A bounded transform approach to self-adjoint operators: Functional calculus and affiliated von Neumann algebras

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Abstract

Spectral theory and functional calculus for unbounded self-adjoint operators on a Hilbert space are usually treated through von Neumann’s Cayley transform. Based on ideas of Woronowicz, we redevelop this theory from the point of view of multiplier algebras and the so-called bounded transform (which establishes a bijective correspondence between closed operators and pure contractions). This also leads to a simple account of the affiliation relation between von Neumann algebras and self-adjoint operators.

1 Introductory overview

The theory of unbounded self-adjoint operators on a Hilbert space was initiated by von Neumann, partly motivated by mathematical problems of quantum mechanics [7]. The monograph by Schmüdgen [10] presents an excellent survey of the present state of the art.

Von Neumann’s approach was based on the Cayley transform and in its subsequent development the notion of a spectral measure played an important role, especially in defining a functional calculus. We consider this route a bit indirect and will avoid both by firstly invoking the bounded transform instead of the Cayley transform, i.e., the formal expressions

\[ S = T \sqrt{I + T^2}^{-1} \]
\[ T = S \sqrt{I - S^2}^{-1} \]

make rigorous sense and provide a bijective correspondence between self-adjoint operators \( T \) and self-adjoint pure contractions \( S \) (i.e., \( \|Sx\| < \|x\| \) for each \( x \in \mathcal{H} \setminus \{0\} \)); cf. [3, 4, 10].
Note that the bounded transform $T \mapsto S$ is an operatorial version of the homeomorphism $\mathbb{R} \cong (-1, 1)$ given by the function $b: \mathbb{R} \to (-1, 1)$ and its inverse $u: (-1, 1) \to \mathbb{R}$, defined by
\begin{align*}
b(x) &= \frac{x}{\sqrt{1 + x^2}}; \\
u(x) &= \frac{x}{\sqrt{1 - x^2}}.
\end{align*}
(1.3) (1.4)

Secondly, we replace spectral measures by simple arguments using multiplier algebras. Our approach is based on the work of Woronowicz [12, 13], whose functional calculus we adopt and to some extent complete, at least in the usual context of operators on a Hilbert space (Woronowicz’s work was mainly intended to deal with problems involving multiplier algebras and, even more generally, with operators on Hilbert C*-modules [5]).

If $T$ is bounded (and, by standing assumption, self-adjoint), it is easy to prove the equality
\begin{equation}
C^*(T) = C^*(S),
\end{equation}
(1.5)
where $C^*(S)$ is the C*-algebra generated within $B(H)$ by $S$ and the unit, etc. Furthermore, the spectral mapping theorem implies that the spectra of $S$ and $T$ are related by
\begin{align*}
\sigma(T) &= \left\{ \mu(1 - \mu^2)^{-\frac{1}{2}} \mid \mu \in \sigma(S) \right\}; \\
\sigma(S) &= \left\{ \lambda(1 + \lambda^2)^{-\frac{1}{2}} \mid \lambda \in \sigma(T) \right\},
\end{align*}
(1.6) (1.7)
preserving point spectra. As to the continuous functional calculus, for $S = S^* \in B(H)$ we have the familiar isomorphism $C(\sigma(S)) \cong C^*(S)$, written $g \mapsto g(S)$, given by the spectral theorem. Assuming $T = T^* \in B(H)$, the same applies to $T$. These calculi are related by
\begin{equation}
f(T) = (f \circ u)(S),
\end{equation}
(1.8)
where $f \in C(\sigma(T))$, so that $f \circ u \in C(\sigma(S))$. Self-adjointness is preserved, in that
\begin{equation}
f(T)^* = f^*(T),
\end{equation}
(1.9)
where $f^*(x) = \overline{f(x)}$. In particular, if $f$ is real-valued, then $f(T)$ is self-adjoint. At the level of von Neumann algebras, defining $W^*(S) = C^*(S)''$ and similarly for $T$, eq. (1.5) gives
\begin{equation}
\end{equation}
(1.10)
The functional calculus $f \mapsto f(T)$ may then be extended to bounded Borel functions $f$ on $\sigma(T)$, in which case it is still given by (1.8). We then have $f(T) \in W^*(T)$, whilst (1.9) remains valid; however, instead of the isometric property $\|f(T)\| = \|f\|_{\infty}$ for continuous $f$, we now have $\|f(T)\| \leq \|f\|_{\infty}$ (where $\| \cdot \|_{\infty}$ is the supremum-norm). See, e.g., [8].
Our aim is to generalize these results to the case where $T$ is unbounded. This indeed turns out to be possible, so that our main results are as follows. Throughout the remainder of this paper we assume that $T^* = T$ is possibly unbounded, with bounded transform $S$.

