THEOREM 1: For \( n \) mutually independent integer-valued rankings, each with equally likely rank scores ranging from 1 to \( k \), the exact probability to obtain pairwise difference \( d \) for any two rank sums equals

\[
P(D = d; k, n) = \left\{ k(k-1) \right\}^{-n} W(D = d; k, n),
\]

where

\[
W(D = d; k, n) = \left\{ k(k-1) \right\}^n \sum_{h=0}^{n} \binom{n}{h} k^{-h}(1-k)^{-h} \sum_{i=0}^{h} \sum_{j=0}^{h} (-1)^{(j-i)} \binom{h}{i} \binom{h}{j} \frac{k(j-i)-d+h-1}{k(j-i)-d-h},
\]

is the number of distinct ways a rank sum difference of \( d \) can arise, with \( d \) having support on \( d = [-n(k-1), n(k-1)] \).

Proof: Our approach exploits the observation that the difference among a pair of rank sums is the sum of rank pair differences. Problem-solving tool is the probability generating function

\[
f(t) = \sum_{r=1}^{k} p_r t^r,
\]

where the probabilities \( p_r = P(R = r) \) of the integer rank values \( r \) are the coefficients of \( t^r \). Denote two distinct ranks as \( r_i \) and \( r_j \), with \( r_i \neq r_j \). For a single ranking (i.e., one block), the generating function of the sequence of probabilities of the random rank differences is given by the finite power series

\[
f(t; k) = \sum_{i=1}^{k} \sum_{r_{i+1}}^{k} \left\{ p_{r_i} p_{r_j} t^{(r_j-r_i)} + p_{r_i} p_{r_j} t^{(r_j-r_i)} \right\}.
\]

According to the null hypothesis each ranking within a block is equally likely. Consequently, rank \( r_i \) has probability \( P(R = r_i) = p_{r_i} = k^{-1} \), and rank \( r_j \) has probability \( P(R = r_j) = p_{r_j} = (k-1)^{-1} \), for \( 1 \leq r_i, r_j \leq k \). The generating function of the probabilities may therefore be expressed as

\[
f(t; k) = \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} \left\{ \frac{1}{k(k-1)} t^{(r_j-r_i)} + \frac{1}{k(k-1)} t^{(r_j-r_i)} \right\} = \left\{ \frac{1}{k(k-1)} \sum_{i=1}^{k} \sum_{j=i+1}^{k} t^{(r_j-r_i)} \right\} - \frac{1}{k-1}.
\]
This function can be put in more compact form by summing the geometric progressions:

\[
f(t; k) = -\frac{1}{k(k-1)} \left( \frac{(t-t^k)(t-t^{k-1})}{(1-t)} \right) = \frac{1}{k(1-k)} \left( \frac{t(1-t^k)(1-t^{k-1})}{(1-t)^2} \right) - \frac{1}{k-1}.
\]

The form of the first equation on the right demonstrates that the generating function is symmetric in \( t \) around zero, that is \( f(t^{-1}; k) = f(t; k) \). From the convolution theorem (e.g., Feller [49]), the probability generating function of the sum of \( n \) mutually independent rankings, each assuming the rank differences \(- (k-1), -1, 1, \ldots, (k-1)\), is given by the \( n \)th power of \( f(t; k) \):

\[
f(t; k, n) = \left( t(1-t^k)(1-t^{k-1}) \right) - \frac{1}{k-1} \right)^n,
\]

which, according to the binomial theorem, can be rewritten as

\[
f(t; k, n) = \sum_{h=0}^{n} \binom{n}{h} \left( \frac{1}{k(1-k)} \right)^h \left( \frac{t(1-t^k)(1-t^{k-1})}{(1-t)^2} \right)^h \left( \frac{1}{1-k} \right)^{n-h}.
\]

