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ON THE APPROXIMATION OF AN INTEGRAL BY A SUM OF RANDOM VARIABLES

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We approximate the integral of a smooth function on \([0, 1]\), where values are only known at \(n\) random points (i.e., a random sample from the uniform-(0, 1) distribution), and at 0 and 1. Our approximations are based on the trapezoidal rule and Simpson's rule (generalized to the non-equidistant case), respectively. In the first case, we obtain an \(n^2\)-rate of convergence with a degenerate limiting distribution; in the second case, the rate of convergence is as fast as \(n^{31/2}\), whereas the limiting distribution is Gaussian then.

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1. Introduction and Main Results

Suppose we (can) only observe the values of a smooth function \(f: [0, 1] \to \mathbb{R}\) at the points \(U_0, U_1, \ldots, U_n, U_{n+1}\), where \(U_1, U_2, \ldots, U_n\) are the order statistics \((U_1 \leq U_2 \leq \ldots \leq U_n)\) of \(n\) independent uniformly-(0, 1) distributed random variables and \(U_0, U_{n+1} = 0, 1\). It is our aim to estimate the integral

\[
I := \int_0^1 f(x) \, dx
\]

from these observations, i.e., by only using \((U_i, f(U_i)), i = 0, 1, \ldots, n + 1\). The first

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estimator we will employ is constructed by using the ‘trapezoidal rule’ on each sub-
interval \([U_{i-1}, U_i]\), \(i = 1, \ldots, n+1\). This rule approximates an integral \(\int_a^b g(x)dx\)
simply by \(\frac{1}{2}(b-a)(g(a) + g(b))\) and it can easily be shown (see, e.g., Isaacson and
Keller [2], p. 304) that
\[
\frac{1}{2}(b-a)(g(a) + g(b)) - \int_a^b g(x)dx = \frac{1}{12}(b-a)^3 g''(\eta),
\]
(2)
where \(\eta \in (a, b)\). Writing \(D_i = U_i - U_{i-1}, i = 1, \ldots, n+1\), for the spacings of the
\(U_i\)'s, our estimator of \(I\) becomes
\[
I_n^* = \sum_{i=1}^{n+1} \frac{1}{2} D_i (f(U_{i-1}) + f(U_i)).
\]
(3)
Using (2), we will prove the following limiting result for the standardized difference of
\(I_n^*\) and \(I\):

**Theorem 1:** If \(|f'''|\) is bounded, then
\[
n^2(I_n^* - I) \xrightarrow{P} \frac{1}{2}(f'(1) - f'(0)), \quad \text{as } n \to \infty.
\]
(4)

A much better and probabilistically more interesting estimator is obtained by
applying a 3-points formula, i.e., for a given \(c \in (a, b)\), we approximate \(\int_a^b g(x)dx\) by
\(w_1 g(a) + w_2 g(c) + w_3 g(b)\) in such a way that the approximation error is zero in the
case \(g\) is a polynomial of second degree. If the 3 points are equidistant, this
approximation is known as Simpson’s rule. It is not hard to show that
\[
w_1 = \frac{1}{6}(b-a)\left(2 - \frac{b-c}{a-c}\right), \quad w_2 = \frac{1}{6} \frac{(b-a)^3}{(c-a)(b-c)}, \quad w_3 = \frac{1}{6}(b-a)\left(2 - \frac{c-a}{b-c}\right),
\]
(5)
and it follows (see again Isaacson and Keller [2], p. 304) that
\[
w_1 g(a) + w_2 g(c) + w_3 g(b) - \int_a^b g(x)dx = -\frac{1}{6} \int_a^b (x-a)(x-c)(x-b)g^{(3)}(\eta)dx,
\]
(6)
where \(\eta = \eta(x) \in (a, b)\). Hence, our estimator of \(I\) in (1), again denoted by \(I_n\),
becomes
\[
I_n = \sum_{i=1}^{\frac{n+1}{2}} \frac{1}{6} (D_{2i-1} + D_{2i}) \{ (2 - \frac{D_{2i}}{D_{2i-1}}) f(U_{2i-1}) - \}
\[
\frac{(D_{2i-1} + D_{2i})^2}{D_{2i-1}D_{2i}} f(U_{2i}) + (2 - \frac{D_{2i-1}}{D_{2i}}) f(U_{2i}) \},
\]
(7)
where, for convenience, \(n\) is taken to be odd. Formula (6) will be used to prove our
main result:

**Theorem 2:** Let \(n\) be odd. If \(|f^{(5)}|\) is bounded, then
\[
n^{\frac{3}{2}} \frac{(I_n - I)}{d} \xrightarrow{d} \sqrt{\frac{35}{3}} \int_0^1 (f^{(3)}(x))^2 dx \ Z, \quad \text{as } n \to \infty,
\]
(8)
where \( Z \) is a standard normal random variable.

