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Electronic Notes in Theoretical Computer Science 319 (2015) 239–253

Electronic Notes in Theoretical Computer Science

www.elsevier.com/locate/entcs

An Effect-Theoretic Account of Lebesgue Integration

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Abstract

Effect algebras have been introduced in the 1990s in the study of the foundations of quantum mechanics, as part of a quantum-theoretic version of probability theory. This paper is part of that programme and gives a systematic account of Lebesgue integration for $[0,1]$-valued functions in terms of effect algebras and effect modules. The starting point is the 'indicator' function for a measurable subset. It gives a homomorphism from the effect algebra of measurable subsets to the effect module of $[0,1]$-valued measurable functions which preserves countable joins.

It is shown that the indicator is free among these maps: any such homomorphism from the effect algebra of measurable subsets can be thought of as a generalised probability measure and can be extended uniquely to a homomorphism from the effect module of $[0,1]$-valued measurable functions which preserves joins of countable chains. The extension is the Lebesgue integral associated to this probability measure. The preservation of joins by it is the monotone convergence theorem.

Keywords: Effect algebra, effect module, Lebesgue integration

1 Introduction

Integration is a fundamental mathematical technique developed to compute quantities such as lengths of curves, areas of surfaces, volumes of solids, averages of distributions, Fourier transforms of functions, solutions to differential equations, and so on. Roughly speaking, the integral assigns to a function the area under its graph (counting the area under the x-axis negatively). The notation $\int f(x)\,\mathrm{d}x$ for the integral of f suggests that it should be thought of as a sum ("$\int$" is an elongated "s") of uncountably many rectangles $f(x)\,\mathrm{d}x$ of infinitesimal width $\mathrm{d}x$. While this makes for an elegant picture, a formal definition of the integral requires a different approach: for instance, by approximating f by basic functions for which the integral is easily determined.

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http://dx.doi.org/10.1016/j.entcs.2015.12.015
1571-0661/

In probability theory, integration is used for calculating probabilities of events and expected values of random variables (among many other things). In the theory of continuous probabilistic computation, integration is used for sequential composition (of Markov kernels, or coalgebras of the Giry monad), see *e.g.* [18,20]. Integration is also used for calculation weakest preconditions of quantitative predicates (random values), see *e.g.* [15].

This paper gives an elementary account of Lebesgue integration, using basic measure theory. It is restricted to measurable functions $X \to [0,1]$ to the unit interval, which may be understood as fuzzy predicates. What distinguishes our account from the traditional one is that it makes systematic use of the notions of effect algebra and effect module, where an effect module is an effect algebra with scalar multiplication, where scalars are taken from $[0,1]$. These effect structures emerged in the foundations of quantum mechanics, as part of a quantum-theoretic version of probability theory (see [7] for an overview). It turns out that the basic notions of Lebesgue integration can be formulated very naturally in terms of ω-(complete)effect algebras and ω-effect modules. For instance, for a measurable space X, with set Σ_X of measurable subsets,

- the σ-algebra Σ_X of measurable subsets is an ω-effect algebra;
- the set $\mathbf{Meas}(X,[0,1])$ of measurable functions $X \to [0,1]$ is an ω-effect module;
- the indicator function gives a map $\mathbf{1}_{(-)} \colon \Sigma_X \to \mathbf{Meas}(X,[0,1])$ which is a homomorphism of ω-effect algebras — where $\mathbf{1}_M(x) = 1$ if $x \in M$ and $\mathbf{1}_M(x) = 0$ if $x \notin M$;
- moreover, this indicator map is *free* in the following sense: for every ω-complete effect module E, and for each probability measure (homomorphism of ω-effect algebras) $\phi \colon \Sigma_X \to E$, there is a unique homomorphism of effect modules $\overline{\phi} \colon \mathbf{Meas}(X,[0,1]) \to E$ with $\overline{\phi} \circ \mathbf{1}_{(-)} = \phi$. This free extension $\overline{\phi}$ precisely is Lebesgue integration! It sends $p \in \mathbf{Meas}(X,[0,1])$ to the integral $\overline{\phi}(p) = \int p \, \mathrm{d}\phi \in E$.

These bullet points summarise the main contributions of the paper. The definition of the integral $\int p \, \mathrm{d}\phi \in E$ proceeds in two stages, as usual, namely first for step functions (using the effect module structure of E), and then for any measurable p function by writing p as an ω-join $\bigvee$ of an ascending chain of step functions (using the ω-completeness of E). Much of the work of the paper goes into verifying that the usual arguments can be adapted to the setting of ω-effect modules.

In the end one may wonder how much of a restriction our use of $[0,1]$-valued functions is. These functions form an effect module. In [17] it is shown that the category of effect modules is equivalent to the category of order unit spaces, via a process called totalisation. By applying such totalisation one obtains the bounded $\mathbb{R}$-valued functions from the $[0,1]$-valued ones. In this way one can extend integration from $[0,1]$-valued functions to bounded $\mathbb{R}$-valued functions.

2 Effect algebras and effect modules

Effect algebras have been introduced in mathematical physics [9] (and also [4,11]), in the investigation of quantum probability, see [7] for an overview. An *effect algebra* is a partial commutative monoid $(M, 0, \varovee)$ with an orthocomplement $(-)^\perp$. One writes $x \perp y$ if $x \varovee y$ is defined. The formulation of the commutativity and associativity requirements are a bit involved, but essentially straightforward. The orthocomplement satisfies $x^{\perp\perp} = x$ and $x \varovee x^\perp = 1$, where $1 = 0^\perp$. There is always a partial order, given by $x \leq y$ iff $x \varovee z = y$, for some z. Then: $x \perp y$ iff $x \leq y^\perp$. The main example is the unit interval $[0,1] \subseteq \mathbb{R}$, where addition $+$ is obviously partial, commutative, associative, and has 0 as unit; moreover, the orthocomplement is $r^\perp = 1 - r$. An ω-effect algebra (also called σ-effect algebra) additionally has joins $\bigvee_n x_n$ of countable ascending chains $x_1 \leq x_2 \leq \cdots$. We write **EA** for the category of effect algebras, with as morphisms maps preserving $\varovee$ and 1 — and thus all other structure. The morphisms in the subcategory ω-**EA** $\hookrightarrow$ **EA** of ω-effect algebras are those that preserve joins of ω-chains.

For each set X, the set $[0,1]^X$ of fuzzy predicates on X is an ω-effect algebra, with pointwise operations. Each Boolean algebra B is an effect algebra with $x \perp y$ iff $x \wedge y = \perp$; then $x \varovee y = x \vee y$. In a quantum setting, the main example is the set of effects $\mathcal{E}f(\mathscr{H})$ on a Hilbert space $\mathscr{H}$ (that is, bounded linear operators $A \colon \mathscr{H} \to \mathscr{H}$ with $0 \leq A \leq I$, see *e.g* [7,14]).

