DERIVED LOGARITHMIC GEOMETRY I

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Abstract. In order to develop the foundations of derived logarithmic geometry, we introduce a model category of logarithmic simplicial rings and a notion of derived log-étale maps and use this to define derived log stacks.

1. Introduction

Before discussing the contents of the present paper, we first want to give some motivation why a solid theory of derived logarithmic geometry is a desirable thing to have.

An important application of logarithmic geometry has been to control degenerations. A typical example is given by a dominant morphism $f\colon X \to C$ from a smooth scheme X to a pointed curve (C, p), where we assume that the restriction $X \setminus X_p \to C \setminus p$ is smooth and the fibre X_p is a normal crossing divisor. If we denote by $j\colon X \setminus X_p \to X$ and $i\colon X_p \to X$ the inclusions, then $i^*(j_*\mathcal{O}^\times_{X\setminus X_p}) \to \mathcal{O}_{X_p}$ defines a log structure on X_p. In the opposite vein, given a normal crossing variety Y, the existence of certain logarithmic structures on Y helps in determining if Y can be obtained as the fibre of morphism $X \to C$ as above (see [ACG$^+$13, Section 5] and the references therein for this point of view).

A further striking example where logarithmic geometry helps to control degenerations is given by the Deligne–Mumford compactification of the moduli space of curves. This compactification can also be obtained by studying the moduli problem of stable log-smooth curves satisfying a certain basicness condition. Since logarithmic geometry incorporates degenerations, the moduli space of log-smooth curves is immediately compact. An overview over these topics can be found in [ACG$^+$13].

On the other hand, derived algebraic geometry has been successfully applied to study hidden smoothness in moduli spaces. A typical example is given by the moduli space of morphisms between a smooth curve and a smooth projective variety. Even for smooth domain and target, this moduli space can be horribly singular and much larger than the expected dimension. Studying the same moduli problem in derived algebraic geometry leads to an interesting "nilpotent" structure on the moduli space [TV08, Corollary 2.2.6.14]. This structure provides the algebraic-geometric counterpart to deforming to transversal intersection. Equipped with this nilpotent structure, the moduli space becomes quasi-smooth, the immediate generalization to derived algebraic geometry of local complete intersection. The quasi-smooth structure induces a 1–perfect obstruction theory and a virtual fundamental class in the expected dimension on the underlying moduli space, which is the key to many enumerative invariants.

Logarithmic and derived algebraic geometry naturally meet in the study of degenerations of moduli spaces. Suppose we are given a morphism $f\colon X \to S$ as above. We would then like to understand how some moduli space attached to a

Date: April 14, 2016.

1991 *Mathematics Subject Classification.* 14A20, 14D23, 13D03.

Key words and phrases. Log structures, derived algebraic geometry, cotangent complex, simplicial commutative rings.

smooth fiber interacts with the corresponding moduli space of the fiber X_s. If the moduli spaces are quasi-smooth, one would ideally want to compute enumerative invariants of the smooth fiber in terms of enumerative invariants of the components of X_s.

In case X_s only consists of two components, this has been indeed carried out by Jun Li in [Li01, Li02]. Instead of using log geometry, Li constructs an explicit degeneration of the moduli space of stable maps. The most difficult part in Li's theory is to find a perfect obstruction theory on the moduli space attached to the fiber X_s. Using this degeneration, he is able to prove a formula for enumerative invariants that since has found many applications.

Gross and Siebert [GS13] have recently observed that one can circumvent these difficulties by working in the category of logarithmic schemes. The moduli space attached to the fiber X_s should just be the corresponding moduli functor taken in the category of logarithmic schemes, where X_s is equipped with its natural logarithmic structure. If on top of this we want the moduli space attached to X_s to carry a 1-perfect obstruction theory, one is naturally led to consider derived logarithmic geometry. The correct functor that combines both the degeneration aspects as well as hidden smoothness is a moduli functor living in the category of derived logarithmic schemes or stacks.

Besides applications to degenerations of quasi-smooth moduli spaces, there are also other areas where such a theory might be useful. Much of the work on logarithmic geometry has been concerned with p-adic and arithmetic aspects. Recently, Beilinson [Bei12] has used derived logarithmic geometry to prove a p-adic Poincaré lemma. Much more material on this can be found in [Bha12]. It may also be interesting to extend the framework of derived log geometry developed here to the homotopy theoretic notion of *logarithmic ring spectra* developed by Rognes in [Rog09] in order to study moduli problems for structured ring spectra.

We hope that now the reader is convinced that it would be desirable to have a solid theory of derived logarithmic geometry. The aim of this work is to begin providing such foundations. The essential starting point for derived algebraic geometry is that the category of simplicial rings forms a well-behaved model category. In Sections 1 to 3 we provide a model category $s\mathcal{L}$ of *logarithmic simplicial rings*. Its objects are simplicial objects in the category of pre-log rings, and the fibrant objects in this model structure satisfy a log condition analogous to that of a log ring. Besides that, we give a model category description of the group completion of simplicial commutative monoids and outline how this leads to a notion of *repletion* for augmented simplicial commutative monoids. Although the repletion is not necessary for setting up the model category $s\mathcal{L}$, it might become relevant for a further development of the theory. All the model structures developed in this part have counterparts in the context of structured ring spectra that complement Rognes' work on *topological logarithmic structures* [Rog09].

In Sections 4 to 6 we develop the theory of étale and smooth morphisms between logarithmic simplicial rings. The key ingredient in defining these notions is the logarithmic cotangent complex. We define the logarithmic cotangent complex as the complex that represents the derived functor of logarithmic derivations. Since for a logarithmic ring (A, M) the category of A-modules is equivalent to the category of abelian objects in the category of strict logarithmic rings over (A, M), this exhibits the logarithmic cotangent complex as the left derived functor of abelianization, which is very close to Quillen's original definition for ordinary rings. This coincides with Gabber's definition in [Ols05, §8], and we prove that it also corresponds to Rognes' definition for structured ring spectra in [Rog09]. We also compare our notions of log-smooth and log-étale maps to the definitions given by Kato in terms of lifting properties with respect to strict square zero extensions.

In Section 7 we glue logarithmic simplicial rings to form derived logarithmic schemes and derived logarithmic n–stacks. We conclude this section with some speculations about the correct notion of log-modules. In Section 8 we explain how to set up a derived version of the logarithmic moduli of stable maps introduced by Gross and Siebert.

Acknowledgments. Bhargav Bhatt has also recently started laying the foundations of derived log geometry in [Bha12, §4]. The third author wishes to thank him for interesting exchanges on the subject. Thanks are also due to Vittoria Bussi for her interest and useful discussions. The authors thank the referee for a quick and careful reading and a large number of suggestions improving the text.

Notations. If k is a base commutative ring, then the category of pre-log k-algebras will consist of triples $(A, M, \alpha\colon M \to (A, \cdot))$ where A is a commutative k-algebra, M is a commutative monoid and α is a morphism of commutative monoids, and the morphisms $(A, M, \alpha\colon M \to (A, \cdot)) \to (B, N, \alpha\colon N \to (B, \cdot))$ will be pairs $(f\colon A \to B, f^\flat : M \to N)$, where f is a map of k-algebras and $f^\flat$ a map of commutative monoids, commuting with the structure maps. When the base ring is $k = \mathbb{Z}$ we will simply speak about pre-log rings.

2. Simplicial commutative monoids

In the following, we let $\mathcal{M}$ be the category of commutative monoids, $\mathcal{AB}$ be the category of abelian groups, and $\mathcal{R}$ be the category of commutative rings. Moreover, $\mathcal{S}$ denotes the category of simplicial sets, and $s\mathcal{M}$, $s\mathcal{AB}$, and $s\mathcal{R}$ denote the categories of simplicial objects in commutative monoids, abelian groups, and commutative rings.

The categories $s\mathcal{M}$, $s\mathcal{AB}$, and $s\mathcal{R}$ are simplicial categories (as for example defined in [GJ99, II.Definition 2.1]). This means that they are enriched, tensored, and cotensored over the category of simplicial sets. In each case, the tensor $X \otimes K$ of an object is the realization of the bisimplicial object $[n] \mapsto \coprod_{K_n} X$ where $\coprod$ is the coproduct in the respective category. The simplicial mapping spaces are given by $\mathbf{Hom}(X, Y)_n = \mathrm{Hom}(X \otimes \Delta^n, Y)$, and the cotensor is defined on the underlying simplicial sets. There exist well-known model structures on these categories:

Proposition 2.1. *The categories of simplicial commutative rings $s\mathcal{R}$, simplicial abelian groups $s\mathcal{AB}$, and simplicial commutative monoids $s\mathcal{M}$ admit proper simplicial cellular model structures. In all three cases, a map is a fibration (resp. weak equivalence) if and only if the underlying map of simplicial sets is a fibration (resp. weak equivalence).*

We refer to these model structures as the *standard* model structures on these categories.

Proof. The existence of these model structures is provided by [Qui67, II.4 Theorem 4] or [GJ99, II.Corollary 5.6]. Right properness is inherited from simplicial sets. Since the cartesian product is the coproduct of commutative monoids, left properness of $s\mathcal{M}$ is a consequence of [Rez02, Theorem 9.1]. Left properness of $s\mathcal{AB}$ may for example be established using the Dold-Kan correspondence. For $s\mathcal{R}$, left properness is verified in [Sch97, Lemma 3.1.2].

Applying the respective free functors from simplicial sets to the usual generating cofibrations and generating acyclic cofibrations for $\mathcal{S}$ shows that all three categories are cofibrantly generated. The argument given in [SS13, Appendix A] can be adopted to show that $s\mathcal{M}$ and $s\mathcal{R}$ are cellular. Cellularity of $s\mathcal{AB}$ can be checked from the definition. $\square$

2.1. **Group completion.** For the rest of this section we focus on the category of simplicial commutative monoids. This category is pointed by the constant simplicial object on the one point monoid. Hence $s\mathcal{M}$ is a pointed simplicial model category, i.e., it is tensored, cotensored and enriched over the category of pointed simplicial sets. The tensor with the pointed simplicial set $S^1 = \Delta^1/\partial\Delta^1$ is isomorphic to the bar construction on a simplicial commutative monoid. It follows that the functors $B(M) = M \otimes S^1$ and $\Omega(M) = M^{S^1}$ form a Quillen adjunction $B\colon s\mathcal{M} \rightleftarrows s\mathcal{M}\colon \Omega$ with respect to the standard model structure.

Definition 2.2. A simplicial commutative monoid M is *grouplike* if the commutative monoid $\pi_0(M)$ is a group.

Forming the adjoint of the fibrant replacement $BM \to (BM)^{\mathrm{fib}}$ of BM in $s\mathcal{M}$ provides a natural transformation

$$\eta_M\colon M \to \Omega((BM)^{\mathrm{fib}}). \tag{2.1}$$

It is immediate that $\Omega((BM)^{\mathrm{fib}})$ is always grouplike. The map η_M is known as the group completion of M. Below we will compare it with two other ways of forming a group completion.

Lemma 2.3. *If M is grouplike, then $M \to \Omega((BM)^{\mathrm{fib}})$ is a weak equivalence.*

Proof. We may assume that M is fibrant. Writing $E_\bullet M = B_\bullet(*, M, M)$ for the bisimplicial set whose realization is the simplicial set EM, an application of the Bousfield–Friedlander theorem [BF78, Theorem B.4] shows that the realization of the degree-wise pullback square

$$\begin{tikzcd} \operatorname{const} M \arrow[r] \arrow[d] & E_\bullet M \arrow[d] \\ * \arrow[r] & B_\bullet M \end{tikzcd}$$

provides a homotopy fiber sequence $M \to EM \to BM$. Since EM is contractible, it follows that $M \to \Omega((BM)^{\mathrm{fib}})$ is a weak equivalence. $\square$

We now let C be the free simplicial commutative monoid on a point, i.e., the simplicial commutative monoid obtained applying the free commutative monoid functor on sets degree-wise to Δ^0. Then we apply $\Omega((B(-))^{\mathrm{fib}})$ to form the group completion of C and choose a factorization

$$C \overset{\xi}{\rightarrowtail} C' \overset{\sim}{\twoheadrightarrow} \Omega((BC)^{\mathrm{fib}}) \tag{2.2}$$

of η_C into a cofibration ξ followed by an acyclic fibration.

Lemma 2.4. *The map $B\xi\colon BC \to BC'$ is a weak equivalence.*

Proof. Since BC and BC' are connected as simplicial sets, it is enough to show that $\Omega((B\xi)^{\mathrm{fib}})$ is a weak equivalence. By construction of ξ, this reduces to showing that $\Omega((B\eta_C)^{\mathrm{fib}})$ is a weak equivalence. The composite of $B\eta_C$ with the adjunction counit $\varepsilon_D\colon B\Omega D \to D$ on $D = (BC)^{\mathrm{fib}}$ is the fibrant replacement of BC. Hence it is enough to show that ε_D becomes a weak equivalence after applying $\Omega((-)^{\mathrm{fib}})$. The composite of $\Omega((\varepsilon_D)^{\mathrm{fib}})$ with the group completion map $\eta_{\Omega D}$ is the weak equivalence $\Omega(D \to D^{\mathrm{fib}})$. Hence it is enough to see that $\eta_{\Omega D}$ is the weak equivalence, and this follows from the last lemma. $\square$

The next lemma shows that we may view $\xi\colon C \to C'$ as the group completion in the universal example. To phrase it, recall that an object X in a simplicial model category $\mathcal{C}$ is local with respect to a cofibration $U \to V$ in $\mathcal{C}$ if X is fibrant and the induced map of simplicial sets $\mathbf{Hom}(V, X) \to \mathbf{Hom}(U, X)$ is an acyclic fibration.

Lemma 2.5. *An object in $s\mathcal{M}$ is ξ-local if and only if it is fibrant and grouplike.*

Proof. Let M be ξ-local. Then

$$s\mathcal{M}(C', M) \cong \mathbf{Hom}(C', M)_0 \to \mathbf{Hom}(C, M)_0 \cong s\mathcal{M}(C, M)$$

is surjective. Hence every map $C \to M$ extends over C'. Passing to connected components, this means that any homomorphism $(\mathbb{N}, +) \cong \pi_0(C) \to \pi_0(M)$ extends over the group completion $(\mathbb{N}, +) \to (\mathbb{Z}, +)$. This implies that M is grouplike.

Now let M be fibrant as a simplicial set and grouplike. Since $M \to \Omega((BM)^{\mathrm{fib}})$ is a weak equivalence by Lemma 2.3, it is enough to show that

$$\mathbf{Hom}(C', \Omega((BM)^{\mathrm{fib}})) \to \mathbf{Hom}(C, \Omega((BM)^{\mathrm{fib}}))$$

is a weak equivalence. By adjunction, this map is isomorphic to

$$\mathbf{Hom}(BC', (BM)^{\mathrm{fib}}) \to \mathbf{Hom}(BC, (BM)^{\mathrm{fib}}),$$

and the claim follows from Lemma 2.4 and the fact that $s\mathcal{M}$ is simplicial. □

The previous lemma enables us to view the group completion of simplicial commutative monoids as a fibrant replacement in an appropriate model structure:

Proposition 2.6. *The category of simplicial commutative monoids $s\mathcal{M}$ admits a left proper simplicial cellular* group completion model structure*.*

The cofibrations in this model structure are the same as in the standard model structure. A map $M \to N$ is a weak equivalence if and only if the induced map $BM \to BN$ is a weak equivalence of simplicial sets. An object is fibrant if and only if it is both fibrant as a simplicial set and grouplike.

The fibrant replacement $M \rightarrowtail M^{\mathrm{gp}}$ in the group completion model structure is weakly equivalent to η_M.

Proof. The desired model structure is defined as the left Bousfield localization of the standard model structure with respect to the single map ξ. The existence of this model structure, the characterization of the cofibrations, and the fact that it is left proper, simplicial, and cellular follow from [Hir03, Theorem 4.1.1]. Lemma 2.5 provides the description of the fibrant objects.