**Remark 1:** The present techniques can be easily adapted to cover the situation where the \( U_i \)'s are the order statistics of \( n \) independent random variables with common distribution function \( G \) (on \((0,1)) \) having a smooth density \( g \). The adaptation is based on the quantile transform, transforming a uniform random variable \( V \) into a random variable \( G^{-1}(V) \) with distribution function \( G \). In this case, under regularity conditions on \( g \), we obtain that the weak limit in Theorem 1 becomes

\[
\frac{1}{2} \int_0^1 \left( f''(x)/g^2(x) \right) dx
\]

instead of \( \frac{1}{2} \int_0^1 f''(x) dx = \frac{1}{2}(f'(1) - f'(0)) \). In Theorem 2, the limiting random variable is again centered normal but now the standard deviation becomes

\[
\sqrt{\frac{35}{3} \int_0^1 \left( \frac{f^{(3)}(x)}{g(x)} \right)^2 dx}.
\]

On the other hand, the uniform distribution seems very relevant because of the following. Since \( \int_0^1 f(x) dx \) can be considered as the mean ‘output’, given that the \( x \)-values are ‘equally important’, it seems desirable to estimate \( \int_0^1 f(x) g(x) dx = \int_0^1 f(G^{-1}(y)) dy \) in the case the random variables are distributed according to \( G \). But if \( G \) is known, we can replace the pairs \((U_i, f(U_i))\) (just below (1)), with \( U_i \)'s being the order statistics from \( G \), by \((G(U_i), f(U_i)) = (G(U_i), f(G^{-1}(G(U_i))))\). This brings us back to the ‘uniform distribution setup’ with \( f \) replaced by \( f \circ G^{-1} \), but that is just the function whose integral we wanted to estimate as argued above!

This idea leads to possible ways of applying the results. Suppose \( U_i \) represents some uncontrollable physical random quantity, like temperature, humidity or light intensity with a known distribution function \( G \) having density \( g \). Suppose also that we can measure \( f \) (the output or yield) only at the \( U_i \) and that we are interested in the mean output \( I_g = \int_0^1 f(x) g(x) dx \). Then one can use our theorems to obtain rapidly converging estimators of \( I_g \). In particular, when measuring the \( f \)-values is hard or expensive, one can get good estimators based on a few observations.

Also note that for the trapezoidal rule in Theorem 1 and \( f'' \) being constant, the uniform distribution is optimal, since

\[
\int_0^1 \frac{1}{g^2(x)} dx \geq \int_0^1 1 dx = 1.
\]  

(This can be easily seen by using Jensen’s inequality:

\[
\int_0^1 \frac{1}{g^2(x)} dx = \int_0^1 \frac{1}{g^3(x)} g(x) dx = E \frac{1}{g^3(X)} \geq \left( E \frac{1}{g(X)} \right)^3 = \left( \int_0^1 \frac{1}{g(x)} g(x) dx \right)^3 = 1,
\]

where \( X \) is a random variable with density \( g \).) A similar remark applies to Theorem 2 with \( f^{(3)} \) being constant.

**Remark 2:** There are various other ways to extend our results, which we will not pursue here, e.g., applying \( m \)-points formulas for \( m > 3 \) (Simpson’s rule is ‘by far the
most frequently used in obtaining approximate integrals', Davis and Rabinowitz [1], p. 45), combining trapezoidal rules to eliminate the bias \( \frac{1}{2}(f'(1) - f'(0)) \), proving a 'second order' limit result for \( n^2(I_n - I) - \frac{1}{2}(f'(1) - f'(0)) \) in Theorem 1, or treating the case \( n \) 'even' in Theorem 2. We are not pursuing these extensions because we believe they are not very interesting and/or they do not give good results.

