
LIGO SCIENTIFIC COLLABORATION AND VIRGO COLLABORATION

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ABSTRACT

This article provides supplemental information for a Letter reporting the rate of (BBH) coalescences inferred from 16 days of coincident Advanced LIGO observations surrounding the transient (GW) signal GW150914. In that work we reported various rate estimates whose 90% confidence intervals fell in the range 2–600 Gpc⁻³ yr⁻¹. Here we give details on our method and computations, including information about our search pipelines, a derivation of our likelihood function for the analysis, a description of the astrophysical search trigger distribution expected from merging BBHs, details on our computational methods, a description of the effects and our model for calibration uncertainty, and an analytic method for estimating our detector sensitivity, which is calibrated to our measurements.

Key words: gravitational waves – stars: black holes

Supporting material: data behind figures

The first detection of a gravitational-wave (GW) signal from a merging binary black hole (BBH) system was described in Abbott et al. (2016d). Abbott et al. (2016g) reported on inference of the local BBH merger rate from surrounding Advanced LIGO observations. This Supplement provides supporting material and methodological details for Abbott et al. (2016g, hereafter referred to as the Letter).

1. SEARCH PIPELINES

Both the pycbc and gstlal pipelines are based on matched filtering against a bank of template waveforms. See Abbott et al. (2016c) for a detailed description of the pipelines in operation around the time of GW150914; here we provide an abbreviated description.

In the pycbc pipeline, the single-detector signal-to-noise ratio (S/N) is re-weighted by a chi-squared factor (Allen 2005) to account for template-data mismatch (Babak et al. 2013); the re-weighted single-detector S/Ns are combined in quadrature to produce a detection statistic for search triggers.

The gstlal pipeline’s detection statistic, however, is based on a likelihood ratio (Cannon et al. 2013, 2015) constructed from the single-detector S/Ns and a signal-consistency statistic. An analytic estimate of the distribution of astrophysical signals in multiple-detector S/N and signal-consistency statistic space is compared to a measured distribution of single-detector triggers without a coincident counterpart in the other detector to form a multiple-detector likelihood ratio.

Both pipelines rely on an empirical estimate of the search background, making the assumption that triggers of terrestrial origin occur independently in the two detectors. The background estimate is built from observations of single-detector triggers over a long time (gstlal) or through searching over a data stream with one detector’s output shifted in time relative to the other’s by an interval that is longer than the light travel time between detectors, ensuring that no coincident astrophysical signals remain in the data (pycbc). For both pipelines it is not possible to produce an instantaneous background estimate at a particular time; this drives our choice of likelihood function, as described in Section 2.

The gstlal pipeline natively determines the functions $p_0(x)$ and $p_1(x)$ for its detection statistic $x$. For this analysis a threshold of $x_{\text{min}} = 5$ was applied, which is sufficiently low that the trigger density is dominated by terrestrial triggers near a threshold. There were $M = 15848$ triggers observed above this threshold in the 17 days of observation time analyzed by gstlal.

For pycbc, the quantity $x'$ is the re-weighted S/N detection statistic. We set a threshold $x'_{\text{min}} = 8$, above which $M' = 270$ triggers remain in the search. We use a histogram of triggers collected from time-shifted data to estimate the terrestrial trigger density, $p_0(x')$, and a histogram of the recovered triggers from the injection sets described in Section 2.2 of the Letter to estimate the astrophysical trigger density, $p_1(x')$. These estimates are shown in Figure 1. The uncertainty in the distribution of triggers from this estimation procedure is much smaller than the uncertainty in the overall rate from the finite number statistics (see, for example, Figure 5). The empirical estimate is necessary to properly account for the interaction of the various single- and double-interferometer thresholds in the pycbc search (Abbott et al. 2016c). At high S/N, where these thresholds are irrelevant, the astrophysical triggers follow an approximately flat-space volumetric density (see Section 3) of

$$p_1(x') \approx \frac{3}{x'^{\frac{3}{2}}} x^{\frac{1}{2}},$$

but they deviate from this at smaller S/N due to threshold effects in the search.