Now let $M \to M^{\mathrm{gp}}$ be a fibrant replacement in the group completion model structure and consider the square

$$\begin{array}{ccc} M & \longrightarrow & \Omega((B(M))^{\mathrm{fib}}) \\ \downarrow & & \downarrow \\ M^{\mathrm{gp}} & \longrightarrow & \Omega((B(M^{\mathrm{gp}}))^{\mathrm{fib}}). \end{array}$$

The bottom horizontal map is a weak equivalence by Lemma 2.3. Using the universal property of the left Bousfield localization [Hir03, Proposition 3.3.18] and Lemma 2.4, it follows that $B(M) \to B(M^{\mathrm{gp}})$ is a weak equivalence. Hence the right vertical map is a weak equivalence. This provides the desired characterization of the fibrant replacement.

By the previous argument and [Hir03, Theorem 3.2.18], a map $M \to N$ is a weak equivalence in the group completion model structure if and only if it becomes a weak equivalence when applying $\Omega((B(-))^{\mathrm{fib}})$. This is the case if and only if $B(M) \to B(N)$ is a weak equivalence. □

Remark 2.7. Simplicial commutative monoids have a topological analog known as *commutative $\mathcal{I}$-space monoids*. These provide strictly commutative models for the more common E_∞ *spaces*. The group completion model structure of the previous proposition is analogous to the group completion model structure for commutative $\mathcal{I}$-space monoids developed in [SS13, Theorem 1.3]. The proofs of Proposition 2.1 and of Lemmas 2.3 and 2.5 closely follow the corresponding statements in [SS13, §5].

The fact that the fibrant replacement $M \to M^{\mathrm{gp}}$ is always a cofibration easily provides the following universal property:

Corollary 2.8. *Every map $M \to N$ of simplicial commutative monoids with N fibrant and grouplike extends over the group completion $M \rightarrowtail M^{\mathrm{gp}}$.* □

Remark 2.9. The example discussed in [BF78, §5.7] shows that the group completion model structure is not right proper.

A different way of group completing a simplicial commutative monoid is to apply the usual group completion of commutative monoids degree-wise. As a functor

$$(-)^{\text{deg-gp}} \colon s\mathcal{M} \to s\mathcal{AB}, \tag{2.3}$$

this construction is left adjoint to the forgetful functor. The resulting natural transformation $M \to M^{\text{deg-gp}}$ is indeed equivalent to the group completion considered above:

Lemma 2.10. *The map $M \to M^{\text{deg-gp}}$ is a weak equivalence with fibrant codomain with respect to the group completion model structure.*

Proof. Since simplicial abelian groups are fibrant as simplicial sets, it follows that $M^{\text{deg-gp}}$ is fibrant in the group completion model structure. Quillen's analysis of $M^{\text{deg-gp}}$ in [FM94, Propositions Q1 and Q2] implies that $BM \to B(M^{\text{deg-gp}})$ is a weak equivalence of simplicial sets. □

Corollary 2.11. *The degree-wise group completion is the left adjoint in a Quillen equivalence between the category of simplicial commutative monoids with the group completion model structure and the category of simplicial abelian groups.*

Proof. Since all objects in $s\mathcal{AB}$ are fibrant as simplicial sets and grouplike, the forgetful functor $U \colon s\mathcal{AB} \to s\mathcal{M}$ preserves fibrant objects. Hence it follows from [Hir03, Proposition 3.3.16] that U preserves fibrations. Since weak equivalences between fibrant objects in the group completion model structure are precisely the underlying weak equivalences of simplicial sets, it follows that a map f in $s\mathcal{AB}$ is a weak equivalence if $U(f)$ is. Hence U is a right Quillen functor. Together with the previous lemma, this implies that $((-)^{\text{deg-gp}}, U)$ is a Quillen equivalence. □

Corollary 2.12. *The homotopy category of simplicial abelian groups is equivalent to the homotopy category of grouplike simplicial commutative monoids.* □

Remark 2.13. We have seen that the derived adjunction unit $M \to \Omega((BM)^{\mathrm{fib}})$, the fibrant replacement $M \to M^{\mathrm{gp}}$ in the group completion model structure, and the degree-wise group completion $M \to M^{\text{deg-gp}}$ provide three equivalent ways of forming group completions of simplicial commutative monoids.

The following result will be used in Section 4:

Corollary 2.14. *A commutative square*

$$\begin{array}{ccc} M & \longrightarrow & P \\ \downarrow & & \downarrow \\ N & \longrightarrow & Q \end{array} \tag{2.4}$$

of simplicial abelian groups is homotopy cocartesian in $s\mathcal{AB}$ if and only if it is homotopy cocartesian when viewed as a square in $s\mathcal{M}$.

Proof. By Lemma 2.10, the left Quillen functor $(-)^{\text{deg-gp}}\colon s\mathcal{M} \to s\mathcal{AB}$ sends weak equivalences between not necessarily cofibrant objects to weak equivalences. So if (2.4) is homotopy cocartesian in $s\mathcal{M}$, applying $(-)^{\text{deg-gp}}$ shows that the square is homotopy cocartesian in $s\mathcal{AB}$. Let (2.4) be homotopy cocartesian in $s\mathcal{AB}$. We choose a factorization $M \rightarrowtail N^c \xrightarrow{\sim} N$ in $s\mathcal{M}$ and deduce that $N^c \coprod_M P \to Q$ is a weak equivalence in $s\mathcal{M}$ since it is a weak equivalence after applying $(-)^{\text{deg-gp}}$ and its domain and codomain are grouplike. □

2.2. **Repletion.** Many of the conditions on commutative monoids that are useful in logarithmic geometry do not appear to provide homotopy invariant notions when imposing them in each level of a simplicial commutative monoids. As explained by Rognes in [Rog09, Remark 3.2], the notion of *repletion* for commutative monoids [Rog09, §3] and for commutative $\mathcal{I}$-space monoids [Rog09, §8] is made to overcome this difficulty in one relevant instance. Repletion has already proved useful for the definition of logarithmic topological Hochschild homology in [Rog09, §8] and [RSS14]. The close relation between repletion and a group completion model structure on commutative $\mathcal{I}$-space monoids explained in [SS13, §5.10] makes it easy to adopt this to simplicial commutative rings.

Definition 2.15. Let $M \to N$ be a map of simplicial commutative monoids and let

$$M \overset{\sim}{\rightarrowtail} M^{\text{rep}} \twoheadrightarrow M$$

be a factorization of this map in the group completion model structure. Then $M \to M^{\text{rep}}$ is the *repletion* of M over N.

General properties of left Bousfield localizations imply that as an object under M and over N, the repletion M^{rep} is well defined up to weak equivalence in the standard model structure on $s\mathcal{M}$.

Definition 2.16. A map of simplicial commutative monoids $M \to N$ is *virtually surjective* if the induced homomorphism $(\pi_0(M))^{\text{gp}} \to (\pi_0(N))^{\text{gp}}$ is surjective (compare [Rog09, Definition 3.6]). It is *exact* [Kat89] if the following square is homotopy cartesian in the standard model structure on $s\mathcal{M}$:

$$\begin{array}{ccc} M & \longrightarrow & M^{\text{gp}} \\ \downarrow & & \downarrow \\ N & \longrightarrow & N^{\text{gp}} \end{array}$$

The next proposition states that for a virtually surjective map, repletion enforces exactness and can be defined by only using group completions.

Proposition 2.17. *Let $M \to N$ be a virtually surjective map of simplicial commutative monoids.*

(i) The canonical map $M^{\text{gp}} \to (M^{\text{rep}})^{\text{gp}}$ is a weak equivalence in the standard model structure.
(ii) The repletion M^{rep} is weakly equivalent to the map from M into the homotopy pullback of $N \to N^{\text{gp}} \leftarrow M^{\text{rep}}$ (with respect to the standard model structure).
(iii) The map $M^{\text{rep}} \to N$ is exact.

Proof. The properties of the group completion model structure imply (i). For (ii), one can use a similar argument as in the proof of [SS13, Proposition 5.16]. The key ingredient is the Bousfield–Friedlander theorem [BF78, Theorem B.4] that compensates for the missing right properness of the group completion model structure. Part (iii) follows from (i) and (ii). □

If M is a simplicial commutative monoid under and over N, then $M \to N$ is automatically virtually surjective, and passing to the repletion ensures exactness of the augmentation.

3. Logarithmic simplicial rings

The functor sending a commutative ring A to its underlying multiplicative monoid $(A, \cdot)$ is right adjoint to the integral monoid ring functor $\mathbb{Z}[-]$ from commutative monoids to commutative rings. Applying this adjunction degree-wise provides an adjunction

$$\mathbb{Z}[-]\colon s\mathcal{M} \rightleftarrows s\mathcal{R}\colon (-, \cdot) \tag{3.1}$$

between the associated categories of simplicial objects. The following definition is the obvious generalization of the pre-log structures introduced by Kato in [Kat89].

Definition 3.1. A *pre-log structure* (M, α) on a simplicial commutative ring R is a simplicial commutative monoid M together with a map of simplicial commutative monoids $\alpha\colon M \to (A, \cdot)$. A simplicial commutative ring R together with a pre-log structure (M, α) is called a *pre-log* simplicial ring. It is denoted by (A, M, α) or simply by (A, M) if α is understood from the context.

A map of simplicial pre-log rings $(A, M) \to (B, N)$ is a pair $(f, f^\flat)$ of maps $f\colon A \to B$ in $s\mathcal{R}$ and $f^\flat\colon M \to N$ in $s\mathcal{M}$ such that the obvious square commutes. We write $s\mathcal{P}$ for the resulting category of simplicial commutative pre-log rings.

Viewing pre-log simplicial rings as simplicial objects in pre-log rings, the same arguments as in the case of $s\mathcal{M}$ and $s\mathcal{R}$ show that $s\mathcal{P}$ is a simplicial category. Since $\mathbb{Z}[-]$ preserves coproducts, it is immediate that $(A, M) \otimes K \cong (A \otimes K, M \otimes K)$. Expressing the compatibility of the two components of a map of pre-log simplicial rings as a pullback, it follows that the mapping spaces in $s\mathcal{P}$ are related to the mapping spaces in $s\mathcal{M}$ and $s\mathcal{R}$ by a pullback square

$$\begin{array}{ccc} \mathbf{Hom}_{s\mathcal{P}}((A, M), (B, N)) & \longrightarrow & \mathbf{Hom}_{s\mathcal{R}}(A, B) \\ \downarrow & & \downarrow \\ \mathbf{Hom}_{s\mathcal{M}}(M, N) & \longrightarrow & \mathbf{Hom}_{s\mathcal{M}}(M, (B, \cdot)) \end{array}$$

3.1. **The pre-log model structures.** Since the adjunction (3.1) is a Quillen adjunction with respect to the model structures from Proposition 2.1, we obtain the two model structures on $s\mathcal{P}$ described in the next two propositions:

Proposition 3.2. *The category of simplicial pre-log rings $s\mathcal{P}$ admits an* injective *proper simplicial cellular model structure where $(f, f^\flat)\colon (A, M) \to (B, N)$ is*

- *a weak equivalence (or a cofibration) if both f and $f^\flat$ are weak equivalences (or cofibrations) in the standard model structures on $s\mathcal{R}$ and $s\mathcal{M}$ and*
- *a fibration if f is a fibration in $s\mathcal{R}$ and the induced map $M \to (A, \cdot) \times_{(B, \cdot)} N$ is a fibration in $s\mathcal{M}$.*

We call this model structure the *injective pre-log model structure* and write $s\mathcal{P}^{\mathrm{inj}}$ for the resulting model category. The fibrant objects are called *pre-fibrant.*

Proof. The existence of this model structure is established by standard lifting arguments. Using Lemma 3.12 below, one can check that the generating cofibrations $I_{s\mathcal{M}}$ and $I_{s\mathcal{R}}$ for $s\mathcal{M}$ and $s\mathcal{R}$ give rise to a set

$$\{(\mathbb{Z}[L], K) \to (\mathbb{Z}[L], L) \,|\, K \to L \in I_{s\mathcal{M}}\} \cup \{(i, id_*) \,|\, i \in I_{s\mathcal{R}}\}$$

of generating cofibrations for $s\mathcal{P}$, and similarly for the generating acyclic cofibrations. □

Similarly, we get a *projective pre-log model structure* denoted by $s\mathcal{P}^{\mathrm{proj}}$:

Proposition 3.3. *The category of simplicial pre-log rings $s\mathcal{P}$ admits a* projective *proper simplicial cellular model structure where $(f, f^\flat)\colon (A, M) \to (B, N)$ is*

- *a weak equivalence (or a fibration) if both f and $f^\flat$ are weak equivalences (or fibrations) in the standard model structures on $s\mathcal{R}$ and $s\mathcal{M}$ and*
- *a cofibration if $f^\flat$ is a cofibration in $s\mathcal{M}$ and the induced map $\mathbb{Z}[N] \otimes_{\mathbb{Z}[M]} A \to B$ is a cofibration in $s\mathcal{R}$.*

Proof. Again this follows by standard lifting arguments. In this case the generating cofibrations $I_{s\mathcal{M}}$ and $I_{s\mathcal{R}}$ for $s\mathcal{M}$ and $s\mathcal{R}$ give rise to a set

$$\{(\mathbb{Z}[i], i) \,|\, i \in I_{s\mathcal{M}}\} \cup \{(i, id_*) \,|\, i \in I_{s\mathcal{R}}\}$$

of generating cofibrations for $s\mathcal{P}$, and similarly for the generating acyclic cofibrations. □

Corollary 3.4. *The identity functor from simplicial pre-log rings with the projective model structure to simplicial pre-log rings with the injective model structure is the left Quillen functor of a Quillen equivalence.* □

Remark 3.5. The corollary implies that the two model structures are equivalent for many purposes. However, as we will see in Section 3.2 below, the fact that the injectively fibrant objects (A, M) have the property that the structure map $M \to (A, \cdot)$ is a fibration makes the injective model structure more convenient for the purpose of *log* structures.

If (A, M, α) is a pre-log simplicial ring, we write $(A, \cdot)^\times$ for the sub simplicial commutative monoid of invertible path components $(A, \cdot)^\times \subset (A, \cdot)$, i.e, the sub simplicial commutative monoid of $(A, \cdot)$ consisting of those simplices whose vertices represent units in the multiplicative monoid $\pi_0(A)$. Using $(A, \cdot)^\times$, we form the following pullback square:

$$\begin{array}{ccc} \alpha^{-1}((A,\cdot)^\times) & \longrightarrow & (A,\cdot)^\times \\ \downarrow & & \downarrow \\ M & \xrightarrow{\alpha} & (A,\cdot) \end{array} \tag{3.2}$$

Definition 3.6. A pre-log structure (M, α) on a simplicial commutative ring A is a *log structure* if the top horizontal map in the square (3.2) is a weak equivalence in the standard model structure on $s\mathcal{M}$. In this case, (A, M, α) is called a *log simplicial ring.*

Corollary 3.7. *If $(A, M) \to (B, N)$ is a weak equivalence of pre-log simplicial rings, then (A, M) is a log simplicial ring if and only if (B, N) is.*

Proof. This uses that the inclusion of path components is a fibration of simplicial sets. □

Remark 3.8. While a pre-log simplicial ring is the same as simplicial object in the category of pre-log rings, it is not true that a log simplicial ring is a simplicial object in the category of log rings: Already in simplicial degree 0, the monoid $((A, \cdot)^\times)_0$ does not need to coincide with its submonoid $((A, \cdot)_0)^\times$. The homotopy invariance statement of the previous corollary would not hold if the log condition was defined using the degree-wise units.

Construction 3.9. If (A, M) is a pre-log simplicial ring, then we may factor the top horizontal map in the square (3.2) as a cofibration $\alpha^{-1}((A, \cdot)^\times) \to G$ followed by an acyclic fibration $G \to (A, \cdot)^\times$ with respect to the standard model structure.

The pushout $M^a = M \coprod_{\alpha^{-1}((A,\cdot)^\times)} G$ of the resulting diagram in $s\mathcal{M}$ comes with a canonical map $\alpha^a \colon M^a \to (A,\cdot)$.

