Helicity Amplitudes for Charmonium Production in Hadron-Hadron and Photon-Hadron Collisions

M.M. Meijer*
Institute for Mathematics, Astrophysics and Particle Physics, Radboud University Nijmegen Toernooiveld 1, 6525 ED Nijmegen, The Netherlands

J. Smith†
C.N. Yang Institute for Theoretical Physics, Stony Brook University Stony Brook, NY 11794-3840 USA

W.L. van Neerven‡
Lorentz Institute, University of Leiden Postbus 9502, 2300 RA Leiden, The Netherlands
(Dated: February 18, 2013)

We present the gluon-gluon and photon-gluon helicity amplitudes for color singlet and octet charmonium production in polarized and unpolarized hadron-hadron and photon-hadron collisions.

PACS numbers: 12.38.Bx, 13.60.Le, 13.85.Ni, 14.40.Gx

I. INTRODUCTION

The amplitudes for the production of charmonia states in hadron-hadron and photon-hadron collisions are usually calculated within the framework of non-relativistic quantum chromodynamics (NRQCD). Several perturbative QCD reactions are required among them being $g+g \to g+$charmonia and $\gamma+g \to g+$charmonia, where g represents a gluon. The latter can either be color singlet or color octet states. Specific results have been presented in [1], [2], [3], [4] and [5], among others. However a close examination of these papers reveals inconsistencies between the published results. Also while the individual helicity amplitudes are available in the color singlet case we could not find the corresponding results for the color octet case. Therefore we have calculated the amplitudes by helicity methods and present our results below. For the benefit of the reader we also give some details of the calculation.

We used the helicty method described in the book by Gastmans and Wu [6] (see also [1]) to calculate processes where three gluons or two gluons and a photon form charmonium. Like Gastmans and Wu we projected out the lowest angular momenta states of the heavy quark pair, namely ${}^1S_0$, ${}^3S_1$, ${}^1P_1$, ${}^3P_0$, ${}^3P_1$ and ${}^3P_2$, using appropriate projection operators (see [7]). We then flipped one of the gluons from incoming to outgoing and with these squared matrix elements calculated the polarized and unpolarized differential cross sections.

II. THREE GLUONS

Gastmans and Wu have presented results for the differential cross section for the production of a color singlet heavy quark pair in angular momentum states ${}^{2S+1}L_J$. They begin with the reaction with three incoming gluons where the momenta and colors of the particles are labelled as

$$g(k_1,a)+g(k_2,b)+g(k_3,c) \to q(p/2+q)+\bar{q}(p/2-q)\,. \quad (1)$$

There are six Feynman diagrams where the three gluons couple directly to the heavy quark line and six diagrams where two gluons couple to the heavy quark line.

There are eight helicity matrix elements which are labelled by assigning either a $+$ or a $-$ to each gluon and which are related by CP conjugation and crossing. All eight can be derived from two, called $|M(+,+,+)|^2$ and $|M(+,+,-)|^2$. We will list them below.

The gluon helicities for the ${}^{2S+1}L_J$ (+,+,+) combination are

$$\begin{aligned}
\not\epsilon_1^+ &= N[\not k_1\not k_2\not k_3(1-\gamma_5)+\not k_3\not k_2\not k_1(1+\gamma_5)] &(2a)\\
\not\epsilon_2^+ &= N[\not k_2\not k_3\not k_1(1-\gamma_5)+\not k_1\not k_3\not k_2(1+\gamma_5)] &(2b)\\
\not\epsilon_3^+ &= N[\not k_3\not k_1\not k_2(1-\gamma_5)+\not k_2\not k_1\not k_3(1+\gamma_5)]\,, &(2c)
\end{aligned}$$

while those for the ${}^{2S+1}L_J$ $(+,+,-)$ combination are

$$\begin{aligned}
\not\epsilon_1^+ &= N[\not k_1\not k_2\not k_3(1-\gamma_5)+\not k_3\not k_2\not k_1(1+\gamma_5)] &(3a)\\
\not\epsilon_2^+ &= -N[\not k_2\not k_1\not k_3(1-\gamma_5)+\not k_3\not k_1\not k_2(1+\gamma_5)] &(3b)\\
\not\epsilon_3^- &= N[\not k_3\not k_1\not k_2(1+\gamma_5)+\not k_2\not k_1\not k_3(1-\gamma_5)]\,, &(3c)
\end{aligned}$$

where $N=[(k_1\cdot k_2)(k_2\cdot k_3)(k_3\cdot k_1)]^{-\frac{1}{2}}/4$ is a normalization factor. In principle there should be extra terms in these expressions but they do not change the answers

*Electronic address: m.meijer@hef.ru.nl
†Electronic address: smith@max2.physics.sunysb.edu
‡Deceased

in this reaction due to the symmetric choice of variables. The helicity amplitudes are functions of the invariants $s=(k_1+k_2)^2$, $t=(k_2+k_3)^2$, $u=(k_3+k_1)^2$ and the mass of the pair $M\approx 2m$ where m is the heavy quark mass. Note that $s+t+u=M^2$, $N=(2stu)^{-\frac{1}{2}}$ and the color singlet projection operator is $\delta_{ij}/\sqrt{3}$. Also they depend on two parameters R_0 and R_1' which are the S-state wave function and the derivative of the P-state wave function evaluated at the origin. The former is defined in terms of the leptonic decay width

$$R_0^2 = M^2\Gamma(^3S_1\rightarrow e^+e^-)/4\alpha^2Q_f^2\,, \tag{4}$$

with $\alpha\approx 1/137$ the fine structure constant and Q_f is the fractional charge of the quarks. R_1' is determined from a fit to the charmonium potential and has the value

$${R_1'}^2/M_\chi^2 \approx 0.006 \quad (\mathrm{GEV})^3\,. \tag{5}$$

We actually need the differential cross section for the reaction

$$g(k_1,a)+g(k_2,b)\rightarrow q(p/2+q)+\bar{q}(p/2-q)+g(k_3,c)\,, \tag{6}$$

where the invariants are now $s=(k_1+k_2)^2$, $t=(k_2-k_3)^2$, $u=(k_1-k_3)^2$.

The squares of the matrix elements for the reaction (6) follow from those in reaction (1) by crossing $k_3\rightarrow -k_3$ and flipping the helicity of the third gluon.

They are denoted by $M|(+,+;+)|^2$, $|M(+,+;-)|^2$, $|M(+,-;-)|^2$ and $|M(-,+;-)|^2$. Note that these are equal to $M|(-,-;-)|^2$, $|M(-,-;+)|^2$, $|M(-,+;+)|^2$ and $|M(+,-;+)|^2$ respectively by CP conjugation. However the kinematic variables require permutations to reflect the crossing of gluon number three. These relations are

$$|M(+,+;+)|^2 = |M(+,+,-)|^2 \tag{7a}$$
$$|M(+,+;-)|^2 = |M(+,+,+)|^2 \tag{7b}$$
$$|M(+,-;-)|^2 = |M(+,+,-)|^2\Big|_{s\leftrightarrow u} \tag{7c}$$
$$|M(-,+;-)|^2 = |M(+,+,-)|^2\Big|_{s\leftrightarrow t}\,. \tag{7d}$$

It is very convenient to use completely symmetric variables which are then invariant under any crossing transformations. Hence we express several results in terms of the variables $M^2=s+t+u$, $P=st+tu+us$ and $Q=stu$, which are invariant under $s\leftrightarrow t$ and $s\leftrightarrow u$. The denominators of the helicity amplitudes are written in these variables while the numerators contain terms in s. Therefore the crossing simply involves changing $s\rightarrow t$ and $s\rightarrow u$ in the numerators of our expressions.

We have also calculated the corresponding amplitudes for the production of a color octet heavy quark pair which requires four additional Feynman diagrams for processes where only one gluon couples to the heavy quark pair. These contain three gluon and four gluon couplings. The color octet projection operator is required so the factor $\delta_{ij}/\sqrt{3}$ in the color singlet case is replaced by $\sqrt{2}T^a_{ij}$. Also the color octet amplitudes cannot be determined from decay processes so they are fit to quarkonium production differential cross sections in proton-proton, proton-antiproton and photo-hadron collisions.