Applying the binomial expansions of \( (1-t^k)^h \) and \( (1-t^{k-1})^h \), the power series expansion of \( (1-t)^{-2h} \), and the \(-1\) transformation and the upper negation identity transformation of the binomial coefficient, i.e.,

\[
\sum_{l=0}^{\infty} \binom{2h}{l} (-t)^l = \sum_{l=0}^{\infty} \binom{-2h}{l} t^l = \sum_{l=0}^{\infty} \binom{1+2h-1}{l} t^l,
\]

the expression becomes

\[
f(t; k, n) = \sum_{h=0}^{n} \binom{n}{h} \left( \frac{1}{k^h(1-k)^n} \right)^t \sum_{l=0}^{h} \sum_{j=0}^{h} \binom{h}{j} \binom{h}{j} \binom{-1}{j-i} \binom{h}{j} \binom{h}{j} \sum_{l=0}^{l+2h-1} \binom{l}{l} t^l.
\]

To calculate the probability \( P(D = d; k, n) \) of rank sum difference \( d \) as the coefficient of the power of \( t \), we first collect all terms involving powers of \( t \), and subsequently make the change of variable \( h - k(j - i) + l = d \), yielding
where the second equation follows by noting that whenever $h - k(j - i) + l = d$ and $l \geq 0$, we have $l + 2h - 1 = k(j - i) + d + h - 1$ and $k(j - i) + d - h \geq 0$, and the third equation follows from the property of symmetry of the generating function. The probability of rank sum difference $d$ is therefore

$$P(D = d; k, n) = \{k(k-1)\}^{-n} W(D = d; k, n),$$

where

$$W(D = d; k, n) = \{k(k-1)\}^n \sum_{h=0}^{n} \left( \sum_{i=0}^{h} \left( \sum_{j=0}^{h} (-1)^{(j-i)} \frac{h}{i} \right) \right) \left( \sum_{i=0}^{h} \left( \sum_{j=0}^{h} (-1)^{(j-i)} \frac{h}{i} \right) \right)(k(j-i) - d + h - 1),$$

$$d = -n(k-1), \ldots, n(k-1),$$

represents the number of compositions of $d$ into $n$ parts, where each part is restricted to the interval $[-(k-1), k-1]$. Hence $W(D = d; k, n)$ gives the number of different ways a rank sum difference of $d$ can arise, for $n$ mutually independent rankings, each with equally likely integer-valued ranks $\{1, 2, \ldots, k\}$. This completes the proof.

The cumulative distribution of rank sum difference $d$ has generating function $f(t; k, n) / (1-t)$ [see, 50-52], and by repeated application of Pascal’s identity

$$\binom{m}{r} + \binom{m}{r-1} = \binom{m+1}{r}, \quad \text{for } 1 \leq r \leq m+1,$$

the expression for the exact $p$-value is obtained as

$$P(D \geq d; k, n) = \sum_{k=0}^{n} \left( \sum_{i=0}^{h} \left( \sum_{j=0}^{h} (-1)^{(j-i)} \frac{h}{i} \right) \right) \left( \sum_{i=0}^{h} \left( \sum_{j=0}^{h} (-1)^{(j-i)} \frac{h}{i} \right) \right)(k(j-i) - d + h - 1),$$

$$d = -n(k-1), \ldots, n(k-1).$$

To illustrate the derivations presented above, Additional file 4 offers a small-sized numerical example ($k=3$, $n=2$). Additional file 5 tabulates the number of compositions of $d$ for combinations of $n=k=2, \ldots, 6$, for inclusion in the OEIS [53].
**Exact \( p \)-values for pairwise comparison of Friedman rank sums, with application to comparing classifiers**

by Eisinga, Heskes, Pelzer & Te Grotenhuis, *BMC Bioinformatics*, 2017

**THEOREM 2:** For nonnegative integers \( d \) and \( k \)

\[
\sum_{i=0}^{h} \sum_{j=0}^{h} (-1)^{(j-i)} \binom{h}{i} \binom{k(j-i) - d + h}{j} = \sum_{s=0}^{h} (-1)^{s} \binom{2h}{h+s} \binom{ks - d + h}{h+s}.
\]