For the pycbc pipeline, a detection statistic $x' \geq 10.1$ corresponds to an estimated search false alarm rate (FAR) of one per century.

2. DERIVATION OF POISSON MIXTURE MODEL LIKELIHOOD

In this section we derive the likelihood function in Equation (3) of the Letter. Consider a search of the type described in

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When quoting pipeline-specific values we distinguish pycbc quantities with a prime.
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Poisson uncertainty in the terrestrial count is uncorrelated in the posterior. There is no meaningful correlation between the two variables. The force the terrestrial count to adjust in a meaningful way and the variables are expected counts \( \Lambda_0 \) and \( \Lambda_1 \) in Equation (5) of the Letter on \( \Lambda_0 \) and \( \Lambda_1 \).

We have checked that using a \( \delta \)-function prior

\[
p(\Lambda_0) = \delta(\Lambda_0 - 270)
\]

in the \texttt{pycbc} analysis that is the result of the pipeline \( \Lambda_0 \) estimate from timeslides\(^{140} \) (Abbott et al. 2016c) and using a looser prior that is the result of a \texttt{gstlal} estimate on a single set of time-sliced data produce no meaningful change in our results. Figure 3 shows our canonical rate posterior inferred with the \texttt{pycbc} \( \Lambda_0 \) prior in Equation (11) and our canonical broad prior.

3. UNIVERSAL ASTROPHYSICAL TRIGGER DISTRIBUTION

Both the \texttt{pycbc} and \texttt{gstlal} pipelines rely on the S/N as part of their detection statistic, \( x \). The S/N of an astrophysical trigger is a function of the detector noise at the time of detection and the parameters of the trigger. Schutz (2011) and Chen & Holz (2014) demonstrate that the distribution of the expected S/N \( \langle \rho \rangle \) in a simple model of a detection pipeline that simply thresholds on S/N, \( \rho \geq \rho_B \), with sources in the local universe is universal, that is, independent of the source properties. It follows

\[
p(\langle \rho \rangle) = \frac{3 \rho_B^4}{\langle \rho \rangle^4}
\]

This result follows from the fact that the expected value of the S/N in a matched-filter search for compact binary coalescence (CBC) signals scales inversely with transverse comoving distance (Hogg 1999):

\[
\langle \rho \rangle = \frac{A(m_1, m_2, a_1, a_2, S(f), z)B(\text{angles})}{D_M},
\]

where \( A \) is an amplitude factor that depends on the intrinsic properties (source-frame masses and spins) of the source, the detector sensitivity expressed as a noise power spectral density \( S(f) \) as a function of observer frequency and redshift \( z \), and \( B \) is an angular factor depending on the location of the source in the sky and the relative orientations of the binary orbit and detector. The redshift enters \( A \) only through shifting the source waveform to lower frequency at higher redshift, changing \( A \) because the sensitivity varies with observer frequency \( f \). For the redshifts to which we are sensitive to BBH in this observation period this effect on \( A \) is small.

If we assume that the distribution of source parameters is constant over the range of distances to which we are sensitive, and ignore the small redshift-dependent sensitivity correction mentioned above, then the distribution of S/N will be governed entirely by the distribution of distances of the sources, which, in the local universe, is approximately

\[
p(D_M) \propto D_M^0,
\]

yielding the distribution of S/N given in Equation (12).

\(^{140} \)While the statistical uncertainty on the pipeline \( \Lambda_0 \) estimate is not precisely zero, \( \sigma_{\Lambda_0}/\Lambda_0 \lesssim 10^{-3} \), it is so small that a \( \delta \)-function prior is appropriate.
recovered injections in each campaign.

For estimation of the Poisson rate parameter in a simple Poisson model, the Jeffreys prior is proportional to \( S / N \) in this pipeline, and the distribution is not sensitive to the complicated thresholding in the pipeline, so we recover Equation (12); for quiet signals the interaction of various single-detector thresholds in the pipeline causes the distribution to deviate from this analytic approximation, but it remains independent of the distribution of sources. Note that the empirical distribution of detection statistics, not the analytic one, forms the basis for \( p_t \), the foreground distribution used in this rate estimation work.