We compare our results with those in previous papers. The differential cross section for unpolarized reactions such as $P+P\rightarrow c\bar{c}+X$ contains the sum of the squares of the helicity amplitudes, $|M(+,+;+)|^2 + |M(+,+;-)|^2 + |M(+,-;-)|^2 + |M(-,+;-)|^2$.

The color singlet case results are given by [6] and [1], which we refer to as GW and GTW respectively. The color octet results are available in the Appendix of Cho and Leibovich [2], which we refer to as CL. The differential cross sections for longitudinally polarized collisions contain the differences $|M(+,+;+)|^2 + |M(+,+;-)|^2 - |M(+,-;-)|^2 - |M(-,+;-)|^2$, and are listed for both the color singlet and the color octet cases in the paper of Klasen, Kniehl, Mihaila and Steinhauser [5], which we refer to as KKMS.

A. Matrix Elements Squared

We now list the results for the squares of the color singlet matrix elements when the heavy quark pair (with mass M) is in the appropriate angular momentum state. However for convenience we rename $R_0^2=\langle R[^1S_0^{(1)}]\rangle = \langle R[^3S_1^{(1)}]\rangle$ and $R'^2=\langle R[^1P_1^{(1)}]\rangle = \langle R[^3P_0^{(1)}]\rangle = \langle R[^3P_1^{(1)}]\rangle = \langle R[^3P_2^{(1)}]\rangle$, where the final superscript indicates the color singlet.

1. *Color Singlet*

For $^1S_0$ we find

$$|M(+,+,+)|^2 = \frac{16g^6\langle R[^1S_0^{(1)}]\rangle}{\pi M}\frac{M^8P^2}{Q(Q-M^2P)^2} \tag{8a}$$
$$|M(+,+,-)|^2 = \frac{16g^6\langle R[^1S_0^{(1)}]\rangle}{\pi M}\frac{s^4P^2}{Q(Q-M^2P)^2}\,, \tag{8b}$$

where the color states of the gluons have been summed over. These results agree with the squares of (8.29) and (8.40) in GW.

For $^3S_1$ we find

$$|M(+,+,+)|^2 = 0 \tag{9a}$$
$$|M(+,+,-)|^2 = \frac{160g^6\langle R[^3S_1^{(1)}]\rangle}{9\pi M}\frac{M^2s^2(s-M^2)^2}{(Q-M^2P)^2}, \tag{9b}$$

where the color states of the gluons have been summed over. The polarization of the spin one charmonium state has also been summed over. The second result agrees with the (8.50) in GW after correcting an obvious typo that the $(t-M^2)$ should read $(t-M^2)^2$.

For $^1P_1$ we find, after summing over colors and polarizations,

$$|M(+,+,+)|^2 = \frac{640g^6\langle R[^1P_1^{(1)}]\rangle}{3\pi M^3} \times\frac{M^{10}(-M^2P+5Q)}{(Q-M^2P)^3} \quad (10a)$$

$$|M(+,+,-)|^2 = \frac{640g^6\langle R[^1P_1^{(1)}]\rangle}{3\pi M^3}\frac{M^2s^2}{(Q-M^2P)^3} \times\left[3M^4Q-M^6P+2Qs^2\right] . \quad (10b)$$

Here we agree with the results (8.55) and (8.57) in GW.

For $^3P_0$ we find, after summing over colors and polarizations,

$$|M(+,+,+)|^2 = \frac{64g^6\langle R[^3P_0^{(1)}]\rangle}{\pi M^3} \times\frac{9M^8P^2(Q-M^2P)^2}{Q(Q-M^2P)^4} \quad (11a)$$

$$|M(+,+,-)|^2 = \frac{64g^6\langle R[^3P_0^{(1)}]\rangle}{\pi M^3}\frac{(s-M^2)^2}{Q(Q-M^2P)^4} \times\left[Q^2-s^2Q(s-3M^2)+3PM^2s^3\right]^2 . \quad (11b)$$

Here we agree with the results in (8.59) in GW.

For $^3P_1$ we find, after summing over colors and polarizations,

$$|M(+,+,+)|^2 = 0 \quad (12a)$$

$$|M(+,+,-)|^2 = \frac{192g^6\langle R[^3P_1^{(1)}]\rangle}{\pi M^3}\frac{(s-M^2)^2s^2}{(Q-M^2P)^4} \times\Big[2Q\big(5M^4P-M^8+P^2 -(4P-2sM^2+4s^2-M^4)(s-M^2)^2\big) -Q^2(15M^2-8s)-4M^2P^3+M^6P^2\Big], \quad (12b)$$

which agrees with (8.63) in GW.

For $^3P_2$ we find, after summing over colors and polarizations,

$$|M(+,+,+)|^2 = 0 \quad (13a)$$

$$\begin{aligned}|M(+,+,-)|^2 = {}& \frac{64g^6\langle R[^3P_2^{(1)}]\rangle}{\pi M^3}\frac{1}{Q(Q-M^2P)^4} \\ &\times\Big[12M^8P^4(3s-M^2)(s-M^2) \\ &-12M^4P^5s(s-3M^2)+2P^2Q^3(s-11M^2) \\ &-3M^6P^3Q(s-M^2)(25s-8M^2) \\ &+12M^2P^4Q(s^2-4M^2s-3M^4) \\ &+M^4P^2Q^2(8s^2+9M^2s-15M^4) \\ &-2P^3Q^2(s^2-5M^2s-30M^4) \\ &+M^2PQ^3(29s^2-51M^2s+18M^4) \\ &-M^2Q^4(9s-11M^2)\Big] , \quad (13b)\end{aligned}$$

which agrees with (8.70) in GW.

2. Color Octet

Now we present the corresponding results for the color octet projections. These results do not seem to be available in the literature. We have only found expressions for the differential cross sections which we will compare to ours later on. We give these results since we need the differences between the helicity combinations to check the octet longitudinally polarized differential cross sections. The constants from the wave functions are now simply renamed as $R^2 \to \langle R\,[^1S_0^{(8)}]\,\rangle$ etc., since there are other definitions in the literature. We will present the relations between the definitions later on.

For $^1S_0$ we find

$$|M(+,+,+)|^2 = \frac{40g^6\langle R\,[^1S_0^{(8)}]\,\rangle}{\pi M}\frac{M^8(P^2-M^2Q)}{Q(Q-M^2P)^2} \quad (14a)$$

$$|M(+,+,-)|^2 = \frac{40g^6\langle R\,[^1S_0^{(8)}]\,\rangle}{\pi M}\frac{s^3}{Q(Q-M^2P)^2} \times\left[PQ+s^3(s-M^2)^2-s^2Q\right] . \quad (14b)$$

For $^3S_1$ we find

$$|M(+,+,+)|^2 = 0 \quad (15a)$$

$$|M(+,+,-)|^2 = \frac{16g^6\langle R\,[^3S_1^{(8)}]\,\rangle}{3\pi M}\frac{s^2(s-M^2)^2}{M^2(Q-M^2P)^2} \times(19M^4-27P) . \quad (15b)$$

For $^1P_1$ we find

$$|M(+,+,+)|^2 = \frac{32g^6\langle R\,[^1P_1^{(8)}]\,\rangle}{\pi M^3}\frac{M^6}{Q(Q-M^2P)^3} \times\Big[217M^4Q^2-54PQ^2+43M^6PQ -27M^2P^2Q-27M^4P^3\Big] \quad (16a)$$

$$\begin{aligned}|M(+,+,-)|^2 = {}& \frac{32g^6\langle R\,[^1P_1^{(8)}]\,\rangle}{\pi M^3}\frac{s^3}{Q(Q-M^2P)^3} \\ &\times\Big[Qs^2(t+u)(-174u^2+26tu-174t^2) \\ &+Qs(-98u^4-278tu^3-468t^2u^2-278t^3u-98t^4) \\ &+Q(t+u)(-38u^4-82tu^3-169t^2u^2-82t^3u-38t^4) \\ &+s^4(-27u^4-152tu^3+10t^2u^2-152t^3u-27t^4) \\ &+t^2u^2(t+u)(-38u^3-60tu^2-60t^2u-38t^3) \\ &+s^5(t+u)(-27u^2-11tu-27t^2)\Big] . \quad (16b)\end{aligned}$$

