{7.162}$$

$$u^{(3)}(\phi) \equiv \cos(\phi/2)\cdot 1_2 + i\sin(\phi/2)\cdot\sigma_3; \tag{7.163}$$

note that $u_z = u^{(3)}(\alpha)$. A calculation similar to the one leading to (7.158) gives

$$\tilde{u}^{(r,m)}(u_z,0)\tilde{\psi}(\phi,\theta) = e^{im\alpha/2}\tilde{\psi}(\phi-\alpha,\theta). \tag{7.164}$$

Comparing (7.158) and (7.164), we see that if $m$ is even, then $n = m/2$ (of course, by convention we may replace $m/2$ in (7.164) by $n$ on the understanding that $n$ may now be half-integral). If $m$ is odd, choosing $\alpha = 2\pi$ we famously obtain

$$\tilde{u}^{(r,m)}(-1_2,0)\tilde{\psi} = -\tilde{\psi}. \tag{7.165}$$

More generally, if we take a closed path $t \mapsto R_{2\pi t}(\mathbf{u})$, $t \in [0,1]$ in $SO(3)$, which starts and ends at $1_3$, and lift it (with respect to the covering projection $\tilde{\pi} : SU(2) \to SO(3)$) to a path $t \mapsto u(t) \equiv \cos(\pi t) + i\sin(\pi t)\mathbf{u}\cdot\sigma$ in $SU(2)$, which now starts at $1_2$ and ends at $-1_2$, then the corresponding representation $\tilde{u}^{(r,m)}(u(t),0)$ takes the wave-function $\tilde{\psi}$ to itself if $m$ is even, whereas it takes $\tilde{\psi}$ to $-\tilde{\psi}$ whenever $m$ is odd (this is an embryonic version of the connection between spin and statistics, fully realized only in quantum field theory).

- $G = L \ltimes \mathbb{R}^{3+1}$, the ***Poincaré group***, where the ***Lorentz group*** $L = O(3,1)$ consists of all real $4 \times 4$ matrices that leave the indefinite quadratic form

$$x^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2 \tag{7.166}$$

invariant; in this context the standard coordinates on $\mathbb{R}^4$ are labeled as $(x_0, x_1, x_2, x_3)$. The Lorentz group has four connected components, which may be identified by the (independent) conditions $\det(\lambda) = \pm 1$ and $\pm\lambda_{00} \geq 1$. For simplicity we restrict ourselves to the connected component $L_+^\uparrow$ of the identity, in which $\det(\lambda) = 1$ and $\lambda_{00} \geq 1$. This group is called the ***proper orthochronous Lorentz group***, which in turn defines the ***proper orthochronous Poincaré group*** $P_+^\uparrow = L_+^\uparrow \ltimes \mathbb{R}^4$. Writing $p^2 = p_0^2 - p_1^2 - p_2^2 - p_3^2$, the $L_+^\uparrow$-orbits in $(\mathbb{R}^4)^* = \mathbb{R}^4$ are seen to be:

1. $\mathscr{O}_0 = \{(0,0,0,0)\}$, with stabilizer $(L_+^\uparrow)_0 = L_+^\uparrow$;
2. $\mathscr{O}_m^\pm = \{p \in \mathbb{R}^4 \mid p^2 = m^2, \pm p_0 \geq 0\}$, $m > 0$, with $(L_+^\uparrow)_0 = SO(3)$;
3. $\mathscr{O}_0^\pm = \{p \in \mathbb{R}^4 \mid p^2 = 0, \pm p_0 \geq 0\}$, with $(L_+^\uparrow)_0 = E(2)$;
4. $\mathscr{O}_{im} = \{p \in \mathbb{R}^4 \mid p^2 = -m^2, \pm p_0 \geq 0\}$, $m > 0$, with $(L_+^\uparrow)_0 = SO(2,1)$.

Here the stabilizers $L_0$ are found by taking the reference points $(\pm m, 0, 0, 0)$ in case 2, $(\pm 1, 0, 0, -1)$ in case 3, and $(0,0,0,m)$ in case 4. The physically relevant cases are probably $\mathscr{O}_{m^2}^+$ and $\mathscr{O}_0^+$. We pass straight to the universal covering group

$$\tilde{P}_+^\uparrow = SL(2,\mathbb{C}) \ltimes \mathbb{R}^4, \tag{7.167}$$

where the covering projection $\tilde{\pi} : SL(2,\mathbb{C}) \to L_+^\uparrow$ is given analogously to the case (5.46). We again start from the four matrices $(\sigma_0, \sigma_1, \sigma_2, \sigma_3)$ in (5.42), and note:

- These form a basis for the (real) vector space of all self-adjoint $2 \times 2$ matrices;
- For any $x \in \mathbb{R}^4$ we have $\det(\sum_{\mu=0}^3 x_\mu \sigma_\mu) = x^2$ as defined in (7.166);
- For any $\tilde{\lambda} \in SL(2,\mathbb{C})$ and $a \in M_2(\mathbb{C})$ we have $\det(\tilde{\lambda} a \tilde{\lambda}^*) = \det(a)$;
- For any $\tilde{\lambda} \in SL(2,\mathbb{C})$ and self-adjoint $a \in M_2(\mathbb{C})$, $\tilde{\lambda} a \tilde{\lambda}^*$ is again self-adjoint.

Taking $a = \sum_\mu x_\mu \sigma_\mu$, it follows that for $\tilde{\lambda} \in SL(2,\mathbb{C})$ and $x \in \mathbb{R}^4$ there must be $\lambda \in O(3,1)$ such that $\tilde{\lambda} \sum_\mu x_\mu \sigma_\mu \tilde{\lambda}^* = \sum_\mu (\lambda \cdot x)_\mu \sigma_\mu$. By continuity and the fact that $SL(2,\mathbb{C})$ is connected it follows that in fact $\lambda \in L_+^\uparrow$, so we put $\tilde{\pi}(\lambda) = \lambda$. As for (5.46), the kernel is $\ker(\tilde{\pi}) = \mathbb{Z}_2 = \{\pm 1_2\}$. This enlarges the stabilizers:

1. For $\mathscr{O}_{m^2}^+$ we now obtain $(\tilde{L}_+^\uparrow)_0 = SU(2)$, leading to a family of unitary irreducible representations $u^{m,j}$ labeled by ***mass*** $m > 0$ and ***spin*** $j = 0, \frac{1}{2}, 1, \ldots$.
2. For $\mathscr{O}_0^+$ the stabilizer $(\tilde{L}_+^\uparrow)_0$ of $(1,0,0,1)$ is a double cover $E(2)'$ of $E(2)$, whose unitary irreducible representations are labeled by either $(0,n)$ with $n \in \mathbb{Z}/2$ (called ***helicity***) or by $r > 0$. The latter case does not occur in nature.

On the one hand, this classification is a triumph of mathematical physics, but on the other hand, it fails to single out which cases actually occur in nature: as far as we know, these are spin $j = 0$ and $j = \frac{1}{2}$ and helicity $n = \pm 1$ and $n = \pm 2$.

- $G = E(3) \ltimes \mathbb{R}^4$, the Galilei group, defined via the following $E(3)$-action on $\mathbb{R}^4$:

$$(R, \mathbf{v}) : (a_0, \mathbf{a}) \mapsto (a_0, R\mathbf{a} + a_0\mathbf{v}). \tag{7.168}$$

Note that $\mathbf{v}$ is physically interpreted as a velocity, whereas earlier $\mathbf{a} \in \mathbb{R}^3 \subset E(3)$ was a position variable. This is clear from the defining $G$-action on $\mathbb{R}^4$, given by

$$(R, \mathbf{v}, a_0, \mathbf{a}) : (t, \mathbf{x}) \mapsto (t + a_0, R\mathbf{x} + \mathbf{a} + t\mathbf{v}), \tag{7.169}$$

which in fact determines the action (7.168). Either way, we obtain the group law

$$(R, \mathbf{v}, a_0, \mathbf{a}) \cdot (R', \mathbf{v}', a_0', \mathbf{a}') = (RR', \mathbf{v} + R\mathbf{v}', a_0 + a_0', \mathbf{a} + R\mathbf{a}' + a_0'\mathbf{v}). \tag{7.170}$$

We therefore see that the role of the Lorentz group $SO(3,1)$ is now played by the Euclidean group $E(3)$. Since from (7.170) the inverse is found to be

$$(R, \mathbf{v}, a_0, \mathbf{a})^{-1} = (R^{-1}, -R^{-1}\mathbf{v}, -a_0, -R^{-1}(\mathbf{a} - a_0\mathbf{v})), \tag{7.171}$$

the dual $E(3)$-action on $(\mathbb{R}^4)^* \cong \mathbb{R}^4$ is given (in non-relativistic notation) by

$$(R, \mathbf{v}) : (E, \mathbf{p}) \mapsto (E - \langle \mathbf{v}, R\mathbf{p} \rangle, R\mathbf{p}). \tag{7.172}$$

Hence the dual $E(3)$-orbits in $\mathbb{R}^4$ are labeled by $E \in \mathbb{R}$ and $r > 0$, as follows:

$$\mathscr{O}_E = \{(E, \mathbf{0})\}; \tag{7.173}$$
$$\mathscr{O}_{(r)} = \{(E, \mathbf{p}), E \in \mathbb{R}, \|\mathbf{p}\| = r\}. \tag{7.174}$$