The induced map $(\alpha^a)^{-1}((A,\cdot)^\times) \to (A,\cdot)^\times$ is isomorphic to $G \to (A,\cdot)^\times$. Hence (M^a, α^a) is a log structure on A. We call it the *associated log structure* of (M, A) and refer to (A, M^a, α^a) as the *logification* of (A, M, α).

The logification comes with a natural map $(A, M, \alpha) \to (A, M^a, \alpha^a)$. The use of the relative cofibrant replacement of $(A,\cdot)^\times$ and the left properness of $s\mathcal{M}$ ensures that logification preserves weak equivalences.

Lemma 3.10. *If (A, M, α) is a log simplicial ring, then $(A, M, \alpha) \to (A, M^a, \alpha^a)$ is a weak equivalence.*

Proof. If (M, α) is a log structure, then $\alpha^{-1}((A,\cdot)^\times) \to G$ is a weak equivalence. This implies that $(A, M, \alpha) \to (A, M^a, \alpha^a)$ is a weak equivalence. □

3.2. **The log model structure.** Our next aim is to express the log condition and the logification in terms of model structures.

Lemma 3.11. *Let (A, M) be fibrant in the injective pre-log model structure. Then (A, M) is a log ring if and only if for every cofibration $K \to L$ in $s\mathcal{M}$ with L grouplike, every commutative square*

$$\begin{tikzcd} K \arrow[r] \arrow[d] & M \arrow[d] \\ L \arrow[r] & (A,\cdot) \end{tikzcd}$$

in $s\mathcal{M}$ admits a lift $L \to M$ making both triangles commutative.

Proof. Let (A, M) be a pre-fibrant log simplicial ring. Then $L \to (A,\cdot)$ factors through the inclusion $(A,\cdot)^\times \to (A,\cdot)$ because L is grouplike, and there exists a lifting in the resulting square

$$\begin{tikzcd} K \arrow[r] \arrow[d] & \alpha^{-1}((A,\cdot)^\times) \arrow[d] \\ L \arrow[r] & (A,\cdot)^\times \end{tikzcd} \tag{3.3}$$

because $K \to L$ is cofibration and $\alpha^{-1}((A,\cdot)^\times) \to (A,\cdot)^\times$ is an acyclic fibration. Composing with $\alpha^{-1}((A,\cdot)^\times) \to M$ gives the desired lift.

For the converse, it is enough to show that for every generating cofibration $K \to L$ in the standard model structure on $s\mathcal{M}$ and every square of the form (3.3) there exists a lift $L \to \alpha^{-1}((A,\cdot)^\times)$. Since $(A,\cdot)^\times$ is grouplike, the map $L \to (A,\cdot)^\times$ extends over the group completion $L \to L^{\mathrm{gp}}$. The composed map $K \to L^{\mathrm{gp}}$ lifts against $M \to (A,\cdot)$. This provides a map $L^{\mathrm{gp}} \to M$ whose composite with $L \to L^{\mathrm{gp}}$, in combination with $L \to (A,\cdot)^\times$, induces the desired lifting $L \to \alpha^{-1}((A,\cdot)^\times)$. □

Every map $K \to L$ in $s\mathcal{M}$ gives rise to a pre-log simplicial ring $(\mathbb{Z}[L], K)$ and a canonical map $(\mathbb{Z}[L], K) \to (\mathbb{Z}[L], L)$ in $s\mathcal{P}$. An adjunction argument shows

Lemma 3.12. *Let $K \to L$ be a map in $s\mathcal{M}$, let $(A, M) \to (B, N)$ be a map in $s\mathcal{P}$, and consider commutative squares*

$$\begin{tikzcd} (\mathbb{Z}[L], K) \arrow[r] \arrow[d] & (A, M) \arrow[d] \\ (\mathbb{Z}[L], L) \arrow[r] & (B, N) \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} K \arrow[r] \arrow[d] & M \arrow[d] \\ L \arrow[r] & N \times_{(B,\cdot)} (A,\cdot) \end{tikzcd}$$

in $s\mathcal{P}$ and $s\mathcal{M}$. Then the universal property of $\mathbb{Z}[-]$ induces a one-to-one correspondence between commutative squares of the first and second type, and the first square admits a lift if and only if the second does. □

Let I be the set of generating cofibrations for the standard model structure on $s\mathcal{M}$, and let

$$S = \{(\mathbb{Z}[L^{\mathrm{gp}}], K) \to (\mathbb{Z}[L^{\mathrm{gp}}], L^{\mathrm{gp}}) \mid (K \to L^{\mathrm{gp}}) = (K \xrightarrow{f} L \to L^{\mathrm{gp}}) \text{ where } f \in I\}$$

be set of maps in $s\mathcal{P}$ obtained by group-completing the codomains of the generating cofibrations for $s\mathcal{M}$ and forming the associated maps of pre-log simplicial rings.

We will say that a map of pre-log simplicial rings is a *log equivalence* if it induces a weak equivalence after logification, and a *log cofibration* if it is a cofibration in the injective pre-log model structure of Proposition 3.2. Moreover, a pre-log simplicial ring is *log fibrant* if it is a pre-fibrant log simplicial ring.

Theorem 3.13. *The log equivalences and the log cofibrations are the weak equivalences and cofibrations of a left proper simplicial cellular* log model structure *on the category of simplicial pre-log rings $s\mathcal{P}$. The log fibrant objects are the fibrant objects in this model structure.*

We write $s\mathcal{L}$ for this model category. By slight abuse of language, we refer to it as the model category of *log simplicial rings*.

Proof. The log model structure is defined to be the left Bousfield localization of the injective pre-log model structure with respect to S. Its existence and most of its properties are provided by [Hir03, Theorem 4.1.1]. Lemma 3.14 provides the characterization of the fibrant objects, and Lemma 3.15 and [Hir03, Theorem 3.2.18] provide the characterization of the weak equivalences. □

Lemma 3.14. *A pre-log simplicial ring (A, M) is S-local if and only if it is a pre-fibrant log simplicial ring.*

Proof. Let (A, M) be a pre-fibrant log simplicial ring. By [Hir03, Proposition 4.2.4], showing that it is S-local is equivalent to showing that $(A, M) \to *$ has the right lifting property with respect to the pushout product map

$$(\mathbb{Z}[L^{\mathrm{gp}}], L^{\mathrm{gp}}) \otimes \partial\Delta^n \textstyle\coprod_{(\mathbb{Z}[K], L^{\mathrm{gp}}) \otimes \partial\Delta^n} (\mathbb{Z}[K], L^{\mathrm{gp}}) \otimes \Delta^n \to (\mathbb{Z}[L^{\mathrm{gp}}], L^{\mathrm{gp}}) \otimes \Delta^n.$$

This map is isomorphic to the map $(\mathbb{Z}[L'], K') \to (\mathbb{Z}[L'], L')$ associated with

$$K' = L^{\mathrm{gp}} \otimes \partial\Delta^n \textstyle\coprod_{K \otimes \partial\Delta^n} L^{\mathrm{gp}} \otimes \Delta^n \to L^{\mathrm{gp}} \otimes \Delta^n = L'.$$

Then L' is grouplike because L^{gp} is grouplike and Δ^n is contractible. Combining Lemmas 3.11 and 3.12 provides the desired lifting.

Now assume that (A, M) is S-local. Then (A, M) is pre-fibrant by definition. Lemma 3.12 and the argument given in the proof of Lemma 3.11 show that (A, M) is a log simplicial ring. □

The next lemma exhibits the logification of Construction 3.9 as an explicit fibrant replacement for the log model structure.

Lemma 3.15. *Let (A, M) be a pre-fibrant pre-log simplicial ring. Then the logification map $(A, M) \to (A, M^a)$ is an S-local equivalence of pre-log simplicial rings.*

Proof. Let $\alpha^{-1}((A, \cdot)^\times) \to G$ be the cofibration used in Construction 3.9. Then we can form the associated map of pre-log simplicial rings and observe that the logification may be obtained as the right vertical map in the pushout square

$$\begin{tikzcd} (\mathbb{Z}[G], \alpha^{-1}((A, \cdot)^\times)) \arrow[r] \arrow[d] & (A, M) \arrow[d] \\ (\mathbb{Z}[G], G) \arrow[r] & (A, M^a). \end{tikzcd}$$

It is enough to show that the left hand vertical map is an S-local equivalence. For this we have to verify that it induces a weak equivalence of simplicial sets when applying the functor $\mathbf{Hom}(-,(B,N))$ where (B,N) is a fibrant object in the log model structure. By adjunction and Lemma 3.12, this is equivalent to showing that

$$\alpha^{-1}((A,\cdot)^{\times})\otimes\Delta^n \coprod_{\alpha^{-1}((A,\cdot)^{\times})\otimes\partial\Delta^n} G\otimes\partial\Delta^n \to G\otimes\Delta^n$$

has the lifting property against $N \to (B,\cdot)$. Since (B,N) is log and G is grouplike, this follows from Lemma 3.11. □

The last lemma and the formal properties of a left Bousfield localization easily imply the following statement.

Corollary 3.16. *The homotopy category* $\mathrm{Ho}(s\mathcal{L})$ *is equivalent to the full subcategory of* $\mathrm{Ho}(s\mathcal{P})$ *consisting of log simplicial rings, and the logification induces an adjoint pair* $(-)^a : \mathrm{Ho}(s\mathcal{P}) \rightleftarrows \mathrm{Ho}(s\mathcal{L}) : i$ *where* i *is the canonical inclusion functor.*

3.3. **The replete model structures.** Rognes' notion of *repletion* discussed in Section 2.2 can also be described in terms of appropriate model structures on $s\mathcal{P}$.

Proposition 3.17. *The category of simplicial pre-log rings* $s\mathcal{P}$ *admits a left proper simplicial* replete pre-log *model structure where*

- *a map* $(f,f^\flat)\colon (A,M)\to(B,N)$ *is cofibration if* f *is a cofibration in* $s\mathcal{R}$ *and* $f^\flat$ *is a cofibration in* $s\mathcal{M}$ *and*
- *an object* (A,M) *is fibrant if* A *is fibrant in* $s\mathcal{R}$*, the structure map* $M\to(A,\cdot)$ *is a fibration in the standard model structure on* $s\mathcal{M}$*, and* M *is grouplike.*

The forgetful functor $s\mathcal{P}\to s\mathcal{M}$ *sending* (A,M) *to* M *is a right Quillen functor with respect to the replete pre-log model structure and the group completion model structure on* $s\mathcal{M}$*.*

Proof. We let $\xi\colon C\to C'$ be the map in $s\mathcal{M}$ introduced in (2.2) and form the left Bousfield localization of the injective pre-log model structure with respect to the associated map $(\mathbb{Z}[\xi],\xi)$ in $s\mathcal{P}$. This model structure has the same cofibrations as the injective pre-log model structure, and the isomorphism $\mathbf{Hom}_{s\mathcal{P}}((\mathbb{Z}[\xi],\xi),(A,M))\cong \mathbf{Hom}_{s\mathcal{M}}(\xi,M)$ shows that the fibrant objects are the pre-fibrant objects (A,M) with ξ-local M. Hence Lemma 2.5 provides the characterization of the fibrant objects. Since $M\mapsto(\mathbb{Z}[M],M)$ is left adjoint to the forgetful functor $s\mathcal{P}\to s\mathcal{M}$, the last statement is a formal consequence of [Hir03, Theorem 3.3.20]. □

If (A,M) is a pre-log simplicial ring, then the group completion of M enables us to form the *trivial locus* $(A[M^{-1}],M^{\mathrm{gp}}) = (\mathbb{Z}[M^{\mathrm{gp}}]\otimes_{\mathbb{Z}[M]} A, M^{\mathrm{gp}})$. Up to a pre-fibrant replacement, this construction can be viewed as a fibrant replacement in the replete pre-log model structure:

Lemma 3.18. *The composite* $(A,M)\to(A[M^{-1}],M^{\mathrm{gp}})^{\mathrm{pre-fib}}$ *of the canonical map* $(A,M)\to(A[M^{-1}],M^{\mathrm{gp}})$ *with a fibrant replacement functor for the injective pre-log model structure provides a fibrant replacement functor for the replete pre-log model structure.*

Proof. Since $i\colon M\to M^{\mathrm{gp}}$ is an acyclic cofibration in the group completion model structure, the associated map $(\mathbb{Z}[i],i)$ is an acyclic cofibration in the replete pre-log model structure. The map $(A,M)\to(A[M^{-1}],M^{\mathrm{gp}})$ is a cobase change of this map and hence also an acyclic cofibration in the replete pre-log model structure. This implies that the map in question is an acyclic cofibration whose codomain is fibrant in the replete pre-log model structure. □

As it is often the case with left Bousfield localizations, we don't have an explicit characterization of general fibrations in the replete pre-log model structure. However, the replete pre-log model structure can be used to guarantee exactness on the underlying monoid map of a fibrant augmented object:

Corollary 3.19. *Let (A, M) be a pre-log simplicial ring and let $s\mathcal{P}_{(A,M)}/(A, M)$ be the category of augmented (A, M)-algebras with the model structure induced by the replete pre-log model structure on $s\mathcal{P}$. If $(A, M) \to (B, N) \to (A, M)$ is fibrant in this model category, then the underlying map $N \to M$ is exact in the sense of Definition 2.16.*

Proof. By Proposition 3.17, the map $N \to M$ is a fibration in the group completion model structure. Hence $N \to N^{\mathrm{rep}}$ is a weak equivalence in the standard model structure, and since $N \to M$ is virtually surjective, Proposition 2.17(iii) shows that $N \to M$ is exact. $\square$

Remark 3.20. By analogy with [Rog09, Definition 3.12], one can define the repletion of a map $(B, N) \to (A, M)$ of pre-log simplicial rings with virtually surjective $N \to M$ as the first map in the factorization

$$(B, N) \to (\mathbb{Z}[N^{\mathrm{rep}}] \otimes_{\mathbb{Z}[N]} B, N^{\mathrm{rep}}) \to (A, M).$$

A similar argument as in Lemma 3.18 shows that the repletion map is an acyclic cofibration in the replete pre-log model structure. However, we do not know if the fact that $N^{\mathrm{rep}} \to M$ is a fibration in the group completion model structure is sufficient to conclude that $(\mathbb{Z}[N^{\mathrm{rep}}] \otimes_{\mathbb{Z}[N]} B, N^{\mathrm{rep}}) \to (A, M)$ gives rise to a fibration in the replete pre-log model structure after replacing it by a fibration of pre-log simplicial rings.

Remark 3.21. The projective pre-log model structure gives rise to a projective version of the replete pre-log model structure with similar properties.

Remark 3.22. Combining the arguments of Proposition 3.17 with the log model structure of Theorem 3.13, we obtain a left proper simplicial *replete* log model structure on $s\mathcal{P}$. Here an object is fibrant if and only if it is injectively fibrant as a pre-log simplicial ring, $M \to (A, \cdot)$ is a log structure, and M is grouplike.

It follows that the fibrant objects in this model structure always carry the trivial log structure. Up to pre-log fibrant replacement, the fibrant replacement of (A, M) in the replete log model structure is given by $(A, M) \to (A[M^{-1}], A[M^{-1}]^{\times})$.

3.4. **Functorialities.**

Definition 3.23. Let (A, M) be a simplicial pre-log ring, and let $f\colon A \to B$ be a morphism of simplicial rings. Then the *inverse image pre-log structure* on B is given by $M \to (A, \cdot) \to (B, \cdot)$ and is denoted by f_*M. The *inverse image log structure* is defined to be the associated log structure. We will denote it by $(f_*M)^a \to (B, \cdot)$.

Definition 3.24. Let (B, N) be a simplicial pre-log ring, and let $f\colon A \to B$ be a morphism of simplicial rings. Then the *direct image pre-log structure* on A is given by the fiber product of simplicial monoids

$$\begin{array}{ccc} f^*N & \longrightarrow & (A, \cdot) \\ \downarrow & & \downarrow \\ N & \longrightarrow & (B, \cdot). \end{array}$$

The associated log structure is denoted by $(f^*N)^a$.