For ${}^3P_0$ we find

$$|M(+,+,+)|^2 = \frac{160g^6\langle R\,[{}^3P_0^{(8)}]\,\rangle}{\pi M^3} \times \frac{9M^8(P^2-M^2Q)(Q-M^2P)^2}{Q(Q-M^2P)^4} \quad (17a)$$

$$|M(+,+,-)|^2 = \frac{160g^6\langle R\,[{}^3P_0^{(8)}]\,\rangle}{\pi M^3}\frac{(s-M^2)^2}{Q(Q-M^2P)^4} \times\Big[Q^4+9s^8M^4(s-M^2)^2 +Qs^5M^2(6s^3-6M^6+33sM^4-42s^2M^2) +Q^2s^2(44s^2M^4+4M^8-18s^3M^2+s^4) +Q^3s(-2(s-M^2)^2+9sM^2)\Big]. \quad (17b)$$

For ${}^3P_1$ we find

$$|M(+,+,+)|^2 = 0 \quad (18a)$$

$$|M(+,+,-)|^2 = \frac{960g^6\langle R\,[{}^3P_1^{(8)}]\,\rangle}{\pi M^3} \times\frac{(s-M^2)^2(P+s^2-sM^2)}{Q(Q-M^2P)^4}\Big[Q^3(s-2M^2) +s^5M^2(M^8-4sM^6+7s^2M^4-6s^3M^2+2s^4) +Qs^3(M^8-4sM^6+11s^2M^4-10s^3M^2+s^4) +Q^2s(M^6+7s^2M^2-2s^3)\Big]. \quad (18b)$$

For ${}^3P_2$ we find

$$|M(+,+,+)|^2 = 0 \quad (19a)$$

$$|M(+,+,-)|^2 = \frac{320g^6\langle R\,[{}^3P_2^{(8)}]\,\rangle}{\pi M^3}\frac{(s-M^2)^2}{Qs^4(Q-M^2P)^4} \times\Big[6s^8M^4(s-M^2)^6+Qs^6M^2(18M^{12}-114sM^{10} +285s^2M^8-354s^3M^6+225s^4M^4-66s^5M^2+6s^6) +Q^2s^4(24M^{12}-132sM^{10}+313s^2M^8-336s^3M^6 +161s^4M^4-30s^5M^2+s^6)+Q^3s^2(18M^{10}-78sM^8 +141s^2M^6-110s^3M^4+25s^4M^2-2s^5) +Q^4\big(s^4-6M^2(s-M^2)^3+6sM^4(s-M^2)\big)\Big]. \quad (19b)$$

B. Unpolarized Differential Cross Sections

These follow from the sum of the squares of the helicity matrix elements $|M(+,+;+)|^2+|M(+,+;-)|^2+|M(+,-;-)|^2+|M(-,+;-)|^2$ with the substitutions $s\to t$ and $s\to u$ as described above. However to sum over all polarization states we have to multiply by 2 to include the CP conjugates. Then one adds the average over the initial gluon colors and polarizations (1/256) and multiplies by an overall factor of $1/(16\pi s^2)$.

1. Color Singlet

These results can be compared with the results in GW and KKMS. The latter authors give the differential cross sections as functions of polarization factors $\xi_a\xi_b$ in the form $a(s,t,u)+\xi_a\xi_b\,b(s,t,u)$. The unpolarized cross sections are obtained by setting $\xi_a\xi_b=0$. We call these the first terms and the coefficients of $\xi_a\xi_b$, which yield the longitudinally polarized differential cross sections, the second terms. Note that, due to the differences in the definitions of the wave functions, our comments concern the polynomial dependence of $a(s,t,u)$ and $b(s,t,u)$ on the invariants. However we will also identify the prefactors. This is possible because their polarized differential cross sections agree with ours.

For ${}^1S_0$ we find

$$\frac{d\sigma}{dt} = \frac{\pi\alpha_s^3\langle\, R[{}^1S_0^{(1)}]\,\rangle}{Ms^2}\frac{P^2}{Q(Q-M^2P)^2} \times\Big[(P-M^4)^2+2M^2Q\Big], \quad (20)$$

which agrees with (8.46) in GW. However it does not agree with the first term in (A.16) in KKMS, who use the notation where $\langle\, R[{}^1S_0^{(1)}]\,\rangle = 4\pi\langle O[{}^1S_0^{(1)}]\rangle$.

For ${}^3S_1$ we find

$$\frac{d\sigma}{dt} = \frac{10\pi\alpha_s^3\langle\, R[{}^3S_1^{(1)}]\,\rangle}{9Ms^2}\frac{M^2(P^2-M^2Q)}{(Q-M^2P)^2}, \quad (21)$$

which agrees with (8.52) in GW and also agrees with the first terms in (A.17) in KKMS, who use the notation where $\langle\, R[{}^3S_1^{(1)}]\,\rangle = 4\pi\langle O[{}^3S_1^{(1)}]\rangle/3$.

For ${}^1P_1$ we find

$$\frac{d\sigma}{dt} = \frac{40\pi\alpha_s^3\langle\, R[{}^1P_1^{(1)}]\,\rangle}{3M^3s^2}\frac{M^2}{(Q-M^2P)^3}\Big[-M^{10}P +M^6P^2+Q(5M^8-7M^4P+2P^2)+4M^2Q^2\Big], \quad (22)$$

which agrees with (8.58) in GW. It also agrees with the first terms in (A.18) in KKMS, who use the notation where $\langle\, R[{}^1P_1^{(1)}]\,\rangle = 4\pi\langle O[{}^1P_1^{(1)}]\rangle/9$.

For ${}^3P_0$ we find

$$\frac{d\sigma}{dt} = \frac{4\pi\alpha_s^3\langle\, R[{}^3P_0^{(1)}]\,\rangle}{M^3s^2}\frac{1}{Q(Q-M^2P)^4}\Big[-2M^8P^2Q^2 +6M^6P^3Q(3P-M^4)-2M^2PQ^3(P-M^4) +P^2(3PM^2-Q)^2(P-M^4)^2+6M^4Q^4\Big], \quad (23)$$

which agrees with (8.60) in GW. It does not agree with the first terms in (A.19) in KKMS, who use the notation where $\langle\, R[{}^3P_0^{(1)}]\,\rangle = 4\pi\langle O[{}^3P_0^{(1)}]\rangle/3$.

For ${}^3P_1$ we find

$$\frac{d\sigma}{dt} = \frac{12\pi\alpha_s^3\langle\, R[{}^3P_1^{(1)}]\,\rangle}{M^3s^2}\frac{P^2}{(Q-M^2P)^4}\Big[-15M^2Q^2 +M^2P^2(M^4-4P)-2Q(M^8-5M^4P-P^2)\Big], \quad (24)$$

which agrees with (8.64) in GW. It does not agree with the first terms in (A.20) in KKMS, who use the notation where $\langle R[^3P_1^{(1)}]\rangle = 4\pi\langle O[^3P_1^{(1)}]\rangle/9$.

For $^3P_2$ we find

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{4\pi\alpha_s^3\langle R[^3P_2^{(1)}]\rangle}{M^3 s^2}\frac{1}{Q(Q-M^2P)^4} \times\Big[12M^4P^4(P-M^4)^2 + M^2PQ^3(16M^4-61P) -3M^2P^3Q(8M^8-M^4P+4P^2)+12M^4Q^4 -2P^2Q^2(7M^8-43M^4P-P^2)\Big], \quad (25)$$

which agrees with (8.71) in GW after correcting a typo. They have $(8M^8-M^4P+P^2)$ which should read $(8M^8-M^4P+4P^2)$. The expression is given correctly in their published paper [1]. Also it does not agree with the first terms in (A.21) in KKMS, who use the notation where $\langle R[^3P_2^{(1)}]\rangle = 4\pi\langle O[^3P_2^{(1)}]\rangle/15$.