The representations of $G$ corresponding to the first type are basically the representations of $E(3)$, whereas in the second case the stability group of say $(0,0,0,r)$ is isomorphic to $E(2)$. None of the ensuing induced representations of $G$ reproduces some recognizable version of non-relativistic quantum mechanics, for which we need to pass to projective representations of $G$. These may be found from Theorem 5.62, which here applies in full glory, since $H^2(\mathfrak{g}, \mathbb{R}) \neq 0$. A (lengthy) computation shows that $H^2(\mathfrak{g}, \mathbb{R})$ has a single generator

$$\varphi((M, \mathbf{v}, a_0, \mathbf{a}), (M', \mathbf{v}', a_0', \mathbf{a}')) = \langle \mathbf{v}, \mathbf{a}' \rangle - \langle \mathbf{v}', \mathbf{a} \rangle, \tag{7.175}$$

where $M \in \mathfrak{so}(3)$, and $(\mathbf{v}, a_0, \mathbf{a}) \in \mathbb{R}^3 \times \mathbb{R}^4 \subset \mathfrak{g} = \mathfrak{so}(3) \oplus \mathbb{R}^3 \oplus \mathbb{R}^4$ are identified with the corresponding Lie group elements. Following the procedure culminating in Theorem 5.62, the central extension $\check{G}$ is found to be (cf. (7.159) and (5.46))

$$\check{G} = \widetilde{E(3)} \ltimes \mathbb{R}^5, \tag{7.176}$$

where, writing $\tilde{\pi}(u) \equiv R(u)$, the covering group $\widetilde{E(3)}$ acts on $\mathbb{R}^5$ through

$$(u, \mathbf{v}) : (a_0, \mathbf{a}, c) \mapsto (a_0, R(u)\mathbf{a} + a_0\mathbf{v}, c + \tfrac{1}{2}a_0\|\mathbf{v}\|^2 + \langle \mathbf{v}, R(u)\mathbf{a} \rangle). \tag{7.177}$$

Consequently, writing $\tilde{x} = (R, \mathbf{v}, a_0, \mathbf{a})$, for the group law in $\check{G}$ we obtain

$$(\tilde{x}, c) \cdot (\tilde{x}', c') = (\tilde{x} \cdot \tilde{x}', c + c' + \langle \mathbf{v}, R(u)\mathbf{a}' \rangle + \tfrac{1}{2} a_0' \|\mathbf{v}\|^2). \tag{7.178}$$

Eq. (7.177) implies the following dual $\widetilde{E(3)}$-action on $(\mathbb{R}^5)^* = \mathbb{R}^5$:

$$(u, \mathbf{v}) : (E, \mathbf{p}, m) \mapsto (E - \langle \mathbf{v}, R(u)\mathbf{p} \rangle + \tfrac{1}{2} m \|\mathbf{v}\|^2, R(u)\mathbf{p} - m\mathbf{v}, m). \tag{7.179}$$

This time, the $\widetilde{E(3)}$-orbits in $\mathbb{R}^5$ are:

1. $\mathscr{O}_E = \{(E, \mathbf{0}, 0)\}$ ($E \in \mathbb{R}$), with stabilizer $\widetilde{E(3)}$;
2. $\mathscr{O}_{(r,0)} = \{(E, \mathbf{p}, 0) \mid E \in \mathbb{R}, \|\mathbf{p}\| = r\}$ ($r > 0$), with stabilizer $E(2)'$;
3. $\mathscr{O}_{U,m} = \{(E, \mathbf{p}, m) \mid E - E_{\mathbf{p}} = U\}$ ($m \in \mathbb{R} \backslash \{0\}$, $U \in \mathbb{R}$), with stabilizer $SU(2)$.

Here $E(2)' \subset \widetilde{E(3)}$ is a double cover of $E(2)$, like the subgroup of $SL(2, \mathbb{C})$ stabilizing the point $(1, 0, 0, 1) \in \mathbb{R}^4$ in the theory of the Poincaré-group. This time we take any point $(E, 0, 0, r, 0) \in \mathbb{R}^5$, which is stabilized by pairs $(u, \mathbf{v}) \in \widetilde{E(3)}$ for which $R(u)$ is a rotation around the $z$-axis and $\mathbf{v} = (v_1, v_2, 0)$; the image of these pairs in $E(3)$ is $E(2) = SO(2) \ltimes \mathbb{R}^2$, where $SO(2) \subset SO(3)$ consists of rotations around the $z$-axis and $\mathbb{R}^2$ is the $x$-$y$ plane. In the third case we write $E_{\mathbf{p}} = \|\mathbf{p}\|^2 / 2m$ and take $(U, \mathbf{0}, m)$, whose stabilizer in $E(3)$ is evidently $SO(3)$.

Thus we have massless as well as massive particles both in relativistic and in non-relativistic quantum physics. The simplest case of all is formed by massive non-relativistic particles, which correspond to the orbits $\mathscr{O}_{U,m}$ above, supplemented with a spin $j$ labelling the underlying irreducible representation $D_j$ of $SU(2)$. Such orbits are diffeomorphic to $\mathbb{R}^3$ under the identification $(U + E_{\mathbf{p}}, \mathbf{p}, m) \leftrightarrow \mathbf{p}$, and a convenient choice of the cross-section $s : \mathscr{O}_{U,m} \to \widetilde{E(3)}$ is $s(\mathbf{p}) = (1_2, -\mathbf{p}/m)$, since in that case the Wigner cocycle simply becomes $s(\mathbf{p})^{-1}(u, \mathbf{v}) s((u, \mathbf{v})^{-1}\mathbf{p}) = u$. Since different values of $U$ turn out to give equivalent representations of $\check{G}$ (in the sense explained at the end of §5.10), we take $U = 0$, and eqs. (7.135) - (7.136) become

$$\tilde{H}^{m,j} = L^2(\mathbb{R}^3) \otimes H_j; \tag{7.180}$$

$$\tilde{u}^{m,j}(u, \mathbf{v}, a_0, \mathbf{a}) \tilde{\psi}(\mathbf{p}) = e^{i(a_0 E_{\mathbf{p}} + \langle \mathbf{a}, \mathbf{p} \rangle)} D_j(u) \tilde{\psi}(R(u)^{-1}(\mathbf{p} + m\mathbf{v})). \tag{7.181}$$

Here $L^2(\mathbb{R}^3)$ simply carries Lebesgue measure $d^3\mathbf{p}$, which is $\widetilde{E(3)}$-invariant.

The massive relativistic case is slightly more involved: we again have $\mathscr{O}_m^+ \cong \mathbb{R}^3$ under $(\omega_{\mathbf{p}}, \mathbf{p}) \leftrightarrow \mathbf{p}$, where $\omega_{\mathbf{p}} = \sqrt{\|\mathbf{p}\|^2 + m^2}$, but the Lorentz-invariant measure on $\mathscr{O}_m^+$ is $d^3\mathbf{p}/\omega_{\mathbf{p}}$. For each $\mathbf{p} \in \mathbb{R}^3$ there is a unique boost $b_{\mathbf{p}} \in L_+^{\uparrow}$ that maps $(m, 0, 0, 0)$ to $(\omega_{\mathbf{p}}, \mathbf{p})$, with pre-image $\tilde{b}_{\mathbf{p}}$ in $SL(2, \mathbb{C})$, so we take $s(\mathbf{p}) = \tilde{b}_{\mathbf{p}}$. The Hilbert space is (*mutatis mutandis*) still given by (7.180), but instead of (7.181) we now obtain

$$\tilde{u}^{m,j}(\tilde{\lambda}), \mathbf{a}) \tilde{\psi}(\mathbf{p}) = e^{i(a_0 \omega_{\mathbf{p}} - \langle \mathbf{a}, \mathbf{p} \rangle)} D_j(\tilde{b}_{\mathbf{p}}^{-1} \tilde{\lambda} \tilde{b}_{\lambda^{-1}\mathbf{p}}) \tilde{\psi}(\lambda^{-1}\mathbf{p}), \tag{7.182}$$

where $a = (a_0, \mathbf{a})$, $\tilde{\lambda} \in SL(2, \mathbb{C})$, and $\lambda \in L_+^{\uparrow}$ the image of $\tilde{\lambda}$ under the covering projection. We leave the corresponding formulae for the massless case to the reader.

## 7.7 Quantization and permutation symmetry

Another interesting application of the quantization theory developed in this chapter is to ***indistinghuishable particles***. Since all elementary particles come in families of indistinghuishable sorts (such as electrons, photons, ...), this topic is obviously of fundamental importance to physics. It is also puzzling, since (as we shall see) mathematically one expects more possibilities than those realized in Nature (namely bosons and fermions). This topic is also interesting philosophically, because it appears to be a testing ground for Leibniz's ***Principle of the Identity of Indiscernibles*** (PII), which states that two different objects cannot have exactly the same properties (in other words, two objects that have exactly the same properties must be identical).

After a period of confusion but growing insight, involving some of the greatest physicists such as Planck, Einstein, Ehrenfest, Fermi, and especially Heisenberg, the modern point of view on quantum statistics was introduced by Dirac.

Using modern notation, and abstracting from his specific example (which involved electronic wave-functions), Dirac's argument is as follows. Let $H$ be the Hilbert space of a single quantum system, called a *particle* in what follows. The two-fold tensor product $H^2 \equiv H \otimes H$ then describes two distinguishable copies of this particle. The permutation group $\mathfrak{S}_2$ on two objects, with nontrivial element $(12)$, acts on the state space $H^2$ by linear extension of $u(12)\psi_1 \otimes \psi_2 = \psi_2 \otimes \psi_1$. Praising Heisenberg's emphasis on defining everything in terms of observable quantities only, Dirac then declares the two particles to be indistinguishable if $u(12)au(12)^* = a$ for any two-particle observable $a$; by unitarity, this is to say that $a$ commutes with $u(12)$. Dirac notes that such operators map symmetrized vectors (i.e. those $\psi \in H \otimes H$ for which $u(12)\psi = \psi$) into symmetrized vectors, and likewise map anti-symmetrized vectors (i.e. those $\psi \in H \otimes H$ for which $u(12)\psi = -\psi$) into anti-symmetrized vectors, and these are the only possibilities; we would now say that under the action of the $\mathfrak{S}_2$-invariant (bounded) operators one has

$$H^2 \cong H^2_+ \oplus H^2_-; \tag{7.183}$$

$$H^2_+ = \{\psi \in H^2 \mid u(12)\psi = \psi\}; \tag{7.184}$$

$$H^2_- = \{\psi \in H^2 \mid u(12)\psi = -\psi\}. \tag{7.185}$$

Arguing that in order to avoid double counting (in that $\psi$ and $u(12)\psi$ should not both occur as independent states) one has to pick one of these two possibilities, Dirac concludes that state vectors of a system of two indistinguishable particles must be either symmetric or anti-symmetric. He then generalizes this to $N$ identical particles: if $(ij)$ is the element of the permutation group $\mathfrak{S}_N$ on $N$ objects that permutes $i$ and $j$ $(i, j = 1, \ldots, N)$, then according to Dirac, $\psi \in H^N \equiv H^{\otimes N}$ should satisfy either $u(ij)\psi = \psi$, in which case $\psi \in H^2_+$, or $u(ij)\psi = -\psi$, in which case $\psi \in H^2_-$, where $u$ is the natural unitary representation of $\mathfrak{S}_N$ on $H^N$, given, on $p \in \mathfrak{S}_N$, by linear (and if necessary continuous) extension of

$$u(p)\psi_1 \otimes \cdots \otimes \psi_N = \psi_{p(1)} \otimes \cdots \otimes \psi_{p(N)}. \tag{7.186}$$

Equivalently, $\psi \in H^2_+$ if it is invariant under all permutations, and $\psi \in H^2_-$ if it is invariant under even permutations and picks up a minus sign under odd permutations.