It is straightforward to check that if (B, N) is a log simplicial ring, then f^*N is again a log structure on (A, M) if $N \to (B, \dot{)}$ or $A^{\times} \to B^{\times}$ is a fibration. On the contrary, the inverse image of a log structure will in general not again be a log structure.

Definition 3.25. Let $(f, f^\flat)\colon (A, M) \to (B, N)$ be a morphism of log simplicial rings. Then $(f, f^\flat)$ is *strict* if $(f_*M)^a \to N$ is an equivalence of simplicial monoids.

If A is a simplicial commutative ring, we will denote by $s\mathcal{P}_A$ the category of pre-log structures on A, i.e., the over-category $s\mathcal{M}/(A,\cdot)$, with its canonical induced injective model structure. Likewise, we denote by $\mathbf{sLog}_A$ the full subcategory of the homotopy category of $s\mathcal{P}_A$ on the objects $M \to (A,\cdot)$ that are log structures.

Proposition 3.26. *A morphism of simplicial commutative rings $f\colon A \to B$ induces a Quillen adjunction*

$$f_*\colon s\mathcal{P}_A \rightleftarrows s\mathcal{P}_B : f^*.$$

On the level of homotopy categories, this adjunction and the logification induce an adjunction

$$f_{*a}\colon \mathbf{sLog}_A \rightleftarrows \mathbf{sLog}_B : f^* = f^{*a}$$

whose left adjoint f_{*a} sends $M \to (A,\cdot)$ to $(f_*M)^a \to B$.

Proof. The first adjunction is immediate, and it is easy to verify that f_* preserves cofibrations and trivial cofibrations. The second adjunction follows using Corollary 3.16. □

Remark 3.27. The adjunction (f_*, f^*) actually induces the structure of a *left Quillen presheaf* over $s\mathcal{R}$ on $s\mathcal{P}$, endowed with the injective pre-log model structure (see for example [Sim12, p. 127] for the notion of left Quillen presheaf).

4. Log-derivations and the log cotangent complex

4.1. **Log-derivations.** We begin by defining derivations in the pre-log context. For this we use that the simplicial model structures discussed in the previous section provide simplicial mapping spaces for the respective categories, and we will write $\mathrm{Map}_{\mathcal{C}}(-,-)$ for the derived mapping spaces in a simplicial model category $\mathcal{C}$.

Let $s\mathcal{P}_{(A,M)//(B,N)}$ denote the category of simplicial pre-log (A,M)-algebras over (B,N).

Definition 4.1. Let $(A,M) \to (B,N)$ be a morphism of simplicial pre-log rings, and let J be a simplicial B-module. Denote by $B \oplus J$ the trivial square zero extension of B by J, and let $N \oplus J$ be the simplicial monoid $N \times J^{\mathrm{add}}$. Define a morphism $N \oplus J \to (B \oplus J, \cdot)$ as the product of the two canonical maps

$$N \longrightarrow (B,\cdot) \longrightarrow (B \oplus J, \cdot),$$

$$(J,+) \hookrightarrow (B \oplus J,\cdot), \quad x \mapsto (1,x).$$

Then $(B \oplus J, N \oplus J)$ is canonically an object in $s\mathcal{P}_{(A,M)//(B,N)}$, and we will call it the *trivial square zero extension of* (B,N) *by* J.

Remark 4.2. In case $(f,f^\flat)\colon (A,M) \to (B,N)$ is a morphism of log simplicial rings, an equivalent definition of the trivial square zero extension is $(B \oplus J, ((s_0)_*N)^a)$, where $s_0\colon B \to B \oplus J$ is the canonical section of the projection $B \oplus J \to B$. See [Rog09, Lemma 11.5].

Definition 4.3. Let $(f,f^\flat)\colon (A,M) \to (B,N)$ be a morphism of simplicial pre-log rings, and let J be a simplicial B-module. The simplicial set of f-linear derivations of (B,N) with values in J is defined as

$$\mathrm{Der}_{(A,M)}((B,N),J) = \mathrm{Map}_{s\mathcal{P}_{(A,M)//(B,N)}}((B,N),(B \oplus J, N \oplus J)).$$

For a morphism of log simplicial rings, it does not make a difference if we compute derivations in the category of log simplicial rings or in the category of simplicial pre-log rings:

Lemma 4.4. *Let $(f, f^\flat)\colon (A, M) \to (B, N)$ be a morphism of log simplicial rings. Then*

$$\mathrm{Der}_{(A,M)}((B,N), J) \simeq \mathrm{Map}_{s\mathcal{L}_{(A,M)//(B,N)}}((B,N), (B \oplus J, N \oplus J))$$

Proof. The map $(B \oplus J, N \oplus J) \to (B, N)$ is a fibration in the projective pre-log model structure since J is fibrant as a simplicial set. So we can model the derived mapping space $\mathrm{Map}_{s\mathcal{P}_{(A,M)//(B,N)}}((B,N), (B \oplus J, N \oplus J))$ by the simplicial mapping space in $s\mathcal{P}_{(A,M)//(B,N)}$ where we use $(B \oplus J, N \oplus J)$ as the target and a cofibrant replacement of $(A, M) \to (B, N) \xrightarrow{=} (B, N)$ in the projective pre-log model structure as the source. Since cofibrations in the projective pre-log model structure are also cofibrations in the injective pre-log model structure, it remains to show that with respect to the injective pre-log model structure, the target is weakly equivalent to a fibration in the log model structure. By Remark 4.2, $(B \oplus J, N \oplus J) \simeq (B \oplus J, ((s_0)_* N)^a)$, showing that $(B \oplus J, N \oplus J)$ is a log simplicial ring. Using [Hir03, Proposition 3.3.16], it follows that the fibrant replacement of $(B \oplus J, N \oplus J)$ in the injective pre-log model structure on $s\mathcal{P}_{(A,M)//(B,N)}$ also provides a fibrant replacement in the log model structure. □

Remark 4.5. In case $(f, f^\flat)\colon (A, M) \to (B, N)$ is a morphism of log simplicial rings, every log-derivation is a strict morphism. One can show that for a morphism of discrete log rings, the functor from B-modules to trivial square zero extensions of (B, N) gives an equivalence of categories between abelian objects in the category of log (A, M)-algebras that are strict over (B, N) and the category of B-modules [Rog09, Lemma 4.13]. The same should also hold for a morphism of log simplicial rings, giving a Quillen equivalence between the categories of simplicial B-modules and the category $(s\mathcal{P}^{\mathrm{str}}_{(A,M)//(B,N)})_{\mathrm{ab}}$ of abelian objects in the category of log (A, M)-algebras that are strict over (B, N).

4.2. **The log cotangent complex.** We have a functor

$$\Omega\colon s\mathcal{P}_{(A,M)//(B,N)} \longrightarrow \mathrm{Mod}_B, \qquad (C, O) \longmapsto \Omega_{(C,O)/(A,M)} \otimes_C B$$

where $\Omega_{(C,O)/(A,M)}$ is defined by level-wise application of the functor of log Kähler differentials (see [Kat89, Section 1.7] for the definition) for discrete pre-log rings. On the other hand, we have the functor of the previous section

$$K\colon \mathrm{Mod}_B \longrightarrow s\mathcal{P}_{(A,M)//(B,N)}, \qquad J \longmapsto (B \oplus J, N \oplus J).$$

Note that K is given by applying the trivial square zero extension functor for discrete pre-log rings levelwise. We then have the following result:

Lemma 4.6. *The pair $\Omega\colon s\mathcal{P}_{(A,M)//(B,N)} \rightleftarrows \mathrm{Mod}_B : K$ is a Quillen adjunction with respect to the projective pre-log model structure on $\mathcal{P}_{(A,M)//(B,N)}$ and the standard model structure on Mod_B.*

Proof. Adjointness follows from the corresponding statement for discrete log rings, since Ω and K are both applied level-wise. Since K clearly preserves fibrations and trivial fibrations, the adjunction is in fact a Quillen adjunction. □

Since Ω is part of a Quillen adjunction we obtain a left derived functor

$$L\Omega\colon \mathrm{Ho}(s\mathcal{P}_{(A,M)//(B,N)}) \longrightarrow \mathrm{Ho}(\mathrm{Mod}_B).$$

Definition 4.7. We define the log cotangent complex $\mathbb{L}_{(B,N)/(A,M)}$ of a morphism of simplicial pre-log rings $(A, M) \to (B, N)$ to be $L\Omega(B, N)$. Here (B, N) is regarded as an object of $s\mathcal{P}_{(A,M)//(B,N)}$.

Thus by definition, the log cotangent complex represents the derivations, since by adjunction we have

$$\begin{aligned}\mathrm{Der}_{(A,M)}((B,N),J) &= \mathrm{Map}_{s\mathcal{P}_{(A,M)//(B,N)}}((B,N),(B\oplus J, N\oplus J))\\ &\simeq \mathrm{Map}_{\mathrm{Mod}_B}(\mathbb{L}_{(B,N)/(A,M)}, J)\end{aligned}$$

Remark 4.8. In case $(A,M)\to(B,N)$ is a morphism of discrete log rings, the above definition recovers Gabber's definition [Ols05, §8] of the log cotangent complex.

Remark 4.9. Note that by Remark 4.5 the log Kähler differentials of a morphism of discrete log rings only depend on the abelian objects in the category of log rings that are strict over (B,N), since the log Kähler differentials explicitly compute the abelianization functor in this category. If we assume that we have the Quillen equivalence between sMod_B and $(s\mathcal{P}^{\mathrm{str}}_{(A,M)//(B,N)})_{\mathrm{ab}}$ mentioned in Remark 4.5, then by Lemma 4.4 the log cotangent complex explicitly computes the left derived of abelianization in the category $s\mathcal{P}^{\mathrm{str}}_{(A,M)//(B,N)}$. As Rognes points out in [Rog09], it should be interesting to investigate the abelianization functor in other categories than log rings that are strict over (B,N). For instance one could replace strictness by the weaker notion of repleteness, or by no condition at all.

The homotopical version of the log cotangent complex used in [Rog09, §11] immediately translates to give the following version of the log cotangent complex of a morphism of log simplicial rings $(f,f^\flat)\colon (A,M)\to(B,N)$, which is defined as the following homotopy pushout:

$$\begin{array}{ccc} B\otimes_{\mathbb{Z}[N]}\mathbb{L}_{\mathbb{Z}[N]/\mathbb{Z}[M]} & \xrightarrow{\psi} & B\otimes N^{\mathrm{gp}}/M^{\mathrm{gp}} \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\bar\phi} \\ \mathbb{L}_{B/A} & \xrightarrow[\bar\psi]{} & \mathbb{L}^{\mathrm{Rog}}_{(B,N)/(A,M)} \end{array}$$

Here ψ is in simplicial degree s defined as the application of the morphism

$$\begin{aligned} B_s\otimes_{\mathbb{Z}[N_s]}\Omega^1_{\mathbb{Z}[N_s]/\mathbb{Z}[M_s]} &\to B_s\otimes N_s^{\mathrm{gp}}/M_s^{\mathrm{gp}}\\ b\otimes dn &\longmapsto b\cdot\beta(n)\otimes\gamma(n)\end{aligned}$$

where γ denotes the canonical morphism to group completion.

Rognes' verification that this complex represents the derived functor of derivations also carries over to the present context:

Proposition 4.10. [Rog09, Proposition 11.21] *There is a natural weak equivalence of mapping spaces*

$$\mathrm{Map}_{\mathrm{Mod}_B}(\mathbb{L}^{\mathrm{Rog}}_{(B,N)/(A,M)}, J)\simeq \mathrm{Der}_{(A,M)}((B,N),J)$$

This allows us to compare the two definitions.

Theorem 4.11. *Let $(f,f^\flat)\colon(A,M)\to(B,N)$ be a morphism of simplicial pre-log rings. Then*

$$\mathbb{L}_{(B,N)/(A,M)}\cong\mathbb{L}^{\mathrm{Rog}}_{(B,N)/(A,M)}$$

in $\mathrm{D}(B)$.

Proof. Let J be a simplicial module. Then the functor mapping J to the logarithmic derivations $\mathrm{Der}_{(A,M)}((B,N),J)$ is representable both by $\mathbb{L}_{(B,N)/(A,M)}$ and $\mathbb{L}^{\mathrm{Rog}}_{(B,N)/(A,M)}$. Using the Yoneda lemma, we thus conclude that $\mathbb{L}_{(B,N)/(A,M)}\cong\mathbb{L}^{\mathrm{Rog}}_{(B,N)/(A,M)}$ in the derived category $D(B)$ of simplicial B-modules. □

Rognes definition leads to simple proofs of the expected properties of the log cotangent complex.

Proposition 4.12. *(i) Let $(A,M) \to (B,N) \to (C,O)$ be maps of simplicial pre-log rings. Then there is a transitivity homotopy cofiber sequence in the homotopy category of simplicial C-modules*

$$C \otimes_B^{\mathbb{L}} \mathbb{L}_{(B,N)/(A,M)} \longrightarrow \mathbb{L}_{(C,O)/(A,M)} \longrightarrow \mathbb{L}_{(C,O)/(B,N)}.$$

(ii) Let

$$\begin{tikzcd} (A,M) \arrow[r] \arrow[d] & (B,N) \arrow[d] \\ (R,P) \arrow[r] & (S,Q) \end{tikzcd}$$

be a homotopy pushout square in $s\mathcal{L}$, then there is an isomorphism in the homotopy category of simplicial S-modules

$$S \otimes_B^{\mathbb{L}} \mathbb{L}_{(B,N)/(A,M)} \simeq \mathbb{L}_{(S,Q)/(R,P)}.$$

Proof. These follow immediately from [Rog09, Propositions 11.28 and 11.29]. □

4.3. **Square zero extensions.**

Definition 4.13. Let $(f,f^\flat)\colon (A,M) \to (B,N)$ be a morphism of log simplicial rings, J a simplicial B-module, and $\eta\colon \Omega_{(B,N)/(A,M)} \to J$ a derivation. We define (B^η, N^η) via the pullback diagram in $s\mathcal{L}_{(A,M)}$

$$\begin{tikzcd} (B^\eta,N^\eta) \arrow[r] \arrow[d, "p_\eta"'] & (B,N) \arrow[d, "s_0"] \\ (B,N) \arrow[r, "\eta"] & (B\oplus J, N\oplus J) \end{tikzcd}$$

and call the map $p_\eta\colon (B^\eta,N^\eta) \to (B,N)$ the *natural projection.*

This defines a functor

$$\Phi\colon (\mathrm{Mod}_B)_{\Omega_{(B,N)/(A,M)}} \to s\mathcal{L}_{(A,M)//(B,N)}$$

from the category of simplicial modules under the log Kähler differentials to the category of log simplicial (A,M)-algebras augmented over (B,N). This functor has a left adjoint Ψ given by mapping an object $(A,M) \to (C,O) \to (B,N)$ to the sequence of differentials $\Omega_{(B,N)/(A,M)} \to \Omega_{(B,N)/(C,O)}$. It is straightforward to verify that this defines a Quillen adjunction, so that we obtain derived functors $L\Psi$ and $R\Phi$.

The following statement is proved in Appendix A:

Proposition 4.14. *Let $\pi\colon R \to S$ be a square zero extension of discrete commutative rings, and let $J = \ker\pi$ be the corresponding square zero ideal. Then there exists a derivation $d \in \pi_0\mathrm{Map}_{R/s\mathcal{A}/S}(S, S\oplus J[1])$ such that there exists an isomorphism in* $\mathrm{Ho}(s\mathcal{R}/S)$*, between $\pi\colon R \to S$ and the canonical projection $p_d\colon S\oplus_d J \to S$, where p_d is defined by the homotopy pullback diagram*

$$\begin{tikzcd} S\oplus_d J \arrow[r] \arrow[d, "p_d"'] & S \arrow[d, "0"] \\ S \arrow[r, "d"] & S\oplus J[1]. \end{tikzcd}$$

Theorem 4.17 below provides the analog of Proposition 4.14 for strict square zero log extensions of discrete log rings. The proof of that theorem will be based on the next two lemmas. For this recall that a monoid P is *integral* if the group completion map $P \to P^{\mathrm{gp}}$ is injective.