In view of these differences we contacted the authors of the KKMS paper. They calculated their results with projection operators for the sums over the gluon polarization states, which required the calculation of additional ghost diagrams. However they inadvertently presented the formulae (A.16), (A.19), (A.20) and (A.21) without the contributions from these ghost terms. They claim that the correct formulae are included in their fortran programs and that their numerical results are therefore correct.

2. *Color Octet*

These can be compared with the results for the squares of the matrix elements in the appendix of CL and with the first parts of the expressions in Appendix A of KKMS.

First we find for $^1S_0$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{5\pi\alpha_s^3\langle R[^1S_0^{(8)}]\rangle}{2Ms^2}\frac{P^2-M^2Q}{Q(Q-M^2P)^2} \times\Big[(P-M^4)^2+2M^2Q\Big], \quad (26)$$

which agrees with (A5a) in CL. It does not agree with the first part of (A.22) in KKMS, who use the notation where $\langle R[^1S_0^{(8)}]\rangle = \pi\langle O[^1S_0^{(8)}]\rangle/2$.

Next, we find for $^3S_1$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{\pi\alpha_s^3\langle R[^3S_1^{(8)}]\rangle}{3Ms^2}\frac{(P^2-M^2Q)(19M^4-27P)}{M^2(Q-M^2P)^2}, \quad (27)$$

which agrees with the sum of (A5b) plus (A5c) in CL. It does not agree with the first terms in (A.23) in KKMS, who use the notation where $\langle R[^3S_1^{(8)}]\rangle = \pi\langle O[^3S_1^{(8)}]\rangle/6$.

The expression for $^1P_1$,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{2\pi\alpha_s^3\langle R[^1P_1^{(8)}]\rangle}{M^3s^2}\frac{1}{Q(Q-M^2P)^3}\Big[179M^4Q^3 +217M^{10}Q^2-27M^2P^5+54M^6P^4-27M^{10}P^3 +135PQ^3+103M^2P^2Q^2-212M^6PQ^2 -124M^8P^2Q+43M^{12}PQ+27P^4Q\Big], \quad (28)$$

is not given in CL. It does not agree with the first terms in (A.24) in KKMS, who use the notation where $\langle R[^1P_1^{(8)}]\rangle = \pi\langle O[^1P_1^{(8)}]\rangle/18$.

Now we turn to the expression for $^3P_0$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{10\pi\alpha_s^3\langle R[^3P_0^{(8)}]\rangle}{M^3s^2}\frac{1}{Q(Q-M^2P)^4} \times\Big[9M^4P^4(P-M^4)^2+3M^{10}P^3Q-6M^2P^5Q +27M^6P^4Q+18M^{12}PQ^2-32M^8P^2Q^2 -9M^{14}P^2Q-4M^4P^3Q^2+5M^4Q^4+P^4Q^2 +11M^6PQ^3-M^2P^2Q^3-13M^{10}Q^3\Big], \quad (29)$$

which agrees with (A5d) in CL. It does not agree with the first terms in (A.25) in KKMS, who use the notation where $\langle R[^3P_0^{(8)}]\rangle = \pi\langle O[^3P_0^{(8)}]\rangle/6$.

For $^3P_1$ we find

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{60\pi\alpha_s^3\langle R[^3P_1^{(8)}]\rangle}{M^3s^2}\frac{1}{(Q-M^2P)^4}\Big[P^4Q+M^{10}Q^2 +M^6P^4-2M^2P^5-2M^8P^2Q+7M^4P^3Q -3M^6PQ^2-9M^2P^2Q^2+6M^4Q^3\Big], \quad (30)$$

which agrees with the sum of (A5e) and (A5f) in CL. It does not agree with the first terms in (A.26) in KKMS, who use the notation where $\langle R[^3P_1^{(8)}]\rangle = \pi\langle O[^3P_1^{(8)}]\rangle/6$.

For $^3P_2$ we find

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{20\pi\alpha_s^3\langle R[^3P_2^{(8)}]\rangle}{M^3s^2}\frac{1}{Q(Q-M^2P)^4} \times\Big[6M^{12}P^4-12M^8P^5+6M^4P^6+11M^4Q^4 +Q(-6M^{14}P^2-3M^{10}P^3+3M^6P^4-6M^2P^5) +Q^2(24M^{12}P-29M^8P^2+41M^4P^3+P^4) +Q^3(-19M^{10}+14M^6P-31M^2P^2)\Big], \quad (31)$$

which agrees with the sum of (A5g) plus (A5h) plus (A5i) in CL, after correcting an obvious typo that the term $-M\hat{s}^2$, which multiplies the second line in (A5i), should read $-M^2\hat{s}$. It does not agree with the first terms in (A.27) in KKMS, who use the notation where $\langle R[^3P_2^{(8)}]\rangle = \pi\langle O[^3P_2^{(8)}]\rangle/30$. The explanation for the difference between our results and (A.22) - (A.27) in

KKMS is again that they inadvertently neglected to include ghost contributions to their amplitudes. However they claim that they did so in their computer programs so their numerical results are correct.

In view of the differences in the above results and before contacting KKMS we recalculated the differential cross sections by summing over the physical polarizations of the external gluons using the covariant expression

$$\sum_{\alpha=+,-} \epsilon^{\mu}(k,\alpha)\epsilon^{\nu}(k,\alpha) = P^{\mu\nu}(n,k)\,, \tag{32}$$

with

$$P^{\mu\nu}(n,k) = -g_{\mu\nu} + (n_\mu k_\nu + k_\mu n_\nu)/n\cdot k\,, \tag{33}$$

where n_μ satisfies $n_\mu P^{\mu\nu} = P^{\mu\nu} n_\nu = 0$ and $n^2 = 0$. One uses this sum for each external gluon and the answer for the square of the matrix elements should be independent of n_μ. This method does not require any ghosts and yielded the same answers we obtained above for the differential cross sections.

C. Polarized Differential Cross Sections

Now we calculate the expressions $|M(+,+;+)|^2 + |M(+,+;-)|^2 - |M(+,-;-)|^2 - |M(-,+;-)|^2$, which yield the longitudinally polarized differential cross sections.

1. Color Singlet

We begin with the color singlet expressions. These are available in KKMS as the second terms, i.e., $b(s,t,u)$, those terms proportional to $\xi_a\xi_b$. The prefactors are identified as in the unpolarized differential cross sections given previously. We repeat them here for convenience.

For ${}^1S_0$ we find

$$\frac{\mathrm{d}\Delta\sigma}{\mathrm{d}t} = \frac{\pi\alpha_s^3\langle R\,[{}^1S_0^{(1)}]\,\rangle}{s^2 M}\frac{P^2}{Qs^2(Q-M^2P)^2}\Big[s^6 - 2Q^2 + 4Qs(s-M^2)^2 + +s^2M^8 - s^2(s-M^2)^4\Big]\,. \tag{34}$$

This is in agreement with the second terms in (A.16) in KKMS, when we make the replacement $\langle\, R[{}^1S_0^{(1)}]\,\rangle = 4\pi\langle O[{}^1S_0^{(1)}]\rangle$.

For ${}^3S_1$ we find

$$\frac{\mathrm{d}\Delta\sigma}{\mathrm{d}t} = \frac{10\pi\alpha_s^3\langle R\,[{}^3S_1^{(1)}]\,\rangle}{9Ms^2}\frac{M^2Q(s^2-P)}{s(Q-M^2P)^2}\,. \tag{35}$$

This is in agreement with the second terms in (A.17) in KKMS, when we make the replacement $\langle\, R[{}^3S_1^{(1)}]\,\rangle = 4\pi\langle O[{}^3S_1^{(1)}]\rangle\,/3$.