**Remark 3:** We briefly compare our results with the deterministic, equidistant case, i.e., \( U_i = \frac{i}{n+1}, \quad i = 0, 1, \ldots, n + 1 \). It is well-known that the limit in Theorem 1 is \( \frac{1}{12}(f'(1) - f'(0)) \) in that case, which means that we loose a factor of 6 by having random \( U_i \)'s. (Essentially, this 6 is coming from the third moment of a standard exponential random variable.) From Theorem 2, it is well-known that in the equidistant case (Simpson's rule), the rate is \( n^4 \). So, there our loss is of order \( n^{1/2} \). Nevertheless, from statistical point of view, \( n^{3.2} \) is a remarkably fast rate of convergence.

## 2. Proofs

The following well-known lemma will be used frequently; it can be found in, e.g., Shorack and Wellner [3], p. 721.

**Lemma 1:** Let \( E_1, \ldots, E_{n+1} \) be independent exponential random variables with mean 1 and \( S_{n+1} \) be their sum. With \( D_i, i = 1, \ldots, n + 1, \) as before, we have

\[
(D_1, \ldots, D_{n+1}) \overset{d}{=} \left( \frac{E_1}{S_{n+1}}, \ldots, \frac{E_{n+1}}{S_{n+1}} \right).
\]

**Proof of Theorem 1:** Using (3), (1) and (2) we see that

\[
n^2(I_n - I) = \frac{n^2}{12} \sum_{i=1}^{n+1} D_i^3 f'''(\tilde{U}_i)
\]

for some \( \tilde{U}_i \in (U_{i-1}, U_i) \), and hence,

\[
n^2(I_n - I) = \frac{n^2}{12} \sum_{i=1}^{n+1} D_i^3 f'''(\frac{i}{n+1}) + \frac{n^2}{12} \sum_{i=1}^{n+1} D_i^3 (\tilde{U}_i - \frac{i}{n+1}) f'''(\tilde{U}_i),
\]

with \( \tilde{U}_i \) between \( \tilde{U}_i \) and \( \frac{i}{n+1} \). From the boundedness of \( |f'''| \) (by \( M \), say) and the weak convergence (to a Brownian bridge) of the uniform quantile process (see, e.g., Shorack and Wellner [3]), it is readily seen that

\[
\left| \frac{n^2}{12} \sum_{i=1}^{n+1} D_i^3 (\tilde{U}_i - \frac{i}{n+1}) f'''(\tilde{U}_i) \right| \leq \frac{n^2}{12} M \sup_{i \in \{1, \ldots, n+1\}} \left| \tilde{U}_i - \frac{i}{n+1} \right| \sum_{i=1}^{n+1} D_i^3 = O_p(n^{1/2}) \sum_{i=1}^{n+1} D_i^3.
\]

But

\[
\sum_{i=1}^{n+1} D_i^3 \overset{d}{=} \frac{1}{S_{n+1}^3} \sum_{i=1}^{n+1} E_i^3,
\]

by Lemma 1, and by two applications of the weak law of large numbers, this last expression is \( O_p(n^{-2}) \). Combining this with (10) and (11) yields that the second term on the right in (9) converges to zero in probability. Hence, it remains to consider the first term.
\[ \frac{n^2}{12} \sum_{i=1}^{n+1} \frac{n+1}{i}^3 f''\left(\frac{i}{n+1}\right) = \frac{1}{12n} \left( \frac{n}{S_{n+1}} \right)^3 \sum_{i=1}^{n+1} f''\left(\frac{i}{n+1}\right) E_i^3, \]

or, since \((n/S_{n+1})^3 \geq 1,\)

\[ \frac{1}{12n} \sum_{i=1}^{n+1} f''\left(\frac{i}{n+1}\right) E_i^3. \]

By Chebysev's inequality, it follows that

\[ \frac{1}{12n} \sum_{i=1}^{n+1} f''\left(\frac{i}{n+1}\right) E_i^3 \leq \frac{1}{2n} \sum_{i=1}^{n+1} f''\left(\frac{i}{n+1}\right) E_i^3. \]

The proof is complete by noting that

\[ \frac{1}{2n} \sum_{i=1}^{n+1} f''\left(\frac{i}{n+1}\right) \rightarrow \frac{1}{2} \int_0^1 f''(x)dx = \frac{1}{2}(f'(1) - f'(0)). \]

The proof of Theorem 2 is heavily based on the following two lemmas.