**Theorem 1.** The (point) spectra of $T$ and its bounded transform $S$ are related by

\[
\sigma(T) = \left\{ \mu (1 - \mu^2)^{-\frac{1}{2}} : \mu \in \tilde{\sigma}(S) \right\}; \\
\sigma(S) = \left\{ \lambda (1 + \lambda^2)^{-\frac{1}{2}} : \lambda \in \sigma(T) \right\}^-, 
\]

where $^-$ denotes the closure in $\mathbb{R}$, and we abbreviate

\[
\tilde{\sigma}(S) = \sigma(S) \cap (-1, 1). 
\]

Note that $\tilde{\sigma}(S) = \sigma(S)$ iff $T$ is bounded (in which case $\sigma(S)$ is a compact subset of $(-1, 1)$, since $\pm 1 \in \sigma(S)$ iff $T$ is unbounded). We define the following operator algebras within $B(\mathcal{H})$:

\[
C^*_\bullet(S) = \left\{ g(S) : g \in C^*_\bullet(\tilde{\sigma}(S)) \right\}, 
\]

where $\bullet$ is $b$, $c$, or $0$, so that we have defined $C^*_c(S)$, $C^*_0(S)$, and $C^*_b(S)$. Notice that $C(\sigma(S))$ consists of all $g \in C_b(\tilde{\sigma}(S))$ for which $\lim_{y \to \pm 1} g(y)$ exists, where this limit is 0 if and only if $g \in C_0(\tilde{\sigma}(S))$. Hence we have the inclusions (of which the first set implies the second)

\[
C_c(\tilde{\sigma}(S)) \subseteq C_0(\tilde{\sigma}(S)) \subseteq C(\sigma(S)) \subseteq C_b(\tilde{\sigma}(S)); \\
C^*_c(S) \subseteq C^*_0(S) \subseteq C^*(S) \subseteq C^*_b(S),
\]

with equalities iff $T$ is bounded. This means that $g(S)$ is defined for $g \in C_0(\tilde{\sigma}(S))$, and hence a fortiori also for $g \in C_c(\tilde{\sigma}(S))$. Consequently, $f(T)$ may be defined by (1.8) whenever $f \in C_0(\sigma(T))$, including $f \in C_c(\sigma(T))$. To pass to the larger class $f \in C_b(\sigma(T))$, we define $C^*_0(S)\mathcal{H}$ as the linear span of all vectors of the form $g(S)\psi$, where $g \in C_0(\tilde{\sigma}(S))$ and $\psi \in \mathcal{H}$. Then $C^*_0(S)\mathcal{H}$ is dense in $\mathcal{H}$ (Lemma 1). In the spirit of Woronowicz [5, 12], we then initially define $f(T)$ for $f \in C_b(\sigma(T))$ on the domain $C^*_0(S)\mathcal{H}$ by linear extension of the formula

\[
f_0(T)h(T)\psi = (fh)(T)\psi, 
\]

where $h \in C_0(\sigma(T))$ and hence also $fh \in C_0(\sigma(T))$, since $C_b(\sigma(T))$ is the multiplier algebra of $C_0(\sigma(T))$. Then $f_0(T)$ is bounded (Lemma 2), and we define $f(T)$ as its closure, i.e.,

\[
f(T) = f_0(T)^-.
\]

This also works for $f \in C(\sigma(T))$, in which case $f_0(T)$ may no longer be bounded, but remains closable (Lemma 3), so that we may once again define $f(T)$ as its closure, cf. (1.18). We have:
Theorem 2. If \( f \in C(\sigma(T)) \) is real-valued, then \( f(T) \) is self-adjoint, i.e., \( f_0(T)^- = f_0(T)^* \); more generally, \( f(T)^* = f^*(T) \). Furthermore, the continuous functional calculus \( f \mapsto f(T) \) restricts to an isometric *-homomorphism from \( C_0(\sigma(T)) \) (with supremum-norm) to \( C^*(S) \).