For ${}^1P_1$ we find

$$\begin{aligned}\frac{\mathrm{d}\Delta\sigma}{\mathrm{d}t} = {} & \frac{40\pi\alpha_s^3\langle R\,[{}^1P_1^{(1)}]\,\rangle}{3M^3s^2}\frac{M^2}{(Q-M^2P)^3} \\ & \times\Big[stu(2u^4+4tu^3+6t^2u^2+4t^3u+2t^4) \\ & +s^2(5tu^4+7t^2u^3+7t^3u^2+5t^4u+t^5+u^5) \\ & +s^3(10tu^3+10t^2u^2+10t^3u+4t^4+4u^4) \\ & +s^4(10tu^2+10t^2u+6t^3+6u^3) \\ & +s^5(4tu+4t^2+4u^2)+s^6(t+u) \\ & +t^2u^5+3t^3u^4+3t^4u^3+t^5u^2\Big]\,.\end{aligned} \tag{36}$$

This is in agreement with the second terms in (A.18) in KKMS, when we make the replacement $\langle\, R[{}^1P_1^{(1)}]\,\rangle = 4\pi\langle O[{}^1P_1^{(1)}]\rangle\,/9$.

For ${}^3P_0$ we find

$$\begin{aligned}\frac{\mathrm{d}\Delta\sigma}{\mathrm{d}t} = {} & \frac{4\pi\alpha_s^3\langle R\,[{}^3P_0^{(1)}]\,\rangle}{M^3s^2}\frac{Q+s^2M^2}{Qs^6(Q-M^2P)^4} \\ & \times\Big[18s^9M^4(s-M^2)^6+9s^{10}M^6(s-M^2)^4 \\ & +Qs^7M^2(66M^{12}-327sM^{10}+684s^2M^8 \\ & -762s^3M^6+462s^4M^4-135s^5M^2+12s^6) \\ & +Q^3s^3(66M^{10}-260sM^8+374s^2M^6-5s^5 \\ & -237s^3M^4+62s^4M^2)+Q^5(6sM^2-s^2-9M^4) \\ & +Q^4s(18M^8-75sM^6+80s^2M^4-31s^3M^2+4s^4) \\ & +Q^2s^5(96M^{12}-422sM^{10}+750s^2M^8 \\ & -663s^3M^6+286s^4M^4-49s^5M^2+2s^6)\Big]\,.\end{aligned} \tag{37}$$

This is in agreement with the second terms in (A.19) in KKMS, when we make the replacement $\langle\, R[{}^3P_0^{(1)}]\,\rangle = 4\pi\langle O[{}^3P_0^{(1)}]\rangle\,/3$.

For ${}^3P_1$ we find

$$\begin{aligned}\frac{\mathrm{d}\Delta\sigma}{\mathrm{d}t} = {} & \frac{12\pi\alpha_s^3\langle R\,[{}^3P_1^{(1)}]\,\rangle}{M^3s^2}\frac{Q(Q+s^2M^2)}{s^5(Q-M^2P)^4}\Big[2s^6(-M^8 \\ & +5sM^6-9s^2M^4+7s^3M^2-2s^4)+Qs^3(M^8-4sM^6 \\ & +11s^2M^4-18s^3M^2+10s^4)+Q^2s(M^6-6sM^4 \\ & +11s^2M^2-8s^3)+2Q^3(s-2M^2)\Big]\,.\end{aligned} \tag{38}$$

This is in agreement with the second terms in (A.20) in KKMS, when we make the replacement $\langle\, R[{}^3P_1^{(1)}]\,\rangle = 4\pi\langle O[{}^3P_1^{(1)}]\rangle\,/9$.

For $^3P_2$ we find

$$\frac{\mathrm{d}\Delta\sigma}{\mathrm{d}t} = \frac{4\pi\alpha_s^3\langle R\,[^3P_2^{(1)}]\,\rangle}{M^3s^2}\frac{Q+s^2M^2}{Qs^6(Q-M^2P)^4} \times\Big[-24s^9M^4(s-M^2)^6-12s^{10}M^6(s-M^2)^4 +Qs^7M^4(-48M^{10}+276sM^8-648s^2M^6 +768s^3M^4-456s^4M^2+108s^5) +Q^4s(24M^8-63sM^6+34s^2M^4-5s^3M^2+8s^4) +Q^3s^3(15s^3M^4-79sM^8+28s^2M^6-2s^4M^2 -10s^5+48M^{10})+Q^5(-12M^4+12sM^2-2s^2) +Q^2s^6(-330sM^8+306s^2M^6-82s^3M^4 -14s^4M^2+4s^5+116M^{10})\Big]. \quad (39)$$

This is in agreement with the second terms in (A.21) in KKMS, when we make the replacement $\langle R[^3P_2^{(1)}]\rangle = 4\pi\langle O[^3P_2^{(1)}]\rangle/15$. Our polarized differential cross sections agree with those in KKMS because their method of calculation does not require ghost contributions.

2. Color Octet

These are only available in the Appendix of KKMS.

We begin with $^1S_0$:

$$\frac{\mathrm{d}\Delta\sigma}{\mathrm{d}t} = \frac{5\pi\alpha_s^3\langle R\,[^1S_0^{(8)}]\,\rangle}{2s^2M}\frac{1}{s^4Q(Q-M^2P)^2} \times\Big[2s^7M^2(M^2-s)^4+s^8M^4(M^2-s)^2 +Qs^5(4M^8-15sM^6+20s^2M^4-12s^3M^2+2s^4) +Q^2s^3(4M^6-12sM^4+14s^2M^2-5s^3) +Q^3s(2M^4-5sM^2+4s^2)-Q^4\Big]. \quad (40)$$

This is in agreements with the second terms in (A.22) in KKMS, if we make the replacement $\langle R\,[^1S_0^{(8)}]\,\rangle = \pi\langle O\,[^1S_0^{(8)}]\,\rangle/2$.

For $^3S_1$ we find

$$\frac{\mathrm{d}\Delta\sigma}{\mathrm{d}t} = \frac{\pi\alpha_s^3\langle R\,[^3S_1^{(8)}]\,\rangle}{3Ms^2}\frac{Q(19M^4-27P)(s^2-P)}{M^2s(Q-M^2P)^2}, \quad (41)$$

which agrees with the second terms in (A.23) in KKMS, if we make the replacement $\langle R\,[^3S_1^{(8)}]\,\rangle = \pi\langle O\,[^3S_1^{(8)}]\,\rangle/6$.

For $^1P_1$ we find

$$\frac{\mathrm{d}\Delta\sigma}{\mathrm{d}t} = \frac{2\pi\alpha_s^3\langle R\,[^1P_1^{(8)}]\,\rangle}{M^3s^2}\frac{M^2-s}{Qs^6(Q-M^2P)^4} \times\Big[(27(M^2-s)^3M^6s^{11}(2M^4-3sM^2+2s^2) +Qs^9(M^2-s)M^4(-621sM^6+864s^2M^4 -567s^3M^2+108s^4+173M^8) +Q^2s^7M^2(-1395sM^8+2307s^2M^6 -1988s^3M^4+621s^4M^2-54s^5+249M^{10}) +Q^3s^5(-1488sM^8+2314s^2M^6-1492s^3M^4 +189s^4M^2+249M^{10})+Q^4s^3(-779sM^6 +1379s^2M^4-449s^3M^2+173M^8) +Q^5s(-162sM^4+373s^2M^2+27s^3+54M^6) -Q^6 27(M^2-s))\Big]. \quad (42)$$

This is in agreement with the second terms in (A.24) in KKMS, if we make the replacement $\langle R\,[^1P_1^{(8)}]\,\rangle = \pi\langle O\,[^1P_1^{(8)}]\,\rangle/18$.

