LOCALIZATION SEQUENCES FOR LOGARITHMIC TOPOLOGICAL HOCHSCHILD HOMOLOGY

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Abstract. We study the logarithmic topological Hochschild homology of ring spectra with logarithmic structures and establish localization sequences for this theory. Our results apply, for example, to connective covers of periodic ring spectra like real and complex topological K-theory.

1. Introduction

Algebraic K-theory provides a powerful invariant encoding deep arithmetic properties. For computations of algebraic K-theory of rings it is often useful to invoke trace maps to topological Hochschild homology (THH) and to topological cyclic homology (TC), since this makes tools from equivariant stable homotopy theory applicable. This method is effective for rings satisfying suitable finiteness conditions, as a consequence of Quillen's work for finite fields [Qui72] and McCarthy's theorem [McC97]. Important examples of this approach are found in papers of Bökstedt–Madsen [BM94, BM95] and of Hesselholt–Madsen [HM97].

There are, however, examples of rings A where the cyclotomic trace map to $\mathrm{TC}(A)$ fails to provide a good approximation to the algebraic K-theory $K(A)$. One explanation for this is that THH and TC do not admit the same localization sequences as algebraic K-theory. We illustrate this with an example from the work of Hesselholt–Madsen [HM03]: If p is a prime and F is a finite field extension of $\mathbb{Q}_p$ with valuation ring A and residue field k, then there is a localization homotopy cofiber sequence of K-theory spectra

$$K(A) \to K(F) \to \Sigma K(k) \tag{1.1}$$

established by Quillen [Qui73]. Replacing K-theory by THH or TC, the corresponding diagrams do not form homotopy cofiber sequences, and the trace maps from $K(F)$ to $\mathrm{THH}(F)$ and $\mathrm{TC}(F)$ detect little information about $K(F)$, compared to the trace maps from $K(A)$ and $K(k)$. In [HM03], Hesselholt and Madsen overcome this by constructing relative forms $\mathrm{THH}(A|F)$ and $\mathrm{TC}(A|F)$ of THH and TC that fit into a localization homotopy cofiber sequence

$$\mathrm{THH}(A) \to \mathrm{THH}(A|F) \to \Sigma\,\mathrm{THH}(k) \tag{1.2}$$

and a corresponding sequence for TC, and they use $\mathrm{TC}(A|F)$ to determine $K(F)$. While the definition of $\mathrm{THH}(A|F)$ given in [HM03] uses linear Waldhausen categories, the homotopy groups of $\mathrm{THH}(A|F)$ and $\mathrm{TC}(A|F)$ exhibit a close connection to a logarithmic de Rham complex and a logarithmic de Rham–Witt complex associated with the direct image logarithmic structure on A inherited from F. This indicates a relation between homotopy theory and logarithmic geometry in the sense of [Kat89]. A systematic investigation of the interplay of these two subjects was taken up in the first author's work on *topological logarithmic structures* [Rog09]. The aim of the present paper is to continue and extend this investigation with a focus on THH of structured ring spectra.

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1.1. **Algebraic K-theory and THH of structured ring spectra.** Trace maps to THH and TC also provide a good tool for computing the algebraic K-theory of structured ring spectra, by Dundas' theorem [Dun97]. They are, however, only directly useful for *connective* ring spectra satisfying suitable finiteness conditions on π_0. Again we illustrate this shortcoming by an example of a localization sequence. Analogously to (1.1), there is a homotopy cofiber sequence

$$K(ku) \to K(KU) \to \Sigma K(\mathbb{Z}) \tag{1.3}$$

relating the algebraic K-theory spectra of the periodic complex topological K-theory spectrum KU, its connective cover ku, and the integers. The existence of this homotopy cofiber sequence was conjectured by Ausoni and Rognes [AR02] and established by Blumberg and Mandell [BM08]. Replacing K-theory by THH in (1.3), the corresponding sequence of spectra fails to be a homotopy cofiber sequence. To obtain a THH localization sequence analogous to (1.2), Blumberg and Mandell [BM11] have constructed a relative THH term $\mathrm{THH}(ku|KU)$ that fits in a homotopy cofiber sequence with $\mathrm{THH}(ku)$ and $\mathrm{THH}(\mathbb{Z})$. Their approach uses a version of THH for simplicially enriched Waldhausen categories, by analogy with [HM03].

In the present paper we offer a different approach to such relative THH terms by defining them as the *logarithmic* THH of certain *logarithmic ring spectra*, to be introduced next. Compared to the relative THH terms defined by Blumberg and Mandell, our approach is more directly connected to logarithmic de Rham and de Rham–Witt complexes, more amenable to homology computations, and applies to new examples including the real topological K-theory spectrum. This answers questions that remained open in [Rog09], and builds on more recent foundational work by the last two authors [SS12, Sag13].

1.2. **Logarithmic ring spectra.** A *pre-log ring* (A, M) is a commutative ring A together with a commutative monoid M and a monoid homomorphism $\alpha\colon M \to (A, \cdot)$ to the multiplicative monoid of A. It is a log ring if the base change $\alpha^{-1}(A^\times) \to A^\times$ of α along the inclusion of the units $A^\times \to (A, \cdot)$ is an isomorphism. If A is any commutative ring, we can use $A^\times \to (A, \cdot)$ to form the *trivial* log ring $(A, A^\times)$. A log ring (A, M) determines a localization $A[M^{-1}]$, and the map of trivial log rings associated with $A \to A[M^{-1}]$ factors through (A, M) in a non-trivial way.

In order to form a homotopical generalization of pre-log rings, one can consider commutative symmetric ring spectra A together with maps of commutative $\mathcal{I}$-space monoids $M \to \Omega^{\mathcal{I}}(A)$ in the sense of [SS12, Section 3]. Here, commutative $\mathcal{I}$-space monoids give one possible model for the more commonly studied E_∞ spaces, and may be viewed as a homotopical counterpart of commutative monoids. The commutative $\mathcal{I}$-space monoid $\Omega^{\mathcal{I}}(A)$ encodes the underlying multiplicative E_∞ space of A. This $\mathcal{I}$-space version of *pre-log ring spectra* was considered in [Rog09]. It has the disadvantage that it appears to be difficult to extend A to a pre-log ring spectrum (A, M) in a sufficiently interesting way if A is not an Eilenberg–Mac Lane spectrum. One reason for this is that commutative $\mathcal{I}$-space monoids and E_∞ spaces are inherently connective, hence cannot be group completed in such a way that positive dimensional homotopy classes are inverted. To remedy this, we proceed as in [SS12, §4.30] and [Sag14] and replace the commutative $\mathcal{I}$-space monoids in the previous definition by the commutative $\mathcal{J}$-space monoids developed by the last two authors.

Let $\mathcal{J}$ be the category $\Sigma^{-1}\Sigma$ given by Quillen's localization construction [Gra76] on the category Σ of finite sets and bijections. A *commutative $\mathcal{J}$-space monoid* is a lax symmetric monoidal functor from $\mathcal{J}$ to the category of unbased spaces $\mathcal{S}$.

Equivalently, it is a commutative monoid object with respect to a convolution product on the functor category $\mathcal{S}^{\mathcal{J}}$. The resulting category $\mathcal{CS}^{\mathcal{J}}$ of commutative $\mathcal{J}$-space monoids admits a model structure making it Quillen equivalent to the category of E_∞ spaces over the underlying additive E_∞ space QS^0 of the sphere spectrum. We therefore think of commutative $\mathcal{J}$-space monoids as a model for (QS^0-)*graded* E_∞ *spaces*.

For a commutative symmetric ring spectrum A, one can form a commutative $\mathcal{J}$-space monoid $\Omega^{\mathcal{J}}(A)$ that is a graded version of the multiplicative E_∞ space of A. There is a sub commutative $\mathcal{J}$-space monoid $\mathrm{GL}_1^{\mathcal{J}}(A)$ of $\Omega^{\mathcal{J}}(A)$ that corresponds to the inclusion of multiplicative units $\pi_*(A)^\times \subset \pi_*(A)$. In contrast, the usual E_∞ space of units of A only corresponds to the inclusion $\pi_0(A)^\times \subset \pi_0(A)$ of units in degree 0.

A *pre-log ring spectrum* is then a commutative symmetric ring spectrum A together with a commutative $\mathcal{J}$-space monoid M and a map $\alpha\colon M \to \Omega^{\mathcal{J}}(A)$ of commutative $\mathcal{J}$-space monoids. It is a *log ring spectrum* if the base change $\alpha^{-1}\,\mathrm{GL}_1^{\mathcal{J}}(A) \to \mathrm{GL}_1^{\mathcal{J}}(A)$ of its structure map α along $\mathrm{GL}_1^{\mathcal{J}}(A) \to \Omega^{\mathcal{J}}(A)$ is a weak equivalence of $\mathcal{J}$-spaces. The easiest example of a log ring spectrum is the trivial log ring spectrum $(A, \mathrm{GL}_1^{\mathcal{J}}(A))$ associated to any commutative symmetric ring spectrum A. A more elaborate example is given by the following construction, which plays an important role in this paper: If $j\colon e \to E$ is the connective cover map of a periodic commutative symmetric ring spectrum E, then we let $j_*\mathrm{GL}_1^{\mathcal{J}}(E)$ be the pullback of the following diagram of commutative $\mathcal{J}$-space monoids:

$$\mathrm{GL}_1^{\mathcal{J}}(E) \to \Omega^{\mathcal{J}}(E) \leftarrow \Omega^{\mathcal{J}}(e).$$

Together with the canonical map $j_*\mathrm{GL}_1^{\mathcal{J}}(E) \to \Omega^{\mathcal{J}}(e)$ from the pullback, this defines a log ring spectrum $(e, j_*\mathrm{GL}_1^{\mathcal{J}}(E))$. In analogy with a similar construction in algebraic geometry, we call this the *direct image* log ring spectrum associated with the trivial log ring spectrum $(E, \mathrm{GL}_1^{\mathcal{J}}(E))$. We note that if we were using the $\mathcal{I}$-space version of pre-log ring spectra the same construction would only provide a trivial log structure, because the map of $\mathcal{I}$-space units associated with $e \to E$ is an equivalence.