See also Theorem 4. In addition, the map \( f \mapsto f(T) \) has the reassuring special cases

\[
\begin{align*}
1_{\sigma(T)}(T) &= I; \\
\text{id}(T) &= T; \\
(id - z)^{-1}(T) &= (T - z)^{-1}, \ z \in \rho(T),
\end{align*}
\]

where \( 1_{\sigma(T)}(x) = 1 \) and \( \text{id}(x) = x \ (x \in \sigma(T)) \), and therefore does what it is supposed to do.

Finding the right analogue of (1.10) for unbounded \( T = T^* \) first requires a redefinition of \( W^*(T) \), which is standard [8]. If \( T \) is unbounded and \( R \in B(H) \), then we say that \( R \) and \( T \) commute, written \( TR \subset RT \), if \( R\psi \in D(T) \) and \( RT\psi = TR\psi \) for any \( \psi \in D(T) \). Let \( \{T\}' \) be the set of all bounded operators that commute with \( T \). If \( T^* = T \), then \( \{T\}' \) is a unital, strongly closed \(*\)-subalgebra of \( B(H) \), and hence a von Neumann algebra [8]. Its commutant

\[
W^*(T) = \{T\}''
\]

is a von Neumann algebra, too. If \( T \) is bounded, then \( W^*(T) \) is the von Neumann algebra generated by \( T \), which coincides with \( C^*(T)'' \). As usual, we call a closed unbounded operator \( X \) affiliated to a von Neumann algebra \( A \subset B(H) \), written \( X \in \eta A \), iff \( XR \subset RX \) for each \( R \in A' \). For example, if \( T^* = T \), then \( T\eta W^*(T) \), and if \( T \in \eta A \), then \( W^*(T) \subseteq A \); in other words, \( W^*(T) \) is the smallest von Neumann algebra such that \( T \) is affiliated to it.

As a result of independent interest as well as a lemma for Theorem 4, we may then adapt [8, Lemma 5.2.8] to the bounded transform:

**Theorem 3.** Let \( A \subset B(H) \) be a von Neumann algebra. Then \( T \in \eta A \) iff \( S \in A \).

Denoting the (Banach) space of (bounded) Borel functions on \( \sigma(T) \) (equipped with the supremum-norm) by \( B_0(\sigma(T)) \), we may still define \( f(T) \) by (1.8) and the usual Borel functional calculus for the bounded transform \( S \).

**Theorem 4.** The map \( f \mapsto f(T) \) is a norm-decreasing *-homomorphism from \( B_0(\sigma(T)) \) to

\[
\]

More generally, if \( f \in B(\sigma(T)) \), then \( f(T) \) is affiliated with \( W^*(T) \).

The remainder of this paper simply consists of the proofs of these theorems.

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2 Proofs

This section contains all proofs. We will not repeat the theorems.

2.1 Proof of Theorem 1

The operator \( \sqrt{1-S^2} \) is a bijection from \( \mathcal{H} \) to \( \mathcal{R}(\sqrt{1-S^2}) = \mathcal{D}(T) \). Let \( \lambda \in \rho(T) \equiv \mathbb{C} \setminus \sigma(T) \), so that \( T - \lambda I \) is a bijection from \( \mathcal{D}(T) \) to \( \mathcal{H} \). Thus by composition we have a bijection \( \mathcal{H} \to \mathcal{H} \); equivalently, \( (T - \lambda I)(\sqrt{1-S^2}) \) is invertible, which in turn is equivalent to invertibility of \( S - \lambda \sqrt{1-S^2} \). Thus \( \lambda \in \rho(T) \iff S - \lambda \sqrt{1-S^2} \) is a bijection, or, expressed contrapositively, \( \lambda \in \sigma(T) \iff S - \lambda \sqrt{1-S^2} \) is not invertible in \( B(\mathcal{H}) \). This is the case if \( S - \lambda \sqrt{1-S^2} \) is not invertible in \( C^*(S) \), which, using the Gelfand isomorphism \( C^*(S) \cong C(\sigma(S)) \), in turn is true if the function \( k_\lambda(x) = x - \lambda \sqrt{1-x^2} \) is not invertible in \( C(\sigma(S)) \), i.e., if \( 0 \in \sigma(k_\lambda) \).

