A bounded transform approach to self-adjoint operators:  
Functional calculus and affiliated von Neumann algebras

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Abstract
Spectral theory and functional calculus for unbounded self-adjoint operators on a  
Hilbert space are usually treated through von Neumann’s Cayley transform. Based on  
ideas of Woronowicz, we redevelop this theory from the point of view of multiplier algebras and the so-called bounded transform (which establishes a bijective correspondence between closed operators and pure contractions). This also leads to a simple account of the affiliation relation between von Neumann algebras and self-adjoint operators.

1 Introductory overview
The theory of unbounded self-adjoint operators on a Hilbert space was initiated by von Neumann, partly motivated by mathematical problems of quantum mechanics [7]. The monograph by Schmüdgen [10] presents an excellent survey of the present state of the art.

Von Neumann’s approach was based on the Cayley transform and in its subsequent development the notion of a spectral measure played an important role, especially in defining a functional calculus. We consider this route a bit indirect and will avoid both by firstly invoking the bounded transform instead of the Cayley transform, i.e., the formal expressions

\[
S = T \sqrt{I + T^2}^{-1} ; \quad (1.1)
\]

\[
T = S \sqrt{I - S^2}^{-1} , \quad (1.2)
\]

make rigorous sense and provide a bijective correspondence between self-adjoint operators \( T \) and self-adjoint pure contractions \( S \) (i.e., \( \|Sx\| < \|x\| \) for each \( x \in H \setminus \{0\} \)); cf. [3, 4, 10].
Note that the bounded transform $T \mapsto S$ is an operatorial version of the homeomorphism $\mathbb{R} \cong (-1,1)$ given by the function $b : \mathbb{R} \to (-1,1)$ and its inverse $u : (-1,1) \to \mathbb{R}$, defined by

$$b(x) = \frac{x}{\sqrt{1+x^2}}; \quad (1.3)$$

$$u(x) = \frac{x}{\sqrt{1-x^2}}. \quad (1.4)$$

Secondly, we replace spectral measures by simple arguments using multiplier algebras. Our approach is based on the work of Woronowicz [12, 13], whose functional calculus we adopt and to some extent complete, at least in the usual context of operators on a Hilbert space (Woronowicz’s work was mainly intended to deal with problems involving multiplier algebras and, even more generally, with operators on Hilbert C*-modules [5]).

If $T$ is bounded (and, by standing assumption, self-adjoint), it is easy to prove the equality

$$C^*(T) = C^*(S), \quad (1.5)$$

where $C^*(S)$ is the $C^*$-algebra generated within $B(\mathcal{H})$ by $S$ and the unit, etc. Furthermore, the spectral mapping theorem implies that the spectra of $S$ and $T$ are related by

$$\sigma(T) = \left\{ \mu (1 - \mu^2)^{-\frac{1}{2}} \mid \mu \in \sigma(S) \right\}; \quad (1.6)$$

$$\sigma(S) = \left\{ \lambda (1 + \lambda^2)^{-\frac{1}{2}} \mid \lambda \in \sigma(T) \right\}, \quad (1.7)$$

preserving point spectra. As to the continuous functional calculus, for $S = S^* \in B(\mathcal{H})$ we have the familiar isomorphism $C(\sigma(S)) \cong C^*(S)$, written $g \mapsto g(S)$, given by the spectral theorem. Assuming $T = T^* \in B(\mathcal{H})$, the same applies to $T$. These calculi are related by

$$f(T) = (f \circ u)(S), \quad (1.8)$$

where $f \in C(\sigma(T))$, so that $f \circ u \in C(\sigma(S))$. Self-adjointness is preserved, in that

$$f(T)^* = f^*(T), \quad (1.9)$$

where $f^*(x) = \overline{f(x)}$. In particular, if $f$ is real-valued, then $f(T)$ is self-adjoint. At the level of von Neumann algebras, defining $W^*(S) = C^*(S)''$ and similarly for $T$, eq. (1.5) gives