Proof: Note that, for nonnegative integers \( d \) and \( k \), the right-most binomial coefficient on the left-hand side equals 0, whenever \( j < i \). We set \( j - i = s \), and rewrite the left side as follows

\[
\sum_{s=0}^{j} \sum_{j=0}^{h} (-1)^{s} \binom{h}{j-s} \binom{h}{j} \binom{ks - d + h}{j} = \sum_{s=0}^{h} (-1)^{s} \binom{ks - d + h}{h+s} \sum_{j=0}^{h} \binom{h}{j} \binom{h}{j-s}.
\]

The upper bound of the outer summation over \( s \) in the first equation is \( j \). However, the same summation result is obtained if the upper bound is \( h \), as in the second equation, since

\[
\binom{h}{j-s}
\]

equals 0, whenever \( s > j \). The second equation additionally takes both the alternating sign term and the right-most binomial coefficient of the first equation out of the summation over \( j \), as they are conditioned by \( s \).

We subsequently simplify the summation

\[
\sum_{j=0}^{h} \binom{h}{j} \binom{h}{j-s}
\]

to a closed-form expression, using a special case of Vandermonde’s convolution identity, stating

\[
\sum_{j=0}^{r} \binom{h}{j} \binom{h}{r-j} = \binom{2h}{r}.
\]

We find that

\[
\sum_{j=0}^{h} \binom{h}{j} \binom{h}{j-s} = \sum_{j=0}^{h} \binom{h}{j} \binom{h}{h+s-j} = \sum_{j=0}^{h+s} \binom{h}{j} \binom{h}{h+s-j} = \binom{2h}{h+s}.
\]
where the third equation follows by noting that the binomial coefficient with lower index \( j \) equals 0, whenever \( j > h \). It is therefore immaterial to the result whether the upper bound of the summation is \( h \) or \( h + s \). Inserting the closed-form expression and rearranging yields

\[
\sum_{i=0}^{h} \sum_{j=0}^{h} (-1)^{(j-i)} \binom{h}{i} \binom{k(j-i) - d + h}{j} = \sum_{s=0}^{h} (-1)^{s} \binom{2h}{h + s} \binom{ks - d + h}{k s - d - h},
\]

as claimed. This completes the proof.

We additionally note that an elegant alternative expression for the summation on the right hand side of Theorem 2 is

\[
\sum_{s=0}^{h} (-1)^{s} \binom{2h}{h + s} \binom{ks - d + h}{k s - d - h} = \sum_{s=0}^{h} (-1)^{s} \binom{2h}{h + s} \binom{ks - d + h}{2h},
\]

and that the integer sequence

\[
(-1)^{s} \binom{2h}{h + s}
\]

is a signed version of the elements in the even-numbered rows of Pascal’s triangle (see Sloane’s [52] series A062344 and A034870). For \( k = 2 \), the sequence

\[
\sum_{s=0}^{h} (-1)^{s} \binom{2h}{h + s} \binom{ks - d + h}{2h}
\]

is a signed version of the OEIS [52] triangle A008949.
Exact p-values for pairwise comparison of Friedman rank sums, with application to comparing classifiers
by Eisinga, Heskes, Pelzer & Te Grotenhuis, BMC Bioinformatics, 2017

The following is a small numerical example to illustrate the calculation. Suppose we want to obtain the rank sum difference distribution for $k = 3$ groups and $n = 2$ blocks. Assuming no ties, the ranks are $r = 1, 2, 3$, and the rank differences are $-2, -1, 1, 2$. The permutation distribution assigns equal probabilities to the three ranks, i.e., $p_r = 1/3$, for $r = 1, 2, 3$, and equal probabilities to the four pairwise rank differences, i.e., $p_{ij} = 1/3 \times 1/2 = 1/6$, for $i, j = 1, 2, 3$, with $i \neq j$. The outcomes are not equally likely however. There are two different ways to obtain rank sum difference value 1, namely 2-1 and 3-2, but there is only one way to obtain a rank sum difference of 2, namely 3-1.