An *effect module* is an 'effect' version of a vector space. It involves an effect algebra E with a scalar multiplication $s \cdot x \in E$, where $s \in [0,1]$ and $x \in E$. This scalar multiplication must preserve $0, \varovee$ in each variable separately. The sets $[0,1]^X$ and $\mathcal{E}f(\mathscr{H})$ are clearly such effect modules. In the subcategory **EMod** $\hookrightarrow$ **EA** of effect modules, maps additionally commute with scalar multiplication. We use ω-**EMod** $\hookrightarrow$ **EMod** for the subcategory of ω-complete effect modules, with effect module maps that preserve joins of ω-chains.

We need the following results about effect modules.

Lemma 2.1 *For elements x, y in an effect module, and for scalars $r, s \in [0,1]$,*

(i) $(r \cdot x)^\perp = (r^\perp \cdot x) \varovee x^\perp$;

(ii) $x \perp y$ *implies* $r \cdot x \perp s \cdot y$.

Proof We obtain $(r \cdot x)^\perp = r^\perp \cdot x \varovee x^\perp$ by uniqueness of orthocomplements:

$$r \cdot x \varovee r^\perp \cdot x \varovee x^\perp = (r \varovee r^\perp) \cdot x \varovee x^\perp = 1 \cdot x \varovee x^\perp = x \varovee x^\perp = 1.$$

Next, if $x \perp y$, then $x \leq y^\perp$, and thus $r \cdot x \leq x \leq y^\perp$. Taking complements, we see that $s \cdot y \leq y \leq (r \cdot x)^\perp$. This means $r \cdot x \perp s \cdot y$.

3 Measurable spaces and functions

A measurable space (X, Σ_X) (or simply X) is a pair consisting of a set X and a σ-algebra $\Sigma_X \subseteq \mathcal{P}(X)$. The latter is a collection of *measurable* subsets closed under

$\emptyset$, complements (negation), and countable unions. The measurable subsets form a Boolean algebra in which countable joins exist — so Σ_X is an ω-effect algebra.

A function $f\colon X \to Y$ between measurable spaces — that is, from (X, Σ_X) to (Y, Σ_Y) — is called *measurable* if $f^{-1}(M) \in \Sigma_X$ for each $M \in \Sigma_Y$. This yields a category **Meas**, which comes with a functor $\Sigma_{(-)}\colon \mathbf{Meas} \to \omega\text{-}\mathbf{EA}^{\mathrm{op}}$. With each topological space X one associates the least σ-algebra containing all open subsets, called the Borel algebra/space on X. In particular the unit interval $[0,1]$ forms a measurable space. Its measurable subsets are generated by the intervals $(q, 1]$, where q is a rational number in $[0,1]$.

Measurable functions have more order structure than continuous ones: they are closed under countable joins.

Lemma 3.1 *Let X be a measurable space, and Y a topological space.*

(i) *The set $\mathbf{Meas}(X, [0,1])$ of measurable functions $X \to [0,1]$ is an ω-effect module. In particular, it is closed under joins of ascending ω-chains.*

(ii) *The set $\mathbf{Top}(Y, [0,1])$ of continuous functions $Y \to [0,1]$ is an effect module, but* not *always an ω-effect module: some ascending ω-chains of continuous functions have no join.*

These mappings $X \mapsto \mathbf{Meas}(X, [0,1])$ and $Y \mapsto \mathbf{Top}(Y, [0,1])$ yield functors:

$$\mathbf{Meas} \longrightarrow \omega\text{-}\mathbf{EMod}^{op} \qquad\qquad \mathbf{Top} \longrightarrow \mathbf{EMod}^{op} \tag{1}$$

Proof The measurable functions $X \to [0,1]$ form an effect module, using pointwise the effect module structure from the unit interval $[0,1]$. To show that they are closed under joins let $p_1 \leq p_2 \leq p_3 \leq \cdots$ be measurable functions $p_n\colon X \to [0,1]$. We must show that the (pointwise) join $p = \bigvee_n p_n$ in $[0,1]^X$ is again measurable. Since subsets of the form $(r, 1]$ with $r \in [0,1]$ generate the Borel σ-algebra on $[0,1]$ it suffices to show that $p^{-1}(\,(r,1]\,)$ is measurable. Note that for $x \in X$ and $r \in [0,1]$ we have $p(x) = \bigvee_n p_n(x) > r$ if and only if there is n with $p_n(x) > r$. Thus $p^{-1}(\,(r,1]\,) = \bigcup_n p_n^{-1}(\,(r,1]\,)$. Since each $p_n^{-1}(\,(r,1]\,)$ is measurable, so is $p^{-1}(\,(r,1]\,]$, and the join $p = \bigvee_n p_n$ is measurable.
We thus get a functor $\mathbf{Meas}(-, [0,1])\colon \mathbf{Meas} \to \omega\text{-}\mathbf{EMod}^{\mathrm{op}}$. The ω-effect module structure is preserved by pre-composition, since it is defined pointwise.

The set $\mathbf{Top}(Y, [0,1])$ of continuous functions $Y \to [0,1]$ is an effect module, but in general has no ω-joins. Take for instance $Y = [0,2]$, and consider the continuous functions $f_1 \leq f_2 \leq \cdots \leq f\colon [0,2] \to [0,1]$ defined by:

$$f_n(y) = \begin{cases} 1 - y^n & \text{if } y \in [0,1) \\ 0 & \text{if } y \in [1,2] \end{cases} \qquad \text{and} \qquad f(y) = \begin{cases} 1 & \text{if } y \in [0,1) \\ 0 & \text{if } x \in [1,2] \end{cases}$$

Since $\lim\limits_{n\to\infty} y^n = 0$ for $y \in [0,1)$ we see that f is the pointwise join of $f_1, f_2, \ldots$. Clearly, this join f is not continuous, and so it cannot be the join of $f_1, f_2, \ldots$ in $\mathbf{Top}(Y, [0,1])$. Even more: we claim there is no least continuous function above f. Thus $f_1, f_2, \ldots$ has no join at all in $\mathbf{Top}(Y, [0,1])$.