To quantify the deviations from universality, we have preformed two-sample Kolmogorov-Smirnov (KS) tests between all six pairings of the sets of detections statistics recovered in the injection campaigns described in Sections 2 and 3 of the Letter and featured in Figure 4. The most extreme KS \( p \)-value occurred with the comparison between the injection set with BBH masses drawn flat in \( \log m \) and the one with masses drawn from a power law (both described in Section 3 of the Letter); this test gave a \( p \)-value of 0.013. Given that we have performed six identical comparisons we cannot reject the null hypothesis that the empirical distributions used for rate estimation from the \( \text{pycbc} \) pipeline are identical even at the relatively weak significance \( \alpha = 0.05 \). Certainly any differences in detection statistic distribution attributable to the BBH population are far too small to matter with the few astrophysical signals in our data set (compared with \( O(1000) \) recovered injections in each campaign).

Because the distribution of detection statistics is, to a very good approximation, universal, we cannot learn anything about the source population from the detection statistic alone; we must instead resort to parameter estimation (PE) follow-up (Veitch et al. 2015; Abbott et al. 2016e) of triggers to determine their parameters. The parameters of the waveform template that produced the trigger can be used to guess the parameters of the source that generated that trigger, but the bias and uncertainty in this estimate are very large compared to the PE estimate. We therefore ignore the parameters of the waveform template that generated the trigger in the assignment of triggers to BBH classes.

### 4. COUNT POSTERIOR

We impose a prior on the \( \Lambda \) parameters of:

\[
p(\Lambda_1, \Lambda_0) \propto \frac{1}{\sqrt{\Lambda_1}} \frac{1}{\sqrt{\Lambda_0}}.
\]

The posterior on expected counts is proportional to the product of the likelihood from Equation (3) of the Letter and the prior from Equation (15):

\[
p(\Lambda_1, \Lambda_0 | \{x_j\}_j = 1, \ldots, M) \propto \prod_{j=1}^M [\Lambda_1 p_t(x_j) + \Lambda_0 p_0(x_j)] \times \exp \left[ -\Lambda_1 - \Lambda_0 \right] \frac{1}{\sqrt{\Lambda_1 \Lambda_0}}.
\]

For estimation of the Poisson rate parameter in a simple Poisson model, the Jeffreys prior is \( 1/\sqrt{\Lambda} \). With this prior, the
The estimated astrophysical and terrestrial trigger rate densities (Equation (1) of the Letter) for pycbc are plotted in Figure 5. We select triggers from a subset of the search parameter space (i.e., our bank of template waveforms) that contains GW150914 as well as the mass range considered for possible alternative populations of BBH binaries in Section 3 of the Letter. There are $M' = 270$ two-detector coincident triggers in this range in the pycbc search (Abbott et al. 2016c). Figure 5 also shows an estimate of the density of triggers that comprise our data set, which agrees well with our inference of the trigger rate.

Based on the probability of astrophysical origin inferred for LVT151012 from the two-component mixture model in Equation (16) and shown in Figure 6, we introduce a third class of signals and use a three-component mixture model with expected counts $\Lambda_0$ (terrestrial), $\Lambda_1$ (GW150914-like), and $\Lambda_2$ (LVT151012-like) to infer rates in Sections 2.1 and 2.2 of the Letter.

We use the Stan and emcee Markov Chain Monte Carlo samplers (Foreman-Mackey et al. 2013; Stan Development Team 2015a, 2015b) to draw samples from the posterior in Equation (5) of the Letter for the two pipelines. We have assessed the convergence and mixing of our chains using empirical estimates of the autocorrelation length in each parameter (Sokal 1996), the Gelman-Rubin $R$ convergence statistic (Gelman & Rubin 1992), and through visual inspection of chain plots. By all measures, the chains appear to be well-converged to the posterior distribution.