**Lemma 2:** Let \(E_1, \ldots, E_{n+1}\), \(n\) odd, be independent exponential random variables with mean 1. Write

\[ X_i = (E_{2i-1} + E_{2i})^3 (E_{2i} - E_{2i-1}), \quad i = 1, 2, \ldots, \frac{n+1}{2}, \]

\[ Y_i = X_i \sum_{j=1}^{2i-2} (E_j - 1), \quad i = 2, 3, \ldots, \frac{n+1}{2}. \]

Then,

\[ \mathbb{E} X_i = 0, \quad \text{Var} X_i = 120960, \quad \mathbb{E} Y_i = 0, \quad \text{Var} Y_i = 120960(2i - 2), \]

\[ \text{Cov}(Y_i, Y_k) = 0, \text{ for } i \neq k. \]

**Proof:** By symmetry, we see that \(\mathbb{E} X_i = 0;\) a straightforward computation yields \(\text{Var} X_i = \mathbb{E} X_i^2 = 120960.\) For the \(Y_i\)'s we have

\[ \mathbb{E} Y_i = \mathbb{E} X_i \sum_{j=1}^{2i-2} (E_j - 1) = 0, \]

\[ \text{Var} Y_i = \mathbb{E} Y_i^2 = \mathbb{E} X_i^2 \sum_{j=1}^{2i-2} (E_j - 1)^2 = \text{Var} X_i \text{Var} \left( \sum_{j=1}^{2i-2} E_j \right) = 120960(2i - 2), \]

and for \(i < k,\)

\[ \text{Cov}(Y_i, Y_k) = \mathbb{E} Y_i Y_k \]

\[ = \mathbb{E} X_k \left( \sum_{j=1}^{2k-2} (E_j - 1) \right) X_i \left( \sum_{j=1}^{2i-2} (E_j - 1) \right) \]

\[ = \mathbb{E} X_k \left( \sum_{j=1}^{2k-2} (E_j - 1) \right) X_i \left( \sum_{j=1}^{2i-2} (E_j - 1) \right) \]

\[ = 0. \]
Lemma 3: Under the conditions of Theorem 2, we have, as \( n \to \infty \),

\[
\left| n^{\frac{3}{2}} (I_n - I) - \frac{n^{\frac{3}{2}}}{2} \sum_{i=1}^{n} f^{(3)} \left( \frac{2i-2}{n+1} \right) (D_{2i-1} + D_{2i})^3 (D_{2i} - D_{2i-1}) \right| = o_p(1).
\]

Proof: By (7), (1) and (6) we have

\[
n^{\frac{3}{2}} (I_n - I) = - \frac{n^{\frac{3}{2}}}{6} \sum_{i=1}^{n} \int_{U_{2i-2}}^{U_{2i}} (x - U_{2i-2}) (x - U_{2i-1}) (x - U_{2i}) f^{(4)}(\tilde{U}_{2i}) dx,
\]

for some \( \tilde{U}_{2i} = \tilde{U}_{2i}(x) \in (U_{2i-2}, U_{2i}) \) and hence for some \( \tilde{U}_{2i} = \tilde{U}_{2i}(x) \in (U_{2i-2}, U_{2i}) \), the right-hand side of (12) is equal to

\[
- \frac{n^{\frac{3}{2}}}{6} \sum_{i=1}^{n} \int_{U_{2i-2}}^{U_{2i}} (x - U_{2i-2}) (x - U_{2i-1}) (x - U_{2i}) f^{(4)}(\tilde{U}_{2i}) dx
\]

\[
= \frac{n^{\frac{3}{2}}}{2} \sum_{i=1}^{n} f^{(3)}(U_{2i-2}) (D_{2i-1} + D_{2i})^3 (D_{2i} - D_{2i-1})
\]

Let \( M \) be a bound on \( |f^{(5)}| \) and all lower order derivatives of \( f \). Then the absolute value of this last term is bounded from above by

\[
M \frac{n^{\frac{3}{2}}}{6} \sum_{i=1}^{n} \int_{U_{2i-2}}^{U_{2i}} (x - U_{2i-2}) |x - U_{2i-1}||((U_{2i} - x)(\tilde{U}_{2i} - U_{2i-2})| dx
\]

\[
\leq M \frac{n^{\frac{3}{2}}}{6} \sum_{i=1}^{n} (D_{2i-1} + D_{2i})^5 \frac{M n^{\frac{3}{2}}}{S_{n+1}^{\frac{1}{2}}} \sum_{i=1}^{n} (E_{2i-1} + E_{2i})^5 = o_p(1),
\]

due to Lemma 1 and two applications of the weak law of large numbers.