It follows from the definition that the map of trivial log ring spectra associated with $e \to E$ factors through $(e, j_*\mathrm{GL}_1^{\mathcal{J}}(E))$. One indication for why $(e, j_*\mathrm{GL}_1^{\mathcal{J}}(E))$ is interesting is the following result proved in [Sag14, Theorem 4.4]: If E is periodic, it can be recovered as the trivial locus of $(e, j_*\mathrm{GL}_1^{\mathcal{J}}(E))$ (see [Rog09, Definition 7.15]) by forming the homotopy pushout of

$$\mathbb{S}^{\mathcal{J}}[(j_*\mathrm{GL}_1^{\mathcal{J}}(E))^{\mathrm{gp}}] \leftarrow \mathbb{S}^{\mathcal{J}}[j_*\mathrm{GL}_1^{\mathcal{J}}(E)] \to e.$$

Here $\mathbb{S}^{\mathcal{J}}$ denotes the graded spherical monoid ring functor that is left adjoint to $\Omega^{\mathcal{J}}$, the right hand map is the adjoint of the structure map of $(e, j_*\mathrm{GL}_1^{\mathcal{J}}(E))$, and the left hand map is induced by the group completion of $j_*\mathrm{GL}_1^{\mathcal{J}}(E)$ as defined in [Sag13].

1.3. **Logarithmic** THH. If A is a commutative symmetric ring spectrum, then its topological Hochschild homology $\mathrm{THH}(A)$ can be defined as the realization of the *cyclic bar construction* $[q] \mapsto A^{\wedge(q+1)}$. For a commutative $\mathcal{J}$-space monoid M, one can define $B^{\mathrm{cy}}(M)$ as the realization of the analogous cyclic bar construction $[q] \mapsto M^{\boxtimes(q+1)}$, where $\boxtimes$ is the convolution product of $\mathcal{J}$-spaces. If (A, M) is a pre-log ring spectrum, then the adjoint structure map $\mathbb{S}^{\mathcal{J}}[M] \to A$ induces a map $\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)] \to \mathrm{THH}(A)$ of commutative symmetric ring spectra.

The last ingredient in the definition of logarithmic THH is the *replete bar construction* $B^{\mathrm{rep}}(M)$. This is a subtle variant of the cyclic bar construction. One motivation for using the replete bar construction is that its algebraic counterpart can be used to define the logarithmic Hochschild homology of log rings, which in the log smooth case agrees with the logarithmic de Rham complex [Rog09, §3, §5]. Using the group completion $M \to M^{\mathrm{gp}}$ for commutative $\mathcal{J}$-space monoids constructed in [Sag13], the commutative $\mathcal{J}$-space monoid $B^{\mathrm{rep}}(M)$ is defined as the homotopy pullback of the diagram $M \to M^{\mathrm{gp}} \leftarrow B^{\mathrm{cy}}(M^{\mathrm{gp}})$. It comes with a canonical *repletion map* $\rho\colon B^{\mathrm{cy}}(M) \to B^{\mathrm{rep}}(M)$.

The logarithmic THH of a pre-log ring spectrum (A, M) is then defined as the homotopy pushout of the following diagram of commutative symmetric ring spectra:

$$\mathbb{S}^{\mathcal{J}}[B^{\mathrm{rep}}(M)] \leftarrow \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)] \to \mathrm{THH}(A).$$

It is not difficult to see that the canonical map $\mathrm{THH}(A) \to \mathrm{THH}(A, M)$ is a stable equivalence if M is grouplike. This applies in particular for the trivial log ring spectrum $(A, \mathrm{GL}_1^{\mathcal{J}}(A))$. A useful but more involved property of log THH is its invariance under logification. This means that the *logification map* $(A, M) \to (A^a, M^a)$, which naturally associates a log ring spectrum (A^a, M^a) to each pre-log ring spectrum (A, M), induces a stable equivalence $\mathrm{THH}(A, M) \to \mathrm{THH}(A^a, M^a)$.

Our main theorem states that under a certain condition on M, the logarithmic THH of a pre-log ring spectrum (A, M) participates in a localization homotopy cofiber sequence, where the two other terms are given by ordinary topological Hochschild homology. To formulate the condition, we use that every $\mathcal{J}$-space inherits a $\mathbb{Z}$-grading from the isomorphism $\pi_0(B\mathcal{J}) \cong \mathbb{Z}$. A commutative $\mathcal{J}$-space monoid M is said to be *repetitive* if it is $\mathcal{J}$-equivalent to the non-negative part of its group completion M^{gp}, and if in addition the positive part of M is nonempty.

Theorem 1.4. *Let (A, M) be a pre-log ring spectrum with M repetitive. Then there is a natural homotopy cofiber sequence*

$$\mathrm{THH}(A) \xrightarrow{\rho} \mathrm{THH}(A, M) \xrightarrow{\partial} \Sigma\,\mathrm{THH}(A/(M_{>0}))$$

of $\mathrm{THH}(A)$*-module spectra with circle action.*

In the theorem, the commutative symmetric ring spectrum $A/(M_{>0})$ is the homotopy pushout of the diagram $A \leftarrow \mathbb{S}^{\mathcal{J}}[M] \to \mathbb{S}^{\mathcal{J}}[M_{\{0\}}]$ induced by the adjoint structure map of (A, M) and the map $\mathbb{S}^{\mathcal{J}}[M] \to \mathbb{S}^{\mathcal{J}}[M_{\{0\}}]$ that collapses the positive degree parts of M.

In examples of interest, we can describe $A/(M_{>0})$ more explicitly. Let E be a commutative symmetric ring spectrum such that $0 \neq 1$ in $\pi_0(E)$. We say that E is *d-periodic* if $\pi_*(E)$ has a unit of positive degree and d is the minimal degree of such a unit. If E is d-periodic and $j\colon e \to E$ is the connective cover, then the commutative $\mathcal{J}$-space monoid $j_*\,\mathrm{GL}_1^{\mathcal{J}}(E)$ participating in the log ring spectrum $(e, j_*\,\mathrm{GL}_1^{\mathcal{J}}(E))$ is repetitive, and $e/(j_*\,\mathrm{GL}_1^{\mathcal{J}}(E)_{>0})$ is stably equivalent to the $(d-1)$-th Postnikov section $e[0, d\rangle$ of e. In this situation Theorem 1.4 leads to the following statement:

Theorem 1.5. *Let E be a d-periodic commutative symmetric ring spectrum with connective cover $j\colon e \to E$. There is a natural homotopy cofiber sequence*

$$\mathrm{THH}(e) \xrightarrow{\rho} \mathrm{THH}(e, j_*\mathrm{GL}_1^{\mathcal{J}}(E)) \xrightarrow{\partial} \Sigma\,\mathrm{THH}(e[0, d\rangle)$$

of $\mathrm{THH}(e)$*-module spectra with circle action.*

The theorem applies, for example, to the 8-periodic real K-theory spectrum KO and its connective cover $j\colon ko \to KO$, where we obtain a homotopy cofiber sequence

$$\mathrm{THH}(ko) \xrightarrow{\rho} \mathrm{THH}(ko, j_*\mathrm{GL}_1^{\mathcal{J}}(KO)) \xrightarrow{\partial} \Sigma\,\mathrm{THH}(ko[0, 8\rangle).$$

The analogy with the homotopy cofiber sequence (1.2) indicates that one may view the Postnikov section $ko[0, 8\rangle$ as a nilpotent extension of the *residue ring (spectrum)* of ko. Consequently, one may wonder about the K-theoretic significance of $ko[0,8\rangle$, and we expect that $ko[0,8\rangle \to H\mathbb{Z}$ will induce an equivalence in G-theory (compare [BL14]).

In the case of the p-local complex topological K-theory spectra $ku_{(p)} \to KU_{(p)}$, Theorem 1.5 provides a homotopy cofiber sequence

$$\mathrm{THH}(ku_{(p)}) \xrightarrow{\rho} \mathrm{THH}(ku_{(p)}, j_*\mathrm{GL}_1^{\mathcal{J}}(KU_{(p)})) \xrightarrow{\partial} \Sigma\,\mathrm{THH}(\mathbb{Z}_{(p)}),$$

and similarly for the Adams summand ℓ of $ku_{(p)}$ and the map $j\colon \ell \to L$ to its periodic counterpart. In a sequel [RSS14] to this paper, we will determine the $V(1)$-homotopy of $\mathrm{THH}(\ell, j_*\mathrm{GL}_1^{\mathcal{J}}(L))$, show that the inclusion of the Adams summand $\ell \to ku_{(p)}$ induces a stable equivalence

$$ku_{(p)} \wedge_\ell \mathrm{THH}(\ell, j_*\mathrm{GL}_1^{\mathcal{J}}(L)) \to \mathrm{THH}(ku_{(p)}, j_*\mathrm{GL}_1^{\mathcal{J}}(KU_{(p)})),$$

and use this to calculate the $V(1)$-homotopy of $\mathrm{THH}(ku_{(p)}, j_*\mathrm{GL}_1^{\mathcal{J}}(KU_{(p)}))$. In this way we complete the conjectural program outlined by Ausoni and Hesselholt for simplifying Ausoni's computation in [Aus05] of the $V(1)$-homotopy of $\mathrm{THH}(ku_{(p)})$.

The construction of the localization sequence in Theorem 1.4 is based on a general principle that also applies to pre-log ring spectra that arise from pre-log rings (B,N) in the algebraic sense. Let B be a commutative ring and let $\beta\colon N \to (B,\cdot)$ be such that N is a free commutative monoid on one generator and β maps that generator to an $x \in B$ that does not divide zero. Then there is a homotopy cofiber sequence

$$\mathrm{THH}(B) \to \mathrm{THH}(B,N) \to \Sigma\,\mathrm{THH}(B/(x)).$$

In Section 5 we calculate the mod p homotopy of $\mathrm{THH}(B,N)$ in the case $B = \mathbb{Z}_{(p)}$ and $x = p$ and show that it agrees with that of Hesselholt and Madsen's construction $\mathrm{THH}(\mathbb{Z}_{(p)}|\mathbb{Q})$.