For $^3P_0$ we find

$$\frac{\mathrm{d}\Delta\sigma}{\mathrm{d}t} = \frac{10\pi\alpha_s^3\langle R\,[^3P_0^{(8)}]\,\rangle}{M^3s^2}\frac{1}{Qs^6(Q-M^2P)^4} \times\Big[9s^{11}M^6(2(M^2-s)^6+sM^2(M^2-s)^4) +3Qs^9M^4(23M^{12}-126sM^{10}+285s^2M^8 -342s^3M^6+228s^4M^4-78s^5M^2+10s^6) +Q^2s^7M^2(117M^{12}-612sM^{10}+1285s^2M^8 -1358s^3M^6+738s^4M^4-184s^5M^2+14s^6) +Q^3s^5(117M^{12}-553sM^{10}+1021s^2M^8+2s^6 -881s^3M^6+352s^4M^4-54s^5M^2) +Q^5s(18M^8-81sM^6+88s^2M^4-33s^3M^2+4s^4) +Q^4s^3(69M^{10}-292sM^8+439s^2M^6-275s^3M^4 +68s^4M^2-5s^5)-Q^6(s-3M^2)^2\Big]. \quad (43)$$

This is in agreement with the second terms in (A.25) in KKMS, if we make the replacement $\langle R\,[^3P_0^{(8)}]\,\rangle = \pi\langle O\,[^3P_0^{(8)}]\,\rangle/6$.

For $^3P_1$ we find

$$\frac{\mathrm{d}\Delta\sigma}{\mathrm{d}t} = \frac{60\pi\alpha_s^3\langle R\,[^3P_1^{(8)}]\,\rangle}{M^3s^2}\frac{Q}{s^5(Q-M^2P)^4} \times\Big[s^7M^2(-M^{10}+5sM^8-10s^2M^6+11s^3M^4+2s^5 -7s^4M^2)+Qs^5(-2M^{10}+8sM^8-14s^2M^6+2s^5 +17s^3M^4-12s^4M^2)+Q^4(-s+2M^2) +Q^2s^3(-2M^8+4sM^6-8s^2M^4+10s^3M^2-5s^4) +Q^3s(-M^6+3sM^4-5s^2M^2+4s^3)\Big]. \quad (44)$$

This is in agreement with the second terms in (A.26) in KKMS, if we make the replacement $\langle R\,[^3P_1^{(8)}]\,\rangle = \pi\langle O\,[^3P_1^{(8)}]\,\rangle/6$.

For $^3P_2$ we find

$$\frac{\mathrm{d}\Delta\sigma}{\mathrm{d}t} = \frac{20\pi\alpha_s^3\langle R\,[^3P_2^{(8)}]\,\rangle}{M^3s^2}\frac{1}{Qs^6(Q-M^2P)^4}$$
$$\times\Big[-12s^{11}M^6(M^2-s)^6-6s^{12}M^8(M^2-s)^4$$
$$+Qs^9M^4(-27M^{12}+177sM^{10}-447s^2M^8$$
$$+567s^3M^6-378s^4M^4+120s^5M^2-12s^6)$$
$$+Q^2s^7M^2(-15M^{12}+138sM^{10}-398s^2M^8$$
$$+481s^3M^6-255s^4M^4+47s^5M^2+2s^6)$$
$$+Q^3s^5(15M^{12}+5sM^{10}-89s^2M^8+115s^3M^6$$
$$-35s^4M^4-12s^5M^2+2s^6)$$
$$+Q^5s(12M^8-39sM^6+25s^2M^4-3s^3M^2+4s^4)$$
$$+Q^4s^3(27M^{10}-55sM^8+37s^2M^6-5s^3M^4$$
$$+2s^4M^2-5s^5)+Q^6(6sM^2-s^2-6M^4)\Big]\,. \quad (45)$$

This is in agreement with the second terms in (A.27) in KKMS, if we make the replacement $\langle R\,[^3P_2^{(8)}]\,\rangle = \pi\langle O\,[^3P_2^{(8)}]\,\rangle/30$.

Again we agree with the KKMS results because, in their method of calculation, the polarized differential cross sections do not require ghost contributions.

III. ONE PHOTON, TWO GLUONS

We also calculate the helicity matrix elements for the reaction

$$\gamma(k_1)+g(k_2,b)+g(k_3,c)\to q(p/2+q)+\bar{q}(p/2-q)\,. \quad (46)$$

Here we can compare our results with those in KKMS as well as those in Yuan, Dong, Hao and Chao [4], which we refer to as YDHC and in Ko, Lee and Soy [3], which we refer to as KLS.

In this case there is no $t\leftrightarrow u$ symmetry. Also we have to change our choice for the helicities. We use

$$\not\epsilon_1^{\pm} = N[\not k_1\not k_2\not k_3(1\mp\gamma_5)-\not k_2\not k_3\not k_1(1\pm\gamma_5)\pm 2k_2.k_3\not k_1\gamma_5] \quad (47a)$$

$$\not\epsilon_2^{\pm} = N[\not k_3\not k_1\not k_2(1\pm\gamma_5)+\not k_2\not k_1\not k_3(1\mp\gamma_5)-2k_1.k_3\not k_2] \quad (47b)$$

$$\not\epsilon_3^{\pm} = N[\not k_1\not k_2\not k_3(1\pm\gamma_5)+\not k_3\not k_2\not k_1(1\mp\gamma_5)-2k_1.k_2\not k_3]\,. \quad (47c)$$

A. Matrix Elements Squared

The squares of the matrix elements now no longer have all the symmetries as in the previous case, so we need the four helicity amplitudes $|M(+,+,+)|^2$, $|M(+,+,-)|^2$, $|M(+,-,+)|^2$ and $|M(-,+,+)|^2$.

When we calculate the differential cross section for

$$\gamma(k_1)+g(k_2,b)\to q(p/2+q)+\bar{q}(p/2-q)+g(k_3,b)\,. \quad (48)$$

we have to cross gluon number three. For the unpolarized differential cross section we need the sum the above terms and for the polarized one we need the difference similar to the three gluon case. Since the sum is over over all the polarization states we have to multiply by 2 to include the CP conjugates. Then divide by 32 to average over the initial gluon colors and initial gluon and photon polarizations. Finally we have to divide by $16\pi s^2$.

1. *Color Singlet*

For $^1S_0$, $^3P_0$, $^3P_1$ and $^3P_2$ we find

$$|M(+,+,+)|^2 = |M(+,+,-)|^2 = |M(+,-,+)|^2 = |M(-,+,+)|^2 = 0\;. \quad (49a)$$

For $^3S_1$ we find

$$|M(+,+,+)|^2 = 0 \quad (50a)$$

$$|M(+,+,-)|^2 = \frac{128g^4e^2\langle R\,[^3S_1^{(1)}]\,\rangle}{3\pi M}\times\frac{M^2s^2(t+u)^2}{(Q-M^2P)^2} \quad (50b)$$

$$|M(+,-,+)|^2 = \frac{128g^4e^2\langle R\,[^3S_1^{(1)}]\,\rangle}{3\pi M}\times\frac{M^2u^2(s+t)^2}{(Q-M^2P)^2} \quad (50c)$$

$$|M(-,+,+)|^2 = \frac{128g^4e^2\langle R\,[^3S_1^{(1)}]\,\rangle}{3\pi M}\times\frac{M^2t^2(u+s)^2}{(Q-M^2P)^2}\,. \quad (50d)$$

Our results agree with the upolarized and polarized results in (A.3) in KKMS if we make the replacement $\langle R\,[^3S_1^{(0)}]\,\rangle = 16\pi\langle O\,[^3S_1^{(0)}]\,\rangle/3$.