A slightly more sophisticated version of this argument often finds runs as follows:

> 'Since, in the case of indistinguishable particles, $\psi \in H^N$ and $u(p)\psi$ must represent the same state for any $p \in \mathfrak{S}_N$, and since two unit vectors represent the same state iff they differ by a phase vector, by unitarity it must be that $u(p)\psi = c(p)\psi$, for some $c(p) \in \mathbb{C}$ satisfying $|c(p)| = 1$. The group property $u(pp') = u(p)u(p')$ then implies that $c(p) = 1$ for even permutations and $c(p) = \pm 1$ for odd permutations. The choice $+1$ in the latter leads to bosons, whereas $-1$ leads to fermions, so these are the only possibilities.'

Alas, where Dirac's argument is incomplete, this one is even inconsistent: the claim that two unit vectors represent the same state iff they differ by a phase vector, presumes that the particles are distinguishable! Indeed, the only physical argument to the effect that two unit vectors $\psi$ and $\psi'$ are equivalent iff $\psi' = z\psi$ with $|z| = 1$, is that it guarantees that expectation values coincide, i.e., that

$$\langle \psi, a\psi \rangle = \langle \psi', a\psi' \rangle, \tag{7.187}$$

for *all* (bounded) operators $a$, i.e., not merely for the permutation-invariant operators (in which case (7.187) does not follow). But, following Heisenberg and Dirac, the whole point of having indistinguishable particles is that an operator $a$ represents a physical observable iff it is invariant under all permutations (acting by conjugation)!

Although the above arguments therefore seem feeble at best, their conclusion that only bosons and fermions can exist seems validated by Nature, despite the mathematical fact that the orthogonal complement of $H^2_+ \oplus H^2_-$ in $H_N$ (describing particles with ***parastatistics***) is non-zero as soon as $N > 2$. This should be a source of concern, and indeed, much research on indistinguishable particles (in $d > 2$) has had the goal of *explaining away parastatistics*. Distinguished by the different actions of $\mathfrak{S}_N$ they depart from, these explanations have traditionally been based on:

- ***Quantum observables.*** $\mathfrak{S}_N$ acts on the C*-algebra $B(H^N)$ of bounded operators on $H^N$ by conjugation of the unitary representation $u(\mathfrak{S}_N)$ on $H^N$, cf. (7.186). One implements permutation invariance by postulating that the physical observables of the $N$-particle system under consideration be the $\mathfrak{S}_N$-invariant operators: with $u$ given by (7.186), the algebra of observables is therefore taken to be

$$M_N = B(H^N)^{\mathfrak{S}_N} \equiv \{a \in B(H^N) \mid [a, u(p)] = 0 \, (p \in \mathfrak{S}_N)\}. \tag{7.188}$$

- ***Quantum states.*** By restriction, $\mathfrak{S}_N$ then also acts on the (normal) state space

$$\mathscr{S}_n(H^N) \cong \mathscr{D}(H^N) \subset B(H^N), \tag{7.189}$$

  from which it is postulated that the physical state space is $\mathscr{D}(H^N)^{\mathfrak{S}_N}$.
- ***Classical states.*** $\mathfrak{S}_N$ acts on $M^N$, the $N$-fold cartesian product of the classical one-particle phase space $M$, by permutation. If $M = T^*Q$ for some configuration space $Q$, we might as well start from the natural action of $\mathfrak{S}_N$ on $Q^N$ (pulled back to $M^N$), and this is indeed what we shall do, often further simplifying to $Q = \mathbb{R}^d$.

Unsurprisingly, the first two approaches equivalent. Define a linear map

$$E_N : B(H^N) \to B(H^N)^{\mathfrak{S}_N}; \tag{7.190}$$

$$a \mapsto \frac{1}{n!} \sum_{p \in \mathfrak{S}_N} u(p) a u(p)^*; \tag{7.191}$$

this is a (normal) ***conditional expectation*** from the von Neumann algebra $B(H^N)$ to the von Neumann algebra $B(H^N)^{\mathfrak{S}_N}$, i.e., $E_N(a^*) = E_N(a)^*$ for all $a \in B(H^N)$, $E_N^2 = E_N$, and $\|E_N\| = 1$. Moreover, $E_N$ preserves positivity as well as the trace, so that it also maps the state space $\mathscr{D}(H^N)$ onto the invariant states $\mathscr{D}(H^N) \subset B(H^N)$. Simple computations also establish the properties

$$\mathrm{Tr}\,(\rho a) = \mathrm{Tr}\,(E_N(\rho) a) \ \ (\rho \in \mathscr{D}(H^N), a \in B(H^N)^{\mathfrak{S}_N}); \tag{7.192}$$

$$\mathrm{Tr}\,(\rho a) = \mathrm{Tr}\,(\rho E_N(a)) \ \ (\rho \in \mathscr{D}(H^N)^{\mathfrak{S}_N}, a \in B(H^N)). \tag{7.193}$$

Finally, the reduction of $H^N$ under $u(\mathfrak{S}_N)$ described below may equally well be described in terms of the state space, since a subspace $eH^N \subset H^N$ (where $e \in \mathscr{P}(H^N)$ is a projection) is stable under $u$ iff $e \in \mathscr{P}(H^N)^{\mathfrak{S}_N}$, in which case it may be described in terms of the associated density operator $\rho = e/\mathrm{Tr}\,(e) \in \mathscr{D}(H^N)^{\mathfrak{S}_N}$. With some more effort, in can be even be shown that $\rho \in \partial_e(\mathscr{D}(H^N)^{\mathfrak{S}_N})$ iff $eH$ is irreducible.

We may therefore focus on the first and the third approaches, starting with the first, based on (7.188). Note that the C*-algebra of invariant *compact* operators, i.e.,

$$A_N = B_0(H^N)^{\mathfrak{S}_N} \equiv \{a \in B_0(H^N) \mid [a, u(p)] = 0\, (p \in \mathfrak{S}_N)\}, \tag{7.194}$$

induces the same decomposition of $H^N$ as $M_N$ does (since $M = A_N''$), so if $H$ is infinite-dimensional one may use $A_N$ rather than $M_N$ as the algebra of quantum observables; this is convenient for comparison with the classical state space approach.

As long as $\dim(H) > 1$ and $N > 1$, the algebras $M_N$ and $A_N$ act reducibly on $H^N$. The reduction of $H^N$ under $M_N$ (and hence of $A_N$ and of $u(H)^N$) is traditionally carried out by ***Schur duality***. This rests on the following concepts.

**Definition 7.11.** • *A* **partition** $\lambda$ *of $N$ is a way of writing*

$$N = n_1 + \cdots + n_k, \ n_1 \geq \cdots \geq n_k > 0, \ k = 1, \ldots, N. \tag{7.195}$$

- *The corresponding* **frame** *(or* **Young diagram***) $F_\lambda$ is a picture of $N$ boxes with $n_i$ boxes in the i'th row, $i = 1, \ldots, k$.*
- *For each frame $F_\lambda$, one has $N!$ possible* **Young tableaux** *$T$, each of which is a particular way of writing all of the numbers 1 to $N$ into the boxes of $F_\lambda$.*
- *A Young tableau is* **standard** *if the entries in each row increase from left to right and the entries in each column increase from top to bottom. The set of all (standard) Young tableaux on $F_\lambda$ is called $\mathscr{T}_\lambda$ ($\mathscr{T}_\lambda^S$).*
- *To each $T \in \mathscr{T}_\lambda$ we associate the subgroup* $\mathrm{Row}(T) \subset \mathfrak{S}_N$ *of all permutations $p \in \mathfrak{S}_N$ that preserve each row (i.e., each row of $T$ is permuted within itself); likewise* $\mathrm{Col}(T) \subset \mathfrak{S}_N$ *consists of all $p \in \mathfrak{S}_N$ that preserve each column.*

The set $\mathrm{Par}(N)$ of all partitions $\lambda$ of $N$ parametrizes the conjugacy classes of $\mathfrak{S}_N$ and hence also the (unitary) dual of $\mathfrak{S}_N$; in other words, up to (unitary) equivalence each (unitary) irreducible representation $u_\lambda$ of $\mathfrak{S}_N$ bijectively corresponds to some partition $\lambda$ of $N$; the dimension of any vector space $V_\lambda$ carrying $u_\lambda$ is $N_\lambda = |\mathscr{T}_\lambda^S|$, that is, the number of different standard Young tableaux on the frame $F_\lambda$.