The following elementary result is well known (see for example the remark after [Kat89, Definition (4.6)]). We include a proof since we were unable to locate one in the literature.

Lemma 4.15. *A strict square zero extension of discrete and integral log rings* $(\pi, \pi^\flat)\colon (R, P, \alpha) \to (S, Q, \beta)$ *is exact, i.e., the diagram*

$$\begin{tikzcd} P \arrow[r] \arrow[d, "\pi^\flat"'] & P^{\mathrm{gp}} \arrow[d, "(\pi^\flat)^{\mathrm{gp}}"] \\ Q \arrow[r] & Q^{\mathrm{gp}} \end{tikzcd}$$

is cartesian in the category of commutative monoids.

If (R, P, α) is a discrete log ring, then we write $c_R\colon R^\times \to P$ for the composite of the inverse of the isomorphism $\alpha^{-1}(R^\times) \to R^\times$ and the canonical map $\alpha^{-1}(R^\times) \to P$.

Proof. Let $J = \ker \pi$. Since π is surjective and $J^2 = 0$, we have $\pi^{-1}(S^\times) \cong R^\times$. Together with the log condition on (R, P, α), this provides isomorphisms $R^\times \cong \alpha^{-1}(R^\times) \cong \alpha^{-1}\pi^{-1}(S^\times)$, and therefore strictness implies that we have a pushout of commutative monoids

$$\begin{tikzcd} R^\times \arrow[r, "c_R"] \arrow[d, "\pi^\times"'] & P \arrow[d, "\pi^\flat"] \\ S^\times \arrow[r, "c_S"] & Q. \end{tikzcd}$$

Since both $R^\times$ and $S^\times$ are abelian groups, we have $Q \cong P \oplus S^\times/\!\sim$ where

$$(x, u) \sim (x', u') \Leftrightarrow \exists\, v \in R^\times \text{ such that } c_R(v)x = x' \text{ and } \pi(v^{-1})u = u'.$$

Given $[x, u] \in P \oplus S^\times/\!\sim$, the square zero condition implies that there exists $v \in R^\times$ such that $\pi(v) = u$. Since $(x, u) \sim (c_R(v)x, 1)$, the morphism $\pi^\flat$ maps the element $c_R(v)x \in P$ to $[x, u] \in Q \cong P \oplus S^\times/\!\sim$. Hence the morphism $\pi^\flat$ is surjective.

Consider the morphism of monoids

$$\varphi\colon P \longrightarrow Q \times_{Q^{\mathrm{gp}}} P^{\mathrm{gp}} \qquad x \longmapsto (\pi^\flat(x), [x, 1]_{\mathrm{gp}})$$

where we have denoted by $x \mapsto [x, 1]_{\mathrm{gp}}$ the canonical map $P \to P^{\mathrm{gp}}$, with P^{gp} implicitly identified, as usual, with a quotient of $P \times P$.

We will prove that φ is bijective, hence a monoid isomorphism. Since P is integral, the map $P \to P^{\mathrm{gp}}$ is injective, and thus so is φ. In order to prove surjectivity of φ, we first observe that if $(y, [x_1, x_2]_{\mathrm{gp}})$ is an arbitrary element in $Q \times_{Q^{\mathrm{gp}}} P^{\mathrm{gp}}$, then there exists $x \in P$ such that $\pi^\flat(x) = y$. It follows that $\pi^\flat(xx_2) = \pi^\flat(x_1)$ since y and $[x_1, x_2]_{\mathrm{gp}}$ have the same image in Q^{gp}, and Q is integral. But under the identification $Q \cong P \oplus S^\times/\!\sim$, we have $\pi^\flat(x') = [x', 1]$ for any $x' \in P$, and hence $[xx_2, 1] = [x_1, 1]$ in $P \oplus S^\times/\!\sim$. By definition of $\sim$, there exists $v \in R^\times$ such that

$$c_R(v)xx_2 = x_1 \quad \text{and} \quad \pi(v^{-1})1 = 1.$$

In particular, we have $\pi(v) = 1$. Now, let us prove that $\varphi((c_R(v)x)) = (y, [x_1, x_2])$. By definition, we have

$$\varphi(c_R(v)x) = (\pi^\flat c_R(v)\pi^\flat(x), [c_R(v)x, 1]_{\mathrm{gp}}).$$

Since $c_R(v)xx_2 = x_1$, we have $[c_R(v)x, 1]_{\mathrm{gp}} = [x_1, x_2]_{\mathrm{gp}}$, and since $\pi^\flat c_R(v) = c_S\pi(v) = 1 \in Q$, we also have $\pi^\flat c_R(v)\pi^\flat(x) = \pi^\flat(x) = y$. This completes the proof of the surjectivity of φ. □

If (R, P) is a discrete log ring, then $c_R^{\mathrm{gp}}\colon R^\times \to P^{\mathrm{gp}}$ denotes the composite of the map $c_R\colon R^\times \to P$ from the log condition and the group completion map $P \to P^{\mathrm{gp}}$.

Lemma 4.16. *Let* $(\pi, \pi^\flat)\colon (R, P, \alpha) \to (S, Q, \beta)$ *be a strict square zero extension of discrete integral log rings, and let* $J = \ker \pi$ *be the corresponding square zero ideal. Then*

$$\exp\colon J \longrightarrow R^\times, \qquad \xi \longmapsto (1 + \xi)$$

and the maps c_R^{gp} and c_S^{gp} induce a commutative diagram

$$\begin{tikzcd} J \arrow[r, "\exp"] \arrow[d] & R^\times \arrow[r, "c_R^{\mathrm{gp}}"] \arrow[d, "\pi^\times"] & P^{\mathrm{gp}} \arrow[d, "(\pi^\flat)^{\mathrm{gp}}"] \\ \{1\} \arrow[r] & S^\times \arrow[r, "c_S^{\mathrm{gp}}"] & Q^{\mathrm{gp}}. \end{tikzcd} \tag{4.1}$$

of constant simplicial commutative monoids where both squares are homotopy cartesian and homotopy cocartesian.

Proof. It is immediate from the square zero condition that the left hand square is cartesian. Since it consists of constant simplicial monoids, it is also homotopy cartesian. Hence $J \to R^\times \to P^{\mathrm{gp}}$ is a homotopy fiber sequence of simplicial abelian groups, and it follows that the left hand square is also homotopy cocartesian as a square of simplicial abelian groups. By Corollary 2.14, it is therefore a homotopy cocartesian square of simplicial commutative monoids.

For the right hand square, we consider the commutative diagram

$$\begin{tikzcd} J \arrow[r, "\exp"] \arrow[d] & R^\times \arrow[r, "c_R"] \arrow[d, "\pi^\times"] & P \arrow[r] \arrow[d, "\pi^\flat"] & P^{\mathrm{gp}} \arrow[d, "(\pi^\flat)^{\mathrm{gp}}"] \\ \{1\} \arrow[r] & S^\times \arrow[r, "c_S"] & Q \arrow[r] & Q^{\mathrm{gp}}. \end{tikzcd} \tag{4.2}$$

Since $(\pi, \pi^\flat)$ is a strict square zero extension between discrete integral log rings, Lemma 4.15 implies that the right hand square in (4.2) is cartesian. The middle square in (4.2) is cartesian since the log conditions on (R, P, α) and (S, Q, β) and the square zero condition on π provide isomorphisms

$$(\pi^\flat)^{-1}(S^\times) \cong (\pi^\flat)^{-1}(\beta^{-1}(S^\times)) \cong \alpha^{-1}\pi^{-1}(S^\times) \cong \alpha^{-1}(R^\times) \cong R^\times.$$

Hence the right hand square in (4.1) is homotopy cartesian. Moreover, since we already observed that the left hand square in (4.2) is cartesian, it follows that the outer square is cartesian. Arguing as above, we deduce that the outer square is homotopy cocartesian as a square of constant simplicial commutative monoids. Since we already know that the left hand square in (4.1) is homotopy cocartesian, it follows that the right hand square in (4.1) is homotopy cocartesian. □

We are now able to prove the promised log-analog of Proposition 4.14.

Theorem 4.17. *Let $(\pi, \pi^\flat)\colon (R, P) \to (S, Q)$ be a strict square zero extension of discrete integral log rings, and let $J = \ker \pi$ be the corresponding square zero ideal. Then there exists a derivation*

$$(d, d^\flat) \in \pi_0 \mathrm{Map}_{(R,P)/s\mathcal{P}/(S,Q)}((S, Q), (S \oplus J[1], Q \oplus J[1]))$$

such that there exists an isomorphism in $\mathrm{Ho}(s\mathcal{P}/(S, Q))$ between $(\pi, \pi^\flat)$ and the canonical projection

$$(p_d, p_d^\flat)\colon (S \oplus_d J, Q \oplus_{d^\flat} J) \to (S, Q),$$

where $(p_d, p_d^\flat)$ is defined by the homotopy pullback diagram

$$\begin{tikzcd} (S \oplus_d J, Q \oplus_{d^\flat} J) \arrow[r] \arrow[d, "(p_d, p_d^\flat)"'] & (S, Q) \arrow[d, "0"] \\ (S, Q) \arrow[r, "(d, d^\flat)"] & (S \oplus J[1], Q \oplus J[1]). \end{tikzcd} \tag{4.3}$$

Proof. In order to simplify the exposition, we do not make explicit some of the cofibrant replacements needed to represent maps in homotopy categories by maps in the relevant model categories. By Proposition 4.14, there exists a ring derivation

$d\colon S \to S \oplus J[1]$ satisfying the statement of that Proposition. Let $\delta\colon S \to J[1]$ denote its $J[1]$ component. By Lemma 4.16, we have a homotopy pushout diagram

$$\begin{tikzcd} R^{\times} \arrow[r, "c_R^{\mathrm{gp}}"] \arrow[d, "\pi^{\times}"'] & P^{\mathrm{gp}} \arrow[d, "(\pi^{\flat})^{\mathrm{gp}}"] \\ S^{\times} \arrow[r, "c_S^{\mathrm{gp}}"] & Q^{\mathrm{gp}}. \end{tikzcd}$$

Using its universal property and $\delta\pi = 0$, it follows that

$$0\colon P^{\mathrm{gp}} \longrightarrow Q \oplus J[1] \qquad \text{and} \qquad S^{\times} \longrightarrow Q \oplus J[1], \quad v \longmapsto (c_S(v), v^{-1}\delta(v))$$

define a homomorphism $\epsilon^{\flat}\colon Q^{\mathrm{gp}} \to Q \oplus J[1]$ whose precomposition with $Q \to Q^{\mathrm{gp}}$ will be denoted by $\delta^{\flat}\colon Q \to Q \oplus J[1]$. Using this map, we define

$$d^{\flat} = (\mathrm{id}_Q, \delta^{\flat})\colon Q \longrightarrow Q \oplus J[1].$$

Since $\beta \circ c_S$ is the canonical inclusion map $S^{\times} \to S$, one can use the restrictions to P^{gp} and $S^{\times}$ in the above homotopy pushout to check that the diagram

$$\begin{tikzcd} Q \arrow[r, "d^{\flat}"] \arrow[d, "\beta"'] & Q \oplus J[1] \arrow[d, "\alpha_{J[1]}"] \\ S \arrow[r, "d"] & S \oplus J[1] \end{tikzcd}$$

commutes. Here $\alpha_{J[1]}\colon Q \oplus J[1] \to S \oplus J[1]$, $(s, \xi) \mapsto (\beta(s), \beta(s)\xi)$ denotes the log-structure map. This implies that $(d, d^{\flat})$ is indeed a log-derivation.

Next we use $(d, d^{\flat})$ and the trivial log derivation $\underline{0}$ to form the homotopy pullback (4.3). Since the forgetful functors from simplicial pre-log rings to simplicial commutative rings and simplicial commutative monoids preserve pullbacks, both the ring and the monoid component of (4.3) are homotopy pullbacks in the respective categories. By Proposition 4.14 we get an isomorphism

$$\begin{tikzcd} R \arrow[rr, "\chi"] \arrow[dr, "\pi"'] & & S \oplus_d J \arrow[dl, "p_d"] \\ & S & \end{tikzcd}$$

in $\mathrm{Ho}(s\mathcal{A}/S)$. Since $d^{\flat} \circ \pi^{\flat} = (\pi^{\flat}, 0) = \underline{0} \circ \pi^{\flat}$, we also get an induced map $\chi^{\flat}\colon P \to Q \oplus_{d^{\flat}} J$ of simplicial monoids over Q, where $Q \oplus_{d^{\flat}} J$ maps to Q via $p_d^{\flat}$. Moreover, since the following diagram commutes

$$\begin{tikzcd} P \arrow[rr, "\pi^{\flat}"] \arrow[dr, "\alpha"] \arrow[dd, "\pi^{\flat}"'] & & Q \arrow[dr, "\beta"] \arrow[dd, "d^{\flat}", near start] & \\ & R \arrow[rr, "\pi", near start] \arrow[dd, "\pi", near start] & & S \arrow[dd, "d"] \\ Q \arrow[rr, "0^{\flat}", near start] \arrow[dr, "\beta"'] & & Q \oplus J[1] \arrow[dr, "\alpha_{J[1]}"] & \\ & S \arrow[rr, "0"'] & & S \oplus J[1], \end{tikzcd}$$

we get that the pair $(\chi, \chi^{\flat})\colon (R, P) \to (S \oplus_d J, Q \oplus_{d^{\flat}} J)$ is indeed a map in $s\mathcal{L}/(S, Q)$.

We already know from Proposition 4.14 that χ is an equivalence of simplicial rings. To prove that $\chi^{\flat}$ is an equivalence, we consider the following commutative diagram of simplicial commutative monoids:

(4.4)
$$\begin{tikzcd} J \arrow[r, "\exp"] \arrow[d] & R^{\times} \arrow[r, "c_R"] \arrow[d, "\pi^{\times}"] & P \arrow[r] \arrow[d, "\pi"] & P^{\mathrm{gp}} \arrow[r] \arrow[d, "\pi^{\flat}"] & \{1\} \arrow[d] \\ \{1\} \arrow[r] & S^{\times} \arrow[r, "c_S"] & Q \arrow[r] & Q^{\mathrm{gp}} \arrow[r, "\epsilon^{\flat}"] & J[1]. \end{tikzcd}$$

The outer square is homotopy cartesian and homotopy cocartesian as a square of simplicial abelian groups. Using Corollary 2.14 and Lemma 4.16, it follows that the rightmost square in (4.4) is homotopy cocartesian. Since it consists of simplicial abelian groups, it is also homotopy cartesian in $s\mathcal{AB}$ and hence also in $s\mathcal{M}$. The third square (spanned by π and $\pi^\flat$) is cartesian by Lemma 4.15. Since it consists of discrete simplicial monoids, it is also homotopy cartesian. It follows that the composite

$$\begin{tikzcd} P \arrow[r] \arrow[d, "\pi^\flat"'] & \{1\} \arrow[d] \\ Q \arrow[r, "\delta^\flat"] & J[1] \end{tikzcd}$$

of the two rightmost squares is homotopy cartesian.