1.6. **Notation and conventions.** We assume some familiarity with model categories, and mostly use Hirschhorn's book [Hir03] as a reference. In particular, we frequently use the notions of homotopy cartesian and cocartesian squares in proper model categories; see e.g. [Hir03, §13]. When working with symmetric spectra, we shall use both the simplicial version introduced in [HSS00] and the topological version discussed in [MMSS01]. Given a symmetric ring spectrum A, we shall use the expression: *a homotopy cofiber sequence*

$$X \xrightarrow{f} Y \xrightarrow{\partial} Z$$

of A-modules, to mean a map $f\colon X \to Y$ of A-module spectra together with an A-module spectrum Z and an implicit chain of stable equivalences of A-module spectra between the mapping cone $C(f)$ and Z, all of this understood internally to the category of symmetric spectra. So, by abuse of notation, ∂ denotes the canonical map $Y \to C(f)$ followed by the chain of stable equivalences. To avoid keeping track of semistability and fibrancy of symmetric spectra, we use the notation $\pi_*(A)$ for the stable homotopy groups of a (positive) fibrant replacement of a symmetric spectrum A.

1.7. **Organization.** We begin in Section 2 with a brief review of $\mathcal{J}$-spaces and their relation to symmetric spectra. In Section 3 we recall the definition of THH in the setting of symmetric ring spectra and introduce the cyclic and replete bar constructions of commutative $\mathcal{J}$-space monoids. Section 4 contains the definition of (pre-)log ring spectra and their log THH, and we prove the invariance of log THH under logification. In Section 5 we study the logarithmic THH of pre-log ring spectra arising from the algebraic version of pre-log rings, and we set up the relevant localization sequences in this case. In Section 6 we turn to repetitive pre-log ring spectra and construct the localization sequences in Theorems 1.4 and 1.5 from the introduction. The final Section 7 contains the proof of the main result about homotopy cofiber sequences needed to prove Theorem 1.4. An Appendix collects homotopy invariance properties of the functor $\mathbb{S}^{\mathcal{J}}$ from $\mathcal{J}$-spaces to symmetric spectra.

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2. Symmetric spectra and $\mathcal{J}$-spaces

In this section we review the definition of $\mathcal{J}$-spaces and commutative $\mathcal{J}$-space monoids from [SS12] and explain their relation to symmetric spectra.

The category of symmetric spectra Sp^{Σ} introduced in [HSS00] is a stable model category whose homotopy category is the stable homotopy category. It has a symmetric monoidal smash product denoted by $\wedge$ whose monoidal unit is the sphere spectrum $\mathbb{S}$. The commutative monoids with respect to the smash product are known as *commutative symmetric ring spectra*. They may be viewed as strictly commutative models for E_∞ ring spectra. We will use that the category of commutative symmetric ring spectra $\mathcal{C}\mathrm{Sp}^{\Sigma}$ inherits a proper simplicial *positive stable model structure* from Sp^{Σ} [MMSS01]. The book project [Sch12] provides extensive background about symmetric spectra.

2.1. **$\mathcal{J}$-spaces and commutative $\mathcal{J}$-space monoids.** We recall from [SS12] how one can use commutative monoid objects in the category of space valued functors on an appropriate indexing category as a model for a graded version of E_∞ spaces.

Definition 2.2 ([SS12, Definition 4.2])**.** Let $\mathcal{J}$ be the category whose objects are pairs $(\mathbf{m_1}, \mathbf{m_2})$ of finite sets $\mathbf{m_i} = \{1, \ldots, m_i\}$ with each $m_i \geq 0$. A morphism

$$(\alpha_1, \alpha_2, \rho) \colon (\mathbf{m_1}, \mathbf{m_2}) \to (\mathbf{n_1}, \mathbf{n_2})$$

in $\mathcal{J}$ consists of two injective functions $\alpha_i \colon \mathbf{m_i} \to \mathbf{n_i}$ and a bijection $\rho \colon \mathbf{n_1} \setminus \alpha_1 \to \mathbf{n_2} \setminus \alpha_2$ identifying the complement of the image of α_1 with the complement of the image of α_2. Consequently, the set of morphisms from $(\mathbf{m_1}, \mathbf{m_2})$ to $(\mathbf{n_1}, \mathbf{n_2})$ is empty unless $m_2 - m_1 = n_2 - n_1$.

It is proven in [SS12, Proposition 4.4] that $\mathcal{J}$ is isomorphic to Quillen's localization construction $\Sigma^{-1}\Sigma$ on the permutative category Σ of finite sets and bijections. Combining this with the Barratt–Priddy–Quillen theorem shows that the classifying space $B\mathcal{J}$ of $\mathcal{J}$ is weakly equivalent to $QS^0 = \Omega^\infty\Sigma^\infty S^0$ as an infinite loop space.

Definition 2.3. A *$\mathcal{J}$-space* is a functor $X \colon \mathcal{J} \to \mathcal{S}$ from $\mathcal{J}$ to the category of unbased simplicial sets $\mathcal{S}$. The functor category of $\mathcal{J}$-spaces is denoted by $\mathcal{S}^{\mathcal{J}}$.

The ordered concatenation $- \sqcup -$ of sets in both entries makes $\mathcal{J}$ a symmetric monoidal category. Its monoidal unit is $(\mathbf{0}, \mathbf{0})$. Defining $X \boxtimes Y$ to be the left Kan extension of the object-wise product along $- \sqcup - \colon \mathcal{J} \times \mathcal{J} \to \mathcal{J}$ makes $\mathcal{S}^{\mathcal{J}}$ a symmetric monoidal category with product $\boxtimes$, unit $U^{\mathcal{J}} = \mathcal{J}((\mathbf{0}, \mathbf{0}), -)$, and symmetry isomorphism $\tau \colon X \boxtimes Y \to Y \boxtimes X$.

Definition 2.4. A *commutative $\mathcal{J}$-space monoid* is a commutative monoid in $(\mathcal{S}^{\mathcal{J}}, \boxtimes, U^{\mathcal{J}}, \tau)$, and $\mathcal{C}\mathcal{S}^{\mathcal{J}}$ denotes the category of commutative $\mathcal{J}$-space monoids.

Although we are mostly concerned with commutative $\mathcal{J}$-space monoids, we will occasionally consider $\mathcal{J}$-space monoids, that is, associative but not necessarily commutative monoid objects in $(\mathcal{S}^{\mathcal{J}}, \boxtimes, U^{\mathcal{J}})$.

Example 2.5. Evaluating a $\mathcal{J}$-space at the object $(\mathbf{d_1}, \mathbf{d_2})$ of $\mathcal{J}$ defines a functor $\mathrm{Ev}^{\mathcal{J}}_{(\mathbf{d_1},\mathbf{d_2})} \colon \mathcal{S}^{\mathcal{J}} \to \mathcal{S}$. It is right adjoint to the free functor $F^{\mathcal{J}}_{(\mathbf{d_1},\mathbf{d_2})} \colon \mathcal{S} \to \mathcal{S}^{\mathcal{J}}$ given by $F^{\mathcal{J}}_{(\mathbf{d_1},\mathbf{d_2})}(K) = \mathcal{J}((\mathbf{d_1}, \mathbf{d_2}), -) \times K$, *the free $\mathcal{J}$-space on K in bidegree* $(\mathbf{d_1}, \mathbf{d_2})$. The values of the free functors for varying $(\mathbf{d_1}, \mathbf{d_2})$ are important examples of $\mathcal{J}$-spaces. We note that 0-simplices $x \in X(\mathbf{d_1}, \mathbf{d_2})$ correspond to $\mathcal{J}$-space maps $\bar{x} \colon F^{\mathcal{J}}_{(\mathbf{d_1},\mathbf{d_2})}(*) \to X$ from the free $\mathcal{J}$-space on a point in bidegree $(\mathbf{d_1}, \mathbf{d_2})$.

If M is a commutative $\mathcal{J}$-space monoid and $x \in M(\mathbf{d_1}, \mathbf{d_2})_0$ is a 0-simplex, then x determines a map of commutative $\mathcal{J}$-space monoids

$$\mathbb{C}\langle \mathbf{d_1}, \mathbf{d_2} \rangle = \coprod_{k \geq 0} F^{\mathcal{J}}_{(\mathbf{d_1},\mathbf{d_2})}(*)^{\boxtimes k}/\Sigma_k \to M.$$

The object $\mathbb{C}\langle \mathbf{d_1}, \mathbf{d_2} \rangle$ defined here is the *free commutative $\mathcal{J}$-space monoid* on a generator in bidegree $(\mathbf{d_1}, \mathbf{d_2})$. It will often be convenient to use the notation $\mathbb{C}\langle x \rangle$ for $\mathbb{C}\langle \mathbf{d_1}, \mathbf{d_2} \rangle$ when discussing that map.

The point of defining the category $\mathcal{J}$ in this way is the following interplay with symmetric spectra:

Lemma 2.6. *There are two adjoint pairs of functors*

$$\mathbb{S}^{\mathcal{J}} \colon \mathcal{S}^{\mathcal{J}} \rightleftarrows \mathrm{Sp}^{\Sigma} \colon \Omega^{\mathcal{J}} \qquad \text{and} \qquad \mathbb{S}^{\mathcal{J}} \colon \mathcal{C}\mathcal{S}^{\mathcal{J}} \rightleftarrows \mathcal{C}\mathrm{Sp}^{\Sigma} \colon \Omega^{\mathcal{J}}. \tag{2.1}$$

The functor $\mathbb{S}^{\mathcal{J}} \colon (\mathcal{S}^{\mathcal{J}}, \boxtimes, U^{\mathcal{J}}) \to (\mathrm{Sp}^{\Sigma}, \wedge, \mathbb{S})$ *is strong symmetric monoidal.* □

For symmetric spectra E and $\mathcal{J}$-spaces X, these functors are given by

$$\Omega^{\mathcal{J}}(E)(\mathbf{n_1}, \mathbf{n_2}) = \Omega^{n_2} E_{n_1} \qquad \text{and} \qquad \mathbb{S}^{\mathcal{J}}[X]_n = \bigvee_{k \geq 0} X(\mathbf{n}, \mathbf{k})_+ \wedge_{\Sigma_k} S^k. \tag{2.2}$$

In particular, the lemma states that every commutative symmetric ring spectrum A gives rise to a commutative $\mathcal{J}$-space monoid $\Omega^{\mathcal{J}}(A)$. Below we indicate why $\Omega^{\mathcal{J}}(A)$ may be viewed as the underlying graded multiplicative E_∞ space of A.