Table 1 contains the full results on expected counts and associated sensitive time-volumes for both pipelines.

5. CALIBRATION UNCERTAINTY

The LIGO detectors are subject to uncertainty in their calibration, in both the measured amplitude and the phase of the GW strain. Abbott et al. (2016b) discussed the methods used to calibrate the strain output of the detector during the 16 days of coincident observations discussed here. Abbott et al. (2016b) estimated that the reported strain is accurate to within 10% in amplitude and 10 degrees in phase between 20 Hz and 1 kHz throughout the observations.

The S/Ns reported by our searches are quadratically sensitive to calibration errors because they are maximized over arrival time, waveform phase, and a template bank of waveforms (Allen 1996; Brown & LIGO Scientific Collaboration 2004). Abbott et al. (2016c) demonstrated that the other search pipeline outputs are also not affected to a significant degree by the calibration uncertainty present during our observing run. Therefore, we ignore the effects of calibration on the pipeline detection statistics $x$ and $x'$ that we use here to estimate rates from the pycbc and gstlal pipelines.

The amplitude calibration uncertainty in the detector results, at leading order, in a corresponding uncertainty between the luminosity distances of sources measured from real detector outputs (Abbott et al. 2016e) and the luminosity distances used to produce injected waveforms used to estimate sensitive time-volumes in this work. A 10% uncertainty in $d_L$ at these redshifts corresponds to an approximately 30% uncertainty in volume. We model this uncertainty by treating $\langle VT \rangle$ as a
parameter in our analysis, and imposing a log-normal prior:

\[ p(\log(\langle VT\rangle)) \propto N(\log \mu, \frac{\sigma^2}{\mu}) \]  

(17)

where \( \mu \) is the Monte Carlo estimate of sensitive time-volume produced from the injection campaigns described in Section 2.2 of the Letter and

\[ \sigma^2 = \sigma^2_{\text{cal}} + \sigma^2_{\text{stat}} \]  

(18)

with \( \sigma_{\text{cal}} = 0.3\mu \) and \( \sigma_{\text{stat}} \) is the estimate of the Monte Carlo uncertainty from the finite number of recovered injections reported above. In all cases \( \sigma_{\text{cal}} \gg \sigma_{\text{stat}} \).

Since the likelihood in Equations (3) or (10) of the Letter does not constrain \( \langle VT \rangle \) independently of \( R \), sampling over \( \langle VT \rangle \) at the same time as \( \Lambda \) and \( R \) has the effect of convolving the log-normal distribution of \( \langle VT \rangle \) with the posterior on \( \Lambda \) in the inference of \( R \).

In spite of the 30% relative uncertainty in \( \langle VT \rangle \) from calibration uncertainty, the counting uncertainty on \( R \) from the small number of detected events dominates the width of the posterior on \( R \).
6. ANALYTIC SENSITIVITY ESTIMATE

As a rough check on our $\langle VT \rangle$ estimates and the integrand $d\langle VT \rangle/dz$, we find that the following approximate, analytic procedure also produces a good approximation to the pycbc Monte Carlo estimate in Table 1.

1. Generate inspiral–merger–ringdown waveforms in a single detector at various redshifts from the source distribution $s(\theta)$ with random orientations and sky positions.

2. Using the high-sensitivity early Advanced LIGO noise power spectral density from Abbott et al. (2016f), compute the S/N in a single detector.

3. Consider a signal found if the S/N is greater than 8.

Employed with the source distributions described above, this approximate procedure yields $\langle VT \rangle_1 \approx 0.107$ Gpc$^3$ yr and $\langle VT \rangle_2 \approx 0.0225$ Gpc$^3$ yr for the sensitivity to the two classes of merging BBH system. Figure 7 shows the sensitive time-volume integrand,

$$d\langle VT \rangle/dz \equiv T \frac{1}{1 + z} \frac{dV_T}{dz} \int d\theta \, s(\theta) f(z, \theta), \quad (19)$$

estimated from this procedure for systems with various parameters superimposed on the Monte Carlo estimates from the injection campaign described above.

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