Since in \( C(X) \) we have \( \sigma(f) = \mathcal{R}(f) \) (with \( X \) a compact Hausdorff space), and \( \sigma(S) \) is indeed compact and Hausdorff because \( S \) is bounded, we obtain \( \lambda \in \sigma(T) \) if \( 0 \in \mathcal{R}(k_\lambda) \). If \( \pm 1 \) lie in \( \sigma(S) \) they cannot give rise to this possibility, since \( k_\lambda(\pm 1) = \pm 1 \) for each \( \lambda \). Hence we have \( 0 \in \mathcal{R}(k_\lambda) \) if \( \lambda = \mu(1 - \mu^2)^{-\frac{1}{2}} \) for some \( \mu \in \sigma(S) \cap (-1,1) \), which yields (1.11).

The same argument shows that \( \mu \in \sigma(S) \cap (-1,1) \) comes from \( \lambda \in \sigma(T) \). But since \( \sigma(S) \) is compact and hence closed in \([-1,1]\) we obtain (1.12). \( \square \)

2.2 Proof of Theorem 2

This proof relies on three lemma's.

**Lemma 1.** Let \( C^*_c(S)\mathcal{H} \) be the linear span of all vectors of the form \( g(S)\psi \), where \( g \in C_c(\sigma(S)) \) and \( \psi \in \mathcal{H} \). Then \( C^*_c(S)\mathcal{H} \) is dense in \( \mathcal{H} \).

**Proof.** Define \( g_n : (-1,1) \to [0,1] \) by putting \( g_n(x) = 0 \) for \( x \in (-1,\frac{1}{n}-1] \cup [1-\frac{1}{n},1) \), \( g_n(x) = 1 \) if \( x \in \left[\frac{2}{n}-1,1-\frac{2}{n}\right] \), and linear interpolation in between. The ensuing sequence converges pointwise to the unit \( 1 \) on \((-1,1)\). Restricting each \( g_n \) to \( \sigma(S) \), the continuous functional calculus gives \( g_n(S) \to 1_{\sigma(S)} \) strongly. Therefore, for any \( \psi \in \mathcal{H} \) we have a sequence \( \psi_n = g_n(S)\psi \in C^*_c(S)\mathcal{H} \) such that \( \psi_n \to \psi \). \( \square \)

**Lemma 2.** For \( f \in C_0(\sigma(T)) \), define an operator \( f_0(T) \) on the domain \( C^*_0(S)\mathcal{H} \) by (1.17). Then \( f_0(T) \) is bounded, with bound
\[
\|f(T)\| \leq \|f\|_\infty.
\] (2.24)

**Proof.** Let \( \varepsilon > 0 \). If \( h \in C_0(\sigma(T)) \), then \( fh \in C_0(\sigma(T)) \), so that we can find a compact subset \( K \subset \sigma(T) \) such that \( |h(x)f(x)| < \varepsilon \) for each \( x \notin K \). Let \( \tilde{h} = h \circ u \), cf. (1.2); then \( \tilde{h} \in C_0(\sigma(S)) \) whenever \( h \in C_0(\sigma(T)) \); in fact, we have an isometric isomorphism
\[
C_0(\sigma(T)) \overset{\cong}{\to} C_0(\sigma(S)), \quad \tilde{h} \mapsto h \circ u.
\] (2.25)
Contractivity of the Borel functional calculus for bounded operators on \( \mathcal{H} \) gives
\[
\| (1_{K^c}fh)(S) \psi \| \leq \| (1_{K^c}fh)(S) \| \| \psi \| \leq \| 1_{K^c}fh \|_\infty \| \psi \| < \varepsilon \| \psi \|.
\]
Using also the homomorphism property of the Borel functional calculus, we then find
\[
\| (fh)(T) \psi \| = \| (fh)(T) \psi \| = \| (1_{K^c}fh)(S) \| + \| (1_{K^c}fh)(S) \| \| \psi \| \leq \| (1_{K^c}fh)(S) \| \| \psi \| + \| (1_{K^c}fh)(S) \| \| \psi \| \leq \| f \|_\infty \| h(T) \psi \| + \varepsilon \| \psi \|,
\]
since \( \| 1_{K^c}f \|_\infty \leq \| f \|_\infty = \| f \|_\infty \). Since the last expression above is independent of \( K \), we may let \( \varepsilon \to 0 \), obtaining boundedness of \( f(T) \) as well as \( 224 \).