$$W^*(T) = W^*(S). \quad (1.10)$$

The functional calculus $f \mapsto f(T)$ may then be extended to bounded Borel functions $f$ on $\sigma(T)$, in which case it is still given by (1.8). We then have $f(T) \in W^*(T)$, whilst (1.9) remains valid; however, instead of the isometric property $\|f(T)\| = \|f\|_\infty$ for continuous $f$, we now have $\|f(T)\| \leq \|f\|_\infty$ (where $\| \cdot \|_\infty$ is the supremum-norm). See, e.g., [8].
Then this also works for closable (Lemma 3), so that we may once again define $f \in h$ hence a fortiori $h$, where $\tilde{\sigma}(S)$ consists of all $g \in C_b(\tilde{\sigma}(S))$ for which $\lim_{y \to \pm 1} g(y)$ exists, where this limit is 0 if and only if $g \in C_0(\tilde{\sigma}(S))$. Hence we have the inclusions (of which the first set implies the second)

\begin{align}
C_c(\tilde{\sigma}(S)) &\subseteq C_0(\tilde{\sigma}(S)) \subseteq C(\sigma(S)) \subseteq C_b(\tilde{\sigma}(S)); \\
C^*_c(S) &\subseteq C^*_0(S) \subseteq C^*(S) \subseteq C^*_b(S),
\end{align}

with equalities iff $T$ is bounded. This means that $g(S)$ is defined for $g \in C_0(\tilde{\sigma}(S))$, and hence a fortiori also for $g \in C_c(\tilde{\sigma}(S))$. Consequently, $f(T)$ may be defined by \eqref{eq:1.8} whenever $f \in C_0(\sigma(T))$, including $f \in C_c(\sigma(T))$. To pass to the larger class $f \in C_b(\sigma(T))$, we define $C^*_0(S)\mathcal{H}$ as the linear span of all vectors of the form $g(S)\psi$, where $g \in C_0(\tilde{\sigma}(S))$ and $\psi \in \mathcal{H}$. Then $C^*_0(S)\mathcal{H}$ is dense in $\mathcal{H}$ (Lemma 1). In the spirit of Woronowicz \cite{5,12}, we then initially define $f(T)$ for $f \in C_b(\sigma(T))$ on the domain $C^*_0(S)\mathcal{H}$ by linear extension of the formula

\begin{equation}
f_0(T)h(T)\psi = (fh)(T)\psi,
\end{equation}

where $h \in C_0(\sigma(T))$ and hence also $fh \in C_0(\sigma(T))$, since $C_b(\sigma(T))$ is the multiplier algebra of $C_0(\sigma(T))$. Then $f_0(T)$ is bounded (Lemma 2), and we define $f(T)$ as its closure, i.e.,

\begin{equation}
f(T) = f_0(T)^-.
\end{equation}

This also works for $f \in C(\sigma(T))$, in which case $f_0(T)$ may no longer be bounded, but remains closable (Lemma 3), so that we may once again define $f(T)$ as its closure, cf. \eqref{eq:1.18}. We have:
\textbf{Theorem 2.} If \( f \in C(\sigma(T)) \) is real-valued, then \( f(T) \) is self-adjoint, i.e., \( f_0(T)^{-} = f_0(T)^* \); more generally, \( f(T)^* = f^*(T) \). Furthermore, the continuous functional calculus \( f \mapsto f(T) \) restricts to an isometric \( \ast \)-homomorphism from \( C_0(\sigma(T)) \) (with supremum-norm) to \( C^*(S) \).

See also Theorem \( \textbf{4} \). In addition, the map \( f \mapsto f(T) \) has the reassuring special cases

\begin{align*}
1_{\sigma(T)}(T) &= I; \\
\text{id}(T) &= T; \\
(id - z)^{-1}(T) &= (T - z)^{-1}, \; z \in \rho(T),
\end{align*}

where \( 1_{\sigma(T)}(x) = 1 \) and \( \text{id}(x) = x \) (\( x \in \sigma(T) \)), and therefore does what it is supposed to do.

Finding the right analogue of \( \textbf{1.10} \) for unbounded \( T = T^* \) first requires a redefinition of \( W^*(T) \), which is standard \( \textbf{[8]} \). If \( T \) is unbounded and \( R \in B(H) \), then we say that \( R \) and \( T \) commute, written \( TR \subset RT \), if \( R\psi \in D(T) \) and \( RT\psi = TR\psi \) for any \( \psi \in D(T) \). Let \( \{T\}' \) be the set of all bounded operators that commute with \( T \). If \( T^* = T \), then \( \{T\}' \) is a unital, strongly closed \( \ast \)-subalgebra of \( B(H) \), and hence a von Neumann algebra \( \textbf{[8]} \). Its commutant