On the other hand, by definition of $Q \oplus_{d^\flat} J$, the outer square in the diagram

$$\begin{tikzcd} Q \oplus_{d^\flat} J \arrow[r] \arrow[d, "p^\flat_d"'] & Q \arrow[r] \arrow[d, "0^\flat"] & \{1\} \arrow[d] \\ Q \arrow[r, "d^\flat"] & Q \oplus J[1] \arrow[r, "\mathrm{proj}"] & J[1] \end{tikzcd}$$

is homotopy cartesian. Now it is enough to observe that, since by definition of $\chi^\flat$, we have $p^\flat \circ \chi^\flat = \pi^\flat$, and the following diagram

$$\begin{tikzcd} P \arrow[rr, "\pi^\flat"] \arrow[dr, "\chi^\flat"] \arrow[dd] & & Q \arrow[dr, "\mathrm{id}"] \arrow[dd, "\delta^\flat"] & \\ & Q \oplus_{d^\flat} J \arrow[rr, "p^\flat"] \arrow[dd] & & Q \arrow[dd, "\delta^\flat"] \\ \{1\} \arrow[rr] \arrow[dr] & & J[1] \arrow[dr, "\mathrm{id}"] & \\ & \{1\} \arrow[rr] & & J[1] \end{tikzcd}$$

commutes. The front and rear faces of this diagram are homotopy pullbacks, hence we conclude that $\chi^\flat$ is an equivalence of simplicial commutative monoids. □

Remark 4.18. Note that the homotopy pullback in Theorem 4.17 is a homotopy pullback in $s\mathcal{L}/(S,Q)$, since the forgetful functor $s\mathcal{L}/(S,Q) \to s\mathcal{L}$ creates homotopy pullbacks. Moreover, Theorem 4.17 holds true if we work in the undercategory $(A,M)/s\mathcal{L}$, where (A,M) is any simplicial log ring, i.e., $(\pi,\pi^\flat)$ is a map in $(A,M)/s\mathcal{L}$ and a strict square zero extension of discrete log rings, and the homotopy pullback defining $(S \oplus_d J, Q \oplus_{d^\flat} J)$ in Theorem 4.17 is taken in $(A,M)/s\mathcal{L}$ (or equivalently in $(A,M)/s\mathcal{L}/(S,Q)$).

5. Derived log-étale maps

Following [Kat89, 3.2], we give the following

Definition 5.1. A morphism $(f, f^\flat)\colon (A,M) \to (B,N)$ of discrete log rings will be called *formally log-étale* if for any strict square zero extension of discrete integral log rings $(g,g^\flat)\colon (R,P) \to (S,Q)$ and every commutative diagram

$$\begin{tikzcd} (S,Q) & (B,N) \arrow[l] \\ (R,P) \arrow[u, "{(g,g^\flat)}"] & (A,M) \arrow[l] \arrow[u, "{(f,f^\flat)}"'] \end{tikzcd}$$

there exists a unique $(h,h^\flat)\colon (B,N) \to (R,P)$ such that the resulting triangles commute.

A morphism $(f, f^\flat)\colon (A, M) \to (B, N)$ of discrete log rings will be called *log-étale* if it is formally log-étale and the underlying map $f\colon A \to B$ is finitely presented.[1]

Remark 5.2. Note that Kato defines étale morphisms only in the category of *fine* log rings, i.e., both $(f, f^\flat)$ and $(g, g^\flat)$ are required to be morphisms of fine log rings. Thus if a map of fine log ring $(A, M) \to (B, N)$ is étale in the sense of Definition 5.1, then it is étale in Kato's sense. Since there seems to be no homotopically meaningful way to define the notion of a fine log structure, we have chosen the previous more general definition, i.e, we just probe using square zero extension of integral, but not necessarily fine, log rings.

For log simplicial rings, we adopt the following

Definition 5.3. A morphism $(f, f^\flat)\colon (A, M) \to (B, N)$ of log simplicial rings will be called *derived formally log-étale* if $\mathbb{L}_{(B,N)/(A,M)}$ is trivial.

A morphism $(f, f^\flat)\colon (A, M) \to (B, N)$ of log simplicial rings will be called *derived log étale* if it is formally derived log-étale, and the underlying morphism f is homotopically finitely presented [TV08, Definition 1.2.3.1].

Proposition 5.4. *The composition of two derived log-étale maps is derived log-étale. If $f\colon (A, M) \to (B, N)$ and $g\colon (A, M) \to (C, O)$ are maps of simplicial pre-log algebras, and f is derived log-étale, then the homotopy base change map*

$$(C, O) \longrightarrow (B, N) \textstyle\coprod^h_{(A,M)} (C, O)$$

is derived log-étale.

Proof. First of all, observe that being of finite presentation is stable under composition and base-change. The remaining statements about the cotangent complex, follow from the transitivity sequence, and from the so-called flat base-change, i.e., from Proposition 4.12. ☐

Since a map of log simplicial rings $(f, f^\flat)\colon (A, M) \to (B, N)$ is strict if and only if the induced map of pre-log rings $(B, f_*(M)) \to (B, N)$ is a weak equivalence in the log model structure, it follows that strict maps are preserved under base change:

Lemma 5.5. *Let*

$$\begin{array}{ccc} (A, M) & \longrightarrow & (B, N) \\ \downarrow & & \downarrow \\ (R, P) & \longrightarrow & (S, Q) \end{array}$$

be a homotopy pushout of log simplicial rings. If $(A, M) \to (B, N)$ is strict, then so is $(R, P) \to (S, Q)$.

Proof. We consider the induced square

$$\begin{array}{ccccc} (A, M) & \longrightarrow & (B, f_*(M)) & \longrightarrow & (B, N) \\ \downarrow & & \downarrow & & \downarrow \\ (R, P) & \longrightarrow & (S, f_*(P)) & \longrightarrow & (S, Q). \end{array}$$

Here the outer square is homotopy cocartesian by assumption, and it follows easily that the left hand square is homotopy cocartesian. Hence the right hand square is homotopy cocartesian. If $(B, f_*(M)) \to (B, N)$ is a weak equivalence in the log model structure, this implies that $(S, f_*(P)) \to (S, Q)$ also has this property. ☐

[1]We here have adopted the convention of Gabber-Ramero [GR14] of not imposing any finiteness condition on the monoid map.

The following Theorem shows that the previous notion of log-étaleness implies the classical one on the truncation. We recall the following notation: Given a model category C and objects A and B of C, we will denote the respective slice model categories by A/C, C/B and $A/C/B$.

Theorem 5.6. *If a morphism $(f, f^\flat)\colon (A, M) \to (B, N)$ of log simplicial rings is derived log-étale and $\pi_0 f^\flat$ is of finite presentation, then the induced morphism $(\pi_0 f, \pi_0 f^\flat)\colon (\pi_0 A, \pi_0 M) \to (\pi_0 B, \pi_0 N)$ is a log-étale morphism of discrete log rings (in the sense of Definition 5.1).*

Proof. First of all observe that the left Quillen functor $\pi_0 : s\mathcal{P} \to \mathcal{P}$ preserves finitely presented objects. So we are left to prove that $(\pi_0 f, \pi_0 f^\flat)$ is formally étale. Let $(\pi, \pi^\flat)\colon (R, P) \to (S, Q)$ be a strict square zero extension of discrete integral log rings under (A, M), with square zero ideal J.

We have to prove that the canonical map

$$\mathrm{Hom}_{(\pi_0 A, \pi_0 M)/\mathcal{P}}((\pi_0 B, \pi_0 N), (R, P)) \longrightarrow \mathrm{Hom}_{(\pi_0 A, \pi_0 M)/\mathcal{P}}((\pi_0 B, \pi_0 N), (S, Q))$$

is bijective. Since (R, P) and (S, Q) are discrete, this is equivalent to showing that the canonical map

$$u\colon \mathrm{Map}_{(A,M)/s\mathcal{L}}((B, N), (R, P)) \longrightarrow \mathrm{Map}_{(A,M)/s\mathcal{L}}((B, N), (S, Q))$$

is a weak equivalence of simplicial sets. This is true if and only if for any 0-simplex $\underline{\varphi} = (\varphi, \varphi^\flat)$ in $\mathrm{Map}_{(A,M)/s\mathcal{L}}((B, N), (S, Q))$, the homotopy fiber

$$\mathrm{hofib}(\underline{\varphi}) = \mathrm{hofib}(u; \underline{\varphi})$$

is non-empty and contractible. In order to establish this, let us write $B_{\underline{\varphi}}$ for (B, N) viewed as an object in $(A, M)/s\mathcal{L}/(S, Q)$ via $\underline{\varphi}$, and let $\varphi^* J[1]$ be $J[1]$ viewed as a B-module via φ. We consider the map

$$\rho_{\underline{\varphi}}\colon \mathrm{Map}_{(A,M)/s\mathcal{L}/(S,Q)}(B_{\underline{\varphi}}, (S, Q) \oplus J[1]) \longrightarrow \mathrm{Map}_{(A,M)/s\mathcal{L}/(S,Q)}((B, N), (B, N) \oplus \varphi^* J[1])$$

induced by sending

$$(D, D^\flat)\colon B_{\underline{\varphi}} \to (S, Q) \oplus J[1]$$

to the map $(B, N) \to (B, N) \oplus \varphi^* J[1]$ whose projection to (B, N) is the identity, and whose projection to $\varphi^* J[1]$ is given, on B and N, respectively, by the composites

$$B \xrightarrow{D} S \oplus J[1] \xrightarrow{\mathrm{pr}_{J[1]}} J[1] \qquad \text{and} \qquad N \xrightarrow{D^\flat} Q \oplus J[1] \xrightarrow{\mathrm{pr}_{J[1]}} J[1]\ .$$

In the notation of Theorem 4.17, we obtain a diagram

$$\begin{tikzcd}
\mathrm{hofib}(\underline{\varphi}) \arrow[r] \arrow[d] & \bullet \arrow[d, "\underline{\varphi}"] \\
\mathrm{Map}_{(A,M)/s\mathcal{L}/(S,Q)}(B_{\underline{\varphi}}, (R, P)) \arrow[r, "p_d"] \arrow[d] & \mathrm{Map}_{(A,M)/s\mathcal{L}/(S,Q)}(B_{\underline{\varphi}}, (S, Q)) \arrow[d, "(d, d^\flat)"] \\
\mathrm{Map}_{(A,M)/s\mathcal{L}/(S,Q)}(B_{\underline{\varphi}}, (S, Q)) \arrow[r, "\underline{0}"] \arrow[d] & \mathrm{Map}_{(A,M)/s\mathcal{L}/(S,Q)}(B_{\underline{\varphi}}, (S, Q) \oplus J[1]) \arrow[d, "\rho_{\underline{\varphi}}"] \\
\bullet \arrow[r, "\underline{0}"'] & \mathrm{Map}_{(A,M)/s\mathcal{L}/(B,N)}((B, N), (B, N) \oplus \varphi^* J[1]).
\end{tikzcd}$$

The top square is homotopy cartesian by definition. The middle one is homotopy cartesian by Theorem 4.17, the fact that Map preserves homotopy cartesian squares, and the fact that the forgetful functor $(A, M)/s\mathcal{L}/(S, Q) \to (A, M)/s\mathcal{L}$ creates homotopy pullbacks, so that the homotopy pullback of Theorem 4.17 is actually a homotopy pullback in $(A,M)/s\mathcal{L}/(S,Q)$ (see Remark 4.18). The bottom square is homotopy cartesian by definition of the map $\rho_{\underline{\varphi}}$. If we set $D_{\underline{\varphi}} = \rho_{\underline{\varphi}} \circ (d, d^\flat) \circ \underline{\varphi}$, we see that $\mathrm{hofib}(\underline{\varphi})$ is non-empty if and only if $D_{\underline{\varphi}}$ is zero in

$$\pi_0 \operatorname{Map}_{(A,M)/s\mathcal{L}/(B,N)}((B,N),(B,N) \oplus \varphi^* J[1]) \cong \operatorname{Ext}^1_B(\mathbb{L}_{(B,N)/(A,M)}, \varphi^* J),$$

and in this case

$$\begin{aligned} \mathrm{hofib}(\underline{\varphi}) &\cong \Omega_0 \operatorname{Map}_{(A,M)/s\mathcal{L}/(B,N)}((B,N),(B,N) \oplus \varphi^* J[1]) \\ &\cong \Omega_0 \operatorname{Map}_{B-\mathrm{Mod}}(\mathbb{L}_{(B,N)/(A,M)}, \varphi^* J[1]). \end{aligned}$$

If $\mathbb{L}_{(B,N)/(A,M)} \cong 0$, $\mathrm{hofib}(\underline{\varphi})$ is then non-empty and contractible. □

6. Derived log-smooth maps

Following [Kat89], we give the following

Definition 6.1. A morphism $(f, f^\beta)\colon (A,M) \to (B,N)$ of discrete log rings, will be called *formally log-smooth* if for any strict square zero extension of discrete integral log rings $(g, g^\flat)\colon (R,P) \to (S,Q)$, the canonical map

$$\operatorname{Hom}_{\mathcal{L}_{(A,M)}}((B,N),(R,P)) \longrightarrow \operatorname{Hom}_{\mathcal{L}_{(A,M)}}((B,N),(S,Q)$$

is surjective. Here $\mathcal{L}_{(A,M)}$ denotes the slice category of $\mathcal{L}$ over (A,M).

A morphism $(f, f^\flat)\colon (A,M) \to (B,N)$ of discrete log rings, will be called *log-smooth* if it is formally log-smooth, and the underlying map $f\colon A \to B$ is finitely presented as a map of commutative algebras.

Remark 6.2. Remark 5.2 applies analogously to log smooth maps: We define smoothness on a larger class of maps than Kato does, and if a map of fine log rings is smooth in the sense of Definition 6.1, then it is smooth in Kato's sense.

Definition 6.3. A morphism $(f, f^\flat)\colon (A,M) \to (B,N)$ of log simplicial rings will be called *derived formally log-smooth* if

$$\operatorname{Hom}_{\mathrm{Ho(B-Mod)}}(\mathbb{L}_{(B,N)/(A,M)}, J) \simeq 0$$

for any simplicial B-module J with $\pi_0 J = 0$.

A morphism $(f, f^\flat)\colon (A,M) \to (B,N)$ of log simplicial rings will be called *derived log-smooth* if it is formally log-smooth and $f\colon A \to B$ is homotopically finitely presented.

The following Theorem shows that the notion of derived log-smoothness implies the classical one on the truncations.

Theorem 6.4. *If $(f, f^\flat)\colon (A,M) \to (B,N)$ of log simplicial rings is derived log-smooth, then the induced morphism $(\pi_0 f, \pi_0 f^\flat)\colon (\pi_0 A, \pi_0 M) \to (\pi_0 B, \pi_0 N)$ is a log-smooth morphism of discrete log rings (in the sense of Definition 6.1).*

Proof. Let $(\pi, \pi^\flat)\colon (R,P) \to (S,Q)$ be a strict square zero extension of discrete integral log rings under (A,M), with square zero ideal J. Following the arguments and the notation in the proof of Theorem 5.6 shows that the map

$$\operatorname{Hom}_{(\pi_0 A, \pi_0 M)/\mathcal{L}}((\pi_0 B, \pi_0 N),(R,P)) \longrightarrow \operatorname{Hom}_{(\pi_0 A, \pi_0 M)/\mathcal{L}}((\pi_0 B, \pi_0 N),(S,Q))$$

is surjective if and only if for any 0-simplex $\underline{\varphi}$ in $\operatorname{Map}_{(A,M)/s\mathcal{L}}((B,N),(S,Q))$, the associated element $D_{\underline{\varphi}}$ is zero in

$$\begin{aligned} &\pi_0 \operatorname{Map}_{(A,M)/s\mathcal{L}/(B,N)}((B,N),(B,N) \oplus \varphi^* J[1]) \\ &\cong \operatorname{Ext}^1_B(\mathbb{L}_{(B,N)/(A,M)}, \varphi^* J) \cong \operatorname{Hom}_{\mathrm{Ho(B-Mod)}}(\mathbb{L}_{(B,N)/(A,M)}, \varphi^* J[1]). \end{aligned}$$

But $\pi_0(\varphi^* J[1]) = 0$, so the result follows. □

7. Derived log stacks

In giving our definitions, we will not mention explicitly the proper choices of universes: the reader will find they are the same as in [TV05].

7.1. **Derived log prestacks.** Throughout we fix a base commutative ring k. If we view k as a constant simplicial ring with the trivial simplicial pre-log structure, then the category of pre-log simplicial k-algebras is the category pre-log simplicial rings under k. It is denoted by $s\mathcal{P}_k$ and inherits an injective and a projective model structure from $s\mathcal{P}$. Likewise, we obtain a model category of log simplicial rings $s\mathcal{L}_k$ from Theorem 3.13 as the comma category $k \downarrow s\mathcal{L}$.