Indeed, if $g\colon [0,2] \to [0,1]$ is continuous and $g \geq f$, then $g^2 \geq f^2 = f$ as well (where $f^2 = f$ because f is $\{0,1\}$-valued). On the other hand g is not $\{0,1\}$-valued because g is continuous at 1. Thus $g^2 < g$. Hence g is not the least continuous function above f, and thus g is not the join of $f_1, f_2, \ldots$. □

For each measurable space (X, Σ) there is the 'indicator' function $\mathbf{1}_{(-)}\colon \Sigma \to \mathbf{Meas}(X,[0,1])$, given by $\mathbf{1}_M(x) = 1$ if $x \in M$ and $\mathbf{1}_M(x) = 0$ if $x \notin M$ where $M \in \Sigma_X$. Then $\mathbf{1}_{(-)}$ is a homomorphism of ω-effect algebras.

The next result neatly organises the situation so far. It turns out that this situation has an additional freeness property that is the essence of Lebesgue integration. This will be elaborated in the next section (see Theorem 4.12).

Lemma 3.2 *Sending a measurable subset M to its indicator function $\mathbf{1}_M$ is a natural transformation in:*

$$\begin{tikzcd} \omega\text{-}\mathbf{EA}^{op} \\ \mathbf{Meas} \arrow[u, bend left=50, "\Sigma_{(-)}"] \arrow[u, bend right=50, "U\mathbf{Meas}(-,[0,1])"'] \arrow[u, phantom, "\overset{\mathbf{1}_{(-)}}{\Longleftarrow}"] \end{tikzcd}$$

where $U\colon \omega$-$\mathbf{EMod} \to \omega$-$\mathbf{EA}$ is the forgetful functor. The (possibly unexpected) direction of the arrow $\Leftarrow$ is explained by the $(-)^{op}$.

Proof Let (X, Σ_X) be a measurable space. We show that the mapping $M \mapsto \mathbf{1}_M$ is a homomorphism of ω-effect algebras $\mathbf{1}_{(-)}\colon \Sigma_X \to \mathbf{Meas}(X,[0,1])$, and leave naturality to the reader. Clearly, the unit is preserved, since $\mathbf{1}_X$ is the constant function $x \mapsto 1$. Also, if $M \perp M'$ in Σ_X, that is, $M \cap M' = \emptyset$, then $\mathbf{1}_{M \ovee M'} = \mathbf{1}_{M \cup M'} = \mathbf{1}_M + \mathbf{1}_{M'} = \mathbf{1}_M \ovee \mathbf{1}_{M'}$. It is easy to see that ω-joins are preserved: $\bigvee_n \mathbf{1}_{M_n} = \mathbf{1}_{\bigcup_n M_n}$. □

Lemma 3.3 *Hom-ing into $[0,1]$ yields an adjunction between ω-effect modules and measurable spaces:*

$$\begin{tikzcd} \omega\text{-}\mathbf{EMod}^{op} \arrow[rr, bend left, "Hom(-,[0,1])"] \arrow[rr, phantom, "\top"] & & \mathbf{Meas} \arrow[ll, bend left, "Hom(-,[0,1])"] \end{tikzcd}$$

Proof In order to do this, we first need to provide the homset ω-$\mathbf{EMod}(E,[0,1])$ with a σ-algebra. We take the least σ-algebra that makes for each $e \in E$ the evaluation map $\mathrm{ev}_e\colon \omega$-$\mathbf{EMod}(E,[0,1]) \to [0,1]$, given by $\mathrm{ev}_e(\omega) = \omega(e)$, measurable. This is functorial, since for $f\colon E \to D$ in ω-$\mathbf{EMod}$, the map $(-) \circ f\colon \omega$-$\mathbf{EMod}(D,[0,1]) \to \omega$-$\mathbf{EMod}(E,[0,1])$ is measurable.

We get an adjunction since there is a natural bijective correspondence:

$$\begin{array}{c} E \xrightarrow{\;f\;} \mathbf{Meas}(X,[0,1]) \qquad \text{in } \omega\text{-}\mathbf{EMod} \\ \hline\hline X \xrightarrow[\;g\;]{} \omega\text{-}\mathbf{EMod}(E,[0,1]) \qquad \text{in } \mathbf{Meas} \end{array}$$

This is done via a simple swapping of arguments. □

Later on, in Corollary 4.14, we shall see that the monad on the category **Meas** induced by this adjunction is the well-known Giry monad [10].

4 Lebesgue integration in ω-effect modules

Our approach to integration is on the one hand more restricted than usual, and on the other hand more general. The restriction lies in the fact that we define integration for $[0,1]$-valued functions, and not for more general functions. The extension involves using probability measures $\phi\colon \Sigma \to E$ into an ω-effect module E, instead of into $[0,1]$ as is commonly done.

Traditionally, a *measure space* consists of a measurable space (X, Σ_X) with a function $\phi\colon \Sigma_X \to [0,\infty]$ which satisfies $\phi(\emptyset) = 0$ and is countably additive:

$$\phi\left(\bigovee_{n\in\mathbb{N}} M_n\right) = \sum_{n\in\mathbb{N}} \phi(M_n) = \bigvee_{n\in\mathbb{N}} \sum_{i\leq n} \phi(M_i), \tag{2}$$

for each pairwise disjoint, countable collection of measurable $M_n \in \Sigma_X$. Here we use $\ovee$ for *disjoint* union, where Σ_X is understood as an effect algebra. Such a measure ϕ is called a *probability measure* if $\phi(X) = 1$, so that ϕ can be restricted to a function $\Sigma_X \to [0,1]$.

Below is a well-known observation (see *e.g.* [22, Thm. 4.4]) that justifies our generalisation of probability measures to other codomains than $[0,1]$.

Lemma 4.1 *Let X be a measurable space, with a function $\phi\colon \Sigma_X \to [0,1]$. The following points are then equivalent:*

(i) *ϕ is a probability measure, that is, $\phi(\emptyset) = 0$ and $\phi(X) = 1$ and ϕ is countably additive as in (2);*

(ii) *ϕ is a homomorphism of ω-effect algebras $\Sigma_X \to [0,1]$.* □

Definition 4.2 Let X be a measurable space, and E a ω-effect module. An *E-valued probability measure*, or simply an *E-probability measure* is a map $\phi\colon \Sigma_X \to U(E)$ in the category ω-**EA** of ω-effect algebras — where $U\colon \omega$-**EMod** $\to \omega$-**EA** is the forgetful functor.

For each element $x \in X$ we write $\eta(x)\colon \Sigma_X \to E$ for the probability measure given by $\eta(x)(M) = 1$ if $x \in M$ and $\eta(x)(M) = 0$ if $x \notin M$.