\begin{equation}
W^*(T) = \{T\}''
\end{equation}

is a von Neumann algebra, too. If \( T \) is bounded, then \( W^*(T) \) is the von Neumann algebra generated by \( T \), which coincides with \( C^*(T)'' \). As usual, we call a closed unbounded operator \( X \) affiliated to a von Neumann algebra \( A \subset B(H) \), written \( X \eta A \), iff \( XR \subset RX \) for each \( R \in A' \). For example, if \( T^* = T \), then \( T\eta W^*(T) \), and if \( T \eta A \), then \( W^*(T) \subseteq A \); in other words, \( W^*(T) \) is the smallest von Neumann algebra such that \( T \) is affiliated to it.

As a result of independent interest as well as a lemma for Theorem \( \textbf{4} \) we may then adapt \( \textbf{[8]} \) Lemma 5.2.8] to the bounded transform:

\textbf{Theorem 3.} Let \( A \subset B(H) \) be a von Neumann algebra. Then \( T \eta A \) iff \( S \in A \).

Denoting the (Banach) space of (bounded) Borel functions on \( \sigma(T) \) (equipped with the supremum-norm) by \( B_b(\sigma(T)) \), we may still define \( f(T) \) by \( \textbf{1.8} \) and the usual Borel functional calculus for the bounded transform \( S \).

\textbf{Theorem 4.} The map \( f \mapsto f(T) \) is a norm-decreasing \( \ast \)-homomorphism from \( B_b(\sigma(T)) \) to

\begin{equation}
\end{equation}

More generally, if \( f \in B(\sigma(T)) \), then \( f(T) \) is affiliated with \( W^*(T) \).

The remainder of this paper simply consists of the proofs of these theorems.

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2 Proofs

This section contains all proofs. We will not repeat the theorems.

2.1 Proof of Theorem 1

The operator $\sqrt{1-S^2}$ is a bijection from $\mathcal{H}$ to $\mathcal{R}(\sqrt{1-S^2}) = \mathcal{D}(T)$. Let $\lambda \in \rho(T) \equiv \mathbb{C} \setminus \sigma(T)$, so that $T - \lambda I$ is a bijection from $\mathcal{D}(T)$ to $\mathcal{H}$. Thus by composition we have a bijection $\mathcal{H} \to \mathcal{H}$; equivalently, $(T - \lambda I)(\sqrt{1-T^2})$ is invertible, which in turn is equivalent to invertibility of $S - \lambda \sqrt{1-S^2}$. Thus $\lambda \in \rho(T) \iff S - \lambda \sqrt{1-S^2}$ is a bijection, or, expressed contrapositively, $\lambda \in \sigma(T) \iff S - \lambda \sqrt{1-S^2}$ is not invertible in $B(\mathcal{H})$. This is the case iff $S - \lambda \sqrt{1-S^2}$ is not invertible in $C^*(S)$, which, using the Gelfand isomorphism $C^*(S) \cong C(\sigma(S))$, in turn is true iff the function $k_\lambda(x) = x - \lambda \sqrt{1-x^2}$ is not invertible in $C(\sigma(S))$, i.e., iff $0 \in \sigma(k_\lambda)$. Since in $C(X)$ we have $\sigma(f) = \mathcal{R}(f)$ (with $X$ a compact Hausdorff space), and $\sigma(S)$ is indeed compact and Hausdorff because $S$ is bounded, we obtain $\lambda \in \sigma(T)$ iff $0 \in \mathcal{R}(k_\lambda)$. If $\pm 1$ lie in $\sigma(S)$ they cannot give rise to this possibility, since $k_\lambda(\pm 1) = \pm 1$ for each $\lambda$. Hence we have $0 \in \mathcal{R}(k_\lambda)$ iff $\lambda = \mu(1-\mu^2)^{-\frac{1}{2}}$ for some $\mu \in \sigma(S) \cap (-1,1)$, which yields (1.11).

The same argument shows that $\mu \in \sigma(S) \cap (-1,1)$ comes from $\lambda \in \sigma(T)$. But since $\sigma(S)$ is compact and hence closed in $[-1,1]$ we obtain (1.12). □

2.2 Proof of Theorem 2

This proof relies on three lemma’s.

