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EULERIAN OPERATORS AND THE JACOBIAN CONJECTURE

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Abstract. In this paper we introduce a new class of polynomial maps, the so-called nice polynomial maps. Using Eulerian operators we show how for these polynomial maps the main results obtained by Bass (Differential structure ofétale extensions of polynomial algebras, Proc. Workshop on Commutative Algebra, MSRI, 1987) can be proved in a very simple and elementary way. Furthermore we show that every polynomial map \( F \) satisfying the Jacobian condition, \( \det JF \in k^* \), is equivalent to a nice polynomial map; more precisely the polynomial map \( F(\lambda)(X) = F(X + \lambda) - F(\lambda) \) is nice for almost all \( \lambda \in k^n \).

Introduction

Let \( F = (F_1, \ldots, F_n): \mathbb{C}^n \to \mathbb{C}^n \) be a polynomial map, i.e., each coordinate function \( F_i \) belongs to the polynomial ring \( \mathbb{C}[X] := \mathbb{C}[X_1, \ldots, X_n] \). The Jacobian Conjecture asserts that \( \det JF \in \mathbb{C}^* \) implies that \( F \) is invertible, i.e., \( \mathbb{C}[X] = \mathbb{C}[F] := \mathbb{C}[F_1, \ldots, F_n] \). In [3] Bass proposed the following idea to attack the Jacobian Conjecture: the condition \( \det JF \in \mathbb{C}^* \) implies that \( \mathbb{C}[F] \) is a polynomial ring and that the derivations \( d/dF_i \) extend uniquely to \( n \) pairwise commuting derivations on \( \mathbb{C}[X] \). In this way \( \mathbb{C}[X] \) becomes a left module over the \( n \)th Weyl algebra \( A_n(F) := \mathbb{C}[F, d/dF_1, \ldots, d/dF_n] \). The derivations \( \varepsilon_{ij} = F_i d/dF_j \) span \( gl_n(\mathbb{C}) \) in \( A_n(F) \). It is shown in [3] that if \( g \) is any Lie subalgebra of \( gl_n(\mathbb{C}) \) of dimension \( > n \), then \( \mathbb{C}[X] \) (and hence \( \mathbb{C}[X]/\mathbb{C}[F] \)) is a torsion module over the universal enveloping algebra \( U(g) \). The strategy proposed in [3] is to show that \( \mathbb{C}[X]/\mathbb{C}[F] \) is a torsionfree \( U(g) \)-module and hence is equal to zero. So, for example, to prove the Jacobian Conjecture for the case \( n = 2 \) it would be sufficient to prove that \( \mathbb{C}[X]/\mathbb{C}[F] \) is a torsionfree module over \( U(b) = \mathbb{C}[\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}] \). In fact, this case is studied extensively in [3] and several partial results are obtained; it is shown that \( \mathbb{C}[X]/\mathbb{C}[F] \) is torsionfree over \( \mathbb{C}[\varepsilon_{11} + 2\varepsilon_{22}, \varepsilon_{12}] \) over \( \mathbb{C}[\partial] \) for all \( \partial \in gl_2(\mathbb{C}) \) and over \( \mathbb{C}[\varepsilon_{11}, \varepsilon_{22}] \). In particular, the proof of the last result is spectacular and rather involved; apart from several algebraic tools it uses Siegel's theorem on algebraic curves with infinitely many integer points and Fabry’s theorem on Gap series.

In this paper we introduce a new class of polynomial maps, the so-called nice polynomial maps. Using the notion of Eulerian operators (as introduced in [2])
we show how for these polynomial maps the results obtained by Bass in [3]
are rather easy to prove. The restriction that $F$ is nice is not essential, since
we show in §3 that every polynomial map satisfying the Jacobian Conjecture
det $JF \in k^*$ is "equivalent" with a nice polynomial map; i.e., we show that
$F(\lambda)(X) := F(X + \lambda) - F(\lambda)$ is nice for almost all $\lambda \in k^n$.