Examples of probability measures with values in an ω-effect module are POVMs: Positive Operator-Valued Measures, see *e.g.* [14, Defn. 3.5]. Such a POVM is a map of ω-effect algebras $\Sigma_X \to \mathcal{E}f(\mathscr{H})$ with the effects of a Hilbert space $\mathscr{H}$ as codomain. We will return to POVMs in Example 4.15 below.

Remark 4.3 While Σ_X and $[0,1]$ are MV-algebras (see [5]), a probability measure $\phi\colon \Sigma_X \to [0,1]$ need not be an homomorphism of MV-algebras, that is, preserve binary joins $\vee$.

Indeed, since in an MV-algebra we have the identity $x \vee y = x + (y^\perp + x)^\perp$ a homomorphism of MV-algebras preserves finite joins. (In fact, a homomorphism of effect algebras between MV-algebras is a homomorphism of MV-algebras precisely when it preserves finite joins.) The standard probability measure μ on $[0,1]$ does not preserve finite joins $\mu([0,\frac{1}{2}] \cup [\frac{1}{2},1]) = 1 \neq \frac{1}{2} = \max\{\, \mu([0,\frac{1}{2}]),\, \mu([\frac{1}{2},1])\,\}$ and is thus not a homomorphism of MV-algebras.

The probability measures $\phi\colon \Sigma_X \to [0,1]$ which preserve joins are in fact quite special. Indeed, for such ϕ we have $\phi(A \cup B) = \max\{\phi(A), \phi(B)\}$ for all $A, B \in \Sigma_X$, and also $\phi(A \cap B) = \min\{\phi(A), \phi(B)\}$. Taking $B = A^{\perp}$, we see that either $\phi(A) = 1$ (and $\phi(A^{\perp}) = 0$) or $\phi(A^{\perp}) = 1$ (and $\phi(A) = 0$). Thus $\{A \in \Sigma_X \colon \phi(A) = 1\}$ is an ultrafilter on Σ_X.

Extending measure to integral is done in two parts, first for step functions, and then for all measurable functions, as joins of ω-chains of step functions.

Definition 4.4 Let X be a measurable space.

(i) A *step function* $X \to [0,1]$ is a function that can be written as finite linear combination $r_1 \cdot \mathbf{1}_{M_1} + \cdots + r_k \cdot \mathbf{1}_{M_k} = \bigovee_i r_i \cdot \mathbf{1}_{M_i} \in \mathbf{Meas}(X,[0,1])$ of indicator functions $\mathbf{1}_{M_i}$ and scalars $r_i \in [0,1]$, where the $M_i \in \Sigma_X$ are pairwise disjoint measurable subsets satisfying $\bigovee_i M_i = X$.

(ii) Let $\phi\colon \Sigma_X \to E$ be a probability measure. The interpretation of $\int s \,\mathrm{d}\phi$ for a step function $s = \bigovee_i r_i \mathbf{1}_{M_i}$ is

$$\int s \,\mathrm{d}\phi \;=\; \bigovee_i r_i \cdot \phi(M_i) \;\in\; E. \tag{3}$$

(There is no ambiguity, see Lemma 4.5 below.)

Since these M_i form a partition, they are k-test in the effect algebra Σ_X. Also, the set of step functions can be described as tensor product $\Sigma_X \otimes U(E)$, where $\otimes$ is the tensor of effect algebras, see [16].

In the second point we use the property that in an effect module $x \perp y$ implies $r \cdot x \perp t \cdot y$ for all scalars $r, t \in [0,1]$, see Lemma 2.1.

We will first show that the integral $\int s \,\mathrm{d}\phi$ in (3) is independent of the representation of the step function s, see *e.g.* [22, Lemma 9.1]. We elaborate the details in order to show that this works in effect modules too.

Lemma 4.5 *Let X be a measurable space, and $\phi\colon \Sigma_X \to E$ a probability measure. Consider two step functions $\bigovee_i r_i \cdot \mathbf{1}_{M_i} \;\leq\; \bigovee_j s_j \cdot \mathbf{1}_{N_j}$ in* $\mathbf{Meas}(X,[0,1])$*. Then $\bigovee_i r_i \cdot \phi(M_i) \;\leq\; \bigovee_j s_j \cdot \phi(N_j)$ in E.*

Proof Since $\bigovee_i M_i = X = \bigovee_j N_j$ by Definition 4.4 we have $M_i = \bigovee_j M_i \cap N_j$ and $N_j = \bigovee_i N_j \cap M_i$. Thus:

$$\begin{aligned} \textstyle\sum_i r_i\phi(M_i) \;=\; \sum_i r_i\phi\big(\bigovee_j M_i \cap N_j\big) &= \textstyle\sum_{i,j} r_i\phi\big(M_i \cap N_j\big) \\ &\leq \textstyle\sum_{i,j} s_j\phi\big(M_i \cap N_j\big) \qquad \text{see below} \\ &= \textstyle\sum_j s_j\phi\big(\bigovee_i M_i \cap N_j\big) \;=\; \sum_j s_j\phi(N_j). \end{aligned}$$

We used the fact that $r_i\phi(M_i \cap N_j) \leq s_j\phi(M_i \cap N_j)$ for all i and j. Indeed, this inequality holds when $M_i \cap N_j = \emptyset$. Otherwise, we have for $x \in M_i \cap N_j$,

$$r_i \;=\; \big(\bigovee_i r_i \mathbf{1}_{M_i}\big)(x) \;\leq\; \big(\bigovee_j s_j \mathbf{1}_{N_j}\big)(x) \;=\; s_j.$$

Thus $r_i\phi(M_i \cap N_j) \;\leq\; s_j\phi(M_i \cap N_j)$. □

A basic observation is that each measurable predicate can be described as join of an ascending ω-chain of step functions (see *e.g.* [22, Thm. 8.8]).

Lemma 4.6 *For each measurable function $p\colon X \to [0,1]$ there is an ω-chain $s_1 \leq s_2 \leq \cdots$ of step functions $s_n \leq p$ with $p = \bigvee s_n$.* □

Lemma 4.6 is the key to the meaning of $\int p\, d\phi$ when p is an arbitrary measurable function in $\mathbf{Meas}(X,[0,1])$. Indeed, we shall have $\int p = \bigvee_n \int s_n$ when $s_1 \leq s_2 \leq \cdots$ are step functions with $\bigvee_n s_n = p$. However, before we can cast this observation into a definition we must check that there is no ambiguity by proving that $\bigvee_n \int s_n = \bigvee_n \int t_n$ whenever $t_1 \leq t_2 \leq \cdots$ and $s_1 \leq s_2 \leq \cdots$ are step functions with $\bigvee_n s_n = \bigvee_n t_n$. This fact will follow from a far more general statement (see Proposition 4.8) about the following notion.