Lemma 1. Let $C^*_c(S)\mathcal{H}$ be the linear span of all vectors of the form $g(S)\psi$, where $g \in C_c(\tilde{\sigma}(S))$ and $\psi \in \mathcal{H}$. Then $C^*_c(S)\mathcal{H}$ is dense in $\mathcal{H}$.

Proof. Define $g_n : (-1,1) \to [0,1]$ by putting $g_n(x) = 0$ for $x \in (-1,\frac{1}{n},-1] \cup [1-\frac{1}{n},1)$, $g_n(x) = 1$ if $x \in \left[\frac{2}{n},1-\frac{2}{n}\right]$, and linear interpolation in between. The ensuing sequence converges pointwise to the unit $1$ on $(-1,1)$. Restricting each $g_n$ to $\tilde{\sigma}(S)$, the continuous functional calculus gives $g_n(S) \to 1_{\tilde{\sigma}(S)}$ strongly. Therefore, for any $\psi \in \mathcal{H}$ we have a sequence $\psi_n = g_n(S)\psi$ in $C^*_c(S)\mathcal{H}$ such that $\psi_n \to \psi$. □

Lemma 2. For $f \in C_0(\sigma(T))$, define an operator $f_0(T)$ on the domain $C^*_0(S)\mathcal{H}$ by (1.17). Then $f_0(T)$ is bounded, with bound

$$\|f(T)\| \leq \|f\|_\infty.$$  \hspace{1cm} (2.24)

Proof. Let $\varepsilon > 0$. If $h \in C_0(\sigma(T))$, then $fh \in C_0(\sigma(T))$, so that we can find a compact subset $K \subset \sigma(T)$ such that $|h(x)f(x)| < \varepsilon$ for each $x \notin K$. Let $\tilde{h} = h \circ u$, cf. (1.4); then $\tilde{h} \in C_0(\tilde{\sigma}(S))$ whenever $h \in C_0(\sigma(T))$; in fact, we have an isometric isomorphism

$$C_0(\sigma(T)) \xrightarrow{\cong} C_0(\tilde{\sigma}(S)), \ h \mapsto h \circ u.$$  \hspace{1cm} (2.25)
Contractivity of the Borel functional calculus for bounded operators on \( \mathcal{H} \) gives
\[
\| \langle 1_{K} \hat{f} h \rangle (S) \psi \| \leq \| \langle 1_{K} \hat{f} h \rangle (S) \| \| \psi \| \leq \| 1_{K} \hat{f} h \|_{\infty} \| \psi \| < \varepsilon \| \psi \|.
\]

Using also the homomorphism property of the Borel functional calculus, we then find
\[
\| (f h)(T) \psi \| = \| (\hat{f} \hat{h})(S) \psi \| \\
= \| (1_{K} \hat{f} h)(S) + (\hat{f} h - 1_{K} \hat{f} h)(S) \psi \| \\
\leq \| (1_{K} \hat{f} h)(S) \psi \| + \| (1_{K} \hat{f} h)(S) \| \psi \| \\
= \| (1_{K} f)(S) \hat{h}(S) \psi \| + \| (1_{K} \hat{f} h)(S) \| \psi \| \\
< \| (1_{K} f) \|_{\infty} \| h(T) \psi \| + \varepsilon \| \psi \|, \\
\leq \| f \|_{\infty} \| h(T) \psi \| + \varepsilon \| \psi \|,
\]

since \( \| (1_{K} f) \|_{\infty} \leq \| \hat{f} \|_{\infty} = \| f \|_{\infty} \). Since the last expression above is independent of \( K \), we may let \( \varepsilon \to 0 \), obtaining boundedness of \( f(T) \) as well as \( 2.24 \). \( \square \)

The last claim in Theorem 2 now follows from the continuous functional calculus for \( S \) and the isometric isomorphism \( 2.24 \). Although isometry may be lost if we go from \( C_{0}(\sigma(T)) \) to \( C_{b}(\sigma(T)) \), it easily follows from \( 1.17 \) - \( 1.18 \) that the map \( f \mapsto f(T) \) at least defines a *-homomorphism \( C_{b}(\sigma(T)) \to B(\mathcal{H}) \). This property will be used after Lemma 3 below.

**Lemma 3.** For \( f \in C(\sigma(T)) \), define an operator \( f_{0}(T) \) on the domain \( C_{c}^{\infty}(S) \mathcal{H} \) by \( 1.17 \). Then \( f_{0}(T) \) is closable. Moreover, if \( f \) is real-valued \(( f^{*} = f \)) \), then \( f_{0}(T) \) is symmetric.