To each $\mathcal{J}$-space X we can associate the space

$$X_{h\mathcal{J}} = \operatorname{hocolim}_{\mathcal{J}} X = \operatorname{diag}\left([s] \mapsto \coprod_{\mathbf{k_0} \leftarrow \ldots \leftarrow \mathbf{k_s}} X(\mathbf{k_s}) \right)$$

given by its Bousfield–Kan homotopy colimit. A map $X \to Y$ of $\mathcal{J}$-spaces is defined to be a *$\mathcal{J}$-equivalence* if the induced map $X_{h\mathcal{J}} \to Y_{h\mathcal{J}}$ is a weak homotopy equivalence. The $\mathcal{J}$-equivalences are the weak equivalences in a cofibrantly generated proper simplicial *positive projective $\mathcal{J}$-model structure* on $\mathcal{S}^{\mathcal{J}}$, where the fibrant objects are the $\mathcal{J}$-spaces X such that each morphism $(\mathbf{m}_1, \mathbf{m}_2) \to (\mathbf{n}_1, \mathbf{n}_2)$ in $\mathcal{J}$ with $m_1 > 0$ induces a weak homotopy equivalence $X(\mathbf{m}_1, \mathbf{m}_2) \to X(\mathbf{n}_1, \mathbf{n}_2)$ between (Kan) fibrant simplicial sets, see [SS12, Proposition 4.8].

A map $M \to N$ of commutative $\mathcal{J}$-space monoids is defined to be a *$\mathcal{J}$-equivalence* if the underlying map of $\mathcal{J}$-spaces is a $\mathcal{J}$-equivalence. These are the weak equivalences in a cofibrantly generated proper simplicial *positive projective $\mathcal{J}$-model structure* on $\mathcal{C}\mathcal{S}^{\mathcal{J}}$, where the fibrant objects are the commutative $\mathcal{J}$-space monoids whose underlying $\mathcal{J}$-spaces are fibrant [SS12, Proposition 4.10]. In the sequel, we will refer to this model structure as the *positive $\mathcal{J}$-model structure*, and the notions of cofibrant or fibrant objects in $\mathcal{C}\mathcal{S}^{\mathcal{J}}$ or of cofibrations or fibrations in $\mathcal{C}\mathcal{S}^{\mathcal{J}}$ will refer to this model structure unless otherwise stated. By construction of the generating cofibrations for $\mathcal{C}\mathcal{S}^{\mathcal{J}}$ in [SS12, Proposition 9.3], the free commutative $\mathcal{J}$-space monoids $\mathbb{C}\langle \mathbf{d_1}, \mathbf{d_2} \rangle$ with $d_1 > 0$ are examples of cofibrant objects in $\mathcal{C}\mathcal{S}^{\mathcal{J}}$.

Lemma 2.7 ([SS12, Proposition 4.23])**.** *The adjunctions* (2.1) *are Quillen adjunctions with respect to the positive $\mathcal{J}$-model structures on $\mathcal{S}^{\mathcal{J}}$ and $\mathcal{C}\mathcal{S}^{\mathcal{J}}$ and the positive stable model structures on* Sp^{Σ} *and* $\mathcal{C}\mathrm{Sp}^{\Sigma}$*, respectively.* □

The functor $(-)_{h\mathcal{J}}\colon (\mathcal{S}^{\mathcal{J}}, \boxtimes, U^{\mathcal{J}}) \to (\mathcal{S}, \times, *)$ is lax monoidal (but not lax *symmetric* monoidal), with monoidal structure map $X_{h\mathcal{J}} \times Y_{h\mathcal{J}} \to (X \boxtimes Y)_{h\mathcal{J}}$ induced by the natural transformation of $\mathcal{J} \times \mathcal{J}$-diagrams

$$X(\mathbf{m_1}, \mathbf{m_2}) \times Y(\mathbf{n_1}, \mathbf{n_2}) \to (X \boxtimes Y)((\mathbf{m_1}, \mathbf{m_2}) \sqcup (\mathbf{n_1}, \mathbf{n_2}))$$

and the functor $- \sqcup -\colon \mathcal{J} \times \mathcal{J} \to \mathcal{J}$. Therefore, the space $M_{h\mathcal{J}}$ associated with a $\mathcal{J}$-space monoid M is a simplicial monoid. If M is commutative, then one can use the fact that $\mathcal{J}$ is a permutative category to show that $M_{h\mathcal{J}}$ is an E_∞ space over the Barratt–Eccles operad. (A closely related statement is proven in [Sch09, Proposition 6.5].)

This observation can be extended to an operadic description of $\mathcal{CS}^{\mathcal{J}}$: By [SS12, Theorem 1.7], the category $\mathcal{CS}^{\mathcal{J}}$ is Quillen equivalent to the category of E_∞ spaces over $B\mathcal{J}$. So commutative $\mathcal{J}$-space monoids correspond to E_∞ spaces over the underlying additive E_∞ space $QS^0 \simeq B\mathcal{J}$ of the sphere spectrum, just as $\mathbb{Z}$-graded monoids in algebra can be defined as monoids over the additive monoid $(\mathbb{Z}, +)$ of the integers. This is one reason why commutative $\mathcal{J}$-space monoids may be viewed as *QS^0-graded E_∞ spaces*. Consequently, we interpret the commutative $\mathcal{J}$-space monoid $\Omega^{\mathcal{J}}(A)$ associated with a commutative symmetric ring spectrum A as the underlying graded E_∞ space of A. This point of view is supported by the fact that the underlying graded multiplicative monoid of $\pi_*(A)$ can be recovered from $\Omega^{\mathcal{J}}(A)$, cf. [SS12, Proposition 4.24].

Since $\mathcal{S}^{\mathcal{J}}$ is a monoidal model category with respect to the $\boxtimes$-product, we know that $X \boxtimes Y$ is homotopically well-behaved if both X and Y are cofibrant. It is often useful that this holds under a weaker cofibrancy condition. To state it, we recall that for an object $(\mathbf{n_1}, \mathbf{n_2})$ in $\mathcal{J}$, the $(\mathbf{n_1}, \mathbf{n_2})$-th *latching space*

$$L_{(\mathbf{n_1}, \mathbf{n_2})}X = \operatorname{colim}_{(\mathbf{m_1}, \mathbf{m_2}) \to (\mathbf{n_1}, \mathbf{n_2})} X(\mathbf{m_1}, \mathbf{m_2})$$

is the colimit over the full subcategory of the comma category $(\mathcal{J} \downarrow (\mathbf{n_1}, \mathbf{n_2}))$ generated by the objects that are not isomorphisms. A $\mathcal{J}$-space X is *flat* if the canonical map $L_{(\mathbf{n_1}, \mathbf{n_2})}X \to X(\mathbf{n_1}, \mathbf{n_2})$ is a cofibration of simplicial sets for each object $(\mathbf{n_1}, \mathbf{n_2})$. A commutative $\mathcal{J}$-space monoid is flat if its underlying $\mathcal{J}$-space is.

Lemma 2.8. *(i) The functor $- \boxtimes Y$ preserves $\mathcal{J}$-equivalences if Y is flat.*
(ii) A $\mathcal{J}$-space that is cofibrant in the positive $\mathcal{J}$-model structure is flat.
(iii) Cofibrant commutative $\mathcal{J}$-space monoids are flat.

Proof. This is proven in [SS12, Propositions 8.2, 6.20 and 4.28]. □ □

3. The cyclic and replete bar constructions

In this section we introduce the cyclic and replete bar constructions of commutative $\mathcal{J}$-space monoids and recall the definition of the topological Hochschild homology of symmetric ring spectra. These are building blocks of the logarithmic topological Hochschild homology to be defined in Section 4.

3.1. **The cyclic bar construction.** As usual Δ denotes the category with objects $[n] = \{0 < \cdots < n\}$ for $n \geq 0$, and order-preserving maps. The category Δ is a subcategory of Connes' cyclic category Λ, cf. [Lod98, Definition 6.1.1]. The latter has the same objects as Δ, and additional morphisms $\tau_n\colon [n] \to [n]$ satisfying $\tau_n^{n+1} = \mathrm{id}$ as well as $\tau_n\delta_i = \delta_{i-1}\tau_{n-1}$ and $\tau_n\sigma_i = \sigma_{i-1}\tau_{n+1}$ for $1 \leq i \leq n$. The induced simplicial and cyclic operators are denoted $d_i = \delta_i^*$, $s_i = \sigma_i^*$ and $t_n = \tau_n^*$, respectively.

Definition 3.2. Let (M, μ, η) be a not necessarily commutative $\mathcal{J}$-space monoid, and let X be an M-bimodule, i.e., a $\mathcal{J}$-space with commuting left and right M-actions. The *cyclic bar construction* $B^{\mathrm{cy}}_\bullet(M, X)$ is the simplicial $\mathcal{J}$-space

$$[n] \longmapsto X \boxtimes M^{\boxtimes n} = X \boxtimes M \boxtimes \cdots \boxtimes M$$

with n copies of M. The 0-th face map d_0 uses the right action $X \boxtimes M \to X$, the i-th face map d_i for $0 < i < n$ uses the multiplication $\mu\colon M \boxtimes M \to M$ of the i-th and $(i+1)$-th factors, and the n-th face map d_n uses the symmetric structure

$$\tau\colon (X \boxtimes M^{\boxtimes n-1}) \boxtimes M \xrightarrow{\cong} M \boxtimes (X \boxtimes M^{\boxtimes n-1}) \tag{3.1}$$

followed by the left action $M \boxtimes X \to X$. The degeneracy map s_i inserts the unit $\eta\colon U^{\mathcal{J}} \to M$ after the i-th factor.

If M is commutative, we say that an M-bimodule X is *symmetric* if the right action on X equals the twist followed by the left action. In this case, there is an *augmentation* $\epsilon\colon B^{\mathrm{cy}}_\bullet(M, X) \to X$, where the codomain is viewed as a constant simplicial object. It is given in simplicial degree n by the n-fold (right) action $X \boxtimes M^{\boxtimes n} \to X$ and restricts to the identity on the 0-simplices of $B^{\mathrm{cy}}_\bullet(M, X)$.

In the special case when $X = M$, with left and right actions given by the multiplication, we write $B^{\mathrm{cy}}_\bullet(M) = B^{\mathrm{cy}}_\bullet(M, M)$. This is a cyclic $\mathcal{J}$-space, with cyclic operator t_n given by the symmetric structure as in (3.1). When M is commutative, $B^{\mathrm{cy}}_\bullet(M)$ is a cyclic commutative $\mathcal{J}$-space monoid and the augmentation $\epsilon\colon B^{\mathrm{cy}}_\bullet(M) \to M$ is a cyclic map to the constant cyclic object M.