Definition 4.7 Let ϕ be an E-valued probability measure on a measurable space X. An *elementary extension* of ϕ is a map $\Phi\colon S \to E$ defined on a collection of measurable functions $S \subseteq \mathbf{Meas}(X,[0,1])$ such that:

(i) $\mathbf{1}_M \in S$ and $\Phi(\mathbf{1}_M) = \phi(M)$ for all $M \in \Sigma_X$;

(ii) S is a sub-effect module of $\mathbf{Meas}(X,[0,1])$, and $\Phi\colon S \to \mathbf{Meas}(X,[0,1])$ is a homomorphism of effect modules.

(iii) $s \cdot \mathbf{1}_M \in S$ for all $M \in \Sigma_X$ and $s \in [0,1]$.

The integral $\int(-)\,\mathrm{d}\phi$, defined on the sub-effect module of step functions is an elementary extension of ϕ. But also integration on all measurable maps will be an elementary extension. This abstraction allows us to apply the following result both to integration of step functions and of all measurable functions.

Proposition 4.8 *Let $\Phi\colon S \to E$ be an elementary extension of an E-valued probability measure ϕ on a measurable space X.*

(i) *Let s and $t_1 \leq t_2 \leq \cdots$ be from S with $s \leq \bigvee t_n$. Then $\Phi(s) \leq \bigvee_n \Phi(t_n)$.*

(ii) *Let $s_1 \leq s_2 \leq \cdots$ and $t_1 \leq t_2 \leq \cdots$ be from S. Then $\bigvee s_n \leq \bigvee t_n$ implies $\bigvee \Phi(s_n) \leq \bigvee \Phi(t_n)$.*

Proof We will only prove point (i) since point (ii) is an easy consequence.

Writing $a_m = 1 - \frac{1}{m} \in [0,1]$ for $m \geq 1$ we have $\bigvee_m a_m = 1$. Thus to prove $\Phi(s) \leq \bigvee_n \Phi(t_n)$ it suffices to show that $a_m \cdot \Phi(s) \leq \bigvee_n \Phi(t_n)$ for all m. Since then $\Phi(s) = 1 \cdot \Phi(s) = (\bigvee_m a_m) \cdot \Phi(s) = \bigvee_m a_m \cdot \Phi(s) \leq \bigvee_n \Phi(t_n)$.

Let m be given. The trick is to consider the sets

$$M_n = \{\, x \in X \mid a_m \cdot s(x) \leq t_n(x) \,\}.$$

It is not difficult to prove that: (1) each subset $M_n \subseteq X$ is measurable (since $s, t_1, t_2, \ldots$ are measurable functions); that (2) the M_n form an ascending chain with $\bigcup_n M_n = X$ (since $a_m \cdot s(x) < s(x) \leq \bigvee t_n(x)$ for each $x \in X$); and that (3) $a_m \cdot (s \cdot \mathbf{1}_{M_n}) \leq t_n$ for all n. The latter implies $a_m \cdot \bigvee_n \Phi(s \cdot \mathbf{1}_{N_n}) \leq \bigvee_n \Phi(t_n)$ in E. So it suffices to prove that $\Phi(s) = \bigvee_n \Phi(s \cdot \mathbf{1}_{M_n})$, or in other words, $\bigwedge_n \Phi(s \cdot \mathbf{1}_{\neg M_n}) = 0$. Since $s \cdot \mathbf{1}_{\neg N_n} \leq \mathbf{1}_{\neg N_n}$ for all n we have $\bigwedge_n \Phi(s \cdot \mathbf{1}_{\neg M_n}) \leq \bigwedge_n \Phi(\mathbf{1}_{\neg M_n}) = \bigwedge_n \phi(\neg M_n) = \bigwedge_n 1 - \phi(M_n) = 1 - \bigvee_n \phi(M_n) = 1 - \phi(\bigvee_n M_n) = 1 - \phi(X) = 1 - 1 = 0$. □

We can now define the (Lebesgue) integral taking its values in an ω-effect algebra, for measurable predicates.

Definition 4.9 Let ϕ be an E-valued probability measure on a measurable space X. For measurable function $p\colon X \to [0,1]$ we define the *integral* by

$$\int p \,\mathrm{d}\phi \;=\; \bigvee_n \int s_n \,\mathrm{d}\phi \;\in E, \tag{4}$$

where $s_1 \leq s_2 \leq \cdots$ is a chain of step functions with $\bigvee_n s_n = p$. Such a chain exists by Lemma 4.6 and there is no ambiguity by Proposition 4.8.

We list some basic well-known properties of integration, formulated here in effect-theoretic terms.

Proposition 4.10 *Let X be a measurable space, together with a probability measure $\phi\colon \Sigma_X \to E$ in an ω-effect module E.*

(i) *$\int(-)\,\mathrm{d}\phi$ on $\mathbf{Meas}(X,[0,1])$ is an elementary extension of ϕ. In particular, sending $p \mapsto \int p \,\mathrm{d}\phi$ yields a homomorphism of effect modules $\mathbf{Meas}(X,[0,1]) \to E$.*

(ii) *('Levi's Theorem') For all $p_1 \leq p_2 \leq \cdots$ in $\mathbf{Meas}(X,[0,1])$,*

$$\int \bigvee_n p_n \,\mathrm{d}\phi \;=\; \bigvee_n \int p_n \,\mathrm{d}\phi.$$

The latter two points say that $\int(-)\,\mathrm{d}\phi$ is a morphism $\mathbf{Meas}(X,[0,1]) \to E$ in the category ω-$\mathbf{EMod}$ of ω-effect modules.

(iii) *For maps $f\colon X \to Y$ in $\mathbf{Meas}$ and $g\colon E \to D$ in ω-$\mathbf{EMod}$,*

$$\int (q \circ f)\,\mathrm{d}\phi = \int q \,\mathrm{d}(\phi \circ f^{-1}) \qquad g\left(\int p \,\mathrm{d}\phi\right) = \int p \,\mathrm{d}(U(g) \circ \phi),$$

where $U\colon \omega$-$\mathbf{EMod} \to \omega$-$\mathbf{EA}$ is the forgetful functor.

(iv) *For each $x \in X$ and $p \in \mathbf{Meas}(X,[0,1])$ one has:*

$$\int p \,\mathrm{d}\eta(x) \;=\; p(x),$$

where $\eta(x)\colon \Sigma_X \to E$ is as described in Definition 4.2.

Proof (i) We only show that $\int(-)\,\mathrm{d}\phi$ is a homomorphism of effect modules. The other requirements for $\int(-)\,\mathrm{d}\phi$ to be an elementary extension of ϕ (see Definition 4.7) are either trivial to verify or follow immediately from the fact that the integral on step functions is an elementary extension of ϕ.