Returning to (7.186), to each $\lambda \in \mathrm{Par}(N)$ and each Young tableau $T \in \mathscr{T}_\lambda$ we associate an operator $e_T$ on $H^N$ by the formula

$$e_T = \frac{N_\lambda}{N!} \sum_{p \in \mathrm{Col}(T)} \mathrm{sgn}(p) u(p) \sum_{p' \in \mathrm{Row}(T)} u(p'), \tag{7.196}$$

which happens to be a projection. Its image $e_T H^N \subset H^N$ is denoted by $H_T^N$, and the restriction of $M_N$ to $H_T^N$ is called $M_N(T)$. One may now write the decomposition of $H^N$ under the action of $M_N$ (up to unitary equivalence) as

$$H^N \cong \bigoplus_{\lambda \in \mathrm{Par}(N)} H_{T_\lambda}^N \otimes V_\lambda, \tag{7.197}$$

$$M_N \cong \bigoplus_{\lambda \in \mathrm{Par}(N)} M_N(T_\lambda) \otimes 1_{V_\lambda}, \tag{7.198}$$

$$u(\mathfrak{S}_N) \cong \bigoplus_{\lambda \in \mathrm{Par}(N)} 1_{H_{T_\lambda}^N} \otimes u_\lambda, \tag{7.199}$$

where the labeling is by the partitions $\lambda$ of $N$, the multiplicity spaces $V_\lambda$ are irreducible $\mathfrak{S}_N$-modules, and $T_\lambda$ is an arbitrary choice of a Young tableau defined on $F_\lambda$. For simplicity we here assume that $\dim(H) \geq N$; if $\dim(H) < N$, then only partitions (7.195) with $k \leq \dim(H)$ occur. For example, the partitions (7.195) of $N = 2$ are $2 = 2$ and $2 = 1 + 1$, each of which admits only one standard Young tableau, which we denote by $S$ and $A$, respectively. With $N_2 = N_{1+1} = 1$ and hence $V_1 \cong V_{1+1} \cong \mathbb{C}$ as vector spaces, this recovers (7.183); the corresponding projections $e_+$ and $e_-$, respectively, are given by $e_+ = \frac{1}{2}(1 + u(12))$ and $e_- = \frac{1}{2}(1 - u(12))$. The bosonic states $\psi_+$, i.e., the solutions of $\psi_+ \in H_+^2$, or $e_+\psi_+ = \psi_+$, are just the symmetric vectors, whereas the fermionic states $\psi_- \in H_-^2$ are the antisymmetric ones. These sectors exist for all $N > 1$ and they always occur with multiplicity one.

However, and this is the bite of the topic, for $N \geq 3$ additional irreducible representations of $M_N$ appear, always with multiplicity greater than one; states in such sectors are said to describe ***paraparticles*** and/or are said to have ***parastatistics***. For example, for $N = 3$ one new partition $3 = 2 + 1$ occurs, with $N_{2+1} = 2$, and hence

$$H^3 \cong H_+^3 \oplus H_-^3 \oplus H_P^3 \oplus H_{P'}^3, \tag{7.200}$$

where $H_P^3$ and $H_{P'}^3$ are the images of the projections $e_P = \frac{1}{3}(1 - u(13))(1 + u(12))$ and $e_{P'} = \frac{1}{3}(1 - u(12))(1 + u(13))$, respectively. The corresponding two classes of ***parastates*** (i.e. states carrying parastatistics) $\psi_P$ and $\psi_{P'}$ then by definition satisfy $e_P\psi_P = \psi_P$ and $e_{P'}\psi_{P'} = \psi_{P'}$, respectively. In other words, the Hilbert spaces carrying each of the four sectors are the following closed linear spans:

$$H^3_+ = \text{span}^-\{\psi_{123}+\psi_{213}+\psi_{321}+\psi_{312}+\psi_{132}+\psi_{231}\}; \quad (7.201)$$
$$H^3_- = \text{span}^-\{\psi_{123}-\psi_{213}-\psi_{321}+\psi_{312}-\psi_{132}+\psi_{231}\}; \quad (7.202)$$
$$H^3_P = \text{span}^-\{\psi_{123}+\psi_{213}-\psi_{321}-\psi_{312}\}; \quad (7.203)$$
$$H^3_{P'} = \text{span}^-\{\psi_{123}+\psi_{321}-\psi_{213}-\psi_{231}\}, \quad (7.204)$$

where $\psi_{ijk} \equiv \psi_i \otimes \psi_j \otimes \psi_k$ and the $\psi_i$ vary over $H$ (and span$^-$ is closed linear span).

For any $N > 2$, let us note that instead of the decomposition (7.197) - (7.198), which is defined up to unitary equivalence, one may alternatively decompose $H^N$ as

$$H^N = \bigoplus_{T\in\mathscr{T}^S_\lambda, \lambda\in\text{Par}(N)} H^N_T; \quad (7.205)$$
$$M_N = \bigoplus_{T\in\mathscr{T}^S_\lambda, \lambda\in\text{Par}(N)} M_N(T), \quad (7.206)$$

which has the advantage over (7.197) - (7.198) that the $H^N_T$ are subspaces of $H^N$. The disadvantage is that $M_N(T)$ is unitarily equivalent to $M_N(T')$ iff $T$ and $T'$ both lie in $\mathscr{T}^S_\lambda$ (i.e., for the same $\lambda$), so that unlike (7.197) - (7.198), the decomposition (7.205) - (7.206) is non-unique (for example, Young tableaux different from standard ones might have been chosen in the parametrization). The analogue of the third line (7.199) in the earlier decomposition would therefore be a mess. Indeed, although $\mathfrak{S}_N$ maps each of the subspaces $H_+$ and $H_-$ into itself (the former is even pointwise invariant under $\mathfrak{S}_N$, whereas elements of the latter at most pick up a minus sign), this is no longer the case for parastatistics. For example, for $N = 3$ some permutations map $H^3_P$ into $H^3_{P'}$, and *vice versa*. This is clear from (7.205) - (7.206): for $\lambda = P$, one has $\dim(V_P) = 2$, and choosing a basis $(\upsilon_1, \upsilon_2)$ of $V_P$ one may identify $H^{\otimes 3}_P$ and $H^{\otimes 3}_{P'}$ in (7.205) with (say) $H^{\otimes 3}_P \otimes \upsilon_1$ and $H^{\otimes 3}_P \otimes \upsilon_2$ in (7.197), respectively. And analogously for $N > 3$, where $\dim(V_\lambda) > 1$ for all $\lambda \neq S, A$.

A (or perhaps *the*) competing approach to permutation invariance in quantum mechanics starts from classical (rather than quantal) data. Let $Q$ be the classical single-particle configuration space, e.g., $Q = \mathbb{R}^d$; to avoid irrelevant complications, we assume that $Q$ is a connected and simply connected manifold. The associated configuration space of $N$ identical but distinguishable particles is $Q^N$. Depending on the assumption of (in)penetrability of the particles, we may define one of

$$\check{Q}_N = Q^N/\mathfrak{S}_N; \quad (7.207)$$
$$Q_N = (Q^N \backslash \Delta_N)/\mathfrak{S}_N, \quad (7.208)$$

as the configuration space of $N$ indistinguishable particles, where $\Delta_N$ is the extended diagonal in $Q^N$, i.e., the set of points $(q_1,\ldots,q_N) \in Q^N$ where $q_i = q_j$ for at least one pair $(i, j)$, $i \neq j$ (so that for $Q = \mathbb{R}$ and $N = 2$ this is the usual diagonal in $\mathbb{R}^2$). At first sight, these two choices should lead to exactly the same quantum theory, based on the Hilbert space $L^2(\check{Q}_N) = L^2(Q_N)$, since $\Delta_N$ is a subset of measure zero for any measure used to define $L^2$ that is locally equivalent to Lebesgue measure.

However, the effect of $\Delta_N$ is noticeable as soon as one represents physical observables as operators on $L^2$ through any serious quantization procedure, which should be sensitive to both the topological and the smooth structure of the underlying configuration space. In the case at hand, $Q_N$ is multiply connected as a topological space, but as a manifold it is smooth and has no singularities. In contrast, $\check{Q}_N$ is simply connected as a topological space, but in the smooth setting it is a so-called *orbifold*. This leads to interesting complications, but following tradition (i.e., in the configuration space approach to indistinguishable particle) we continue with $Q_N$.

To quantize $Q_N$ we use the language of Lie groupoids and their C*-algebras, cf. §§C.16–C.17. Let $Q$ be any (possibly) multiply connected manifold, with universal covering space $\tilde{Q}$. In particular, the first homotopy group $\pi_1(Q)$ acts (say from the right) on $\tilde{Q}$ in such a way that $Q = \tilde{Q}/\pi_1(Q)$. We denote the canonical projection by $\pi : \tilde{Q} \to Q$. One may have the example $Q = \mathbb{T}$, $\tilde{Q} = \mathbb{R}$, $\pi_1(Q) = \mathbb{Z}$ in mind here.

As a variation on the pair groupoid $G = Q \times Q$, we now consider the Lie groupoid

$$\tilde{G}_Q = \tilde{Q} \times_{\pi_1(Q)} \tilde{Q}, \tag{7.209}$$

whose elements are equivalence classes $[\tilde{q}_1, \tilde{q}_2]$ in $\tilde{Q} \times \tilde{Q}$ under the equivalence relation $\sim$ defined by $(\tilde{q}_1, \tilde{q}_2) \sim (\tilde{q}_1', \tilde{q}_2')$ iff $\tilde{q}_1 = \tilde{q}_1' x$ and $\tilde{q}_2 = \tilde{q}_2' x$ for some $x \in \pi_1(Q)$; the source and target projections are $s([\tilde{q}_1, \tilde{q}_2]) = \pi(\tilde{q}_2)$ and $t([\tilde{q}_1, \tilde{q}_2]) = \pi(\tilde{q}_1)$, respectively, the inverse is $[\tilde{q}_1, \tilde{q}_2]^{-1} = [\tilde{q}_2, \tilde{q}_1]$, and multiplication is the obvious one borrowed from the pair groupoid $\tilde{Q} \times \tilde{Q}$ over $\tilde{Q}$ (which is well defined on $\tilde{G}_Q$). The tangent groupoid $\tilde{G}_Q^T$ of $\tilde{G}_Q$ (cf. Proposition C.117) has the following fiber at $\hbar = 0$:

$$(\tilde{G}_Q)_0^T = TQ, \tag{7.210}$$

to be contrasted with the corresponding fiber $G_0^T = T\tilde{Q}$ of the pair groupoid on the covering space $\tilde{Q}$. In particular, for our configuration space $Q = Q_N$ we have

$$\tilde{G}_{Q_N} = \tilde{Q}_N \times_{\pi_1(Q_N)} \tilde{Q}_N; \tag{7.211}$$

$$(\tilde{G}_{Q_N})_0^T = TQ_N, \tag{7.212}$$

which gives the fibers of the corresponding continuous bundle of C*-algebras as

$$A_0 = C_0(T^*Q_N) \quad (\hbar = 0); \tag{7.213}$$

$$A_\hbar = C^*(\tilde{G}_Q) \qquad (0 < \hbar \leq 1), \tag{7.214}$$

cf. §C.19. This gives a generalization of the fibers (7.17) - (7.18) for $Q = \mathbb{R}^n$, and also now we have an example of Definition 7.1: the fibers (7.213) - (7.214) combine to form a continuous bundle of C*-algebras with total C*-algebra $A = C^*(\tilde{G}_Q^T)$, yielding a deformation quantization of the Poisson manifold $T^*Q_N$ (i.e., the usual phase space defined by the configuration space $Q_N$). We now define the ***inequivalent quantizations*** of $Q_N$ as the inequivalent irreducible representations of the corresponding C*-algebra of quantum observables $C^*(\tilde{G}_{Q_N})$, as follows.