So, it suffices to show the convergence to zero in probability of

\[
\frac{n^{\frac{3}{2}}}{72} \sum_{i=1}^{n} \left( f^{(3)}(U_{2i-2}) - f^{(3)} \left( \frac{2i-2}{n+1} \right) \right) (D_{2i-1} + D_{2i})^3 (D_{2i} - D_{2i-1})
\]

\[
= \frac{n^{\frac{3}{2}}}{72} \sum_{i=2}^{n} \left( U_{2i-2} - \frac{2i-2}{n+1} \right) f^{(4)} \left( \frac{2i-2}{n+1} \right) (D_{2i-1} + D_{2i})^3 (D_{2i} - D_{2i-1})
\]
for some $\bar{U}_{2i-2}$ between $U_{2i-2}$ and $\frac{2i-2}{n+1}$. By the weak convergence of the uniform quantile process,

$$|T_{2,n}| \leq \frac{3\sqrt{2}}{144} \sum_{i \in \{2, 3, \ldots, \frac{n+1}{2}\}} \left( U_{2i-2} - \frac{2i-2}{n+1} \right)^2 \sum_{i=2}^{n+1} (D_{2i-1} + D_{2i})^4$$

By Lemma 1 and twice the weak law of large numbers, this last expression is easily seen to be $O_p(1)$. Hence, the proof of Lemma 3 is complete if we show $T_{1,n} = O_p(1)$.

From Lemma 1 we obtain

$$T_{1,n} \overset{d}{=} \frac{3\sqrt{2}}{2n^2} \sum_{i=2}^{n+1} \left( \frac{2i-2}{S_{n+1}} - \frac{2i-2}{n+1} \right) f^{(4)} \left( \frac{2i-2}{n+1} \right) \left( \frac{E_{2i-1} + E_{2i}}{S_{n+1}} \right)^3 \left( \frac{E_{2i} - E_{2i-1}}{S_{n+1}} \right)$$

$$= \frac{3\sqrt{2}}{2n^2} S_{n+1} \sum_{i=2}^{n+1} \left( 1 - \frac{S_{n+1}}{n+1} \right) \sum_{i=2}^{n+1} (2i-2) f^{(4)} \left( \frac{2i-2}{n+1} \right) (E_{2i-1} + E_{2i})^3 (E_{2i} - E_{2i-1})$$

$$= : T_{3,n} + T_{4,n}.$$
Now, Chebysev's inequality yields $T_{3,n} = o_p(1)$ and hence $T_{1,n} = o_p(1)$. □

**Proof of Theorem 2:** Given the lemmas, especially Lemma 3, the proof of Theorem 2 is rather easy. If $\int_0^1 (f^{(3)}(x))^2 \, dx = 0$, then $f^{(3)}(x) = 0$ for all $x \in [0,1]$ and hence trivially $I_n = I$, because $f$ is a polynomial of second degree. Therefore, we assume now $\int_0^1 (f^{(3)}(x))^2 \, dx > 0$. Using Lemma 1 we have

$$
\frac{n^{3/2}}{72} \sum_{i=1}^{n+1} f^{(3)}\left(\frac{2i-2}{n+1}\right) (D_{2i-1} + D_{2i})^3 (D_{2i} - D_{2i-1})
$$

By the weak law of large numbers and Lemma 3, it now remains to show Theorem 2 with $n^{3/2} (I_n - I)$ replaced by $W_n$. By Lemma 2, we see that $E W_n = 0$ and

$$
\text{Var } W_n = \frac{1}{2(72)^2} \sum_{i=1}^{n+1} \left( f^{(3)}\left(\frac{2i-2}{n+1}\right) \right)^2 120960 \to \frac{35}{3} \int_0^1 (f^{(3)}(x))^2 \, dx.
$$

Now, the Lindeberg central limit theorem applies, because of the boundedness of $|f^{(3)}|$, and it yields the result. □

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**References**


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