Definition 7.1. The category of *derived log affines over* k is the opposite category $\mathrm{dLogAff}_k$ of $s\mathcal{L}_k$, and we let

$$\mathrm{SPr}(\mathrm{dLogAff}_k) := \mathcal{S}^{\mathrm{dLogAff}_k^{\mathrm{op}}} = \mathcal{S}^{s\mathcal{L}_k}$$

be the category of simplicial presheaves on derived log affines over k.

Note that $\mathrm{dLogAff}_k$ is a simplicial model category, and that $\mathrm{SPr}(\mathrm{dLogAff}_k)$ is simplicially enriched by

$$\mathrm{Hom}_{\mathrm{SPr}(\mathrm{dLogAff}_k)}(F,G)_n := \mathrm{Hom}_{\mathrm{SPr}(\mathrm{dLogAff}_k)}(F \times \Delta^n, G)$$

where

$$(F \times \Delta^n)(A,M) := F(A,M) \times \Delta^n.$$

Proposition 7.2. *The category* $\mathrm{SPr}(\mathrm{dLogAff}_k)$ *admits a left proper cellular model structure where the weak equivalences and the fibrations are defined object-wise.*

Proof. This is [TV05, Propositions A.1.3(1) and A.2.5]. □

Consider the Yoneda functor

$$\mathrm{dLogAff}_k \longrightarrow \mathrm{SPr}(\mathrm{dLogAff}_k), \quad X \longmapsto h_X := \mathrm{Hom}_{\mathrm{dLogAff}_k}(-,X),$$

and define

$$h_W := \{h_w : h_X \to h_Y \,|\, w : X \to Y \text{ a weak equivalence in } \mathrm{dLogAff}_k\}.$$

Definition 7.3. The category of *log prestacks over* k is the model category $\mathrm{dLogAff}_k^{\wedge}$ obtained as the left Bousfield localization of $\mathrm{SPr}(\mathrm{dLogAff}_k)$ with respect to h_W.

Remark 7.4. In the notations of [TV05, Definitions 2.3.3 and 4.1.4], the model category $\mathrm{dLogAff}_k^{\wedge}$ (with the appropriate choice of universes) is denoted as $(\mathrm{dLogAff}_k, S)^{\wedge}$, where S stands for the weak equivalences in $\mathrm{dLogAff}_k$.

Note that, by standard properties of left Bousfield localizations (see e.g. [Hir03]), $\mathrm{Ho}(\mathrm{dLogAff}_k^{\wedge})$ can be identified with the full subcategory of $\mathrm{Ho}(\mathrm{SPr}(\mathrm{dLogAff}_k))$ consisting of functors $F\colon \mathrm{dLogAff}_k^{\mathrm{op}} \longrightarrow \mathcal{S}$ preserving weak equivalences.

We are now able to define a derived log analog of the spectrum functor.

Definition 7.5. We define the *derived log spectrum* functor Spec as follows

$$\mathrm{Spec}\colon \mathrm{Ho}(\mathrm{dLogAff}_k) \longrightarrow \mathrm{Ho}(\mathrm{dLogAff}_k^{\wedge}), \quad (A,M) \longmapsto \mathbf{Hom}_{s\mathcal{P}}(Q(A,M), R(-))$$

where $Q(-)$ (respectively, $R(-)$) denotes a cofibrant (resp., fibrant) replacement functor in the model category $s\mathcal{P}^{\mathrm{proj}}$, and $\mathbf{Hom}_{s\mathcal{P}}(-,-)$ the simplicial enrichment in $s\mathcal{P}$.

Equivalently, we could have defined Spec as in [TV05, Definition 4.2.5]. The model category version of the Yoneda lemma ([TV05, Corollary 4.2.4]), tells us that

Proposition 7.6. *The Spec functor is fully faithful, and for any* $(A,M) \in s\mathcal{P}$, *and any* $F \in \mathrm{dLogAff}_k^{\wedge}$, *we have a canonical isomorphism in Ho*$(\mathcal{S})$,

$$\mathrm{Map}_{dLogAff_k^{\wedge}}(\mathrm{Spec}(A,M), F) \simeq F(A,M).$$

7.2. **Derived log stacks.**

Definition 7.7. A family $\{(A,M) \to (A_i, M_i)\}_{i\in \mathrm{I}}$ of morphisms in $s\mathcal{P}_k$ is called a *strict log-étale covering family* of (A,M) in $\mathrm{dLogAff}_k$ if

- each $(A,M) \longrightarrow (A_i, M_i)$ is a strict log-étale morphism (of simplicial pre-log k-algebras), and
- there exists a finite subset $\mathrm{J} \subseteq \mathrm{I}$ such that the family of base-change functors $\{- \otimes^{\mathbb{L}}_A A_j \colon \mathrm{Ho}(s\mathrm{Mod}_A) \longrightarrow \mathrm{Ho}(s\mathrm{Mod}_{A_j})\}_{j\in \mathrm{J}}$ is conservative.

Proposition 7.8. *The collection of strict log-étale covering families form a model pre-topology on the model category* $\mathrm{dLogAff}_k$ *in the sense of* [TV05, Definition 4.3.1].

Proof. This follows immediately from stability of strict log-étale maps with respect to composition and homotopy pullbacks (Proposition 5.4 and Lemma 5.5). □

Definition 7.9. We denote by *str-log-ét* both the model pre-topology, given by strict log-étale covering families, on $\mathrm{dLogAff}_k$, and the Grothendieck topology on $\mathrm{Ho}(\mathrm{dLogAff}_k)$ generated by the induced pre-topology.

To any $F \in \mathrm{SPr}(\mathrm{dLogAff}_k)$, we can associate the *sheaf* of connected components $\pi_0(F)$ on the strict log étale (usual) site $(\mathrm{Ho}(\mathrm{dLogAff}_k), \text{str-log-ét})$. And, for any $i > 0$, any fibrant $X \in \mathrm{dLogAff}_k$, and any $s \in F(X)_0$, we can consider the *sheaf* $\pi_i(F, s)$ on the comma site $(\mathrm{Ho}(\mathrm{dLogAff}_k /X), \text{str-log-ét})$ ([TV05, Definition 4.5.3.]).

Definition 7.10. A map $f : F \longrightarrow G$ in $\mathrm{SPr}(\mathrm{dLogAff}_k)$ is called a π_**-isomorphism* if the induced maps of sheaves

$$\pi_0(F) \longrightarrow \pi_0(G),$$

$$\pi_i(F, s) \longrightarrow \pi_i(G, f(s))$$

are isomorphisms, for any $i > 0$, any fibrant X, and any $s \in F(X)_0$.

Theorem 7.11. *There is a model structure on* $\mathrm{SPr}(\mathrm{dLogAff}_k)$ *in which the cofibrations are the same as those in* $\mathrm{dLogAff}_k^{\wedge}$*, and the weak equivalences are* π_**-isomorphisms.*

Proof. This follows from [TV05, Theorem 4.6.1]. □

Definition 7.12. The model category structure on $\mathrm{SPr}(\mathrm{dLogAff}_k)$ given by Theorem 7.11 will be called the *model category of derived log stacks*, and its homotopy category will be simply denoted by dLogSt_k

It follows from the proof of Theorem 7.11, and from basic properties of left Bousfield localizations, that dLogSt_k can be identified with the full subcategory of $\mathrm{Ho}(\mathrm{SPr}(\mathrm{dLogAff}_k))$ consisting of functors $F \colon \mathrm{dLogAff}_k^{\mathrm{op}} \longrightarrow \mathcal{S}$ such that F preserves weak equivalences and F satisfies *strict log-étale hyperdescent*, i.e., the canonical map

$$F(X) \longrightarrow \mathrm{holim}_{\Delta^{op}} F(H_\bullet) := \mathrm{holim}_{\Delta^{op}} \mathrm{Map}_{\mathrm{dLogAff}_k^{\wedge}}(H_\bullet, F)$$

is an isomorphism in $\mathrm{Ho}(\mathcal{S})$, for any strict log-étale pseudo-representable hypercover $H_\bullet \to h_X$ of X (see [TV05, Definition 4.6.5]).

In particular, we will say that an object $F \in \mathrm{Ho}(\mathrm{dLogAff}_k^{\wedge})$ *is* a derived log stack, if it satisfies the strict log-étale hyperdescent condition.

Proposition 7.13. *The strict log-étale model pre-topology on the model category* $\mathrm{dLogAff}_k$ *is sub-canonical (*[TV08, Definition 1.3.1.3]*), i.e., Spec*(A, M) *is a derived log stack, for any* $(A, M) \in s\mathcal{P}$*.*

Proof. We will only prove the case of a strict log-étale representable hypercover, leaving to the reader the general case of a a strict log-étale pseudo-representable hypercover (as in the proofs of Lemma 2.2.2.13 and Lemma 1.3.2.3 (2) in [TV08]). By using finite products, we can assume that we are working with a strict log-étale covering family given by a single map $(A, M) \to (B, N)$ in $s\mathcal{P}$. We have to show that the morphism $(A, M) \to |(B, N)_\bullet|$ is an isomorphism in $\mathrm{Ho}(s\mathcal{L}_k)$. Let $\mathrm{Ho}(s\mathcal{L}_k)^{\mathrm{str}}$ denote the sub-category of $\mathrm{Ho}(s\mathcal{L}_k)$ spanned by log simplicial rings with strict morphisms. Since strictness is preserved under homotopy colimits, $(A, M) \to |(B, N)_\bullet|$ gives a morphism in $\mathrm{Ho}(s\mathcal{L}_k)^{\mathrm{str}}$. Let $U\colon s\mathcal{L}_k \to s\mathrm{Alg}_k$ denote the functor that forgets the log structure. By strictness, the induced functor $U\colon \mathrm{Ho}(s\mathcal{L}_k)^{\mathrm{str}} \to \mathrm{Ho}(s\mathrm{Alg}_k)$ is conservative. The claim then follows from the string of isomorphisms in $\mathrm{Ho}(s\mathrm{Alg}_k)$

$$U(A, M) \to |U(B, N)_\bullet| \to U(|(B, N)|_\bullet)$$

where the first isomorphism comes from descent for the étale topology on dAff_k, and the second isomorphism holds because U commutes with homotopy colimits. □

By Proposition 7.13, the Spec functor factors as a fully faithful functor

$$\mathrm{Spec}\colon \mathrm{Ho}(\mathrm{dLogAff}_k) \longrightarrow \mathrm{dLogSt}_k.$$

Remark 7.14. One might also consider the not necessarily strict log-étale model pre-topology on the model category $\mathrm{dLogAff}_k$. The problem with this model topology is that it is very likely that it is not subcanonical. This is closely related to the fact that the log-étale topology on general (i.e not necessarily fs) log schemes is probably also not subcanonical.

7.3. **Geometric derived log stacks.** By following the same path as in [TV08], we give the following inductive definition

Definition 7.15. A derived log stack is (-1)-*geometric* if it is representable, i.e., isomorphic in dLogSt_k to $\mathrm{Spec}(A, M)$ for some simplicial pre-log k-algebra (A, M). Let $n \geq 0$ be an integer.

- A derived log stack $F \in \mathrm{dLogSt}_k$ is n-*geometric* if
  - the diagonal map $F \longrightarrow F \times F$ is $(n-1)$-representable
  - There exists a family $\{\mathrm{Spec}(A_i, M_i)\}_{i\in \mathrm{I}}$ of representable derived stacks, and a morphism

    $$p\colon \textstyle\coprod_i \mathrm{Spec}(A_i, M_i) \longrightarrow F,$$

    called an *atlas* for F, such that
    - ∗ the sheafification of $\pi_0(p)$ is an epimorphism of sheaves of sets on the site $(\mathrm{dLogSt}_k, \text{str-log-ét})$;
    - ∗ the induced morphism $p_i\colon \mathrm{Spec}(A_i, M_i) \longrightarrow F$ is log-smooth, for any $i \in \mathrm{I}$.
- A morphism $f : F \longrightarrow G$ in dLogSt_k is n-*representable* if for any representable X and any morphism $X \longrightarrow G$, the derived log stack $F \times_G X$ is n-geometric.
- An n-representable morphism $f : F \longrightarrow G$ in dLogSt_k is *log-smooth* if for any representable X and any morphism $X \longrightarrow G$, there exists an atlas $\coprod_i Y_i \longrightarrow F \times_G X$ for $F \times_G X$ such that each induced map $Y_i \longrightarrow X$ is log-smooth between representable derived stacks.

The statement of the Artin property for derived log stacks, and the corresponding version of Lurie's representability criterion will be treated in a sequel to this paper.

Remark 7.16. (Pre log and log modules.) If (A, M) be a simplicial pre-log algebra, there is an obvious category $\mathbf{PreLogMod}_{(A,M)}$ of pre-log modules over (A, M), whose objects are triples $(S, P, \varphi \colon S \to P)$ where S is a simplicial M-module (i.e., a simplicial set endowed with an action of the simplicial monoid M), P is a simplicial A-module , and φ is a map of simplicial sets that is equivariant with respect to the structure map $\alpha \colon M \to A$, i.e., such that the following diagram commutes

$$\begin{array}{ccc} M \times S & \longrightarrow & M \\ {\scriptstyle \alpha\times\varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ A \times P & \longrightarrow & P \end{array}$$

and whose morphisms are the natural ones. There is a model structure on $\mathbf{Mod}_{(A,M)}$ where weak equivalences (resp. fibrations) are pairs (f, g) where f is a weak equivalence (resp. a fibration) of simplicial sets, and g is a weak equivalence (resp. a fibration) of simplicial A-modules. Direct and inverse image functors define a Quillen pair, and there is a natural monoidal structure on $\mathbf{PreLogMod}_{(A,M)}$ such that algebras in $\mathbf{Mod}_{(k,1)}$ are exactly pre-log k-algebras. However, $\mathbf{PreLogMod}_{(A,M)}$ is very much non additive. This is reflected by the fact that we have functors

$$\mathbf{AbGrps}((k, 1)/\mathcal{P}/(A, M)) \hookrightarrow \mathbf{AbMonoids}((k, 1)/\mathcal{P}/(A, M)) \longrightarrow \mathbf{Mod}_{(A,M)}$$

where the left one is not an equivalence (while it is in the non pre-log case) and the right one is not essentially surjective (while it is an equivalence in the non pre-log case). One might however use [Mar13] to define a notion of flat topology on pre-log algebras (viewed as algebras in $\mathbf{Mod}_{(k,1)}$). Unfortunately, these flat maps have flat underlying maps of schemes, so they are not very interesting.

When (A, M) is a simplicial pre-log algebra with structure map α, there is a log variant $\mathbf{LogMod}_{(A,M)}$ of $\mathbf{PreLogMod}_{(A,M)}$, where we only consider those pre log modules $(S, P, \varphi \colon S \to P)$ such that the map $\alpha^{-1}(A_P^\times) \to A_P^\times$ is a weak equivalence (here $A_P^\times$ denotes the connected components of A acting as equivalences on P). We have not fully investigated the homotopy and monoidal structures on this category.

From a general point of view, in order to get an alternative theory of derived log geometry along these lines, we think it might be interesting to proceed as follows. Embed the category of (pre) log rings in the category of arrows between commutative monoids. This embedding is not full so something new is obtained. Then we may use the approach sketched in [TV09, §5.3] and [Mar13] to build a Zariski, flat or smooth topology for arrows between $\mathbb{S}_1$-derived schemes (i.e., the geometric objects of derived geometry over the monoidal model category of simplicial sets), and explore the derived geometry of objects arising via gluing (pre) log rings. This would roughly correspond classically to partially disregard the fact that there is an underlying scheme of a log scheme. This work remains to be done, and we feel like it is a worthwhile task since it might yield a new insight in the foundations of classical log geometry, too.

8. An example

This section provides an example of a non-trivial derived log stack. We construct a derived version of the logarithmic moduli of stable maps introduced by Gross and Siebert.