Let us finally sketch how to prove that $M := \mathbb{C}[X]/\mathbb{C}[F]$ has no $\mathbb{C}[\epsilon_{11}, \epsilon_{22}]$-
torsion (the difficult case in [3]). So assume $F$ is nice. We show first that for
such an $F$ we have a canonical inclusion $O(F_2)/\mathbb{C}[F_1] \hookrightarrow \mathbb{C}[F_1]/\mathbb{C}[F_1]$, where
$O(F_2) = \mathbb{C}[X]/F_2 \mathbb{C}[X]$. Suppose that $M$ has $\mathbb{C}[\epsilon_{11}, \epsilon_{22}]$-torsion. Then it follows
easily that $M/F_2M$ has $\mathbb{C}[\epsilon_{11}]$-torsion. Since $M/F_2M \approx O(F_2)/\mathbb{C}[F_1]$, the
above inclusion implies that $\mathbb{C}[F_1]/\mathbb{C}[F_1]$ has $\mathbb{C}[\epsilon_{11}]$-torsion. This is a
contradiction since all nonzero operators of $\mathbb{C}[\epsilon_{11}]$ are Eulerian (an operator
$P \in A_1 = \mathbb{C}[F_1, d/dF_1]$ is called Eulerian if $\mathbb{C}[F_1]/\mathbb{C}[F_1]$ has no $P$-torsion).

1. Eulerian operators

In this section we collect some facts concerning (linear) Eulerian operators
(see also [2]). Let $k$ be a field of characteristic zero, $k[x] := k[x_1, \ldots, x_n]$
the polynomial ring in $n$ variables over $k$, $\mathcal{O} := k[[X]] = k[[X_1, \ldots, X_n]]$ the
ring of formal power series over $k$, and $A_n := k[X][\partial_1, \ldots, \partial_n]$ the $n$th Weyl
algebra over $k$, i.e., the ring of differential operators over the polynomial ring
$k[X]$. According to [2], an element $P \in A_n$ is called Eulerian if and only if for
every polynomial $p \in k[X]$ every formal power series solution $g \in \mathcal{O}$ of the
equation $Pg = p$ belongs to $k[X]$. In other words, the $A_n$-module $\mathcal{O}/k[X]$
has no $P$-torsion. From this definition we obtain immediately

(1.1) If $P_1$ and $P_2$ are Eulerian, so is $P_1P_2$.
(1.2) If $P_1P_2$ is Eulerian, so is $P_2$.

The Eulerian operators in case $n = 1$ are easy to describe.

**Proposition 1.3** [1, Remarque 2.7]. Let $Q \in A_1$. Then $Q$ is Eulerian if and
only if $Q = X^rP(X_1\partial_1)$ for some $r \in \mathbb{Z}$ and $0 \neq P(X_1\partial_1) \in k[X_1\partial_1]$.

An example of an Eulerian operator in $n$ variables is the Euler operator
$e := X_1\partial_1 + \cdots + X_n\partial_n$. More generally, every nonzero polynomial
$P(e) \in k[e]$ is also Eulerian. In fact, this is a special case of the following result: let $P \in
k[X_1, \ldots, X_n]$ be a nonzero polynomial. To it we associate the differential
operator $\tilde{P} := P(X_1\partial_1, \ldots, X_n\partial_n)$.

**Proposition 1.4** [2, Proposition 2.3]. $\tilde{P}$ is Eulerian if and only if the equation
$P(x) = 0$ has only a finite number of solutions in $\mathbb{N}^n$.

This result reveals a relationship between Eulerian operators and Diophantine
geometry. It gives a large class of nontrivial Eulerian operators, namely, all
operators $\tilde{P}$ associated to curves with a finite number of integer solutions. In
particular, the Fermat conjecture is equivalent with: $(X_1\partial_1 + 1)^n + (X_2\partial_2 + 1)^n
- (X_3\partial_3 + 1)^n$ is Eulerian for all $n \geq 3$. From these examples it is evident that
it is extremely difficult to describe Eulerian operators in the case $n \geq 2$. A
more modest approach, therefore, is to study first Eulerian operators of small
order (an operator $0 \neq P \in A_n$ is called of order $d \geq 0$ if $P = \sum_{|\alpha| \leq d} a_\alpha \partial^\alpha$
with $\sum_{|\alpha| = d} a_\alpha \partial^\alpha \neq 0$). As we will show below, pre-established understanding
of Eulerian operators of order zero is very useful.
Now we will describe all Eulerian operators of order zero. Let \( a \in k[X] \) be a nonconstant polynomial (a nonzero constant is obviously Eulerian). Write \( a = p_1^{e_1} \cdots p_r^{e_r} \), the prime factor decomposition of \( a \). So each \( p_i \) is irreducible in \( k[X] \) and \( e_i \geq 1 \) for every \( i \). From (1.1) and (1.2) it follows that \( a \) is Eulerian if and only if each \( p_i \) is Eulerian. So it remains to describe which irreducible polynomials in \( k[X] \) are Eulerian.