For $^1P_1$ we find

$$|M(+,+,+)|^2 = \frac{1024g^4e^2\langle R\,[^1P_1^{(1)}]\,\rangle}{\pi M^3}\times\frac{M^{10}(5Q-M^2P)}{(Q-M^2P)^3} \quad (51a)$$

$$|M(+,+,-)|^2 = \frac{1024g^4e^2\langle R\,[^1P_1^{(1)}]\,\rangle}{\pi M^3}\frac{M^2s^2}{(Q-M^2P)^3}\times[5M^4Q-4PQ-M^6P-2Q(t^2+u^2)] \quad (51b)$$

$$|M(+,-,+)|^2 = \frac{1024g^4e^2\langle R\,[^1P_1^{(1)}]\,\rangle}{\pi M^3}\frac{M^2u^2}{(Q-M^2P)^3}\times[5M^4Q-4PQ-M^6P-2Q(s^2+t^2)] \quad (51c)$$

$$|M(-,+,+)|^2 = \frac{1024g^4e^2\langle R\,[^1P_1^{(1)}]\,\rangle}{\pi M^3}\frac{M^2t^2}{(Q-M^2P)^3}\times[5M^4Q-4PQ-M^6P-2Q(s^2+u^2)]\,. \quad (51d)$$

When we calculate the differential cross sections they agree with the unpolarized and polarized results in (A.4) in KKMS if we make the replacement $\langle R\,[^1P_1^{(0)}]\,\rangle = 8\pi\langle O\,[^1P_1^{(0)}]\,\rangle/9$.

2. Color Octet

For $^1S_0$ we find

$$|M(+,+,+)|^2 = \frac{96g^4e^2\langle R\,[^1S_0^{(8)}]\,\rangle}{\pi M}\frac{usM^8}{t(Q-M^2P)^2} \quad (52a)$$

$$|M(+,+,-)|^2 = \frac{96g^4e^2\langle R\,[^1S_0^{(8)}]\,\rangle}{\pi M}\frac{uss^4}{t(Q-M^2P)^2} \quad (52b)$$

$$|M(+,-,+)|^2 = \frac{96g^4e^2\langle R\,[^1S_0^{(8)}]\,\rangle}{\pi M}\frac{usu^4}{t(Q-M^2P)^2} \quad (52c)$$

$$|M(-,+,+)|^2 = \frac{96g^4e^2\langle R\,[^1S_0^{(8)}]\,\rangle}{\pi M}\frac{ust^4}{t(Q-M^2P)^2}\,. \quad (52d)$$

When we calculate the unpolarized differential cross section, it does not agree with the first terms in (A.5) in KKMS. However the polarized differential cross section agrees with the second terms in (A.5) in KKMS if we make the replacement $\langle R\,[^1S_0^{(8)}]\,\rangle = 2\pi\langle O\,[^1S_0^{(8)}]\,\rangle$. The reason for the different results is again caused by the fact that ghost contributions were not included in the KKMS analytic answers but are included in the KKMS fortran programs. Both the sum and the difference agree with (A1) and (A2) in YDHC. The sum agrees with (A1) in KLS.

For $^3S_1$ we find

$$|M(+,+,+)|^2 = 0 \quad (53a)$$

$$|M(+,+,-)|^2 = \frac{320g^4e^2\langle R\,[^3S_1^{(8)}]\,\rangle}{3\pi M}\frac{s^2M^2(t+u)^2}{(Q-M^2P)^2} \quad (53b)$$

$$|M(+,-,+)|^2 = \frac{320g^4e^2\langle R\,[^3S_1^{(8)}]\,\rangle}{3\pi M}\frac{u^2M^2(t+s)^2}{(Q-M^2P)^2} \quad (53c)$$

$$|M(-,+,+)|^2 = \frac{320g^4e^2\langle R\,[^3S_1^{(8)}]\,\rangle}{3\pi M}\frac{t^2M^2(s+u)^2}{(Q-M^2P)^2}\,. \quad (53d)$$

The sum and the difference both agree with (A.6) in KKMS if we make the replacement $\langle R\,[^3S_1^{(8)}]\,\rangle = 2\pi\langle O\,[^3S_1^{(8)}]\,\rangle/3$. They also agree with (A4) and (A5) in YDHC.

For $^1P_1$ we find

$$|M(+,+,+)|^2 = \frac{2560g^4e^2\langle R\,[^1P_1^{(8)}]\,\rangle}{\pi M^3}\times\frac{M^{10}(5Q-M^2P)}{(Q-M^2P)^3} \quad (54a)$$

$$|M(+,+,-)|^2 = \frac{2560g^4e^2\langle R\,[^1P_1^{(8)}]\,\rangle}{\pi M^3}\frac{M^2s^2}{(Q-M^2P)^3}\times\Big[3M^4Q-M^6P+2Qs^2\Big] \quad (54b)$$

$$|M(+,-,+)|^2 = \frac{2560g^4e^2\langle R\,[^1P_1^{(8)}]\,\rangle}{\pi M^3}\frac{M^2u^2}{(Q-M^2P)^3}\times\Big[3M^4Q-M^6P+2Qu^2\Big] \quad (54c)$$

$$|M(-,+,+)|^2 = \frac{2560g^4e^2\langle R\,[^1P_1^{(8)}]\,\rangle}{\pi M^3}\frac{M^2t^2}{(Q-M^2P)^3}\times\Big[3M^4Q-M^6P+2Qt^2\Big]\,. \quad (54d)$$

The sum and the difference both agree with (A.7) in KKMS if we make the replacement $\langle R\,[^1P_1^{(8)}]\,\rangle = \pi\langle O\,[^1P_1^{(8)}]\,\rangle/9$.

For $^3P_0$ we find

$$|M(+,+,+)|^2 = \frac{128g^4e^2\langle R\,[^3P_0^{(8)}]\,\rangle}{\pi M^3}\frac{9usM^8(M^2-s)^2}{t(Q-M^2P)^4}\times\Big[t^2u^2+2Q(M^2-s)+s^2(M^2-s)^2+2sQ+2s^3(M^2-s)+s^4\Big] \quad (55a)$$

$$|M(+,+,-)|^2 = \frac{128g^4e^2\langle R\,[^3P_0^{(8)}]\,\rangle}{\pi M^3}\frac{uss^2(M^2-s)^2}{t(Q-M^2P)^4}\times\Big[4t^2u^2(M^2-s)^2+4Qtu(M^2-s)+Q^2-12sQ(M^2-s)^2-18s^2Q(M^2-s)+9s^6+9s^4(M^2-s)^2-6s^3Q+18s^5(M^2-s)\Big] \quad (55b)$$

$$|M(+,-,+)|^2 = \frac{128g^4e^2\langle R\,[^3P_0^{(8)}]\,\rangle}{\pi M^3}\frac{usu^2(M^2-u)^2}{t(Q-M^2P)^4}\times\Big[4t^2s^2(M^2-u)^2+4Qts(M^2-u)+Q^2-12uQ(M^2-u)^2-18u^2Q(M^2-u)+9u^6+9u^4(M^2-u)^2-6u^3Q+18u^5(M^2-u)\Big] \quad (55c)$$

$$|M(-,+,+)|^2 = \frac{128g^4e^2\langle R\,[^3P_0^{(8)}]\,\rangle}{\pi M^3}\frac{ust^4(M^2-t)^2}{t(Q-M^2P)^4}\times\Big[4(M^2-t)^4+4su(M^2-t)^2+s^2u^2+28t(M^2-t)^3+14Q(M^2-t)+25t^4+69t^2(M^2-t)^2+10tQ+70t^3(M^2-t)\Big]\,, \quad (55d)$$

and only the difference agrees with (A.8) in KKMS if we make the replacement $\langle R\,[^3P_0^{(8)}]\,\rangle = 2\pi\langle O\,[^3P_0^{(8)}]\,\rangle$.

The sum and the difference agree with (A6) and (A7) in YDHC once a typo is corrected; the last term in these equations should have been $(t+u)^{-2}$ instead of $(t+s)^{-2}$. The sum agrees with (A2) in KLS.