Applying the diagonal functor from bisimplicial to simplicial sets object-wise defines a realization functor $|-|$ from simplicial objects in $\mathcal{CS}^{\mathcal{J}}$ to $\mathcal{CS}^{\mathcal{J}}$.

Definition 3.3. The *cyclic bar construction* $B^{\mathrm{cy}}(M, X)$ (resp. $B^{\mathrm{cy}}(M)$) is the realization of $B^{\mathrm{cy}}_\bullet(M, X)$ (resp. $B^{\mathrm{cy}}_\bullet(M)$).

When M is commutative and X is an M-module, it follows from the definition that $B^{\mathrm{cy}}(M, X)$ is a $B^{\mathrm{cy}}(M)$-module.

The realization functor from simplicial objects in $\mathcal{S}^{\mathcal{J}}$ to $\mathcal{S}^{\mathcal{J}}$ sends degree-wise $\mathcal{J}$-equivalences to $\mathcal{J}$-equivalences (this follows from [Hir03, Corollary 18.7.5]). By Lemma 2.8, $B^{\mathrm{cy}}(M, X)$ captures a well-defined homotopy type as soon as M is flat.

The cyclic bar construction admits a different description: The category of commutative $\mathcal{J}$-space monoids is tensored over unbased simplicial sets by setting

$$M \otimes K = \left| [n] \mapsto M^{\boxtimes K_n} \right| .$$

This uses that the $\boxtimes$-product is the coproduct in $\mathcal{CS}^{\mathcal{J}}$. The multiplication and unit of M give the simplicial structure maps. This tensor is part of the structure that makes $\mathcal{CS}^{\mathcal{J}}$ a simplicial model category (as defined for example in [Hir03, Definition 9.1.6]). The compatibility with the model structure lifts from $\mathcal{S}^{\mathcal{J}}$ because the cotensor is the same for $\mathcal{S}^{\mathcal{J}}$ and $\mathcal{CS}^{\mathcal{J}}$ [SS12, Proposition 9.9]. Using $\Delta[1]/\partial\Delta[1]$ as a model for S^1, we obtain:

Lemma 3.4. *There is a natural isomorphism $B^{\mathrm{cy}}(M) \cong M \otimes S^1$ in $\mathcal{CS}^{\mathcal{J}}$. The augmentation $\epsilon\colon B^{\mathrm{cy}}(M) \to M$ corresponds to the collapse map $S^1 \to *$.*

Proof. For $0 \leq k \leq n+1$, we let $a_{n,k}\colon [n] \to [1]$ be the n-simplex of $\Delta[1]$ with $a_{n,k}(i) = 0$ for $i < k$ and $a_{n,k}(i) = 1$ if $i \geq k$. Passing to the quotient $S^1 = \Delta[1]/\partial\Delta[1]$ identifies the constant maps $a_{n,0}$ and $a_{n,n+1}$ and gives an isomorphism $S^1_n \cong \{a_{n,0}, \dots, a_{n,n}\}$. The indicated ordering of S^1_n induces an isomorphism $M^{\boxtimes S^1_n} \xrightarrow{\cong} M^{\boxtimes(1+n)} = B^{\mathrm{cy}}_n(M)$. One can check that this is an isomorphism of simplicial objects. For example, $\delta_2\colon [1] \to [2]$ induces $d_2(a_{2,0}) = a_{1,0}$,

$d_2(a_{2,1}) = a_{1,1}$, and $d_2(a_{2,2}) = a_{1,2} = a_{1,0}$. Hence $d_2 \colon M^{\boxtimes S^1_2} \to M^{\boxtimes S^1_1}$ coincides with $d_2 \colon B^{\mathrm{cy}}_2(M) \to B^{\mathrm{cy}}_1(M)$ under the specified isomorphism. □ □

Remark 3.5. The previous description of $B^{\mathrm{cy}}(M)$ also reflects its cyclic structure: As explained for example in [Lod98, 7.1.2], $S^1 = \Delta[1]/\partial\Delta[1]$ extends to a cyclic set. Using this identification, it is easy to see that $[n] \mapsto M^{\boxtimes S^1_n}$ and $B^{\mathrm{cy}}_\bullet(M)$ are isomorphic as cyclic objects in $\mathcal{C}\mathcal{S}^{\mathcal{J}}$.

3.6. **Topological Hochschild homology.** Let A be a commutative symmetric ring spectrum. Implementing the cyclic bar construction in the context of symmetric spectra provides a cyclic commutative symmetric ring spectrum $B^{\mathrm{cy}}_\bullet(A) = \{[n] \mapsto A^{\wedge(1+n)}\}$, with cyclic structure maps given as in Definition 3.2.

Definition 3.7. Let A be a cofibrant commutative symmetric ring spectrum. Then we write $\mathrm{THH}_\bullet(A) = B^{\mathrm{cy}}_\bullet(A)$, and define the *topological Hochschild homology* $\mathrm{THH}(A)$ to be the realization of this cyclic object.

In this definition, the term "realization" can have two different meanings, both of which will be relevant for us. On the one hand, applying the diagonal functor from bisimplicial based sets to simplicial based sets in each spectrum degree of $\mathrm{THH}_\bullet(A)$ we get a realization internal to $\mathcal{C}\mathrm{Sp}^\Sigma$. On the other hand, we may first form the geometric realization of the smash powers $A^{\wedge(1+n)}$ to get a cyclic object $[n] \mapsto |A^{\wedge(1+n)}|$ in the category of symmetric spectra of topological spaces. The geometric realization of this cyclic object is then a commutative symmetric ring spectrum of topological spaces that comes equipped with an action of the circle group. It will always be clear from the context (or not important) whether we view the realization $\mathrm{THH}(A)$ as a symmetric spectrum internal to simplicial sets or topological spaces.

Remark 3.8. The reason for the cofibrancy condition in Definition 3.7 is that we want THH to be a homotopy invariant construction. Since the coproduct of cofibrant objects in a general model category is homotopy invariant, and the realization of simplicial objects in symmetric spectra sends degreewise stable equivalences to stable equivalences, a stable equivalence $A \to B$ of cofibrant commutative symmetric ring spectra induces a stable equivalence $\mathrm{THH}(A) \to \mathrm{THH}(B)$. For a commutative symmetric ring spectrum A that is not cofibrant, one should first choose a cofibrant replacement $A^{\mathrm{cof}} \xrightarrow{\simeq} A$, and then apply the cyclic bar construction to A^{cof}.

Using the tensor structure of commutative symmetric ring spectra we can identify $\mathrm{THH}(A)$ with $A \otimes S^1$, the tensor of A with the simplicial set $S^1 = \Delta[1]/\partial\Delta[1]$, in analogy with Lemma 3.4.

We noted in Lemmas 2.6 and 2.7 that $\mathbb{S}^{\mathcal{J}} \colon \mathcal{S}^{\mathcal{J}} \to \mathrm{Sp}^\Sigma$ is strong symmetric monoidal and that the induced functor of commutative monoids is a left Quillen functor. This immediately gives the next proposition.

Proposition 3.9. *There is a natural isomorphism* $\mathrm{THH}(\mathbb{S}^{\mathcal{J}}[M]) \cong \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)]$ *for each cofibrant commutative* $\mathcal{J}$*-space monoid* M*.*

3.10. **The replete bar construction.** We now discuss an extension of the cyclic bar construction of a commutative $\mathcal{J}$-space monoid that will play a role in our definition of logarithmic THH in Section 4.

Definition 3.11. A (commutative) $\mathcal{J}$-space monoid M is *grouplike* if the simplicial monoid $M_{h\mathcal{J}}$ is grouplike.

We recall from [Sag13, §5] that the usual group completion of homotopy commutative simplicial monoids lifts to commutative $\mathcal{J}$-space monoids. To formulate this, we use that a commutative $\mathcal{J}$-space monoid M gives rise to an associative simplicial monoid $M_{h\mathcal{J}}$, and write $B(M_{h\mathcal{J}}) = B(*, M_{h\mathcal{J}}, *)$ for the usual bar construction of $M_{h\mathcal{J}}$ with respect to the cartesian product.

Proposition 3.12 ([Sag13, Theorem 1.6]). *The category $\mathcal{CS}^{\mathcal{J}}$ admits a* group completion model structure. *The cofibrations are the same as in the positive $\mathcal{J}$-model structure, and $M \to N$ is a weak equivalence if and only if the induced map $B(M_{h\mathcal{J}}) \to B(N_{h\mathcal{J}})$ is a weak equivalence. An object is fibrant if and only if it is grouplike and positive $\mathcal{J}$-fibrant.* □

An important consequence of the group completion model structure is that its fibrant replacement provides a functorial group completion $\eta_M \colon M \rightarrowtail M^{\mathrm{gp}}$ for commutative $\mathcal{J}$-space monoids: The commutative $\mathcal{J}$-space monoid M^{gp} is grouplike, and η_M induces a group completion $M_{h\mathcal{J}} \to (M^{\mathrm{gp}})_{h\mathcal{J}}$ of E_∞ spaces in the usual sense. We emphasize that the map η_M is assumed to be a cofibration, so that M^{gp} is automatically cofibrant if M is.

Example 3.13 ([Sag13, Example 5.8]). Let $\mathbb{C}\langle \mathbf{d_1}, \mathbf{d_2}\rangle$ be the free commutative $\mathcal{J}$-space monoid on a generator in bidegree $(\mathbf{d_1}, \mathbf{d_2})$ with $d_1 > 0$, as defined in Example 2.5. The map $\mathbb{C}\langle \mathbf{d_1}, \mathbf{d_2}\rangle_{h\mathcal{J}} \to (\mathbb{C}\langle \mathbf{d_1}, \mathbf{d_2}\rangle^{\mathrm{gp}})_{h\mathcal{J}}$ is weakly equivalent to the usual group completion map of E_∞ spaces $\coprod_{k\geq 0} B\Sigma_k \to QS^0$.