**Proposition 1.5.** Let \( p \in k[X] \) be irreducible. Then \( p \) is Eulerian if and only if \( p(0) = 0 \).

**Proof.** \( \rightarrow \) If \( p(0) \neq 0 \) then \( p \) is a unit in \( \hat{\mathcal{O}} \). So \( p^{-1} \) exists in \( \hat{\mathcal{O}} \). Obviously \( p^{-1} \notin k[X] \) (for otherwise \( p \in k^* \)). Then \( p(p^{-1} + k[X]) = 0 \) in \( \hat{\mathcal{O}}/k[X] \), a contradiction since \( p \) is Eulerian.

\( \leftarrow \) We need to show \( p\hat{\mathcal{O}} \cap k[X] = pk[X] \). Since \( k[X]_{(X)} \subset \hat{\mathcal{O}} \) is faithfully flat, we get \( pk[X]_{(X)} \hat{\mathcal{O}} \cap k[X]_{(X)} = pk[X]_{(X)} \), whence \( p\hat{\mathcal{O}} \cap k[X] = pk[X]_{(X)} \cap k[X] \). Since the prime ideal \( pk[X] \) does not meet the multiplicatively closed set \( k[X] \setminus \{0\} \), we have \( pk[X]_{(X)} \cap k[X] = pk[X] \), as desired.

2. Some useful inclusions

In this section we consider the following situation: \( k \) is a field of characteristic zero and \( F := (F_1, \ldots, F_n) : k^n \rightarrow k^n \) is a polynomial map, i.e., each \( F_i \) belongs to \( k[X] \). We assume that \( \det JF \in k^* \) and \( F(0) = 0 \). The local inverse function theorem implies that \( k[[X]] = k[[F]] := k[[F_1, \ldots, F_n]] \), the formal power series ring in \( F_1, \ldots, F_n \) over \( k \). Consequently,

\[
k[[X]]/F_nk[[X]] \cong k[[F_1, \ldots, F_{n-1}]].
\]

We identify these two rings and get a canonical map

\[
k[X] \rightarrow k[[X]]/F_nk[[X]] = k[[F_1, \ldots, F_{n-1}]];
\]

its kernel equals \( k[X] \cap F_nk[[X]] \). Now assume that \( F_n \) is irreducible in \( k[X] \). Then Proposition 1.5 implies that we get the inclusion

\[
\mathcal{O}(F_n) := k[X]/F_nk[X] \hookrightarrow k[[F_1, \ldots, F_{n-1}]].
\]

Furthermore, we have the canonical map \( k[F_1, \ldots, F_{n-1}] \rightarrow \mathcal{O}(F_n) \) with kernel \( k[F_1, \ldots, F_{n-1}] \cap F_nk[X] \). This kernel is zero since it is contained in \( k[F_1, \ldots, F_{n-1}] \cap F_nk[[F]] = (0) \). Hence, we get the inclusion

\[
k[F_1, \ldots, F_{n-1}] \hookrightarrow \mathcal{O}(F_n).
\]

Summarizing, we get

**Proposition 2.3.** Let \( F : k^n \rightarrow k^n \) be a polynomial map such that \( \det JF \in k^* \), \( F_n \) irreducible in \( k[X] \), and \( F(0) = 0 \). Then we have inclusions (2.1) and (2.2). In particular, \( \mathcal{O}(F_n)/k[F_1, \ldots, F_{n-1}] \hookrightarrow k[[F_1, \ldots, F_{n-1}]]/k[F_1, \ldots, F_{n-1}] \).