**Theorem 7.12.** *1. Let $Q$ be multiply connected. The inequivalent irreducible representations $\pi^{\lambda}$ of the C*-algebra $C^*(\tilde{G}_Q)$ bijectively correspond to the inequivalent irreducible unitary representations $u_{\lambda}$ of the first homotopy group $\pi_1(Q)$.*
*2. Each representation $\pi^{\lambda}$ has a natural realization on the Hilbert space*

$$H^{\lambda} = L^2(Q) \otimes H_{\lambda}, \tag{7.215}$$

*where $H_{\lambda}$ is a specific carrier space for the representation $u_{\lambda}$. More fancifully, one may use the Hilbert space $L^2(Q, E_{\lambda})$ of $L^2$-sections of the vector bundle*

$$E_{\lambda} = \tilde{Q} \times_{\pi_1(Q)} H_{\lambda} \tag{7.216}$$

*associated to the principal bundle $\pi : \tilde{Q} \to Q$ by the representation $u_{\lambda}$.*

Provided one accepts (7.208), this theorem in principle gives a complete solution to the problem of quantizing multiply connected configuration spaces, and hence, taking $Q = Q_N$, of the problem of quantizing systems of indistinguishable particles.

*Proof.* We just prove Theorem 7.12 in the case we need, where $\pi_1(Q)$ is finite. Then

$$C^*(\tilde{Q} \times_{\pi_1(Q)} \tilde{Q}) \cong B_0(L^2(\tilde{Q}))^{\pi_1(Q)}; \tag{7.217}$$
$$B_0(L^2(\tilde{Q}))^{\pi_1(Q)} \cong B_0(L^2(Q)) \otimes C^*(\pi_1(Q)), \tag{7.218}$$

where (in our usual notation) $B_0(L^2(\tilde{Q}))^{\pi_1(Q)}$ is the C*-algebra of $\pi_1(Q)$-invariant compact operators on $L^2(\tilde{Q})$, and $C^*(\pi_1(Q))$ is the group C*-algebra of $\pi_1(Q)$ (which is finite-dimensional and hence nuclear, given the assumption that $\pi_1(Q)$ is finite, so that the choice of the C*-algebraic tensor product does not matter).

To prove (7.217), we first exploit finiteness of $\pi_1(Q)$ in order to identify functions $\tilde{a} \in C_c^{\infty}(\tilde{G}_Q)$ with constrained $C_c^{\infty}$ functions $a$ on $\tilde{Q} \times \tilde{Q}$ that satisfy

$$a(\tilde{q}h, \tilde{q}'h) = a(\tilde{q}, \tilde{q}') \ \ (h \in \pi_1(Q)). \tag{7.219}$$

This identification is explicitly given by

$$a(\tilde{q}, \tilde{q}') = \tilde{a}([\tilde{q}, \tilde{q}']), \tag{7.220}$$

where $[\tilde{q}, \tilde{q}']$ denotes the equivalence class of $(\tilde{q}, \tilde{q}') \in \tilde{Q} \times \tilde{Q}$ under the diagonal action of $\pi_1(Q)$. This makes the space $C_c^{\infty}(\tilde{G}_Q)$ a dense subset of $C^*(\tilde{G}_Q)$. We write $a \in C_c^{\infty}(\tilde{Q} \times \tilde{Q})^{\pi_1(Q)}$; for (7.208) this just means that $a$ is a permutation-invariant kernel. Second, we equip $\tilde{Q}$ with some measure $d\tilde{q}$ that is locally equivalent to the Lebesgue measure, and in addition is $\pi_1(Q)$-invariant under the regular action $R$ of $\pi_1(Q)$ on functions on $\tilde{Q}$, given, as usual, by $R_h \tilde{\psi}(\tilde{q}) = \tilde{\psi}(\tilde{q}h)$. In that case, one also has a measure $dq$ on $Q$ that is locally equivalent to the Lebesgue measure, so that the measures $d\tilde{q}$ and $dq$ on $\tilde{Q}$ and $Q$, respectively, are related by

$$\int_{\tilde{Q}} d\tilde{q}\, f(\tilde{q}) = \frac{1}{|\pi_1(Q)|} \sum_{h \in \pi_1(Q)} \int_Q dq\, f(s(q)h). \tag{7.221}$$

Here $f \in C_c(\tilde{Q})$, $|\pi_1(Q)|$ is the number of elements of $\pi_1(Q)$, and $s : Q \to \tilde{Q}$ is any (measurable) cross-section of $\tau : \tilde{Q} \to Q$. We may then define a Hilbert space $L^2(\tilde{Q})$ with respect to $d\tilde{q}$, on which elements $a$ of $C_c^\infty(\tilde{Q} \times \tilde{Q})^{\pi_1(Q)}$ act faithfully by

$$a\tilde{\psi}(\tilde{q}) = \int_{\tilde{Q}} d\tilde{q}'\, a(\tilde{q}, \tilde{q}')\,\tilde{\psi}(\tilde{q}'). \tag{7.222}$$

The product of two such operators is given by the multiplication of the kernels on $\tilde{Q}$, and involution is defined as expected, too, namely by hermitian conjugation:

$$a^*(\tilde{q}, \tilde{q}') = \overline{a(\tilde{q}', \tilde{q})}. \tag{7.223}$$

The norm-closure of $C_c^\infty(\tilde{Q} \times \tilde{Q})^{\pi_1(Q)}$, represented as operators on $L^2(\tilde{Q})$ by (7.222), is then given by $B_0(L^2(\tilde{Q}))^{\pi_1(Q)}$. This proves (7.217).

Eq. (7.218) is a special case of the following: let $X$ be a manifold carrying a *free* action of a *compact* group $G$. If $L^2(X)$ is defined by some $G$-invariant "locally Lebesgue" measure on $X$, as in the construction above, then one has an isomorphism

$$B_0(L^2(X))^G \cong B_0(L^2(X/G) \otimes C^*(G). \tag{7.224}$$

This is proved in a similar way, realizing $B_0(H)$ as the norm-completion of the Hilbert–Schmidt operators $B_2(H)$ (for general $H$), and, in the $L^2$-case at hand, identifying $B_2(L^2(X))$ with the algebra of operators with kernels in $L^2(X \times X)$.

Part 2 of the theorem now follows from the fact that for any Hilbert space $H$ the C*-algebra $B_0(H)$ of compact operators on $H$ has exactly one irreducible representation (up to unitary equivalence), i.e. the defining one (this can be proved in many ways, e.g. from Rieffel's theory of Morita equivalence of C*-algebras), combined with the bijective correspondence between continuous unitary representations $u$ of any locally compact group $G$ and non-degenerate representations of its associated group C*-algebra $C^*(G)$; see §C.18, Definition C.119 etc. □

As mentioned in Theorem 7.12, there are two ways of realizing the Hilbert space $H^\lambda$, where $\lambda$ labels some irreducible representation of $\pi_1(Q)$. This is very similar to the discussion in §7.5, so we will be relatively brief here. The first realization corresponds to having constrained wave-functions defined on the covering space $\tilde{Q}$; for example, the usual description of bosonic or fermonic wave-functions is of this sort. The second realization uses unconstrained wave-functions on the actual configuration space $Q$ (*bad hombres* confusingly call such functions "multi-valued").

1. The space $C^\infty(Q, E_\lambda)$ of *smooth* cross-sections of $E_\lambda$ may be given by the smooth maps $\tilde{\psi} : \tilde{Q} \to H_\lambda$ satisfying the equivariance condition ("constraint")

$$\tilde{\psi}(\tilde{q}h) = u_\lambda(h^{-1})\tilde{\psi}(\tilde{q}), \tag{7.225}$$

for all $h \in \pi_1(Q)$, $\tilde{q} \in \tilde{Q}$. The Hilbert space

$$H^\lambda = L^2(\tilde{Q}, H_\lambda)^{\pi_1(Q)}, \tag{7.226}$$

then, is defined as the usual $L^2$-completion of the space of all $\tilde{\psi} \in \Gamma(Q, E_\lambda)$ for which $\langle \tilde{\psi}, \tilde{\psi} \rangle < \infty$. The irreducible representation $\pi^\lambda(C^*(G_Q))$ is then given on elements $\tilde{a}$ of the dense subspace $C_c^\infty(G_Q)$ of $C^*(G_Q)$ by the expression

$$\pi^\lambda(\tilde{a})\psi(\tilde{q}) = \int_{\tilde{Q}} d\tilde{q}'\, \tilde{a}([\tilde{q}, \tilde{q}'])\psi(\tilde{q}'); \tag{7.227}$$

any $\pi_1(Q)$-invariant operator on $L^2(\tilde{Q})$ acts on $H^\lambda$ in this way (ignoring $H_\lambda$). If $\pi_1(Q)$ is finite, then two simplifications occur. Firstly, $H_\lambda$ is finite-dimensional, and secondly each Hilbert space $H^\lambda$ may be regarded as a subspace of $L^2(\tilde{Q})$; the above action of $C^*(G_Q)$ on $H^\lambda$ is then simply given by restriction of its action on $L^2(\tilde{Q})$. In that case one may equivalently realize this irreducible representation in terms of the right-hand side of (7.217), in which case the action of $\pi^\lambda(a)$ on $H^\lambda$ as defined in (7.226) is given by

$$\pi^\lambda(a)\psi(\tilde{q}) = \int_{\tilde{Q}} d\tilde{q}'\, a(\tilde{q}, \tilde{q}')\psi(\tilde{q}'). \tag{7.228}$$

This is true as it stands if $a \in C_c^\infty(\tilde{Q} \times \tilde{Q})^{\pi_1(Q)}$, cf. (7.219), and may be extended to general $\pi_1(Q)$-invariant compact operators $a \in B_0(L^2(\tilde{Q}))^{\pi_1(Q)}$ by norm continuity, and, furthermore, even to $B(L^2(\tilde{Q}))^{\pi_1(Q)}$ by strong or weak continuity.