Construction 3.14. Let M be a commutative $\mathcal{J}$-space monoid and let

$$M \overset{\sim}{\rightarrowtail} M' \twoheadrightarrow M^{\mathrm{gp}}$$

be a functorial factorization of the group completion map η_M into an acyclic cofibration followed by a fibration, in the positive $\mathcal{J}$-model structure. The natural augmentation from the cyclic bar construction to the constant cyclic object functor induces a commutative diagram of cyclic objects

$$\begin{array}{ccccc} B^{\mathrm{cy}}_\bullet(M) & \longrightarrow & B^{\mathrm{rep}}_\bullet(M) & \longrightarrow & B^{\mathrm{cy}}_\bullet(M^{\mathrm{gp}}) \\ \downarrow & & \downarrow & & \downarrow \\ M & \longrightarrow & M' & \longrightarrow & M^{\mathrm{gp}}, \end{array}$$

where $B^{\mathrm{rep}}_\bullet(M)$ is defined as the pullback of $M' \to M^{\mathrm{gp}} \leftarrow B^{\mathrm{cy}}_\bullet(M^{\mathrm{gp}})$ and the map $B^{\mathrm{cy}}_\bullet(M) \to B^{\mathrm{rep}}_\bullet(M)$ is given by the universal property of the pullback.

Definition 3.15. Let M be a commutative $\mathcal{J}$-space monoid. The *replete bar construction* $B^{\mathrm{rep}}(M)$ is the realization of the cyclic object $B^{\mathrm{rep}}_\bullet(M)$, and the induced map $\rho\colon B^{\mathrm{cy}}(M) \to B^{\mathrm{rep}}(M)$ is called the *repletion map*.

By definition, the replete bar construction $B^{\mathrm{rep}}(M)$ is a functorial model for the homotopy pullback of $M \to M^{\mathrm{gp}} \leftarrow B^{\mathrm{cy}}(M^{\mathrm{gp}})$. The $\mathcal{I}$-space version of this definition was considered in [Rog09, Definition 8.10].

The fact that $M \to M^{\mathrm{gp}}$ is a $\mathcal{J}$-equivalence if M is grouplike implies the following statement.

Lemma 3.16. *The repletion map $\rho\colon B^{\mathrm{cy}}(M) \to B^{\mathrm{rep}}(M)$ is a $\mathcal{J}$-equivalence if M is a grouplike cofibrant commutative $\mathcal{J}$-space monoid.* □

3.17. **General repletion.** We now introduce a more general notion of *repletion*, which can be viewed as a relative version of the group completion. Repleteness is a topological adaption of the algebraic notion of an *exact homomorphism* of integral monoids [Kat89, Definition 4.6], compare [Rog09, Definition 3.6].

Definition 3.18. Let $\epsilon\colon N \to M$ be a map of commutative $\mathcal{J}$-space monoids. The *repletion* $N^{\mathrm{rep}} \to M$ of N over M is defined by factoring ϵ in the group completion model structure as an acyclic cofibration followed by a fibration:

$$N \overset{\sim}{\rightarrowtail} N^{\mathrm{rep}} \twoheadrightarrow M\ .$$

We write $\rho_N\colon N \to N^{\mathrm{rep}}$ for the *repletion map*, defined by the factorization.

Since the group completion model structure is a left Bousfield localization of the positive $\mathcal{J}$-model structure, it follows from [Hir03, Proposition 3.3.5] that N^{rep} is well-defined up to $\mathcal{J}$-equivalence under N and over M. Repletion relative to the terminal object in $\mathcal{CS}^{\mathcal{J}}$ is group completion.

The replete bar construction introduced above can be viewed as a special case of the general repletion:

Proposition 3.19. *There is a chain of $\mathcal{J}$-equivalences under $B^{\mathrm{cy}}(M)$ and over M' connecting the replete bar construction $B^{\mathrm{rep}}(M)$ to the repletion $B^{\mathrm{cy}}(M)^{\mathrm{rep}}$ of the augmentation $B^{\mathrm{cy}}(M) \to M$.*

We prove the proposition at the end of this section. The reason why we do not simply define the replete bar construction in terms of the general repletion is that Construction 3.14 provides a cyclic object $B^{\mathrm{rep}}_{\bullet}(M)$ with realization $B^{\mathrm{rep}}(M)$. This extra structure on $B^{\mathrm{rep}}(M)$ is not visible on $B^{\mathrm{cy}}(M)^{\mathrm{rep}}$. The general notion of repletion is nonetheless useful, for example for the proofs in Section 4.27.

In general, fibrations and acyclic cofibrations in a left Bousfield localization such as the group completion model structure are difficult to understand. However, we can give a simpler description of the repletion of maps that are *virtually surjective*, in the sense of the following $\mathcal{J}$-space variant of [Rog09, Definition 8.1].

Definition 3.20. A map $\epsilon\colon N \to M$ of commutative $\mathcal{J}$-space monoids is *virtually surjective* if it induces a surjective homomorphism of abelian groups $\pi_0(N^{\mathrm{gp}})_{h\mathcal{J}} \to \pi_0(M^{\mathrm{gp}})_{h\mathcal{J}}$.

Lemma 3.21. *Let $\epsilon\colon N \to M$ be a virtually surjective map of commutative $\mathcal{J}$-space monoids, and consider the diagram of solid arrows*

$$\begin{tikzcd} N \arrow[r, "\rho_N"] \arrow[d, "\eta_N"'] & N^{\mathrm{rep}} \arrow[r] \arrow[d, dashed] & M \arrow[d, "\eta_M"] \\ N^{\mathrm{gp}} \arrow[r, tail, "\sim"] & (N^{\mathrm{gp}})' \arrow[r, two heads] & M^{\mathrm{gp}}, \end{tikzcd} \tag{3.2}$$

where the bottom row is a factorization in the positive $\mathcal{J}$-model structure. Then there exists a map $N^{\mathrm{rep}} \to (N^{\mathrm{gp}})'$ such that the diagram commutes, and for any such map the right hand square is homotopy cartesian with respect to the positive $\mathcal{J}$-model structure.

Proof. The map $(N^{\mathrm{gp}})' \to M^{\mathrm{gp}}$ is a fibration in the group completion model structure by [Hir03, Proposition 3.3.16]. Hence the lifting axioms in the group completion model structure provide the desired map $N^{\mathrm{rep}} \to (N^{\mathrm{gp}})'$. The base change of $(N^{\mathrm{gp}})' \to M^{\mathrm{gp}}$ along η_M provides a map $N' \to M$ that is also a fibration in the group completion model structure. Since $N^{\mathrm{rep}} \to M$ has this property by construction, it follows from [Hir03, Proposition 3.3.5] that the induced map $N^{\mathrm{rep}} \to N'$ is a $\mathcal{J}$-equivalence as soon as it is a weak equivalence in the group completion model structure. The two out of three axiom for weak equivalences reduces to showing that $N' \to (N^{\mathrm{gp}})'$ is a weak equivalence in the group completion model structure. We claim that an application of the Bousfield–Friedlander Theorem [BF78, Theorem B.4], similar to the proof of [BF78, Lemma 5.3], shows that the induced square

$$\begin{tikzcd} B(N'_{h\mathcal{J}}) \arrow[r] \arrow[d] & B(M_{h\mathcal{J}}) \arrow[d, "\sim"] \\ B((N^{\mathrm{gp}})'_{h\mathcal{J}}) \arrow[r] & B((M^{\mathrm{gp}})_{h\mathcal{J}}) \end{tikzcd}$$

is homotopy cartesian. For this we note that the square in question results from a pointwise homotopy cartesian square of bisimplicial sets, that the bisimplicial sets $B_\bullet((N^{\mathrm{gp}})'_{h\mathcal{J}})$ and $B_\bullet((M^{\mathrm{gp}})_{h\mathcal{J}})$ satisfy the π_*-Kan condition because the simplicial monoids $(N^{\mathrm{gp}})'_{h\mathcal{J}}$ and $(M^{\mathrm{gp}})_{h\mathcal{J}}$ are grouplike, and that the virtual surjectivity of ϵ implies that $B_\bullet((N^{\mathrm{gp}})'_{h\mathcal{J}}) \to B_\bullet((M^{\mathrm{gp}})_{h\mathcal{J}})$ induces a Kan fibration on vertical path components. Hence [BF78, Theorem B.4] applies, and the claim of the lemma follows. □ □

The next corollary relates the repletion defined here to the $\mathcal{J}$-space version of the notion used in [Rog09, §8], compare also the discussion in [SS13, §5.10].

Corollary 3.22. *Let $\epsilon\colon N \to M$ be a virtually surjective map in $\mathcal{CS}^{\mathcal{J}}$. Then the repletion N^{rep} is $\mathcal{J}$-equivalent to the homotopy pullback of $N^{\mathrm{gp}} \to M^{\mathrm{gp}} \leftarrow M$ with respect to the positive $\mathcal{J}$-model structure.* □

We now return to the cyclic bar construction and prepare for the proof of Proposition 3.19.

Lemma 3.23. *The commutative $\mathcal{J}$-space monoids $B^{\mathrm{cy}}(M^{\mathrm{gp}})$ and $B^{\mathrm{cy}}(M)^{\mathrm{gp}}$ are $\mathcal{J}$-equivalent as commutative $\mathcal{J}$-space monoids under $B^{\mathrm{cy}}(M)$ and over M^{gp}.*

Proof. As observed in Lemma 3.4, there is an isomorphism $B^{\mathrm{cy}}(N) \cong N \otimes S^1$. The group completion and the collapse map $S^1 \to *$ induce the outer square in the commutative diagram

$$\begin{tikzcd}
M \otimes S^1 \arrow[r] \arrow[dd] & (M \otimes S^1)^{\mathrm{gp}} \arrow[d, tail, "\sim"] \\
 & (M \otimes S^1)' \arrow[d, two heads] \\
M^{\mathrm{gp}} \otimes S^1 \arrow[r] \arrow[ur, dashed] & M^{\mathrm{gp}}.
\end{tikzcd}$$