The last claim in Theorem 2 now follows from the continuous functional calculus for \( S \) and the isometric isomorphism \( 225 \). Although isometry may be lost if we go from \( C_0(\sigma(T)) \) to \( C_b(\sigma(T)) \), it easily follows from \( 1.17 \) - \( 1.18 \) that the map \( f \to f(T) \) at least defines a \( \ast \)-homomorphism \( C_b(\sigma(T)) \to B(\mathcal{H}) \). This property will be used after Lemma 3 below.

**Lemma 3.** For \( f \in C(\sigma(T)) \), define an operator \( f_0(T) \) on the domain \( C_c^\infty(S)\mathcal{H} \) by \( 1.17 \). Then \( f_0(T) \) is closable. Moreover, if \( f \) is real-valued \( (f^* = f) \), then \( f_0(T) \) is symmetric.

**Proof.** Suppose that \( h_1(T)\psi_1 \) and \( h_2(T)\psi_2 \) lie in \( D(f_0(T)) \). Then we may compute:
\[
\langle h_2(T)\psi_2, f_0(T)h_1(T)\psi_1 \rangle = \langle \psi_2, (\overline{h_2} f h_1)(T)\psi_1 \rangle = \langle \psi_2, (\overline{h_2} f h_1)(T)\psi_1 \rangle; \tag{2.26}
\]
\[
\langle h_2(T)\psi_2, h_1(T)\psi_1 \rangle = \langle \psi_2, (\overline{h_2} f h_1)(T)\psi_1 \rangle = \langle \psi_2, (\overline{h_2} f h_1)(T)\psi_1 \rangle. \tag{2.27}
\]
This implies that \( D(f_0(T)) \subseteq D(f_0(T)^*) \) since \( D(f_0(T)) \) is dense, so is, \( D(f_0(T)^*) \), which implies that \( f_0(T) \) is closable. The second claim is obvious from \( 2.26 \) - \( 2.27 \).

**Proof.** To prove Theorem 2 we use a well-known result of Nelson [6]; see also [9] (this step was suggested to us by Nigel Higson). For convenience we recall this result (without proof):

**Lemma 4.** Let \( \{U(t)\}_{t \in \mathbb{R}} \) be a strongly continuous unitary group of operators on a Hilbert space \( \mathcal{H} \). Let \( R : D(R) \to \mathcal{H} \) be densely defined and symmetric. Assume that \( D(R) \) is invariant under \( \{U(t)\}_{t \in \mathbb{R}} \), i.e. \( U(t) : D(R) \to D(R) \) for each \( t \), and also that \( \{U(t)\}_{t \in \mathbb{R}} \) is strongly differentiable on \( D(R) \). Then \( -idU(t)/dt \) is essentially self-adjoint on \( D(R) \) and its closure is the self-adjoint generator of \( \{U(t)\}_{t \in \mathbb{R}} \) (given by Stone’s Theorem). In particular, if \( (dU(t)/dt)\psi = iRU(t)\psi \) for each \( \psi \in D(R) \), then \( R \) is essentially self-adjoint.
Set $R = f_0(T)$ for $f \in C(\sigma(T))$, so that

$$\mathcal{D}(R) = C^*_c(S)\mathcal{H},$$

(2.28)

and for each $t \in \mathbb{R}$ define $U(t)$ via the (bounded) function $x \mapsto \exp(itf(x))$ on $\sigma(T)$, that is, for $h \in C_c(\sigma(T))$ and $\psi \in \mathcal{H}$, we initially define

$$U_0(t)h(T)\psi = (e^{itf}h)(T)\psi.$$  

(2.29)

Then $U_0$ bounded by Lemma 2, and we define $U(t)$ as the closure of $U_0(t)$. The remark before Lemma 3 then implies that $t \mapsto U(t)$ defines a unitary representation of $\mathbb{R}$ on $\mathcal{H}$. Strong continuity of this representation follows from an $\varepsilon / 3$ argument. First, for $\psi = h(T)\psi$,

(2.30)

assuming $\|\psi\| = 1$ for simplicity, eqs. (2.29) and (2.24) give

$$\|U(t)\varphi - \varphi\| \leq \|e^{itf}h - h\|_\infty \leq \|h\|_\infty \|e^{itf} - 1\|_\infty^{(K)},$$