The generating function of the rank sum differences for a single block generates the probabilities as

$$f(t; 3) = \frac{1}{6} t^2 + \frac{2}{6} t^{-1} + \frac{2}{6} t^1 + \frac{1}{6} t^2 = \frac{1}{6} \left( t^2 + 2t^{-1} + 2t^1 + t^2 \right).$$

The coefficients of the powers of $t$ in the first equation on the right are the probabilities assigned to the rank sum differences, indicated by the powers -2,-1,1,2 of $t$. The coefficients of the powers of $t$ within brackets in the second equation are the number of different ways to obtain the rank sum difference in question. For example, rank sum difference -1 has 2 configurations (1-2 and 2-3) and the associated probability is 2/6. The probability generating function of the sum of the rank differences for two blocks, assuming independence of the blocks, is the product of the two generation functions, i.e.,

$$f(t; 3, 2) = \left( \frac{1}{6} t^2 + \frac{2}{6} t^{-1} + \frac{2}{6} t^1 + \frac{1}{6} t^2 \right)^2,$$

which is written compactly as

$$f(t; 3, 2) = \left( \frac{1}{6} \frac{t(1-t^3)(1-t^3)}{(1-t)^2} - \frac{1}{2} \right)^2.$$

Expanding and collecting terms yields the generating function
f(t;3, 2) = \sum_{d=-4}^{4} \left[ \sum_{h=0}^{2} \frac{2}{h} \frac{1}{3^h (-2)^2} \sum_{i=0}^{h} \sum_{j=0}^{h} (-1)^{(i+j)} \left( \binom{h}{i} \binom{h}{j} \frac{3(j-i) - d + h - 1}{3(j-i) - d - h} \right) \right] t^{-d} \\
= \sum_{d=-4}^{4} \left[ \frac{1}{6} \right]^2 W(D=d;3,2) t^{-d} \\
= \sum_{d=-4}^{4} ((P=d;3,2)) t^{-d},

where we can read the probability \((P=d;3,2)\) as the coefficient of \(t^{-d}\). The probability mass distribution is obtained as

\[ P(D=d;3,2) = \sum_{h=0}^{2} \frac{2}{h} \frac{1}{3^h (-2)^2} \sum_{i=0}^{h} \sum_{j=0}^{h} (-1)^{(i+j)} \left( \binom{h}{i} \binom{h}{j} \frac{3(j-i) - d + h - 1}{3(j-i) - d - h} \right), \]

and the cumulative distribution as

\[ P(D \geq d;3,2) = \sum_{h=0}^{2} \frac{2}{h} \frac{1}{3^h (-2)^2} \sum_{i=0}^{h} \sum_{j=0}^{h} (-1)^{(i+j)} \left( \binom{h}{i} \binom{h}{j} \frac{3(j-i) - d + h}{3(j-i) - d - h} \right) \quad \text{for } d = -4, \ldots, 4. \]

As explained in the main text, the \(p\)-value of non-negative \(d\) may also be obtained from the simplified expression

\[
P(D \geq |d|;3,2) = \begin{cases} \\
2 \sum_{h=0}^{2} \frac{2}{h} \frac{1}{3^h (-2)^2} \sum_{s=0}^{h} (-1)^s \left( \frac{2h}{h+s} \frac{3s - d + h}{3s - d - h} \right) & \text{for } d = 1, \ldots, 4 \\
1 & \text{for } d = 0.
\end{cases}
\]

Table S1 summarizes the results for \(k = 3\) groups and \(n = 2\) blocks. In this setting, there are a total of 9 possible (positive and negative) values for the differences in rank sums.
In practical settings we are testing absolute differences and, therefore, we use only the right tail of the symmetric distribution. It is immediately clear from the figures in the bottom line of the table that an absolute rank sum difference of 4 would be significant at the (unadjusted) .10 level, as $P(D \geq 4; 3, 2) < .10$, but not at an alpha level of .05. The R code to compute the results presented in the table is included in Additional file 3. The R script `pexactfrsd` computes the exact $p$-value presented in the bottom row of Table S1 and, optionally, all statistics presented in the bottom panel of the table.