Since the positive $\mathcal{J}$-model structure is simplicial, it follows from [Hir03, Theorem 4.1.1 (4)] that the group completion model structure is also simplicial. So the left hand vertical map is an acyclic cofibration in the group completion model structure. The object $(M \otimes S^1)'$ is defined by forming the indicated factorization in the group completion model structure. Then $(M \otimes S^1)'$ is also grouplike. The model category axioms in the group completion model structure provide the lift $M^{\mathrm{gp}} \otimes S^1 \to (M \otimes S^1)'$. The two out of three axiom implies that the lift is a weak equivalence in the group completion model structure. To see that it is a $\mathcal{J}$-equivalence, it is enough to show that $M^{\mathrm{gp}} \otimes S^1$ is also grouplike. For this we note that the monoids of zero-simplices of $(M^{\mathrm{gp}})_{h\mathcal{J}}$ and $(M^{\mathrm{gp}} \otimes S^1)_{h\mathcal{J}}$ coincide, since they are both given by disjoint union of the sets of zero-simplices of $M^{\mathrm{gp}}(\mathbf{m_1}, \mathbf{m_2})$ over all objects $(\mathbf{m_1}, \mathbf{m_2})$ of $\mathcal{J}$. If two 0-simplices of $(M^{\mathrm{gp}})_{h\mathcal{J}}$ become equivalent in $\pi_0((M^{\mathrm{gp}})_{h\mathcal{J}})$, they also become equivalent in $\pi_0((M^{\mathrm{gp}} \otimes S^1)_{h\mathcal{J}})$. Hence the latter monoid is a group if the former one is. □ □

of Proposition 3.19. By the previous lemma, $B^{\mathrm{rep}}(M)$ is $\mathcal{J}$-equivalent to the homotopy pullback of $M \to M^{\mathrm{gp}} \leftarrow B^{\mathrm{cy}}(M)^{\mathrm{gp}}$. Since $B^{\mathrm{cy}}(M) \to M$ has a multiplicative section, it is virtually surjective, and so it follows from Corollary 3.22 that the homotopy pullback of $M \to M^{\mathrm{gp}} \leftarrow B^{\mathrm{cy}}(M)^{\mathrm{gp}}$ is $\mathcal{J}$-equivalent to the repletion of the map $B^{\mathrm{cy}}(M) \to M$. □ □

4. Logarithmic THH

In this section we define pre-log and log (symmetric) ring spectra and introduce their topological Hochschild homology.

Definition 4.1. A *pre-log structure* (M, α) on a commutative symmetric ring spectrum A is a commutative $\mathcal{J}$-space monoid M and a commutative $\mathcal{J}$-space monoid map $\alpha\colon M \to \Omega^{\mathcal{J}}(A)$. A *pre-log ring spectrum* (A, M, α) is a commutative symmetric ring spectrum A with a choice of pre-log structure (M, α). A morphism $(f, f^\flat)\colon (A, M, \alpha) \to (B, N, \beta)$ is a pair of morphisms $f\colon A \to B$ in $\mathcal{C}\mathrm{Sp}^{\Sigma}$ and $f^\flat\colon M \to N$ in $\mathcal{CS}^{\mathcal{J}}$ such that $\Omega^{\mathcal{J}}(f)\alpha = \beta f^\flat$.

Specifying α is equivalent to specifying its adjoint, the commutative symmetric ring spectrum map $\bar{\alpha}\colon \mathbb{S}^{\mathcal{J}}[M] \to A$. We often omit α from the notation.

As suggested by the terminology, there is also the notion of a log ring spectrum. It will be defined in Section 4.20 below.

Remark 4.2. Throughout, *log* is short for *logarithmic*. Our pre-log ring spectra were called *graded pre-log symmetric ring spectra* in [SS12, §4.30] and [Sag14], to distinguish them from the earlier notion of pre-log symmetric ring spectra introduced in [Rog09]. When the latter reference was written the theory of $\mathcal{J}$-spaces was not yet properly developed, so only the "ungraded" version of E_∞ spaces known as $\mathcal{I}$-spaces was considered. That restricted theory suffers from a lack of really interesting examples for log structures on ring spectra that are not Eilenberg–Mac Lane spectra, which is alleviated by the passage to the more general context of $\mathcal{J}$-spaces. It now seems sensible to shift the terminology, so that the most interesting objects (commutative symmetric ring spectra with pre-log structures given by commutative $\mathcal{J}$-space monoids) have the simplest name.

Example 4.3. (i) Let M be a commutative $\mathcal{J}$-space monoid. The adjunction unit $\zeta\colon M \to \Omega^{\mathcal{J}}(\mathbb{S}^{\mathcal{J}}[M])$ defines the *canonical pre-log structure* (M, ζ) on $\mathbb{S}^{\mathcal{J}}[M]$, with adjoint the identity map of $\mathbb{S}^{\mathcal{J}}[M]$.

(ii) Let A be a commutative symmetric ring spectrum. A map $x\colon S^{d_2} \to A_{d_1}$ defines a 0-simplex $x \in \Omega^{\mathcal{J}}(A)(\mathbf{d_1}, \mathbf{d_2})_0$. As explained in Example 2.5, the map x induces a map $\mathbb{C}\langle x\rangle \to \Omega^{\mathcal{J}}(A)$ from the free commutative $\mathcal{J}$-space monoid on a point in bidegree $(\mathbf{d_1}, \mathbf{d_2})$ to $\Omega^{\mathcal{J}}(A)$. This defines the *free pre-log structure* on A generated by x.

(iii) Pre-log rings in the algebraic sense give rise to pre-log ring spectra. We study this in detail in Section 5.

The following definition is an important source of interesting pre-log structures:

Definition 4.4. Let $j\colon A \to B$ be a map of commutative symmetric ring spectra and let $N \to \Omega^{\mathcal{J}}(B)$ be a pre-log structure. The pullback of $N \to \Omega^{\mathcal{J}}(B) \leftarrow \Omega^{\mathcal{J}}(A)$ defines a pre-log structure $j_*N = N \times_{\Omega^{\mathcal{J}}(B)} \Omega^{\mathcal{J}}(A)$ on A that we refer to as the *direct image pre-log structure.*

In order to ensure that the pullback in the definition captures a well-defined homotopy type, we will only consider direct image pre-log structures when $j\colon A \to B$ or $N \to \Omega^{\mathcal{J}}(B)$ are positive fibrations, and B is positive fibrant.

Now we turn to the definition of logarithmic topological Hochschild homology. Our strategy will be to first define it on pre-log ring spectra satisfying a suitable cofibrancy condition, and then extend the definition to all pre-log ring spectra by precomposing with a cofibrant replacement functor.

Definition 4.5. A pre-log ring spectrum (A, M, α) is *cofibrant* if M is a cofibrant commutative $\mathcal{J}$-space monoid and the adjoint structure map $\bar{\alpha}\colon \mathbb{S}^{\mathcal{J}}[M] \to A$ is a cofibration of commutative symmetric ring spectra.

We note that if (A, M, α) is cofibrant, then A is cofibrant as a commutative symmetric ring spectrum. It follows from standard model category arguments that the cofibrant pre-log ring spectra are the cofibrant objects in a cofibrantly generated projective model structure where a map $(f, f^\flat)$ is a fibration or a weak equivalence if and only if both f and $f^\flat$ have this property. This implies that we may choose a cofibrant replacement functor $(A^{\mathrm{cof}}, M^{\mathrm{cof}}, \alpha^{\mathrm{cof}}) \to (A, M, \alpha)$ for pre-log ring spectra. Thus, once we define log THH for cofibrant pre-log ring spectra below, the definition can easily be extended to all pre-log ring spectra by precomposing with this cofibrant replacement functor. Such cofibrant replacements were used implicitly in the formulation of Theorems 1.4 and 1.5 from the introduction.

Definition 4.6. Let (A, M, α) be a cofibrant pre-log ring spectrum. Its *logarithmic topological Hochschild homology* is the commutative symmetric ring spectrum

$$\mathrm{THH}(A, M) = \mathrm{THH}(A) \wedge_{\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)]} \mathbb{S}^{\mathcal{J}}[B^{\mathrm{rep}}(M)]$$

given by the pushout in the following diagram

$$\begin{array}{ccc} \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)] & \xrightarrow{\rho} & \mathbb{S}^{\mathcal{J}}[B^{\mathrm{rep}}(M)] \\ \downarrow & & \downarrow \\ \mathrm{THH}(A) & \xrightarrow{\rho} & \mathrm{THH}(A, M) \end{array} \tag{4.1}$$

of commutative symmetric ring spectra. The upper horizontal arrow is given by applying $\mathbb{S}^{\mathcal{J}}$ to the repletion map $\rho\colon B^{\mathrm{cy}}(M) \to B^{\mathrm{rep}}(M)$ and the left hand vertical map is obtained by applying the functor THH to the adjoint pre-log structure map $\bar{\alpha}\colon \mathbb{S}^{\mathcal{J}}[M] \to A$, under the identification $\mathrm{THH}(\mathbb{S}^{\mathcal{J}}[M]) \cong \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)]$.

It is clear from the construction that $\mathrm{THH}(A, M)$ is isomorphic to the realization of the cyclic commutative symmetric ring spectrum $\mathrm{THH}_\bullet(A, M)$ defined by the pushout of the diagram

$$\mathrm{THH}_\bullet(A) \leftarrow \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}_\bullet(M)] \to \mathbb{S}^{\mathcal{J}}[B^{\mathrm{rep}}_\bullet(M)]$$

in cyclic commutative symmetric ring spectra. Hence the geometric realization of $\mathrm{THH}_\bullet(A, M)$ becomes a commutative symmetric ring spectrum with circle action, which we shall also denote by $\mathrm{THH}(A, M)$. It will always be clear from the context (or not important) whether we think of $\mathrm{THH}(A, M)$ as a symmetric spectrum of simplicial sets or topological spaces.

Remark 4.7. The point of the cofibrancy condition on (A, M, α) in Definition 4.6 is that the adjoint structure map $\bar{\alpha}\colon \mathbb{S}^{\mathcal{J}}[M] \to A$ being a cofibration implies that $\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}_\bullet(M)] \to \mathrm{THH}_\bullet(A)$ is a cofibration in every simplicial degree and that the realization $\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)] \to \mathrm{THH}(A)$ is a cofibration of commutative symmetric ring spectra. This ensures that the pushout squares defining $\mathrm{THH}(A, M)$ and $\mathrm{THH}_\bullet(A, M)$ are homotopy pushout squares. In fact, $\mathrm{THH}(A, M)$ also represents the left derived balanced smash product of $\mathrm{THH}(A)$ and $\mathbb{S}^{\mathcal{J}}[B^{\mathrm{rep}}(M)]$ thought of as $\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)]$-module spectra. This follows by applying the next lemma to the cofibrant pre-log structure on $\mathrm{THH}(A)$ defined by $\mathrm{THH}(\bar{\alpha})$.