(2.31)

where $K$ is the (compact) support of $h$ in $\sigma(T)$. Since the exponential function is uniformly convergent on any compact set, this gives $\lim_{t \to 0} \|U(t)\varphi - \varphi\| = 0$ for $\varphi$ of the form (2.30); taking finite linear combinations thereof gives the same result for any $\varphi \in C^*_c(S)\mathcal{H}$. Thus for any $\varepsilon > 0$ we can find $\delta > 0$ so that $\|U(t)\varphi - \varphi\| < \varepsilon / 3$ whenever $|t| < \delta$. For general $\psi' \in \mathcal{H}$, we find $\varphi \in C^*_c(S)\mathcal{H}$ such that $\|\varphi - \psi'\| < \varepsilon / 3$, and estimate

$$\|U(t)\psi' - \psi'\| \leq \|U(t)\psi' - U(t)\varphi\| + \|U(t)\varphi - \varphi\| + \|\varphi - \psi'\| \leq \varepsilon / 3 + \varepsilon / 3 + \varepsilon / 3 = \varepsilon,$$

since $\|U(t)\psi' - U(t)\varphi\| = \|\psi' - \varphi\|$ by unitarity of $U(t)$. Thus $\lim_{t \to 0} \|U(t)\psi - \psi\| = 0$ for any $\psi \in \mathcal{H}$, so that the unitary representation $t \mapsto U(t)$ is strongly continuous. Similarly,

$$\left\| \frac{U(t+s)\varphi - U(t)\varphi}{s} - iRU(t)\varphi \right\| \leq \left\| \frac{e^{isf}h - h}{s} - ifh \right\|_\infty,$$

(2.32)

assuming (2.30), so that by the same argument as in (2.31) we obtain

$$\frac{dU(t)}{dt}\varphi = iRU(t)\varphi,$$

(2.33)

initially for any $\varphi$ of the form (2.30), and hence, taking finite sums, for any $\varphi \in \mathcal{D}(R)$, cf. (2.28). The final part of Lemma 4 then shows that $f_0(T)$ is essentially self-adjoint on its domain $C^*_c(S)\mathcal{H}$. Its closure $f(T)$ is therefore self-adjoint, and Theorem 2 is proved. $\square$
We now prove the examples (1.19) - (1.21), of which the first is trivial. Writing $T_0$ for the operator $\text{id}_0(T)$, the definition (1.17) gives
\[ T_0 \varphi = T \varphi \]
for $\varphi \in \mathcal{D}(T_0) = C_0^*(S)\mathcal{H}$. Let $\psi \in \mathcal{D}(T_0^{-1})$, so that there is a sequence $(\varphi_n)$ in $\mathcal{D}(T_0)$ such that $\varphi_n \to \varphi$ and $(T_0 \varphi_n)$ converges. Since $T$ is closed, it follows that $T_0 \varphi_n \to T \varphi$, so that $\varphi \in \mathcal{D}(T)$. Hence $T_0^{-1} \subset T$. Since both operators are self-adjoint, this implies $T_0^{-1} = T$, which proves (1.20).

The proof of (1.21) is easier since $(T - z)^{-1}$ is bounded: writing
\[ f(x) = (x - z)^{-1}, \]
where $z \notin \sigma(T)$ is fixed and $x \in \sigma(T)$, we have
\[ f_0(T)h(T)\psi = (fh)(T)\psi = (T - z)^{-1}h(T)\psi, \]
and hence
\[ f_0(T)\varphi = (T - z)^{-1}\varphi \]
for any $\varphi \in \mathcal{D}(f_0(T)) = C_0^*(S)\mathcal{H}$. So if $\varphi_n \to \varphi$ for $\varphi \in \mathcal{H}$ and $\varphi_n \in \mathcal{D}(f_0(T))$, boundedness and hence continuity of the resolvent implies
\[ f(T)\varphi = \lim_{n \to \infty} f_0(T)\varphi_n = \lim_{n \to \infty} (T - z)^{-1}\varphi_n = (T - z)^{-1}\varphi. \]

### 2.3 Proof of Theorem 3

The first step consists in the observation that $T \eta A \text{ iff } TU \subset UT$ (or, equivalently, $UTU^* = T$) merely for each unitary $U \in A'$, which is well known [11].