**Proof.** Suppose that \( h_{1}(T) \psi_{1} \) and \( h_{2}(T) \psi_{2} \) lie in \( \mathcal{D}(f_{0}(T)) \). Then we may compute:
\[
\langle h_{2}(T) \psi_{2}, f_{0}(T) h_{1}(T) \psi_{1} \rangle = \langle \psi_{2}, \overline{f_{0}(T)}(fh_{1})(T) \psi_{1} \rangle = \langle \psi_{2}, (\overline{f_{0}}h_{1})(T) \psi_{1} \rangle; \quad (2.26)
\]
\[
\langle (h_{2}^{T})(T) \psi_{2}, h_{1}(T) \psi_{1} \rangle = \langle \psi_{2}, \overline{h_{2}^{T}}(h_{1})(T) \psi_{1} \rangle = \langle \psi_{2}, (\overline{h_{2}})h_{1}(T) \psi_{1} \rangle. \quad (2.27)
\]

This implies that \( \mathcal{D}(f_{0}(T)) \subseteq \mathcal{D}(f_{0}(T)^{*}) \) Since \( \mathcal{D}(f_{0}(T)) \) is dense, so is, \( \mathcal{D}(f_{0}(T)^{*}) \), which implies that \( f_{0}(T) \) is closable. The second claim is obvious from \( 2.26 \) - \( 2.27 \). \( \square \)

**Proof.** To prove Theorem 2 we use a well-known result of Nelson [6]; see also [9] (this step was suggested to us by Nigel Higson). For convenience we recall this result (without proof):

**Lemma 4.** Let \( \{ U(t) \}_{t \in \mathbb{R}} \) be a strongly continuous unitary group of operators on a Hilbert space \( \mathcal{H} \). Let \( R : \mathcal{D}(R) \to \mathcal{H} \) be densely defined and symmetric. Assume that \( \mathcal{D}(R) \) is invariant under \( \{ U(t) \}_{t \in \mathbb{R}}, \) i.e. \( U(t) : \mathcal{D}(R) \to \mathcal{D}(R) \) for each \( t \), and also that \( \{ U(t) \}_{t \in \mathbb{R}} \) is strongly differentiable on \( \mathcal{D}(R) \). Then \( -idU(t)/dt \) is essentially self-adjoint on \( \mathcal{D}(R) \) and its closure is the self-adjoint generator of \( \{ U(t) \}_{t \in \mathbb{R}} \) (given by Stone’s Theorem). In particular, if \( (dU(t)/dt) \psi = iRU(t) \psi \) for each \( \psi \in \mathcal{D}(R) \), then \( R \) is essentially self-adjoint.
Set $R = f_0(T)$ for $f \in C(\sigma(T))$, so that
\[
D(R) = C^*_c(S)\mathcal{H},
\] (2.28)
and for each $t \in \mathbb{R}$ define $U(t)$ via the (bounded) function $x \mapsto \exp(itf(x))$ on $\sigma(T)$, that is, for $h \in C_c(\sigma(T))$ and $\psi \in \mathcal{H}$, we initially define
\[
U_0(t)h(T)\psi = (e^{itf}h)(T)\psi.
\] (2.29)
Then $U_0$ bounded by Lemma 2, and we define $U(t)$ as the closure of $U_0(t)$. The remark before Lemma 3 then implies that $t \mapsto U(t)$ defines a unitary representation of $\mathbb{R}$ on $\mathcal{H}$. Strong continuity of this representation follows from an $\varepsilon/3$ argument. First, for $
abla = h(T)\psi$,
\[
\|U(t)\varphi - \varphi\| \leq \|e^{itf}h - h\|_{\infty} \leq \|h\|_{\infty}\|e^{itf} - 1\|_{\infty},
\] (2.30)
assuming $\|\psi\| = 1$ for simplicity, eqs. (2.29) and (2.24) give
\[
\|U(t)\psi' - \psi'\| \leq \|e^{itf}h - h\|_{\infty} \leq \|h\|_{\infty}\|e^{itf} - 1\|_{\infty},
\] (2.31)
where $K$ is the (compact) support of $h$ in $\sigma(T)$. Since the exponential function is uniformly convergent on any compact set, this gives $\lim_{t \to 0} \|U(t)\varphi - \varphi\| = 0$ for $\varphi$ of the form (2.30); taking finite linear combinations thereof gives the same result for any $\varphi \in C^*_c(S)\mathcal{H}$. Thus for any $\varepsilon > 0$ we can find $\delta > 0$ so that $\|U(t)\varphi - \varphi\| < \varepsilon/3$ whenever $|t| < \delta$. For general $\psi' \in \mathcal{H}$, we find $\varphi \in C^*_c(S)\mathcal{H}$ such that $\|\varphi - \psi'\| < \varepsilon/3$, and estimate
\[
\|U(t)\psi' - \psi'\| \leq \|U(t)\psi' - U(t)\varphi\| + \|U(t)\varphi - \varphi\| + \|\varphi - \psi'\| \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon,
\] since $\|U(t)\psi' - U(t)\varphi\| = \|\psi' - \varphi\|$ by unitarity of $U(t)$. Thus $\lim_{t \to 0} \|U(t)\psi - \psi\| = 0$ for any $\psi \in \mathcal{H}$, so that the unitary representation $t \mapsto U(t)$ is strongly continuous. Similarly,
\[
\left\| \frac{U(t+s)\varphi - U(t)\varphi}{s} - iRU(t)\varphi \right\| \leq \left\| \frac{e^{isf}h - h}{s} - ifh \right\|_{\infty},
\] (2.32)
assuming (2.30), so that by the same argument as in (2.31) we obtain
\[
\frac{dU(t)}{dt}\varphi = iRU(t)\varphi,
\] (2.33)
initially for any $\varphi$ of the form (2.30), and hence, taking finite sums, for any $\varphi \in D(R)$, cf. (2.28). The final part of Lemma 2 then shows that $f_0(T)$ is essentially self-adjoint on its domain $C^*_c(S)\mathcal{H}$. Its closure $f(T)$ is therefore self-adjoint, and Theorem 2 is proved. \qed
We now prove the examples (1.19) - (1.21), of which the first is trivial. Writing \(T_0\) for the operator \(\text{id}_0(T)\), the definition (1.17) gives