2. Elements of the Hilbert space $L^2(\tilde{Q}, H_\lambda)^{\pi_1(Q)}$ are typically (equivalence classes of) *discontinuous* cross-sections of $E_\lambda$. Possibly discontinuous cross-sections may simply be given directly as functions $\psi : Q \to H_\lambda$, with inner product

$$\langle \psi, \varphi \rangle = \int_Q dq\, \langle \psi(q), \varphi(q) \rangle_{H_\lambda}. \tag{7.229}$$

This specific realization of $L^2(Q, E_\lambda)$ will be denoted by $L^2(Q) \otimes H_\lambda$. If $H_\lambda = \mathbb{C}$,

$$L^2(Q) \otimes H_\lambda \cong L^2(Q). \tag{7.230}$$

These equivalent descriptions of $\pi^\lambda$ may be related once a (typically discontinuous) cross-section $\sigma : Q \to \tilde{Q}$ of the projection $\tau : \tilde{Q} \to Q$ has been chosen (i.e., $\tau \circ \sigma = \mathrm{id}_Q$), in which case $\psi(q) = \tilde{\psi}(\sigma(q))$. We formalize this in terms of a unitary

$$u : L^2(\tilde{Q}, H_\lambda)^{\pi_1(Q)} \to L^2(Q) \otimes H_\lambda \tag{7.231}$$
$$u\tilde{\psi}(q) = \tilde{\psi}(\sigma(q)); \tag{7.232}$$
$$u^{-1}\psi(\tilde{q}) = u_\lambda(h)\psi(q), \tag{7.233}$$

where $q = \tau(\tilde{q})$, and $h$ is the unique element of $\pi_1(Q)$ for which $\tilde{q}h = \sigma(q)$. The action $\pi_\sigma^\lambda(a) = u\pi^\lambda(a)u^{-1}$ on $L^2(Q) \otimes H_\lambda$ now follows from (7.228) - (7.233): If $a$ is a $\pi_1(Q)$-invariant kernel on $L^2(\tilde{Q})$, then using (7.221) we obtain

$$\pi_\sigma^\lambda(a)\psi(q) = \sum_{h \in \pi_1(Q)} \int_Q dq'\, a(\sigma(q), \sigma(q')h)u_\lambda(h)\psi(q'). \tag{7.234}$$

We now apply this formalism to $N$ indistinguishable particles moving on the (single-particle) configuration space $\mathbb{R}^3$. Eq. (7.208) then gives the $N$-particle space

$$Q_N = ((\mathbb{R}^3)^N - \Delta_N)/\mathfrak{S}_N. \tag{7.235}$$

The universal covering space of this multiply connected space is

$$\tilde{Q}_N = \mathring{\mathbb{R}}^{3N} \equiv (\mathbb{R}^3)^N - \Delta_N, \tag{7.236}$$

which (unlike its counterpart in $d = 2$) is connected and simply connected, so that

$$\pi_1(Q_N) = \mathfrak{S}_N. \tag{7.237}$$

It follows from (7.217) and (7.237) that the algebra of observables is given by

$$C^*(\tilde{G}_{Q_N}) = B_0(L^2(\mathbb{R}^3)^{\otimes N})^{\mathfrak{S}_N}. \tag{7.238}$$

Comparing (7.238) with (7.194), we obtain a complete equivalence between the "quantum observables" approach and the deformation quantization approach based on Theorem 7.12, in that the configuration space approach through the representation theory of the groupoid C*-algebra $C^*(\tilde{G}_{Q_N})$ leads to the same classification as the "quantum observables" approach based in (7.188) above, cf. (7.197) - (7.199).

We discuss a few interesting special cases.

**N = 1.** Here $\tilde{Q}_1 = Q_1 = \mathbb{R}^3$ and $\pi_1(Q_1) = \{e\}$, so the algebra of observables is

$$C^*(\tilde{G}_{Q_1}) = B_0(L^2(\mathbb{R}^3)), \tag{7.239}$$

which has a unique irreducible representation on $L^2(\mathbb{R}^3)$.

**N = 2.** This time, the pertinent homotopy group is

$$\pi_1(Q_2) = \mathfrak{S}_2 = \mathbb{Z}_2 = \{e, (12)\}, \tag{7.240}$$

which has two irreducible representations: firstly, $u_B(p) = 1$ for both $p \in \mathfrak{S}_2$, and secondly, $u_F(e) = 1$, $u_F(12) = -1$, each realized on $H_\lambda = \mathbb{C}$. Hence with $q = (x, y, z) \in \mathbb{R}^3$, eq. (7.225) yields

$$H_B^2 = \{\psi \in L^2(\mathbb{R}^3)^2 \mid \psi(q_2, q_1) = \psi(q_1, q_2)\}; \tag{7.241}$$
$$H_F^2 = \{\psi \in L^2(\mathbb{R}^3)^2 \mid \psi(q_2, q_1) = -\psi(q_1, q_2)\}. \tag{7.242}$$

Here $L^2(\mathbb{R}^3)^2 \equiv L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3) \cong L^2(\mathbb{R}^6)$. The C*-algebra

$$C^*(\tilde{G}_{Q_2}) = B_0(L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3))^{\mathfrak{S}_2} \cong B_0(L^2(\mathbb{R}^3 \times \mathbb{R}^3))^{\mathfrak{S}_2} \tag{7.243}$$

consists of all $\mathfrak{S}_2$-invariant compact operators on $L^2(\mathbb{R}^3 \times \mathbb{R}^3)$, acting on $H_B^2$ or $H_F^2$ in the same way as they do on $L^2(\mathbb{R}^6)$; cf. (7.228), noting that the constraints in (7.241) and (7.242) are preserved due to the $\mathfrak{S}_2$-invariance of $A \in C^*(\tilde{G}_{Q_2})$. This recovers Dirac's description of statistics given earlier in this section.

**N = 3.** Here we have a non-abelian homotopy group

$$\pi_1(Q_3) = \mathfrak{S}_3, \tag{7.244}$$

which, besides the irreducible boson and fermion representations on $\mathbb{C}$, has an irreducible ***parafermionic*** representation $u_P$ on $H_P = \mathbb{C}^2$. This representation is most easily obtained explicitly by reducing the natural action of $\mathfrak{S}_3$ on $\mathbb{C}^3$. Define an orthonormal basis of the latter by

$$e_0 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix};\ e_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\end{pmatrix};\ e_2 = \frac{1}{\sqrt{6}}\begin{pmatrix}-2\\1\\1\end{pmatrix}. \tag{7.245}$$

It follows that $\mathbb{C}\cdot e_0$ carries the trivial representation of $\mathfrak{S}_3$, whereas the linear span of $e_1$ and $e_2$ carries a two-dimensional irreducible representation $u_P$, given on the generators $(12)$, $(13)$, and $(23)$ of $\mathfrak{S}_3$ by

$$u_P(12) = \tfrac{1}{2}\begin{pmatrix}1 & -\sqrt{3}\\ -\sqrt{3} & -1\end{pmatrix};\ u_P(13) = \tfrac{1}{2}\begin{pmatrix}1 & \sqrt{3}\\ \sqrt{3} & -1\end{pmatrix};\ u_P(23) = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}. \tag{7.246}$$

We already gave realizations of the Hilbert space $H_P^3$ of three parafermions in (7.203) and (7.204),where it emerged as a subspace of $L^2(\mathbb{R}^3)\otimes L^2(\mathbb{R}^3)\otimes L^2(\mathbb{R}^3) \cong L^2(\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3)$. An equivalent realization $H^P \equiv \tilde{H}_P^3$ may be given on the basis of (7.225), according to which $H^P$ is the subspace of $L^2(\mathbb{R}^3)^3\otimes\mathbb{C}^2 \cong L^2(\mathbb{R}^9)\otimes\mathbb{C}^2$ that consists of doublet wave-functions $\psi_i$ $(i=1,2)$ that satisfy

$$\psi_i(q_{p(1)},q_{p(2)},q_{p(3)}) = \sum_{j=1}^{2} u_{ij}(p)\psi_j(q_1,q_2,q_3), \tag{7.247}$$

for any permutation $p\in\mathfrak{S}_3$, where $u\equiv u_P$, cf. (7.246). I.e., the parafermionic wave-functions in this realization of $H_P^3$ are constrained by the conditions

$$\psi_1(q_2,q_1,q_3) = \tfrac{1}{2}\psi_1(q_1,q_2,q_3) - \tfrac{1}{2}\sqrt{3}\,\psi_2(q_1,q_2,q_3); \tag{7.248}$$
$$\psi_2(q_2,q_1,q_3) = -\tfrac{1}{2}\sqrt{3}\,\psi_1(q_1,q_2,q_3) - \tfrac{1}{2}\psi_2(q_1,q_2,q_3); \tag{7.249}$$
$$\psi_1(q_3,q_2,q_1) = \tfrac{1}{2}\psi_1(q_1,q_2,q_3) + \tfrac{1}{2}\sqrt{3}\,\psi_2(q_1,q_2,q_3); \tag{7.250}$$
$$\psi_2(q_3,q_2,q_1) = \tfrac{1}{2}\sqrt{3}\,\psi_1(q_1,q_2,q_3) - \tfrac{1}{2}\psi_2(q_1,q_2,q_3); \tag{7.251}$$
$$\psi_1(q_1,q_3,q_2) = -\psi_1(q_1,q_2,q_3); \tag{7.252}$$
$$\psi_2(q_1,q_3,q_2) = \psi_2(q_1,q_2,q_3). \tag{7.253}$$

The algebra of observables $C^*(\tilde{G}_{Q_3})$ of three indistinguishable particles without internal degrees of freedom, i.e., then acts on $H^P\subset L^2(\mathbb{R}^3)^3\otimes\mathbb{C}^2$ as in (7.234), identifying $a\in C^*(\tilde{G}_{Q_3})$ with $a\otimes 1_2$ (so that $a$ ignores the internal degree of freedom $\mathbb{C}^2$). This representation $\pi^P$ is irreducible by Theorem 7.12.