Since $\mathbf{1}_X$ is a step function, and $\int(-)\,\mathrm{d}\phi$ extends the integral on step functions, and we already know that that the integral on step functions is a homomorphism of effect modules, we get $\int \mathbf{1}_X\,\mathrm{d}\phi = 1$.

Let $p, q \in \mathbf{Meas}(X, [0,1])$ with $p \perp q$. We must show that $\int p \ovee q\,\mathrm{d}\phi = \int p\,\mathrm{d}\phi \ovee \int q\,\mathrm{d}\phi$. By Lemma 4.6 there are step functions $s_0 \leq s_1 \leq \cdots$ and $t_1 \leq t_2 \leq \cdots$ such that $p = \bigvee s_n$ and $q = \bigvee t_n$. Then $\int p\,\mathrm{d}\phi = \bigvee_n \int s_n\,\mathrm{d}\phi$ and $\int q\,\mathrm{d}\phi = \bigvee_n \int t_n\,\mathrm{d}\phi$ by Definition 4.9. Then $s_n \perp t_n$ for all n and $p \ovee q = \bigvee_n s_n \ovee t_n$ so $\int p \ovee q\,\mathrm{d}\phi = \bigvee_n \int s_n \ovee t_n\,\mathrm{d}\phi$. Since s_n and t_n are step functions, we already know that $\int s_n \ovee t_n\,\mathrm{d}\phi = \int s_n\,\mathrm{d}\phi \ovee \int t_n\,\mathrm{d}\phi$. Thus,

$$\begin{aligned}\textstyle\int p\,\mathrm{d}\phi \ovee \int q\,\mathrm{d}\phi &= \big(\textstyle\bigvee_n \int s_n\,\mathrm{d}\phi\big) \ovee \big(\bigvee_n \int t_n\,\mathrm{d}\phi\big)\\ &= \textstyle\bigvee_n \big(\int s_n\,\mathrm{d}\phi \ovee \int t_n\,\mathrm{d}\phi\big) \;=\; \bigvee_n \int (s_n \ovee t_n)\,\mathrm{d}\phi \;=\; \int p \ovee q\,\mathrm{d}\phi.\end{aligned}$$

By a similar reasoning using that scalar multiplication preserves suprema of ω-joins we get $\int r \cdot p\,\mathrm{d}\phi = r \cdot \int p\,\mathrm{d}\phi$ for all $p \in \mathbf{Meas}(X, [0,1])$, $r \in [0,1]$. Thus $\int(-)\,\mathrm{d}\phi$ is a homomorphism of effect modules.

(ii) This is a consequence of Proposition 4.8 since $\int(-)\,\mathrm{d}\phi$ is an elementary extension of ϕ.

(iii) For measurable $f\colon X \to Y$ and step function $s = \bigovee_i r_i \mathbf{1}_{N_i}$ in $\mathbf{Meas}(Y, [0,1])$ we have:

$$\begin{aligned}\textstyle\int (s \circ f)\,\mathrm{d}\phi &= \textstyle\int \big(\bigovee_i r_i \mathbf{1}_{f^{-1}(N_i)}\big)\,\mathrm{d}\phi && \text{by naturality of } \mathbf{1}_{(-)}\text{, see Lemma 3.2}\\ &= \textstyle\bigovee_i r_i \cdot \phi\big(f^{-1}(N_i)\big)\\ &= \textstyle\int \big(\bigovee r_i \mathbf{1}_{N_i}\big)\,\mathrm{d}(\phi \circ f^{-1})\\ &= \textstyle\int s\,\mathrm{d}(\phi \circ f^{-1}).\end{aligned}$$

The required result for an arbitrary predicate $p \in \mathbf{Meas}(X, [0,1])$ now follows like in (i) using that $(-) \circ f$ preserves suprema of ω-chains.

The second equation is also first obtained for step functions.

(iv) For a step function $s = \bigovee_i r_i \mathbf{1}_{M_i}$ we have:

$$\textstyle\int s\,\mathrm{d}\eta(x) \;=\; \bigovee_i r_i \eta(x)(M_i) \;=\; \sum_i r_i \mathbf{1}_{M_i}(x) \;=\; s(x).$$

Hence for a join $p = \bigvee_n s_n$ of step functions s_n we get:

$$\textstyle\int p\,\mathrm{d}\eta(x) \;=\; \bigvee_n \int s_n\,\mathrm{d}\eta(x) \;=\; \bigvee_n s_n(x) \;=\; p(x). \qquad \square$$

Remark 4.11 Let $\phi\colon \Sigma_X \to E$ be a probability measure on a measurable space X where E is an ω-effect module. Many optional features of ϕ carry over to the integral $\overline{\phi} = \int(-)\,\mathrm{d}\phi$. We give two examples.

(i) If E is an MV-algebra and ϕ is a homomorphism of MV-algebras, then $\overline{\phi} = \int(-)\,\mathrm{d}\phi\colon \mathbf{Meas}(\Sigma_X, [0,1]) \to E$ is a homomorphism of MV-modules. We sketch a proof, but leave the details to the reader.

Note that $\overline{\phi}$ is a homomorphism of MV-algebras iff it preserves binary meets. Given a step function $s \equiv \bigovee_i s_i\mathbf{1}_{M_i}$ the sets M_i are pairwise disjoint and so $s = \bigvee_i s_i\mathbf{1}_{M_i}$. Thus, by distributivity of $\wedge$ over $\vee$, we see that for step functions $s \equiv \bigovee_i s_i\mathbf{1}_{M_i}$ and $t \equiv \bigovee_j t_j\mathbf{1}_{N_j}$ we have $s \wedge t = \bigvee_{i,j}(s_i \wedge t_j)\mathbf{1}_{M_i \cap N_j} = \bigovee_{i,j}(s_i \wedge t_j)\mathbf{1}_{M_i \cap N_j}$. To see that the integral on step functions preserves binary meets, integrate, use the fact that ϕ preserves binary meets, and rewrite.

To see that $\int(-)\,\mathrm{d}\phi$ preserves binary meets, note that for step functions $s_1 \leq s_2 \leq \cdots$ and $t_1 \leq t_2 \leq \cdots$ we have $(\bigvee_n s_n) \wedge (\bigvee_m t_m) = \bigvee_n s_n \wedge t_n$. Now, integrate, use that the integral on step functions preserves binary meets, rewrite, and finish the proof with an appeal to Lemma 4.6.