3. Nice polynomial maps

Let \( k \) be an algebraically closed field of characteristic zero and \( F : k^n \rightarrow k^n \) a polynomial map. Put \( M := k[X]/k[F] \). So \( M \) is a \( k[F] \)-module. Let \( 1 \leq p \leq n \). We call \( F \) \( p \)-nice if \( F_p \) is irreducible in \( k[X] \), \( \bigcap_{i \geq 1} F_p^i M = (0) \), and \( \det JF \in k^* \). If \( F \) is \( p \)-nice for all \( 1 \leq p \leq n \), we call \( F \) nice. For each \( \lambda \in k^n \) we define the polynomial map \( F_{(\lambda)} \) by \( F_{(\lambda)}(X) := F(X + \lambda) - F(\lambda) \). So \( F_{(\lambda)}(0) = 0 \).
Proposition 3.1. Let $\det JF \in k^*$. Then $F(\lambda)$ is nice for almost all $\lambda \in k^n$ (i.e., for all $\lambda$ in a Zariski open set of $k^n$).

Proof. By Lemma 3.2 and Corollary 3.4 it follows that $F - \mu$ is nice for almost all $\mu \in k^n$, i.e., for all $\mu$ outside some hypersurface $f^{-1}(0)$. So $F - F(\lambda)$ is nice for almost all $\lambda$ outside $(f \circ F)^{-1}(0)$. Since $\phi$ defined by $\phi(X) = X + \lambda$ is a polynomial automorphism of $k[X]$, the composition $(F - F(\lambda)) \circ \phi$ is also nice for almost all $\lambda \in k^n$, which completes the proof.

Lemma 3.2. Let $f \in k[X_1, \ldots, X_n]$ be such that $1 \in (\partial f/\partial X_1, \ldots, \partial f/\partial X_n)$. Then there exists a finite subset of $E$ of $k$ such that $f - \lambda$ is irreducible in $k[X]$ for all $\lambda \in k \setminus E$.

Proof. Consider $f(X) - Z \in k[Z, X]$. This polynomial is irreducible in $k[Z][X]$ and hence in $k(Z)[X]$ by Gauss's lemma. Furthermore $\deg_Z f(X) - Z = 1$, so by Bertini's theorem (see [5, §11, Theorem 18] and observe that in the proof given there the hypothesis “$F(x, \lambda^*)$ is irreducible in $k$ for every $\lambda^*$” can be replaced by: for infinitely many $\lambda^*$ in $k$, we obtain that if $f(X) - \lambda$ is reducible for infinitely many $X \in k$, then there exist polynomials $\phi, \chi \in k[X], m \in Z$, and $a_i(Z) \in k[Z]$ such that

\begin{align}
(3.1) & \quad f(X) - Z = a_0(Z)\phi^m + a_1(Z)\phi^{m-1}\chi + \cdots + a_m(Z)\chi^m \\
& \quad \text{and} \quad \deg_X f > \max(\deg_X \phi, \deg_X \chi).
\end{align}

Consequently, $m \geq 2$, for if $m = 1$ then $f(X) - Z = a_0(Z)\phi + a_1(Z)\chi$ and then $\deg_X f \leq \max(\deg_X \phi, \deg_X \chi)$, which contradicts (2). Comparing the coefficients of $Z^0$ and $Z^1$ in (1), we get

\begin{align}
(3.3) & \quad f(X) = \sum_{i=0}^m a_i(0)\phi^{m-i}\chi^i \\
& \quad \text{and} \quad -1 = \sum_{i=0}^m a_i(0)\phi^{m-i}\chi^i = H(\phi, \chi)
\end{align}