\[
T_0 \varphi = T \varphi
\]

for \(\varphi \in \mathcal{D}(T_0) = C^*_c(S)\mathcal{H}\). Let \(\psi \in \mathcal{D}(T_0^-)\), so that there is a sequence \((\varphi_n)\) in \(\mathcal{D}(T_0)\) such that \(\varphi_n \to \varphi\) and \((T_0 \varphi_n)\) converges. Since \(T\) is closed, it follows that \(T_0 \varphi_n \to T \varphi\), so that \(\varphi \in \mathcal{D}(T)\). Hence \(T_0^- \subset T\). Since both operators are self-adjoint, this implies \(T_0^- = T\), which proves (1.20).

The proof of (1.21) is easier since \((T - z)^{-1}\) is bounded: writing

\[
f(x) = (x - z)^{-1},
\]

where \(z \notin \sigma(T)\) is fixed and \(x \in \sigma(T)\), we have

\[
f_0(T)h(T)\psi = (fh)(T)\psi = (T - z)^{-1}h(T)\psi,
\]

and hence

\[
f_0(T)\varphi = (T - z)^{-1}\varphi
\]

for any \(\varphi \in \mathcal{D}(f_0(T)) = C^*_c(S)\mathcal{H}\). So if \(\varphi_n \to \varphi\) for \(\varphi \in \mathcal{H}\) and \(\varphi_n \in \mathcal{D}(f_0(T))\), boundedness and hence continuity of the resolvent implies

\[
f(T)\varphi = \lim_{n \to \infty} f_0(T)\varphi_n = \lim_{n \to \infty} (T - z)^{-1}\varphi_n = (T - z)^{-1}\varphi.
\]

### 2.3 Proof of Theorem 3

The first step consists in the observation that \(T \eta A\) iff \(TU \subset UT\) (or, equivalently, \(UTU^* = T\)) merely for each unitary \(U \in A'\), which is well known [11].