The second step is to show that $TU \subset UT$ if $SU = US$ for any unitary $U$. This is a simple computation. First suppose that $UTU^* = T$. Then:
\[
U(1 + T^2)^{-1}U^* = (U(1 + T^2)U^*)^{-1} = ((U + UT^2)U^*)^{-1} = (UU^* + UT^2U^*)^{-1} = (1 + UTU^*UTU^*)^{-1} = (1 + T^2)^{-1}.
\]

If $R$ is bounded and positive, then $UR = RU$ if $U \in C^*(R)'$, and since $\sqrt{R} \in C^*(R)$ by the continuous functional calculus, we also have $U\sqrt{R} = \sqrt{RU}$. Consequently,
\[
USU^* = U(T\sqrt{(1 + T^2)^{-1}})U^* = (UTU^*)\left(U\sqrt{(1 + T^2)^{-1}}U^*\right) = T\sqrt{(1 + T^2)^{-1}} = S.
\]
Similarly, if $SU = US$, then

$$UTU^* = US\sqrt{1 - S^2}^{-1} U^* = SU\sqrt{1 - S^2}^{-1} U^* = S\left(U\sqrt{1 - S^2}U^*\right)^{-1} = S\sqrt{1 - S^2}^{-1} = T.$$ 

Thirdly, as in the first step, $SU = US$ for any unitary $U \in A'$ iff $S \in A'' = A$. \hfill $\square$

### 2.4 Proof of Theorem 4

Eq. (1.23) in Theorem 4 follows from Theorem 3 taking $A = W^*(T)$, so that $T\eta_A$, yields $S \in W^*(T)$, and hence $W^*(S) \subseteq W^*(T)$. On the other hand, taking $A = W^*(S)$, in which case $S \in A$, gives $T\eta W^*(S)$, and hence $W^*(T) \subseteq W^*(S)$.

Similar to (2.25), we have an isometric isomorphism

$$B_h(\sigma(T)) \cong B_h(\hat{\sigma}(S)), \quad h \mapsto h \circ u,$$

so that the first claim of Theorem 4 follows from the Borel functional calculus for the bounded operator $S$ [8]. The proof of the last one is, mutatis mutandis, practically the same as in [8] Theorem 5.3.8], so we omit the details; see [2]. \hfill $\square$

As explained in [8] §5.3, there exists a Borel measure $\mu$ on $\sigma(T)$ such that the map $f \mapsto f(T)$ may also be seen as a so-called essential *-homomorphism from $B(\sigma(T))/\mathcal{N}(\sigma(T))$ into the *-algebra of normal operators affiliated with $W^*(T)$, where $\mathcal{N}(\sigma(T))$ is the set of $\mu$-null functions on $\sigma(T)$. This remains true in our approach, with the same proof [2].

### 3 Epilogue

Let us finally note that although this paper was inspired by the work of Woronowicz, the $C^*$-algebraic affiliation relation he defines in [12] (as did, independently, also Baaj and Julg [1]) has not been used here. If we call his relation $\eta'$ to avoid confusion with the $W^*$-algebraic relation $\eta$ we do use, if $A \subset B(H)$ we have $T\eta'A \Rightarrow T \in A$ (and hence $T$ is bounded), cf. [12] Prop. 1.3. Woronowicz does not define a $C^*$-algebraic counterpart of the von Neumann algebra $W^*(T)$, but it might be reasonable to define $C^*(T)$ as the smallest $C^*$-algebra $A$ in $B(H)$ such that $T\eta'A$. It follows from [12] Example 4] that this would give $C^*(T) = C_0^*(S)$, as defined in [1,4]. This $C^*$-algebra contains $S$ (and hence $T$) if and only if $T$ is bounded, in which case $C_0^*(S) = C^*(S)$ and hence $C^*(T) = C^*(S)$, as in our approach, cf. [1,5]. Also in general (i.e., if $T$ is possibly unbounded), the bicommutant $C^*(T)^{''}$ coincides with $W^*(T)$ as defined in the usual way (1.22) this follows from $C_0^*(S)^{''} = C^*(S)^{''} = W^*(S)$ and (1.10).

Of course, we could also redefine $\eta'$, now calling it $\eta''$, by stipulating that $T\eta''A$ whenever $S \in A$, and redefine $C^*(T)$ accordingly (i.e., as the smallest $C^*$-algebra $A$ in $B(H)$ such that $T\eta''A$). This would give (1.5) even if $T$ is unbounded, though in a somewhat empty way.
References