We begin by producing an inclusion functor from the category of stacks over discrete log rings to the category of derived stacks over log simplicial rings. To accomplish this, we endow the category of discrete log rings with the trivial model structure. Then the inclusion functor

$$i \colon \mathcal{L}_k \to s\mathcal{L}_k$$

from the category of log rings under a base ring k to the category of log simplicial rings under a base ring k is a right Quillen functor. As a consequence we obtain a Quillen adjunction for the categories of pre-stacks

$$i_!\colon \mathcal{S}^{\mathcal{L}_k} \rightleftarrows \mathrm{dLogAff}_k^{\wedge}\colon i^*.$$

Here $\mathcal{S}^{\mathcal{L}_k}$ is the category of simplicial pre-sheaves on $\mathcal{L}_k$ equipped with the projective model structure.

We equip the category $\mathcal{L}_k$ with the strict étale topology, and using the same construction as in Theorem 7.11 we can define the model category of higher log stacks (see [TV08, Section 2.1] for the construction of (non-derived) higher stacks in the non-logarithmic context).

To verify that the above adjunction descends to the category of stacks we have to check that i preserves coproducts, equivalences and hypercovers. The only non-trivial part is to verify that i preserves strict étale morphisms, since in the discrete case étaleness is characterized by the vanishing of the one-truncated cotangent complex, whereas in the non-discrete case the full cotangent complex must vanish.

Lemma 8.1. *Let $(A, M) \to (B, N)$ be a strict étale morphism of discrete log rings. Then $\mathbb{L}_{(B,N)/(A,M)} \simeq 0$.*

Proof. Since the morphism is strict, we have an equivalence $\mathbb{L}_{(B,N)/(A,M)} \simeq \mathbb{L}_{B/A}$, so that we deduce that $\tau_{\leq 1}\mathbb{L}_{B/A} \simeq 0$. But by [Ill71, Prop. 3.1.1] this implies that $\mathbb{L}_{B/A} \simeq 0$. We conclude using again the equivalence $\mathbb{L}_{(B,N)/(A,M)} \simeq \mathbb{L}_{B/A}$. □

In practice, one usually deals with stacks not defined over the entire category of log rings, but only the category of fine and saturated log rings. Such a log stack over the category of fine and saturated log rings is usually defined as a category fibred in groupoids over this category. Using the inclusion functor from groupoids to simplicial sets and the Grothendieck construction, we can view every such category fibred in groupoids as a simplicial set valued functor on the category of fine and saturated log rings. The main example we have in mind is the following.

Example 8.2. [GS13, Def. 1.3] Assume our base is a separably closed field k. Denote by $M_{g,n}^{\log,\mathrm{pre}}$ the functor that assigns to every fine and saturated log ring (A, M) the groupoid of proper log-smooth and integral morphisms $f\colon (C, \mathcal{M}) \to \mathrm{Spec}(A, M)$ together with n sections $s_i\colon \mathrm{Spec}(A, M) \to (C, \mathcal{M})$ such that every fibre of f is a reduced and connected curve of genus g, and if $U \subset C$ is the non-critical locus of f, then $\mathcal{M}|_U \simeq f^*M \oplus \bigoplus_i (s_i)_*\mathbb{N}_A$.

Remark 8.3. Note that since we are in the relative situation over a separably closed field k and since in the above examples the log schemes are assumed to be fine and saturated, this ensures that the geometric fibres have at worst nodal singularities by [Kat00, Theorem 1.3].

If we now let $\mathcal{L}_k^{\mathrm{fs}}$ denote the category of fine and saturated log rings, we then have an inclusion $j\colon \mathcal{L}_k^{\mathrm{fs}} \to \mathcal{L}_k$. Arguing as above, we obtain an adjunction

$$j_!\colon \mathcal{S}^{\mathcal{L}_k^{\mathrm{fs}}} \rightleftarrows \mathcal{S}^{\mathcal{L}_k}\colon j^*$$

between the categories of simplicial pre-sheaves equipped with the projective model structures, and this again descends to the categories of stacks with respect to the strict étale topology.

Using the composition $i_! \circ j_!$ we can regard any category fibred in groupoids over the category of fine and saturated log schemes as a derived log stack. By combining this composition and Example 8.2 we can construct the derived moduli of stable maps over a fine and saturated base log k-scheme $(S, \mathcal{M}_S)$.

Definition 8.4. Let $(S, \mathcal{M})$ be a fine and saturated log scheme over a separably closed field k, and denote by $\mathrm{dLogSt}_{(S,\mathcal{M})}$ the comma category of dLogSt_k over $(S, \mathcal{M})$. Let C denote the universal curve over $M_{g,n}^{\log,\mathrm{pre}}$, and let X be a derived affine log scheme over $(ij)_!S$. We then defined the derived moduli of stable maps as

$$M(X) = \mathrm{Map}_{\mathrm{dLogSt}_{(S,\mathcal{M})}/(ij)_! M_{g,n}^{\log,\mathrm{pre}}}\left((ij)_!C, X \times (ij)_! M_{g,n}^{\log,\mathrm{pre}}\right)$$

Note that we have not proven that $M(X)$ is algebraic. We hope to return to this in a future paper. If an Artin-Lurie type representability theorem [Art74, Lur12b] for derived log stacks were available, this would be an immediate consequence. Once algebraicity is proven one can compute the cotangent complex of the derived moduli of stable maps. This will coincide with the perfect obstruction theory used in [GS13]. The functoriality of the cotangent complex would be the major advantage of working with derived moduli, as similar statements for the perfect obstruction theory are in general difficult to obtain.

An important problem outlined in Gross-Siebert is to identify interesting quasi-compact substacks of the derived moduli of stable log maps. As the topology of the derived and the underived moduli are the same our approach does not suggest anything on this problem.

Appendix A.

For the readers' convenience, we will give a proof of Proposition 4.14.

Proof of Proposition 4.14. Let $\pi\colon R \to S$ be a square zero extension of discrete commutative rings, and let $J = \ker \pi$ be the corresponding square zero ideal. Then we have to show that there exists a derivation

$$d \in \pi_0 \mathrm{Map}_{R/s\mathcal{A}/S}(S, S \oplus J[1])$$

such that there exists an isomorphism in $\mathrm{Ho}(s\mathcal{A}/S)$, between $\pi\colon R \to S$ and the canonical projection $p_d\colon S \oplus_d J \to S$, where p_d is defined by the homotopy pullback diagram

$$\begin{tikzcd} S \oplus_d J \arrow[r] \arrow[d, "p_d"'] & S \arrow[d, "0"] \\ S \arrow[r, "d"'] & S \oplus J[1]. \end{tikzcd}$$

We will give two proofs, one working in any characteristic and the other, considerably simpler, working in characteristic zero. We begin with the general case.

Let $\pi\colon R \to S$ be a surjection of commutative algebras with square zero ideal $J = \ker \pi$. As a first step, we apply the functor $- \otimes_R^{\mathbb{L}} S$ to the cofiber sequence

$$R \xrightarrow{\pi} S \longrightarrow J[1],$$

and obtain a split fiber sequence. The splitting map gives a map

$$\psi\colon S \otimes_R^{\mathbb{L}} S \longrightarrow S \oplus J[1]$$

in $\mathrm{Ho}(S/s\mathcal{A}/S)$, where

$$\begin{tikzcd} & S & \\ S \otimes_R^{\mathbb{L}} S \arrow[ur, "\mu"] \arrow[rr] & & S \oplus J[1] \arrow[ul, "\mathrm{pr}_1"'] \\ & S \arrow[ul, "j_1"] \arrow[ur, "0"'] & \end{tikzcd}$$

commutes, μ being induced by the product map, and j_1 being induced by $y \longmapsto y \otimes 1$. By computing the action of ψ on homotopy groups, we see that

$$\psi_{\leq 1} := \tau_{\leq 1}(\psi)\colon \tau_{\leq 1}(S \otimes_R^{\mathbb{L}} S) \longrightarrow \tau_{\leq 1}(S \oplus J[1]) \simeq S \oplus J[1]$$

is an isomorphism in $\mathrm{Ho}(S/s\mathcal{A}/S)$.

As a second step, we define $d\colon S \longrightarrow S \oplus J[1]$ as the composite

$$S \xrightarrow{j_2} S \otimes^{\mathbb{L}}_R S \longrightarrow \tau_{\leq 1}(S \otimes^{\mathbb{L}}_R S) \xrightarrow{\psi_{\leq 1}} S \oplus J[1]$$

where j_2 is induced by $y \longmapsto 1 \otimes y$. Observe that, by the first step, d is a section of the projection $\mathrm{pr}_1\colon S \oplus J[1] \longrightarrow S$.

As a third step, we observe that since the two composites

$$R \xrightarrow{\pi} S \xrightarrow{j_1} S \otimes^{\mathbb{L}}_R S\ , \quad R \xrightarrow{\pi} S \xrightarrow{j_2} S \otimes^{\mathbb{L}}_R S$$

coincide, we get an induced canonical map $\alpha\colon R \longrightarrow S \oplus_d J$, where $S \oplus_d J$ is defined by the homotopy pullback diagram

$$\begin{tikzcd} S \oplus_d J \arrow[r] \arrow[d, "p"'] & S \arrow[d, "0"] \\ S \arrow[r, "d"] & S \oplus J[1]. \end{tikzcd}$$

Moreover, if we view $S \oplus_d J$ as an object in $\mathrm{Ho}(s\mathcal{A}/S)$ via p, then α is a morphism in $\mathrm{Ho}(s\mathcal{A}/S)$.

By computing the action of $\alpha\colon R \longrightarrow S \oplus_d J$ on homotopy groups, it is easy to check that it is an isomorphism in $\mathrm{Ho}(s\mathcal{A}/S)$.

We now give an alternative proof in characteristic zero. As above, let $\pi\colon R \to S$ be a surjection of commutative algebras with square zero ideal $J = \ker \pi$. If the base commutative ring k is a $\mathbb{Q}$-algebra, the homotopy theories of simplicial commutative k-algebras and of differential non-positively graded commutative k-algebras (cdga's for short) are equivalent. So we are allowed to work with cdga's. Note that $S \oplus J[1]$ can then be represented by the cdga

$$0 \longrightarrow J \xrightarrow{0} S \longrightarrow 0$$

where S sits in degree 0. The 0 derivation is then represented by the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] \arrow[d, "0"] & S \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & J \arrow[r, "0"] & S \arrow[r] & 0 \end{tikzcd}$$

Observe that we may represent S also by the cdga

$$0 \longrightarrow J \xrightarrow{i} R \longrightarrow 0,$$

where i denotes the inclusion map. Then we can define a derivation d by the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r, "i"] \arrow[d, "\mathrm{id}"] & R \arrow[r] \arrow[d, "\pi"] & 0 \\ 0 \arrow[r] & J \arrow[r, "0"] & S \arrow[r] & 0, \end{tikzcd}$$

and remark that d is a fibration of cdga's. Since the model category of cdga's is proper, the ordinary pullback of the zero derivation and of d computes the homotopy pullback $S \oplus_d J$. But the ordinary pullback is given by just

$$0 \longrightarrow 0 \longrightarrow R \longrightarrow 0$$

(i.e., by just R sitting in degree 0). So we conclude that there is an isomorphism $R \simeq S \oplus_d J$ in the homotopy category of cdga's$/S$. □

References

[ACG+13] D. Abramovich, Q. Chen, D. Gillam, Y. Huang, M. Olsson, M. Satriano, and S. Sun, *Logarithmic geometry and moduli*, Handbook of moduli. Volume II., 2013, pp. 1–62.

[Art74] M. Artin, *Versal deformations and algebraic stacks*, Invent. Math. **27** (1974), 165–189. MR0399094 (53 #2945)

[Bei12] A. Beilinson, *p-adic periods and derived de Rham cohomology*, J. Amer. Math. Soc. **25** (2012), no. 3, 715–738. MR2904571

[BF78] A. K. Bousfield and E. M. Friedlander, *Homotopy theory of Γ-spaces, spectra, and bisimplicial sets*, Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill., 1977), II, 1978, pp. 80–130. MR513569 (80e:55021)

[Bha12] B. Bhatt, *p-adic derived de Rham cohomology*, 2012. `arXiv:1204.6560`.

[FM94] E. M. Friedlander and B. Mazur, *Filtrations on the homology of algebraic varieties*, Mem. Amer. Math. Soc. **110** (1994), no. 529, x+110. With an appendix by Daniel Quillen. MR1211371 (95a:14023)

[GJ99] P. G. Goerss and J. F. Jardine, *Simplicial homotopy theory*, Progress in Mathematics, vol. 174, Birkhäuser Verlag, Basel, 1999. MR1711612 (2001d:55012)

[GR14] O. Gabber and L. Ramero, *Foundations for almost ring theory*, 2014.

[GS13] M. Gross and B. Siebert, *Logarithmic Gromov-Witten invariants*, J. Amer. Math. Soc. **26** (2013), no. 2, 451–510. MR3011419

[Hir03] P. S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs, vol. 99, American Mathematical Society, Providence, RI, 2003. MR1944041 (2003j:18018)

[Ill71] L. Illusie, *Complexe cotangent et déformations. I*, Lecture Notes in Mathematics, Vol. 239, Springer-Verlag, Berlin, 1971. MR0491680 (58 #10886a)

[Kat00] F. Kato, *Log smooth deformation and moduli of log smooth curves*, Internat. J. Math. **11** (2000), no. 2, 215–232. MR1754621 (2001d:14016)

[Kat89] K. Kato, *Logarithmic structures of Fontaine-Illusie*, Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), 1989, pp. 191–224. MR1463703 (99b:14020)

[Li01] J. Li, *Stable morphisms to singular schemes and relative stable morphisms*, J. Differential Geom. **57** (2001), no. 3, 509–578. MR1882667 (2003d:14066)

[Li02] ———, *A degeneration formula of GW-invariants*, J. Differential Geom. **60** (2002), no. 2, 199–293. MR1938113 (2004k:14096)

[Lur12a] J. Lurie, *Higher algebra*, 2012. Preprint, available at `http://www.math.harvard.edu/~lurie/`.

[Lur12b] ———, *Representability theorems*, 2012. Preprint, available at `http://www.math.harvard.edu/~lurie/`.

[Mar13] F. Marty, *Smoothness in Relative Geometry*, J. K-Theory **12** (2013), no. 3, 461–491. MR3165184

[Ols05] M. C. Olsson, *The logarithmic cotangent complex*, Math. Ann. **333** (2005), no. 4, 859–931. MR2195148 (2006j:14017)

[Qui67] D. G. Quillen, *Homotopical algebra*, Lecture Notes in Mathematics, No. 43, Springer-Verlag, Berlin, 1967. MR0223432 (36 #6480)

[Rez02] C. Rezk, *Every homotopy theory of simplicial algebras admits a proper model*, Topology Appl. **119** (2002), no. 1, 65–94. MR1881711 (2003g:55033)

[Rog09] J. Rognes, *Topological logarithmic structures*, New topological contexts for Galois theory and algebraic geometry (BIRS 2008), 2009, pp. 401–544. MR2544395 (2010h:14029)

[RSS14] J. Rognes, S. Sagave, and C. Schlichtkrull, *Localization sequences for logarithmic topological Hochschild homology*, 2014. `arXiv:1402.1317`.

[Sch97] S. Schwede, *Spectra in model categories and applications to the algebraic cotangent complex*, J. Pure Appl. Algebra **120** (1997), no. 1, 77–104. MR1466099 (98h:55027)

[Sim12] C. Simpson, *Homotopy theory of higher categories*, New Mathematical Monographs, vol. 19, Cambridge University Press, Cambridge, 2012. MR2883823 (2012m:18019)

[SS13] S. Sagave and C. Schlichtkrull, *Group completion and units in I-spaces*, Algebr. Geom. Topol. **13** (2013), no. 2, 625–686. MR3044590

[TV05] B. Toën and G. Vezzosi, *Homotopical algebraic geometry. I. Topos theory*, Adv. Math. **193** (2005), no. 2, 257–372. MR2137288 (2007b:14038)

[TV08] ———, *Homotopical algebraic geometry. II. Geometric stacks and applications*, Mem. Amer. Math. Soc. **193** (2008), no. 902, x+224. MR2394633 (2009h:14004)

[TV09] B. Toën and M. Vaquié, *Au-dessous de* $\operatorname{Spec}\mathbb{Z}$, J. K-Theory **3** (2009), no. 3, 437–500. MR2507727 (2010j:14006)

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