For $^3P_1$ we find

$$|M(+,+,+)|^2 = 0 \tag{56a}$$

$$\begin{aligned}|M(+,+,-)|^2 &= \frac{1152 g^4 e^2 \langle R\,[^3P_1^{(8)}]\,\rangle}{\pi M^3} \frac{s^2(M^2-s)^2}{(Q-M^2P)^4} \\ &\times\Big[s^5(t+u)^2 + s(9t^2u^4+20t^3u^3+13t^4u^2+2t^5u) \\ &+s^2(tu^4+26t^2u^3+38t^3u^2+13t^4u+t^5+u^5) \\ &+s^3(6tu^3+34t^2u^2+22t^3u+3t^4-u^4) \\ &+s^4(9tu^2+13t^2u+3t^3-u^3)+t^2u^2(t+u)^3\Big]\end{aligned} \tag{56b}$$

$$\begin{aligned}|M(+,-,+)|^2 &= \frac{1152 g^4 e^2 \langle R\,[^3P_1^{(8)}]\,\rangle}{\pi M^3} \frac{u^2(M^2-u)^2}{(Q-M^2P)^4} \\ &\times\Big[u^5(s+t)^2 + u(2st^5+13s^2t^4+20s^3t^3+9s^4t^2) \\ &+u^2(13st^4+38s^2t^3+26s^3t^2+s^4t+s^5+t^5) \\ &+u^3(22st^3+34s^2t^2+6s^3t-s^4+3t^4) \\ &+u^4(13st^2+9s^2t-s^3+3t^3)+s^2t^2(t+s)^3\Big]\end{aligned} \tag{56c}$$

$$\begin{aligned}|M(-,+,+)|^2 &= \frac{1152 g^4 e^2 \langle R\,[^3P_1^{(8)}]\,\rangle}{\pi M^3} \frac{t^2(M^2-t)^2}{(Q-M^2P)^4} \\ &\times\Big[ts^2u^2\big(5(u+s)^2-2su\big)-4s^3u^3(u+s) \\ &+t^2\big((u+s)^5+8s^2u^2(u+s)\big)+s^2u^2(u+s)^3 \\ &+t^3\big(3(u+s)^4-2su(s^2+u^2)\big)+t^5(s^2+u^2) \\ &+t^4\big(3(u+s)^3-4su(u+s)\big)\Big],\end{aligned} \tag{56d}$$

and only the difference agrees with (A.9) in KKMS if we make the replacement $\langle R\,[^3P_1^{(8)}]\,\rangle = \pi\langle O\,[^3P_1^{(8)}]\,\rangle/4$. The sum and the difference agree with (A8) and (A9) respectively in YDHC. The sum also agrees with (A3) in KLS once the factor $(s^2-u^2)^2$ is replaced by $(s^2-u^2)^4$.

For $^3P_2$ we find that only the difference of the results below agrees with (A.10) in KKMS if we make the replacement $\langle R\,[^3P_2^{(8)}]\,\rangle = 16\pi\langle O\,[^3P_2^{(8)}]\,\rangle/15$. The difference also agrees with (A10) and (A11) in YDHC. The sum from our results does not agree agree with the answer in (A4) in KLS as well as with (A10) in YDHC. The diferent results in (A.5), (A.8), (A.9) and (A.10) of the KKMS paper are again due to the inadvertent omission of gluon contributions to their answers. They say they have included these contributions in their fortran programs.

$$|M(+,+,+)|^2 = 0 \tag{57a}$$

$$\begin{aligned}|M(+,+,-)|^2 &= \frac{48 g^4 e^2 \langle R\,[^3P_2^{(8)}]\,\rangle}{\pi M^3} \frac{s^2u(M^2-s)^2}{Q(Q-M^2P)^4} \\ &\times\Big[(12t^2u^7+48t^3u^6+72t^4u^5+48t^5u^4+12t^6u^3) \\ &+s(24tu^7+96t^2u^6+171t^3u^5+177t^4u^4+105t^5u^3 \\ &+27t^6u^2)+s^2(72tu^6+140t^2u^5+187t^3u^4+200t^4u^3 \\ &+111t^5u^2+18t^6u+12u^7)+s^3(51tu^5+59t^2u^4 \\ &+134t^3u^3+162t^4u^2+63t^5u+3t^6+24u^6) \\ &+s^4(-3tu^4+26t^2u^3+102t^3u^2+78t^4u+9t^5 \\ &+12u^5)+s^5(-3tu^3+27t^2u^2+39t^3u+9t^4) \\ &+s^6(3tu^2+6t^2u+3t^3)\Big]\end{aligned} \tag{57b}$$

$$\begin{aligned}|M(+,-,+)|^2 &= \frac{48 g^4 e^2 \langle R\,[^3P_2^{(8)}]\,\rangle}{\pi M^3} \frac{su^2(M^2-u)^2}{Q(Q-M^2P)^4} \\ &\times\Big[(12s^3t^6+48s^4t^5+72s^5t^4+48s^6t^3+12s^7t^2) \\ &+u(27s^2t^6+105s^3t^5+177s^4t^4+171s^5t^3+96s^6t^2 \\ &+24s^7t)+u^2(18st^6+111s^2t^5+200s^3t^4+187s^4t^3 \\ &+140s^5t^2+72s^6t+12s^7)+u^3(3t^6+63st^5+162s^2t^4 \\ &+134s^3t^3+59s^4t^2+51s^5t+24s^6)+u^4(9t^5+78st^4 \\ &+102s^2t^3+26s^3t^2-3s^4t+12s^5)+u^5(9t^4+39st^3 \\ &+27s^2t^2-3s^3t)+u^6(3t^3+6st^2+3s^2t)\Big]\end{aligned} \tag{57c}$$

$$\begin{aligned}|M(-,+,+)|^2 &= \frac{48 g^4 e^2 \langle R\,[^3P_2^{(8)}]\,\rangle}{\pi M^3} \frac{su(M^2-t)^2}{Q(Q-M^2P)^4} \\ &\times\Big[24Qsu(u^5+5su^4+10s^2u^3+10s^3u^2+5s^4u+s^4) \\ &+12Qt(u^6+10su^5+29s^2u^4+40s^3u^3+29s^4u^2 \\ &+10s^5u+s^6)+3Qt^2(16u^5+89su^4+183s^2u^3 \\ &+183s^3u^2+89s^4u+16s^5)+Qt^3(92u^4+367su^3 \\ &+552s^2u^2+367s^3u+92s^4)+t^5(3u^5+119su^4 \\ &+334s^2u^3+334s^3u^2+119s^4u+3s^5)+t^6(9u^4 \\ &+102su^3+182s^2u^2+102s^3u+9s^4)+t^7(9u^3 \\ &+47su^2+47s^2u+9s^3)+t^8(3u^2+8su+3s^2) \\ &+12s^3u^3(u^4+4su^3+6s^2u^2+4s^3u+s^4)\Big],\end{aligned} \tag{57d}$$

IV. CONCLUSIONS

We have calculated the gluon-gluon and photon-gluon amplitudes for the production of color singlet and color octet charmonium production. These amplitudes are required for the QCD analysis of charmonium production in polarized and unpolarized hadron-hadron and photon-hadron collisions. Our calculations clarify several inconsistencies in previously published results.

V. ACKNOWLEDGMENTS

The work of J. Smith was partially supported by the National Science Foundation grant PHY-0354776. He would also like to thank Professor P. van Baal of the Lorentz Institute, University of Leiden for hospitality and support. He would also like to thank Michael Klasen and Luminita Mihaila for correspondence regarding the formulae in the Appendix of the KKMS paper.

The work of M.M. Meijer was supported by a scholarship from the Huygens Scholarship Programme of the Dutch Ministry of Education, Culture and Science.

[1] R. Gastmans, W. Troost, and T. Wu, Nucl. Phys. **B291**, 731 (1987).

[2] P. Cho and A. Leibovich, Phys. Rev. **D53**, 6203 (1996), hep-ph/9511315.

[3] P. Ko, J. Lee, and H. Song, Phys. Rev. **D54**, 4315 (1996), hep-ph/9602223.

[4] F. Yuan, H.-S. Dong, L.-K. Hao, and K. Chao, Phys. Rev. **D61**, 114013 (2000), hep-ph/9909221.

[5] M. Klasen, B. Kniehl, L. Mihaila, and M. Steinhauser, Phys. Rev **D68**, 034017 (2003), hep-ph/0306080.

[6] R. Gastmans and T. Wu, *The Ubiquitous Photon* (Oxford Science Publishers, 1990).

[7] B. Gubernina, J. Kuhn, R. Peccei, and R. Ruckl, Nuc. Phys. **B174**, 317 (1980).