where $H(U_1, U_2) = \sum_{i=0}^m a_i(0)U_1^{m-i}U_2^i$ is a homogeneous polynomial in $U_1, U_2$. Write $H = \prod_{i=1}^m (\alpha_i U_1 + \beta_i U_2)$ with $\alpha_i, \beta_i \in \overline{k}$ (an algebraic closure of $k$). Then $H(\phi, \chi) = -1$ implies that $\alpha_i \phi + \beta_i \chi \in \overline{k}^*$ for all $i$. So if, for example $\alpha_1 \neq 0$ then $\phi = \beta_1 \chi + \lambda$ for some $\beta, \lambda \in \overline{k}$. Then by (3) $f(X) = \sum_{i=0}^m a_i(0)(\beta_1 \chi + \lambda)^{m-i}\chi^i = g(\chi)$ for some $g(T) \in \overline{k}[T]$. Now observe that $\deg_T g(T) \geq 2$, for if $\deg_T g(T) \leq 1$, say $g(T) = \mu_1 T + \mu_0$, then $f(X) = \mu_1 \chi + \mu_0$, so $\deg_X f(X) \leq \deg_X \chi$, which contradicts (2). So $f(X) = g(\chi)$ with $g(T) \in \overline{k}[T]$, $\deg g(T) \geq 2$. But then $\partial f/\partial X_i = g'(\chi)\partial \chi/\partial X_i$. Since $\deg g'(T) \geq 1$, $g'(z)(z)$ for some $z \in \overline{k}$. Then take $x \in \overline{k}^n$ with $\chi(x) = z$. We get $g'(\chi(x)) = 0$, hence $\partial f(x)/\partial X_i = 0$ for all $i$, which is a contradiction with $1 \in (\partial f/\partial X_1, \ldots, \partial f/\partial X_n)$. So the hypothesis, $f(X) - \lambda$ is reducible for infinitely many $\lambda \in k$, leads to a contradiction, which proves the lemma.

Lemma 3.3. Let $\det JF \in k^*$. There exists $0 \neq f \in k[F]$ such that the following holds: if $I$ is an ideal in $k[F]$ with $\bigcap_{p \geq 1} I^p M \neq (0)$, then $f \in r(I)$.
Proof. Let $d = |k(X) : k(F)|$. By the primitive element theorem there exists an element $g$ in $k(X)$, which we can assume to be integral over $k[F]$, such that $k(X) = k(F)[g]$. So there exists $0 \neq f \in k[F]$ such that $f \cdot X_i \in k[F][g]$ for all $i$. Since $g^d \in \sum_{i=0}^{d-1} k[F]g^i$ ($g$ being integral over $k[F]$), it follows that $k[X] \subset \bigoplus_{i=0}^{d-1} k[F][g]$, and since $M$ has no $k[F]$ torsion (because $k[X] \cap k(F) = k[F]$ [3, Corollary 1.3]), we conclude that $M \subset M_f \subset \bigoplus_{i=0}^{d-1} k[F][g]$. Now assume that $\bigcap_{p \geq 1} I_p M \neq (0)$. It follows that $\bigcap_{p \geq 1} I_p \bigoplus_{i=0}^{d-1} k[F][g] \neq (0)$ and hence that $\bigcap_{p \geq 1} I_p [F] \neq (0)$. By Krull's intersection theorem we conclude that $I k[F] = k[F]/f$, which implies that $f \in r(I)$.

Corollary 3.4. Let $detJF \in k^*$. Then $I(\lambda) := \bigcap_{n=1}^{\infty} (F_n - \lambda)^i M = (0)$ for all $\lambda \in k$ outside some finite subset of $k$.

Proof. Let $f$ be as in Lemma 3.3. Let $\lambda \in k$ be such that $I(\lambda) \neq (0)$ and $F_n - \lambda$ irreducible. Then Lemma 3.3 implies that $F_n - \lambda$ divides $f$. Since $f$ has only a finite number of irreducible factors, and since $F_n - \lambda$ is irreducible for all $\lambda \in k$ outside a finite subset of $k$ [by Lemma 3.2, since $1 \in (\partial F_1/\partial X_1, \ldots, \partial F_n/\partial X_n)$ because $detJF \in k^*$], the corollary follows.

4. Applications to the case $n = 2$

Let again $F: k^n \to k^n$ be a polynomial map with $detJF \in k^*$ and $k$ an algebraically closed field of characteristic zero. First recall (see [3] or [4]) that the Jacobian condition $detJF \in k^*$ implies that the derivations $d/dF_i$ on $k[F]$ can be extended uniquely to pairwise commuting derivations on $k[X]$. In other words, $k[X]$ becomes a left $(A_n(F) = k[F, d/dF_1, \ldots, d/dF_n])$-module. The derivations $e_{ij} := F_i d/dF_j$ span $gl_n(k) \subset A_n(F)$. The strategy proposed in [3] is to attack the Jacobian Conjecture by showing that $M := k[X]/k[F]$ is a torsionfree module over the enveloping algebra $U(g)$ for some Lie subalgebra $g$ of $gl_n(k)$ of dimension $> n$.