Table S1. Rank sum difference distribution for $k = 3$ and $n = 2$

<table>
<thead>
<tr>
<th>$d$</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W(D = d; 3, 2)$</td>
<td>1</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>10</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>$W(D \geq d; 3, 2)$</td>
<td>36</td>
<td>35</td>
<td>31</td>
<td>27</td>
<td>23</td>
<td>13</td>
<td>9</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>$P(D \geq d; 3, 2)$</td>
<td>1</td>
<td>35/36</td>
<td>31/36</td>
<td>27/36</td>
<td>23/36</td>
<td>13/36</td>
<td>9/36</td>
<td>5/36</td>
<td>1/36</td>
</tr>
</tbody>
</table>

| $W(D = |d|; 3, 2)$ | 10 | 8  | 8  | 8  | 2  |
| $W(D \geq |d|; 3, 2)$ | 36 | 26 | 18 | 10 | 2  |
| $P(D = |d|; 3, 2)$ | 10/36 | 8/36 | 8/36 | 8/36 | 2/36 |
| $P(D \geq |d|; 3, 2)$ | 1  | .722 | .500 | .278 | .056 |
Exact \( p \)-values for pairwise comparison of Friedman rank sums, with application to comparing classifiers
by Eisinga, Heskes, Pelzer & Te Grotenhuis, *BMC Bioinformatics*, 2017

The entries in the tables below are the number of compositions of \( d \), referred to in the main text as \( W(D=d;k,n) \), for \( k,n = 2,\ldots,6 \). The tabulated figures are for the right half of the distribution (i.e., non-negative \( d \) values) only. The entries for positive \( d \) (i.e., \( d > 0 \) only) should be doubled if the number of compositions of the absolute \( d \) values are required.

\[
\begin{array}{cccccc}
\multicolumn{6}{c}{k=2} \\
\hline
\multicolumn{6}{c}{n} \\
\hline
\multicolumn{1}{c}{d} & 2 & 3 & 4 & 5 & 6 \\
\hline
0 & 2 & 0 & 6 & 0 & 20 \\
1 & 0 & 3 & 0 & 10 & 0 \\
2 & 1 & 0 & 4 & 0 & 15 \\
3 & 1 & 0 & 5 & 0 &  \\
4 & 1 & 0 & 6 & 0 &  \\
5 & 1 & 0 &  \\
6 & 1 &  \\
\end{array}
\]

\[
\begin{array}{cccccc}
\multicolumn{6}{c}{k=3} \\
\hline
\multicolumn{6}{c}{n} \\
\hline
\multicolumn{1}{c}{d} & 2 & 3 & 4 & 5 & 6 \\
\hline
0 & 10 & 24 & 198 & 880 & 5380 \\
1 & 4 & 36 & 152 & 940 & 4920 \\
2 & 4 & 27 & 136 & 810 & 4440 \\
3 & 4 & 14 & 120 & 600 & 3832 \\
4 & 1 & 12 & 68 & 480 & 2799 \\
5 & 6 & 40 & 312 & 1980 &  \\
6 & 1 & 24 & 165 & 1324 &  \\
7 & 8 & 90 & 732 &  \\
8 & 1 & 40 & 366 &  \\
9 & 10 & 172 &  \\
10 & 1 & 60 &  \\
11 & 12 &  \\
12 & 1 &  \\
\end{array}
\]
### $k=4$

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