The second step is to show that \(TU \subset UT\) iff \(SU = US\) for any unitary \(U\). This is a simple computation. First suppose that \(UTU^* = T\). Then:

\[
U(1 + T^2)^{-1}U^* = (U(1 + T^2)U^*)^{-1} = ((1 + UT^2)U^*)^{-1} = (UU^* + U^2T^2)^{-1} = (1 + UT^2 U^*)^{-1} = (1 + T^2)^{-1}.
\]

If \(R\) is bounded and positive, then \(UR = RU\) iff \(U \in C^*(R)'\), and since \(\sqrt{R} \in C^*(R)\) by the continuous functional calculus, we also have \(U \sqrt{R} = \sqrt{RU}\). Consequently,

\[
USU^* = U\left(T\sqrt{(1 + T^2)^{-1}}\right)U^* = (UTU^*)\left(U\sqrt{(1 + T^2)^{-1}U^*}\right) = T\sqrt{(1 + T^2)^{-1}} = S.
\]
Similarly, if \( SU = US \), then
\[
UTU^* = US\sqrt{1 - S^2}^{-1}U^* = SU\sqrt{1 - S^2}^{-1}U^* = S\left(U\sqrt{1 - S^2}U^*\right)^{-1} = S\sqrt{1 - S^2}^{-1} = T.
\]

Thirdly, as in the first step, \( SU = US \) for any unitary \( U \in A' \) iff \( S \in A'' = A \). \( \Box \)

2.4 Proof of Theorem 4

Eq. (1.23) in Theorem 4 follows from Theorem 3 taking \( A = W^*(T) \), so that \( T_\eta A \), yields \( S \in W^*(T) \), and hence \( W^*(S) \subseteq W^*(T) \). On the other hand, taking \( A = W^*(S) \), in which case \( S \in A \), gives \( T_\eta W^*(S) \), and hence \( W^*(T) \subseteq W^*(S) \).

Similar to (2.25), we have an isometric isomorphism
\[
B_h(\sigma(T)) \xrightarrow{\sim} B_h(\hat{\sigma}(S)), \ h \mapsto h \circ u,
\]
so that the first claim of Theorem 4 follows from the Borel functional calculus for the bounded operator \( S \) [8]. The proof of the last one is, mutatis mutandis, practically the same as in [8] Theorem 5.3.8], so we omit the details; see [2]. \( \Box \)

As explained in [8] §5.3, there exists a Borel measure \( \mu \) on \( \sigma(T) \) such that the map \( f \mapsto f(T) \) may also be seen as a so-called essential \(^*\)-homomorphism from \( B(\sigma(T))/N(\sigma(T)) \) into the \(^*\)-algebra of normal operators affiliated with \( W^*(T) \), where \( N(\sigma(T)) \) is the set of \( \mu \)-null functions on \( \sigma(T) \). This remains true in our approach, with the same proof [2].

3 Epilogue

Let us finally note that although this paper was inspired by the work of Woronowicz, the \( C^* \)-algebraic affiliation relation he defines in [12] (as did, independently, also Baaj and Julg [1]) has not been used here. If we call his relation \( \eta' \) to avoid confusion with the \( W^* \)-algebraic relation \( \eta \) we do use, if \( A \subset B(H) \) we have \( T_\eta' A \Rightarrow T \in A \) (and hence \( T \) is bounded), cf. [12] Prop. 1.3. Woronowicz does not define a \( C^* \)-algebraic counterpart of the von Neumann algebra \( W^*(T) \), but it might be reasonable to define \( C^*(T) \) as the smallest \( C^* \)-algebra in \( B(H) \) such that \( T_\eta' A \). It follows from [12] Example 4] that this would give \( C^*(T) = C_0^*(S) \), as defined in [1.4]. This \( C^* \)-algebra contains \( S \) (and hence \( T \)) if and only if \( T \) is bounded, in which case \( C_0^*(S) = C^*(S) \) and hence \( C^*(T) = C^*(S) \), as in our approach, cf. [1.5]. Also in general (i.e., if \( T \) is possibly unbounded), the bicommutant \( C^*(T)'' \) coincides with \( W^*(T) \) as defined in the usual way (1.22) this follows from \( C_0^*(S)'' = C^*(S)'' = W^*(S) \) and (1.10).

Of course, we could also redefine \( \eta' \), now calling it \( \eta'' \), by stipulating that \( T_\eta'' A \) whenever \( S \in A \), and redefine \( C^*(T) \) accordingly (i.e., as the smallest \( C^* \)-algebra \( A \) in \( B(H) \) such that \( T_\eta'' A \)). This would give (1.5) even if \( T \) is unbounded, though in a somewhat empty way.
References