(ii) If E is endowed with a suitable product $\odot$ and ϕ preserves 'products' (that is, $\phi(A \cap B) = \phi(A) \odot \phi(B)$ for all $A, B \in \Sigma_X$), then $\overline{\phi} = \int(-)\,\mathrm{d}\phi$ preserves products. We leave the proof to the reader.

By a suitable product, we mean an associative map $\odot\colon E \times E \to E$ such that for every $a \in E$ the maps $a \odot (-)$ and $(-) \odot a$ preserve sum $\ovee$, scalar multiplication, and countable joins, and $1 \odot a = a = a \odot 1$.

(Hint for the proof: $\mathbf{1}_M \cdot \mathbf{1}_N = \mathbf{1}_{M \cap N}$ for all $M, N \in \Sigma_X$.)

As will be explained in Remark 4.13 below, the core of the following result occurs as [12, Theorem 6.8], where integration is described via a tensor product with scalars, and thus as a free effect module.

Theorem 4.12 *Let X be a measurable space and E be an ω-effect module.*

(i) *For every ω-effect algebra homomorphism $\phi\colon \Sigma_X \to E$ (that is, E-probability measure on X) there is a unique ω-effect module homomorphism $\overline{\phi}\colon \mathbf{Meas}(X, [0,1]) \to E$ such that $\overline{\phi} \circ \mathbf{1}_{(-)} = \phi$. In a diagram:*

$$\begin{tikzcd}
\Sigma_X \arrow[r, "\mathbf{1}_{(-)}"] \arrow[dr, "\forall\phi"'] & \mathbf{Meas}(X,[0,1]) \arrow[d, "\exists!\overline{\phi}"] \\
& E
\end{tikzcd}$$

(ii) *There is a bijective correspondence between:*

$$\frac{\Sigma_X \xrightarrow{\ \phi\ } U(E) \qquad \text{in } \omega\text{-}\mathbf{EA}}{\mathbf{Meas}(X,[0,1]) \xrightarrow[\ I\ ]{} E \qquad \text{in } \omega\text{-}\mathbf{EMod}}$$

This correspondence is natural in X and in E.

(iii) *The natural transformation $\mathbf{1}_{(-)}\colon \Sigma_{(-)} \Rightarrow U\mathbf{Meas}(-, [0,1])$ from Lemma 3.2 is universal, in the following sense. For each functor $F\colon \mathbf{Meas} \to \omega\text{-}\mathbf{EMod}$ with a natural transformation $\tau\colon \Sigma_{(-)} \Rightarrow UF$, there is a unique*

$\overline{\tau}\colon \mathbf{Meas}(-,[0,1]) \Rightarrow F$ *with* $U\overline{\tau} \circ \mathbf{1}_{(-)} = \tau$*, in:*

$$
\begin{tikzcd}
\Sigma_{(-)} \arrow[r, Rightarrow, "\mathbf{1}_{(-)}"] \arrow[dr, Rightarrow, "\tau"'] & U\mathbf{Meas}(-,[0,1]) \arrow[d, Rightarrow, "U\overline{\tau}"] \\
& UF
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{Meas}(-,[0,1]) \arrow[d, Rightarrow, "\overline{\tau} = \int(-)\,\mathrm{d}\phi"] \\
F
\end{tikzcd}
$$

Proof (i) Take $\overline{\phi} = \int(-)\,\mathrm{d}\phi$; it is a homomorphism of ω-effect modules with $\int \mathbf{1}_M\,\mathrm{d}\phi = \phi(M)$ for all $M \in \Sigma_X$ by Proposition 4.10.

For uniqueness, let $\xi\colon \mathbf{Meas}(X,[0,1]) \to E$ be a homomorphism of ω-effect modules such that $\xi(\mathbf{1}_M) = \phi(M)$ for all $M \in \Sigma_X$. Let $p \in \mathbf{Meas}(X,[0,1])$ be given. We must show that $\xi(p) = \int p\,\mathrm{d}\phi$.

We prove this first for step functions. If $p = \bigovee_i r_i \mathbf{1}_{M_i}$ then we have that $\xi(p) = \bigovee_i r_i \xi(\mathbf{1}_{M_i}) = \bigovee_i r_i \phi(M_i) = \int p\,\mathrm{d}\phi$.

Now, for arbitrary p there are step functions $s_1 \leq s_2 \leq \cdots$ with $p = \bigvee_n s_n$ by Lemma 4.6 and so we have $\xi(p) = \bigvee_n \xi(s_n) = \bigvee_n \int s_n\,\mathrm{d}\phi = \int p\,\mathrm{d}\phi$.

(ii) This follows categorically from point (i), see [19, Thm. IV.1.2 (ii)].

(iii) Define $\overline{\tau}_X = \overline{\tau_X}$, given by $\overline{\tau_X}(p) = \int p\,\mathrm{d}\tau_X$ as in point (i). $\square$

Remark 4.13 Theorem 4.12 says that for a measurable space (X,Σ_X), the predicates $\mathbf{Meas}(X,[0,1])$ form the *free* ω-effect module on the ω-effect algebra Σ_X. In essence, for $[0,1]$-valued functions, Lebesgue integration is thus the extension of a map of ω-effect algebras $\Sigma_X \to E$ to a map of ω-effect modules $\mathbf{Meas}(X,[0,1]) \to E$.

Such free modules are obtained by tensoring $[0,1]\otimes(-)$ with the scalars involved. This description is used by Gudder in [12, Theorem 6.8]. He proves an isomorphism $[0,1] \otimes \Sigma_X \cong \mathbf{Meas}(X,[0,1])$, which implies that the predicates $\mathbf{Meas}(X,[0,1])$ form the free ω-effect module on Σ_X. There are a few more things to say.

- Gudder does not use ω-effect *modules*, only ω-effect *algebras*. He proves that there is a suitable bihomomorphism of ω-effect algebras $[0,1] \times \Sigma_X \to \mathbf{Meas}(X,[0,1])$. He does not prove the existence of tensors of ω-effect algebras in general. He only shows the existence of this particular one in $[0,1] \otimes \Sigma_X \cong \mathbf{Meas}(X,[0,1])$.
- Gudder does not use categorical language, and so the formulation of integration as free construction (as in Theorem 4.12) does not occur in [12].

The most common instance of the ω-effect module E in Theorem 4.12 uses $E = [0,1]$. But as we shall see later, we can also use the effects of a Hilbert space or of a von Neumann algebra. Thus, the generality of using E-valued probability measures $\Sigma_X \to E$ pays off.