**N > 3**. The above construction may be generalized to any $N > 3$. There will now be many parafermionic representations $u_\lambda$ of $\mathfrak{S}_N$ (given by Young tableaus), each of which induces an irreducible representation of the C*-algebra (7.238).

The question now arises whether parastatistics is to be found in Nature—or, indeed, if this question is even well defined! As a warm-up to the case $N = 3$, where the question first plays a role, let us give an alternative realization of $\pi^F(C^*(\tilde{G}_{Q_2}))$, cf. Theorem 7.12. Take two isospin doublet bosons (which by definition transform under the defining spin-$\frac{1}{2}$ representation $D_{1/2}$ of $SU(2)$ on $\mathbb{C}^2$). With

$$H^{(2)} = (L^2(\mathbb{R}^3) \otimes \mathbb{C}^2)^{\otimes 2}, \tag{7.254}$$

and using indices $a_1, a_2 = 1, 2$, the Hilbert space of these bosons is

$$H_B^{(2)} = \{\psi \in H^{(2)} \mid (\psi_{a_2 a_1}(q_2, q_1) = \psi_{a_1 a_2}(q_1, q_2)\}, \tag{7.255}$$

with corresponding projection $e_B^{(2)} : H^{(2)} \to H_B^{(2)}$ given by

$$e_B^{(2)} \psi_{a_1 a_2}(q_1, q_2) = \tfrac{1}{2}(\psi_{a_2 a_1}(q_2, q_1) + \psi_{a_1 a_2}(q_1, q_2)). \tag{7.256}$$

Subsequently, define a partial isometry $w : H^{(2)} \to L^2(\mathbb{R}^3)^{\otimes 2}$ by

$$w\psi(q_1, q_2) \equiv \psi_0(q_1, q_2) = \frac{1}{\sqrt{2}}(\psi_{12}(q_1, q_2) - \psi_{21}(q_1, q_2)). \tag{7.257}$$

Physically, this singles out an isospin singlet Hilbert subspace $H^{(0)} = e_0 H^{(2)}$ within $H^{(2)}$, where $e_0 = w^* w$ (which is a projection). This singlet subspace may be constrained to the bosonic sector by passing to

$$H_B^{(0)} = e_0 e_B^{(2)} H^{(2)}; \tag{7.258}$$

note that $e_0$ and $e_B^{(2)}$ commute. Now extend the defining representation of $C^*(\tilde{G}_{Q_2})$ on $L^2(\mathbb{R}^3)^{\otimes 2}$ to $H^{(2)}$ by ignoring the indices $a_1, a_2$ (i.e., isospin is deemed unobservable). This extended representation commutes with $e_0$ and with $e_B^{(2)}$, and hence is well defined on $H_B^{(0)} \subset H^{(2)}$. Let us denote this representation of $\tilde{G}_{Q_2}$ by $\pi_B^{(0)}$. It is then immediate from the property $\psi_0(q_2, q_1) = -\psi_0(q_1, q_2)$ that:

**Proposition 7.13.** *The representations $\pi_B^{(0)}(C^*(\tilde{G}_{Q_2}))$ on $H_B^{(0)}$ and $\pi^F(C^*(\tilde{G}_{Q_2}))$ on $H^F$ are unitarily equivalent.*

In other words, two *fermions* without internal degrees of freedom are equivalent to the singlet state of two *bosons* with an isospin degrees of freedom, at least if the observables are isospin-blind. Similarly, two *bosons* without internal degrees of freedom are equivalent to the singlet state of two *fermions* with isospin, and two fermions without internal degrees of freedom are equivalent to the isospin triplet state of two fermions (this corresponds to the Schur decomposition of $(\mathbb{C}^2)^{\otimes 2}$ under the commuting actions of $\mathfrak{S}_2$ and $SU(2)$).

For $N = 3$ we may carry out a similar trick as for $N = 2$, and replace parafermions without (further) degrees of freedom by either bosons or fermions. We discuss the former and leave the explicit description of the various alternative descriptions to the reader. We proceed as for $N = 2$, *mutatis mutandis*. We have a Hilbert space

$$H^{(3)} = (L^2(\mathbb{R}^3) \otimes \mathbb{C}^2)^{\otimes 3}, \tag{7.259}$$

of three distinguishable isospin doublets, containing the Hilbert space $H_B^{(3)}$ of three bosonic isospin doublets as a subspace, that is,

$$H_B^{(3)} = \{\psi \in H^{(3)} \mid \psi_{a_{p(1)}a_{p(2)}a_{p(3)}}(q_{p(1)}, q_{p(2)}, q_{p(3)}) = \psi_{a_1a_2a_3}(q_1, q_2, q_3)\, (p \in \mathfrak{S}_3)\}. \tag{7.260}$$

The corresponding projection, denoted by $e_B^{(3)} : H^{(3)} \to H_B^{(3)}$, will not be written down explicitly. Define an $SU(2)$ doublet $(\psi_1, \psi_2)$ within the space $H^{(3)}$ through a partial isometry

$$w : H^{(3)} \to L^2(\mathbb{R}^3)^{\otimes 3} \otimes \mathbb{C}^2; \tag{7.261}$$

$$w\psi_1(q_1, q_2, q_3) = \frac{1}{\sqrt{2}}(\psi_{121}(q_1, q_2, q_3) - \psi_{112}(q_1, q_2, q_3)); \tag{7.262}$$

$$w\psi_2(q_1, q_2, q_3) = \frac{1}{\sqrt{6}}(-2\psi_{211}(q_1, q_2, q_3) + \psi_{121}(q_1, q_2, q_3) + \psi_{112}(q_1, q_2, q_3)). \tag{7.263}$$

Defining a projection $e_2 = w^*w$ on $H^{(3)}$, the Hilbert space $H^{(3)}$ contains a closed subspace $H_B^{(2)} = e_2 e_B^{(3)} H^{(3)}$, which is stable under the natural representation of $C^*(\tilde{G}_{Q_3})$ (since $e_2$ and $e_B^{(3)}$ commute). We call this representation $\pi_B^{(2)}$. An easy calculation then establishes:

**Proposition 7.14.** *The representations $\pi_B^{(2)}(C^*(\tilde{G}_{Q_3}))$ on $H_B^{(2)}$ and $\pi^P(C^*(\tilde{G}_{Q_3}))$ on $H^P$ (as defined by Theorem 7.12) are unitarily equivalent.*

In other words, three *parafermions* without internal degrees of freedom are quivalent to an isospin doublet formed by three identical *bosonic* isospin doublets (corresponding to the Schur decomposition of $(\mathbb{C}^2)^{\otimes 3}$ under the commuting actions of $\mathfrak{S}_3$ and $SU(2)$; in this decomposition, the spin 3/2 representation of $SU(2)$ couples to the bosonic representation of $\mathfrak{S}_3$, whilst the spin-$\frac{1}{2}$ representation of $SU(2)$ couples to the parafermionic representation of $\mathfrak{S}_3$), at least if the observables of the latter are isospin-blind. Many other realizations of parafermions in terms of fermions or bosons with an internal degree of freedom can be constructed in a similar way.

For $N > 3$ we similarly find that the representation of the C*-algebra (7.238) induced by some parafermionic representations $u_\chi$ of $\mathfrak{S}_N$ is unitarily equivalent to a representation on some $SU(n)$ multiplet of bosons with an internal degree of freedom; the appropriate multiplet is the one coupled to $u_\chi$ in the Schur reduction of $(\mathbb{C}^n)^{\otimes N}$ with respect to the natural and commuting actions of $\mathfrak{S}_N$ and $SU(n)$.

The moral of this story is that one cannot tell from glancing at some Hilbert space whether the world consists of fermions or bosons or parafermions; what matters is the Hilbert space *as a carrier of some (irreducible) representation of the algebra of observables*. From that perspective we already see for $N = 2$ that being bosonic or fermionic is not an invariant property of such representations, since one may freely choose between fermions/bosons *without* internal degrees of freedom and bosons/fermions *with* internal degrees of freedom. In a more systematic discussion using superselection theory one may impose some physical selection criterion in order to restrict attention to "physically interesting" sectors. Such criteria (which, for example, would have the goal of excluding parastatistics) should be formulated with reference to some algebra of observables. Such issues cannot be settled at the level of quantum mechanics and instead require quantum field theory, where parastatistics can always be removed in terms of either bose- or fermi-statistics, in somewhat similar vein to our discussion. For (nonlocal) charges in gauge theories there are no rigorous results, but historically a similar goal played a role in the road to quantum chromodynamics (QCD), which is one of the ingredients of the Standard Model.