Observe that if $\lambda \in k^n$, then $F$ is invertible if and only if the polynomial morphism $F_\lambda := F(x + \lambda) - F(x)$ is invertible. So in order to prove the Jacobian Conjecture we may replace $F$ by $F_\lambda$, and hence by Proposition 3.1 we assume from now on: $F$ is nice and $F(0) = 0$. Now we will show how under these conditions the result obtained by Bass in §6 of [3] easily follows from the inclusion in Proposition 2.3 and the fact that each operator of $k[F_i d/dF_i] \setminus \{0\}$ is Eulerian.

So let $n = 2$. To simplify the notations we put $x = F_1$, $y = F_2$, $X = X_1$, $Y = X_2$, $e_x = x \partial_x$, $e_y = y \partial_y$. First observe that from Proposition 2.3 and the assumption that $F$ is nice and $F(0) = 0$, it follows that $\bigcap \{x^p M \} = (0)$, $M/yM \subset k[[x]]/k[x]$, and $M/xM \subset k[[y]]/k[y]$.

Now we are able to prove the main result of this paper.

Theorem 4.1. Let $P = \sum_{i \geq 0} (y \partial_x)^i P_i (e_x, e_y) \in A_2$ with $P_0(e_x, r) \neq 0$ for all $r \in \mathbb{N}$. If $D \in A_2$ is such that $M$ has no $D$-torsion, then $M$ has no $DP$-torsion.

Proof. It suffices to show that $M$ has no $P$-torsion. So let $0 \neq m \in M$ with $Pm = 0$. Since $\bigcap y^p M \neq (0)$, there exists $r \in \mathbb{N}$ with $m = y^r M$ with $m \notin yM$. Observe that $Py^r = y^r \tilde{P}$ in $A_2$, where $\tilde{P} = \sum (y \partial_x)^i P_i (e_x, e_y + r)$. Since
$M$ has no $y$-torsion, it follows that $\tilde{P}m = 0$. Observe that $\tilde{P} = P_0(\epsilon_x, r) + yQ$ for some $Q \in A_2$. So $P_0(\epsilon_x, r)m = 0$ in $M/yM$, where $m = \hat{m} + yM$, which is nonzero since $\hat{m} \notin yM$. Since $M/yM \subset k[[x]]/k[x]$ and $P_0(\epsilon_x, r)$ is Eulerian by Proposition 1.3, it follows that $m \equiv 0$, which is a contradiction.

**Corollary 4.2.** $M$ has no torsion over $k[\epsilon_x, \epsilon_y]$, $k[a\epsilon_x + b\epsilon_y, y\partial_x]$ $(a, b \in k, a \neq 0)$, and $k[\partial]$ where $\partial \in gl_2(k)$.

**Proof.** (i) From Theorem 4.1 it follows in particular that $M$ has no $\partial_x$ torsion and no $\partial_x - \lambda$ torsion for all $\lambda \in k$. Interchanging the roles of $x$ and $y$ the same holds for $\partial_y$ and $\epsilon_y - \lambda$.

(ii) Let $0 \neq P(\epsilon_x, \epsilon_y) \in k[\epsilon_x, \epsilon_y]$ and assume $Pm = 0$. Dividing the polynomial $P(X, Y)$ by possible factors of the form $(Y - r)^e$, $e \geq 1$, $r \in \mathbb{N}$, we can write $P(X, Y) = \prod (Y - r_i)^{e_i} \tilde{P}(X, Y)$ with $\tilde{P}(X, r) \neq 0$ for all $r \in \mathbb{N}$. Then by (i) $Pm = 0$ implies $\tilde{P}(\epsilon_x, \epsilon_y)m = 0$. Then apply Theorem 4.1, which gives a contradiction, proving the first case.

(iii) Since $M$ has no $y\partial_x$-torsion (by (i)), the second case follows readily from Theorem 4.1, observing that $P_0(a\epsilon_x + bY) \neq 0$ for all $r \in \mathbb{N}$ if $P_0(aX + bY) \neq 0$.

(iv) After a linear change of coordinates in $k[x, y]$ we can assume that $\partial$ is in Jordan canonical form. Then either $\partial = a(\epsilon_x + \epsilon_y) + y\partial_x$ or $\partial = a\epsilon_x + b\epsilon_y$, $a, b \in k$. Then use the second case proved before. □

**References**


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