Corollary 4.14 *The monad* $X \mapsto \omega\text{-}\mathbf{EMod}\big(\mathbf{Meas}(X,[0,1]),[0,1]\big)$ *on the category* $\mathbf{Meas}$ *of measureable spaces induced by the adjunction* $\omega\text{-}\mathbf{EMod}^{op} \leftrightarrows \mathbf{Meas}$ *from Lemma 3.3 is (isomorphic to) the Giry monad* $\mathcal{G}$ *(from [10]).*

Proof Since by Theorem 4.12, with $E = [0,1]$ we have:

$$\mathcal{G}(X) \stackrel{\text{defn}}{=} \omega\text{-}\mathbf{EA}\big(\Sigma_X, [0,1]\big) \cong \omega\text{-}\mathbf{EMod}\big(\mathbf{Meas}(X,[0,1]), [0,1]\big). \qquad \square$$

(A relation between integration and the Giry monad has been described before in a different language, see *e.g.* [1].)

Example 4.15 One application of the general mechanism of Theorem 4.12 is the formulation of the spectral theorem for effects on a Hilbert space $\mathscr{H}$. Recall that a bounded self-adjoint linear map A on $\mathscr{H}$ is called an effect when $0 \leq \langle Ax, x\rangle \leq 1$ for all $x \in \mathscr{H}$. These effects form an ω-effect module, which we denote by $\mathcal{E}\!f(\mathscr{H})$.

Let $A \in \mathcal{E}\!f(\mathscr{H})$ be an effect. The spectrum σ_A of A (*i.e.* all $\lambda \in \mathbb{C}$ such that $A - \lambda \cdot I$ is not invertible) inherits the topology of $\mathbb{C}$. Since A is an effect, we get $\sigma_A \subseteq [0,1]$. Endow σ_A with the σ-algebra of Borel measurable subsets of $\mathbb{C}$, so that σ_A becomes a measurable space.

Recall that an ω-effect algebra homomorphism $\phi\colon \Sigma_{\sigma_A} \to \mathcal{E}\!f(\mathscr{H})$ is called a POVM (positive operator valued measure). We are interested in POVMs $\phi\colon \Sigma_{\sigma_A} \to \mathcal{E}\!f(\mathscr{H})$ such that $\phi(M)$ is a projection for all $M \in \Sigma_{\sigma_A}$. Such a ϕ is called a *spectral measure* on σ_A, and by Theorem 4.12, it has a unique extension to an ω-effect module homomorphism $\int(-)\,\mathrm{d}\phi\colon \mathbf{Meas}(\sigma_A,[0,1]) \to \mathcal{E}\!f(\mathscr{H})$.

(Note that while ϕ is projection-valued the integral $\int(-)\,\mathrm{d}\phi$ is not: $\int \frac{1}{2}\mathbf{1}_X\,\mathrm{d}\phi = \frac{1}{2}I$ is not a projection. Also, the set of projections does not form an ω-effect module.)

The **spectral theorem** states that there is a unique spectral measure ϕ on σ_A which satisfies the following requirements (see [13, §43 and §39]).

(i) $A = \int \mathrm{id}\,\mathrm{d}\phi$ where $\mathrm{id}\colon \sigma_A \to [0,1]$ is given by $\mathrm{id}(x) = x$ for $x \in \sigma_A$. This means that the effect A has a 'spectral decomposition' as an integral over projections.

(ii) For any open subset G of σ_A with $\phi(G) = 0$ we have $G = \emptyset$.

In fact, we may replace the latter requirement by the following weaker form.

(ii′) The complement of $\bigcup\{G \mid G \subseteq [0,1] \text{ open and } \phi(G) = 0\}$ in $[0,1]$ is compact.

Moreover, such a spectral measure ϕ has the following properties.

(iii) $(\int f\,\mathrm{d}\phi) \cdot (\int g\,\mathrm{d}\phi) = \int f \cdot g\,\mathrm{d}\phi$ for all $f, g \in \mathbf{Meas}(\sigma_A,[0,1])$, see [13, §37, Thm. 3].

(iv) Let B be a bounded linear operator on $\mathscr{H}$; then B commutes with A if and only if B commutes with $\phi(M) \in \mathcal{E}\!f(\mathscr{H})$ for all $M \in \Sigma_{\sigma_A}$, via a combination of Theorem 2 from §41 and Theorem 4 from §37 of [13].

The spectral theorem is one of the great achievements of 20th century mathematics. It reveals that effects behave somewhat like measurable functions to $[0,1]$; the integral $\int(-)\,\mathrm{d}\phi$ provides the translation from measurable functions to effects.

5 Perspectives and future work

By Theorem 4.12 an E-probability measure can be extended to an integral. But how does one obtain an E-probability measure? Carathéodory's extension theorem guarantees that given a measurable space (X, Σ_X) any homomorphism of effect algebras $\mu\colon S \to [0,1]$ defined on a Boolean subalgebra S of a σ-algebra Σ_X on a set X can be extended uniquely to a probability measure $\tilde{\mu}\colon \Sigma_X \to [0,1]$ provided that $\mu(\bigcup_n A_n) = \bigvee_n \mu(A_n)$ for all $A_1 \subseteq A_2 \subseteq \cdots$ from Σ_X with $\bigcup_n A_n \in \Sigma_X$

We do not know if a similar theorem holds for E-valued homomorphisms μ where E is an arbitrary ω-effect module. Our attempts to generalise existing proofs are blocked by the potential lack of a complete metric on E, which leads us to the following problem.

Problem 5.1 *Let E be an Archimedean ω-effect module. Is the metric on E complete? (See [17], Equation (10) for the definition of the metric on E.)*

Other questions remain: for example, can we fit Fubini (integration over product spaces) in our general framework?

Of the numerous generalisations of the formal definition of integral given by Riemann our work is perhaps most similar in setup and breadth to the vector valued variations on the Lebesgue integral studied by Bochner [3] and Pettis [21]. Their integrals takes values from a Banach space while our integral takes values from an ω-effect module. They exploit the uniform structure on a Banach space, while we use the order structure of an ω-effect module. An order-theoretic approach to integration has also been considered by Alfsen (for real-valued lattice valuations, see [2]), the second author (for lattice valuations taking their values from a suitable lattice-ordered abelian group, see [24]), and others [6,23].

Traditionally, countable chains take centre stage in the theory of measure and integral as opposed to the directed sets of domain theory. To see why, note that the Lebesgue measure on $[0, 1]$ does not preserve joins of directed sets as any (measurable) set is the union of the directed set of its finite (and thus negligible) subsets. Nevertheless, there are connections between integration and domain theory. For example, the measurable subsets on $[0, 1]$ modulo negligibility form a complete lattice, and the real-valued Riemann integration of continuous functions on a compact metric space can be related to the probabilistic power domain (see [8]).

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