A different argument against parastatistics arises from the state space approach based on the compact convex set $\mathscr{D}(H^N)^{\mathfrak{S}_N}$ studied at the beginning of this section. The extreme boundary $\partial_e\left(\mathscr{D}(H^N)^{\mathfrak{S}_N}\right)$ consists of one part that is contained in $\partial_e\mathscr{D}(H^N) = \mathscr{P}_1(H^N)$, and one that is not. The first part consists of those one-dimensional invariant projections $e \in \mathscr{P}_1(H^N)^{\mathfrak{S}_N}$ whose image $eH^N$ belongs to either the bosonic subspace $H^N_+$ (in which case $u(p)e = e$ for each $p \in \mathfrak{S}_N$) or the fermionic subspace $H^N_-$ of $H^N$ (in which case $u(p)e = \mathrm{sgn}(p)e$ for each $p \in \mathfrak{S}_N$); in other words, pure bosonic on fermionic states on $B(H^N)^{\mathfrak{S}_N}$ are also pure on $B(H^N)$. The second part, then, consists of parastatistical pure states on $B(H^N)^{\mathfrak{S}_N}$, which are therefore mixed on $B(H^N)$. Furthermore, pure bosonic or fermionic states on $B(H^N)^{\mathfrak{S}_N}$ both extend and restrict to pure bosonic or fermionic states on $B(H^{N+1})^{\mathfrak{S}_{N+1}}$ and $B(H^{N-1})^{\mathfrak{S}_{N-1}}$, respectively, whereas parastatistical pure states turn out to have neither property and hence are "isolated" at the given value of $N$.

Finally, in $d = 2$ the equivalence between the operator and configuration space approaches breaks down, because $\mathfrak{S}_N \neq \pi_1(Q_N) = B_N$, i.e., the braid group on $N$ strings. Even defining the operator quantum theory on $H_N = L^2(\tilde{Q}_N)$, with algebra of observables $M_N = B(L^2(\tilde{Q}_N))^{B_N}$, fails to rescue the equivalence, because the decomposition of $H_N$ under $M_N$ by no means contains all irreducible representations of $B_N$. In this case deformation quantization gives many more sectors than the improved operator approach (which already gave more sectors than the approach using 'multi-valued' scalar wave-functions).

## Notes

The quotations in the preamble are from Dirac (1947), p. 87. Similarly, the *Dreimännerarbeit* (Born, Heisenberg, & Jordan, 1926) bluntly states (in Ch. 1, §1) that:

> 'one can see from eq. (5) [i.e., $pq - qp = -i\hbar \cdot 1_H$, cf. our eq. (7.5)] that in the limit $\hbar = 0$, the new theory would converge to the classical theory, as is physically required.'

§7.1. **Deformation quantization**

In the wake of Dirac's famous insight on the analogy between the Poisson bracket and the commutator in quantum mechanics, the idea of deformation quantization (in the form of what we now call *star products*) may be traced back to Groenewold (1946) and Moyal (1949). The mathematical (physics) literature on the subject started with Berezin (1975) and Bayen et al (1978), who introduced what we now call ***formal deformation quantization***, in which $\hbar$ is not a real number but a formal parameter occurring in formal power series. The C*-algebraic setting for deformation quantization we use was introduced by Rieffel (1989, 1994); see also Landsman (1998a), Chapter 2, for a detailed treatment.

§7.2. **Quantization and internal symmetry**

This section is based on Rieffel (1990) and Landsman (1998a), Chapter 3.

§7.3. **Quantization and external symmetry**

§7.4. **Intermezzo: The Big Picture**

§7.5. **Induced representations and the imprimitivity theorem**

§7.6. **Representations of semi-direct products**

The action Poisson bracket (7.58) was introduced by Krishnaprasad & Marsden (1987); see also Marsden & Ratiu (1994).

Systems of imprimitivity and their applications to representation theory, semi-direct products, and quantum mechanics are due to Mackey (1958, 1968), who was inspired by Weyl (1927, 1928), von Neumann (1932), and Wigner (1939). As Mackey (1978, 1992) describes, he saw his work as the development of what he calls ***Weyl's Program***. Weyl (1927) posed two questions in quantum mechanics:

1. 'How to construct the matrix of Hermitian form[1] that represents some quantity given in the context of a known physical system?'[2]
2. 'Given this Hermitian form, what is their physical meaning, and which physical statements can we make about it?'[3]

Weyl considered the second question to have been resolved by von Neumann's recent work, and so he concentrated on the first, which he tried to answer using group theory. The main achievement of Weyl (1927), elaborated in his subsequent

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[1] Like Hilbert himself, Weyl at the time still thought of operators in terms of matrices or Hermitian forms, rather than abstractly, like von Neumann. Also cf. our Introduction.

[2] 'Wie komme ich zu der Matrix, der Hermiteschen Form, welche eine gegebene Größe in einem seiner Konstitution nach bekannten physikalischen System repräsentiert?' (Weyl, 1927, p. 1)

[3] 'Wenn einmal die Hermitesche Form gewonnen ist, was ist ihre physikalische Bedeutung, was für physikalische Aussagen kann ich ihr entnehmen?' (*ibid.*)

book Weyl (1928), was a reformulation of the canonical commutation relations $i[p,q] = \hbar \cdot 1_H$ in terms of projective unitary representations of the additive group $\mathbb{R}^2$ (or, equivalently, of unitary representations of the associated Heisenberg group). He also introduced the formula (7.21) in an equivalent form where the (classical) Fourier expansion of $f$, i.e.,

$$f(p,q) = \int_{\mathbb{R}^2} dadb\, e^{iap+ibq} \hat{f}(a,b), \tag{7.264}$$

is "quantized" by the operator in which $\exp(iap + ibq)$ in the above formula is replaced by the (projective) unitary representative $u(a,b)$ of $(a,b) \in \mathbb{R}^2$ just mentioned, i.e., the real numbers $p$ and $q$ are replaced by the corresponding operators $\hat{p}$ and $\hat{q}$, as in (7.3) - (7.4). In particular, Weyl treated $p$ and $q$ symmetrically.

In his development of Weyl's Program, Mackey broke the symmetry between $p$ and $q$, in that he saw the momentum operator $\hat{p}$ as the ("infinitesimal") generator of a unitary representation of the additive group $\mathbb{R}$, whereas the position operator $\hat{q}$ was replaced by a projection-valued measure on the real line; this is equivalent to a nondegenerate representation of the commutative C*-algebra $C_0(Q)$, as in our discussion in §7.3. This way of tearing $p$ and $q$ apart was the key to the general case of quantizing group actions on configuration space discussed in §7.3.

In their independent elaboration of Weyl's ideas, Groenewold (1946) and Moyal (1949) emphasized the deformation aspect of quantization (including the classical limit) rather than its group-theoretical underpinning; the former aspect is completely absent in Mackey's work. "The Big Picture" (Landsman, 1998a, Ch. 3; Landsman & Ramazan, 2001; Landsman, 2007) is an attempt to have the best of both worlds, in that the role of Lie groupoids delivers the symmetry aspect of quantization, whereas our (i.e. Rieffel's) very definition of quantization puts the deformation aspect in the front seat. The underlying theory of Lie groupoids and Lie algebroids may be found in Moerdijk & Mrčun (2003) or Mackenzie (2005); see also Landsman (1998a).

A comprehensive study of the Mackey–Glimm dichotomy may be found in Williams (2007), which contains a wealth of information on crossed product C*-algebras and induced representations in general.

The representation theory of the Poincaré-group was first studied (using somewhat heuristic methods) by Wigner (1939) using induced representations. The entire subject was subsequently taken up and finished by Mackey. For treatments in the spirit of (mathematical) physics see e.g. Simms (1968), Niederer & O'Raifeartaigh (1974), and Barut & Raçka (1977). Lemma 7.10 is proved by Bargmann (1954).

Among the known elementary particles, the case $j = 0$ (and $m > 0$) corresponds to the Higgs boson, whereas $j = \frac{1}{2}$ gives all known fermionic particles (i.e., electrons, quarks, neutrino's, and their antiparticles). If one counts the gauge bososn $W_{\pm}$ and $Z_0$ as massive, they provide the case $j = 1$, but in the fundamental Lagrangian they are massless and correspond to helicity $n = \pm 1$, like the photon. Helicity $\pm 2$ gives the graviton. We discard particles predicted by supersymmetry, which evidently does not exist in nature (this evidence seems lost on string theorists).

§7.7. **Quantization and permutation symmetry**

This section is based on Landsman (2016a). The literature on indistinguishable particles is enormous, initiated by Heisenberg (1926) and Dirac (1926). What we call the "quantum observables" approach goes back to Messiah & Greenberg (1964); see also Drühl, Haag, & Roberts (1970). Key papers in the configuration space approach are Souriau (1967), Laidlaw & DeWitt-Morette (1971) and Leinaas & Myrheim (1977). More generally, for the quantization of multiply connected space see Dowker (1972), Schulman (1981), Isham (1984), Horvathy, Morandi, & Sudarshan (1989), Morchio & Strocchi (2007), and Morandi (1992). The state space approach to indistinguishable particles was proposed by Bach (1997), who proves (7.192) - (7.193), as well as the claim following these equations to the effect that $\rho \in \partial_e \mathscr{D}(H^N)^{\mathfrak{S}_N}$ iff $eH$ is irreducible. The state space arguments against parastatistics given near the end of this section are also due to Bach (1997).

The representation theory used in this section may be found in many books, such as Weyl (1928), Fulton (1997), or Goodman & Wallach (2000).

The groupoid (7.209) is a special case of the so-called ***gauge groupoid*** defined by a principal $H$-bundle $P \xrightarrow{\pi} Q$, where $G_1 = P \times_H P$ (which stands for $(P \times P)/H$ with respect to the diagonal $H$-action on $P \times P$), $G_0 = Q$, and the operations are

$$s([p,q]) = \pi(q),\ t([p,q]) = \pi(p),\ [x,y]^{-1} = [y,x],\ [p,q][q,r] = [p,r];$$

here $[p,q][q',r]$ is defined whenever $\pi(q) = \pi(q')$, but to write down the product one picks some element $q \in \pi^{-1}(q')$.

Recent philosophical literature on indistinguishable particles includes French & Krause (2006), Earman (2010), Caulton & Butterfield (2012), Saunders (2013), and Baker, Halvorson, & Swanson (2015). This philosophical literature stills needs to be integrated with the mathematical approach launched in this section, and it was indeed the goal of Earman, Halvorson, & Landsman (2013ish) to do so. Alas!