Lemma 4.8. *Let (A, M) be a cofibrant pre-log ring spectrum. Extension of scalars along the adjoint structure map $A \wedge_{\mathbb{S}^{\mathcal{J}}[M]} (-)\colon \mathrm{Mod}_{\mathbb{S}^{\mathcal{J}}[M]} \to \mathrm{Mod}_A$ preserves stable equivalences between not necessarily cofibrant objects.*

Proof. We consider more generally a cofibration $E \to F$ of commutative symmetric ring spectra. By [Shi04, Proposition 4.1], F is cofibrant as a flat E-module, where our use of the term "flat" is synonymous with the term "E-cofibrant" used in [Shi04]. A cell induction argument reduces the claim to the following statement: If Y is an E-module that is obtained from an E-module X by attaching a generating cofibration of flat E-modules $(K \to L) \wedge E$, then $Y \wedge_E (-)$ preserves stable equivalences if $X \wedge_E (-)$ does. Since the smash products with the flat symmetric spectra K and L preserve stable equivalences and the smash product with any symmetric spectrum sends flat cofibrations to level cofibrations, the gluing lemma for stable equivalences and level cofibrations of symmetric spectra shows the claim. □ □

Proposition 4.9. *If $(f, f^\flat)\colon (A, M) \to (B, N)$ is a map of cofibrant pre-log ring spectra such that $f\colon A \to B$ is a stable equivalence and $f^\flat\colon M \to N$ is a $\mathcal{J}$-equivalence, then the induced map $\mathrm{THH}(A, M) \to \mathrm{THH}(B, N)$ is a stable equivalence.*

Proof. The cofibrancy conditions imply that f gives rise to a stable equivalence $\mathrm{THH}(A) \to \mathrm{THH}(B)$ and that $f^\flat$ gives rise to $\mathcal{J}$-equivalences $B^{\mathrm{cy}}(M) \to B^{\mathrm{cy}}(N)$ and $B^{\mathrm{rep}}(M) \to B^{\mathrm{rep}}(N)$. Although $B^{\mathrm{rep}}(M)$ and $B^{\mathrm{rep}}(N)$ are not necessarily cofibrant, it follows from Corollary A.8 that the induced maps

$$\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)] \to \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(N)], \qquad \mathbb{S}^{\mathcal{J}}[B^{\mathrm{rep}}(M)] \to \mathbb{S}^{\mathcal{J}}[B^{\mathrm{rep}}(N)]$$

are stable equivalences. Hence the result follows from left properness of the positive stable model structure on $\mathcal{C}\mathrm{Sp}^{\Sigma}$. □ □

This result implies in particular that we obtain a homotopy invariant functor if we precompose our log THH functor with a cofibrant replacement functor.

Proposition 4.10. *For a cofibrant commutative $\mathcal{J}$-space monoid M, the natural map*

$$\mathbb{S}^{\mathcal{J}}[B^{\mathrm{rep}}(M)] \xrightarrow{\cong} \mathrm{THH}(\mathbb{S}^{\mathcal{J}}[M], M)$$

is an isomorphism, and the canonical map $(\mathbb{S}^{\mathcal{J}}[M], M) \to (A, M)$ induces a natural pushout square

$$\begin{tikzcd} \mathrm{THH}(\mathbb{S}^{\mathcal{J}}[M]) \arrow[r, "\rho"] \arrow[d, "\bar{\alpha}"'] & \mathrm{THH}(\mathbb{S}^{\mathcal{J}}[M], M) \arrow[d] \\ \mathrm{THH}(A) \arrow[r, "\rho"] & \mathrm{THH}(A, M) \end{tikzcd}$$

of commutative symmetric ring spectra. □

Remark 4.11. If $(f, f^\flat)\colon (B, N) \to (A, M)$ is a map of pre-log ring spectra, then the repletion $N^{\mathrm{rep}} \to M$ extends to a map of pre-log ring spectra

$$(B \wedge_{\mathbb{S}^{\mathcal{J}}[N]} \mathbb{S}^{\mathcal{J}}[N^{\mathrm{rep}}], N^{\mathrm{rep}}) \to (A, M).$$

We call this map the repletion of $(f, f^\flat)$.

The adjoints of the vertical maps in (4.1) define pre-log ring spectra

$$(\mathrm{THH}(A), B^{\mathrm{cy}}(M)) \qquad \text{and} \qquad (\mathrm{THH}(A, M), B^{\mathrm{rep}}(M)).$$

The augmentation of the cyclic bar construction induces an augmentation

$$(\mathrm{THH}(A), B^{\mathrm{cy}}(M)) \to (A, M),$$

and it follows from Proposition 3.19 that the repletion of this map is stably equivalent to $(\mathrm{THH}(A, M), B^{\mathrm{rep}}(M))$.

4.12. **Log THH and localization.** We now explain how the logarithmic THH of (A, M) lies between THH of A and THH of the localization of A away from M.

Definition 4.13 ([Rog09, Definition 7.15])**.** Let (A, M) be a pre-log ring spectrum. The commutative symmetric ring spectrum given by the pushout

$$A[M^{-1}] = A \wedge_{\mathbb{S}^{\mathcal{J}}[M]} \mathbb{S}^{\mathcal{J}}[M^{\mathrm{gp}}]$$

is the *localization* of (A, M), and the pre-log ring spectrum $(A[M^{-1}], M^{\mathrm{gp}})$ is the *trivial locus* of (A, M).

We note that since $\eta_M\colon M \to M^{\mathrm{gp}}$ is a cofibration, the pre-log ring spectrum $(A[M^{-1}], M^{\mathrm{gp}})$ is cofibrant if (A, M) is.

Example 4.14. The trivial locus of the pre-log ring spectrum $(\mathbb{S}^{\mathcal{J}}[M], M)$ from Example 4.3(i) is $(\mathbb{S}^{\mathcal{J}}[M^{\mathrm{gp}}], M^{\mathrm{gp}})$.

Example 4.15 ([Sag14, Proposition 3.19])**.** If the map $x\colon S^{d_2} \to A_{d_1}$ represents a homotopy class $[x] \in \pi_{d_2-d_1}(A)$ of a positive fibrant commutative symmetric ring spectrum A, then the map $A \to A[\mathbb{C}\langle x\rangle^{-1}]$ induces the localization homomorphism $\pi_*(A) \to (\pi_*(A))[1/[x]]$ at the level of homotopy groups.

Proposition 4.16. *Let (A, M) be a cofibrant pre-log ring spectrum. Then there is a natural factorization of the localization map* $\mathrm{THH}(A) \to \mathrm{THH}(A[M^{-1}])$ *through the repletion map* $\mathrm{THH}(A) \to \mathrm{THH}(A, M)$*.*

Proof. The claim follows by observing that the maps $B^{\mathrm{cy}}(M) \to B^{\mathrm{rep}}(M)$ and $B^{\mathrm{rep}}(M) \to B^{\mathrm{cy}}(M^{\mathrm{gp}})$ induce pushout squares

$$\begin{tikzcd}
\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)] \arrow[r] \arrow[d] & \mathbb{S}^{\mathcal{J}}[B^{\mathrm{rep}}(M)] \arrow[r] \arrow[d] & \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M^{\mathrm{gp}})] \arrow[d] \\
\mathrm{THH}(A) \arrow[r] & \mathrm{THH}(A, M) \arrow[r] & \mathrm{THH}(A[M^{-1}])
\end{tikzcd}$$

of commutative symmetric ring spectra. □ □

4.17. **Homotopy cofiber sequences for log** THH**.** We shall use the following proposition as a general principle for setting up homotopy cofiber sequences involving log THH. This will be used to construct localization sequences for discrete rings in Section 5 and for periodic ring spectra in Section 6.

Proposition 4.18. *Let (A, M) be a cofibrant pre-log ring spectrum and suppose that P is a cofibrant commutative $\mathcal{J}$-space monoid such that there is a map of commutative symmetric ring spectra $\mathbb{S}^{\mathcal{J}}[M] \to \mathbb{S}^{\mathcal{J}}[P]$ and a homotopy cofiber sequence*

$$\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)] \xrightarrow{\rho} \mathbb{S}^{\mathcal{J}}[B^{\mathrm{rep}}(M)] \xrightarrow{\partial} \Sigma\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(P)]$$

of $\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)]$-modules with circle action. Then there is a homotopy cofiber sequence

$$\mathrm{THH}(A) \xrightarrow{\rho} \mathrm{THH}(A, M) \xrightarrow{\partial} \Sigma\,\mathrm{THH}(A \wedge_{\mathbb{S}^{\mathcal{J}}[M]} \mathbb{S}^{\mathcal{J}}[P])$$

of $\mathrm{THH}(A)$*-modules with circle action.*

Proof. Applying base change along $\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)] \to \mathrm{THH}(A)$ to the homotopy cofiber sequence of $\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)]$-modules in the proposition, we get a homotopy cofiber sequence

$$\mathrm{THH}(A) \to \mathrm{THH}(A, M) \to \Sigma\,\mathrm{THH}(A) \wedge_{\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)]} \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(P)]$$

of $\mathrm{THH}(A)$-modules with circle action. This uses that the functor in question takes stable equivalences of $\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)]$-modules to stable equivalences of $\mathrm{THH}(A)$-modules, as follows from Lemma 4.8 applied to the cofibrant pre-log ring spectrum $(\mathrm{THH}(A), \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)])$. To get the statement in the proposition we identify the last term via the sequence of isomorphisms

$$\begin{aligned}
\mathrm{THH}(A) \wedge_{\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)]} \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(P)] &\cong |B^{\mathrm{cy}}_{\bullet}(A) \wedge_{B^{\mathrm{cy}}_{\bullet}(\mathbb{S}^{\mathcal{J}}[M])} B^{\mathrm{cy}}_{\bullet}(\mathbb{S}^{\mathcal{J}}[P])| \\
&\cong |B^{\mathrm{cy}}_{\bullet}(A \wedge_{\mathbb{S}^{\mathcal{J}}[M]} \mathbb{S}^{\mathcal{J}}[P])| = \mathrm{THH}(A \wedge_{\mathbb{S}^{\mathcal{J}}[M]} \mathbb{S}^{\mathcal{J}}[P])
\end{aligned}$$

of $\mathrm{THH}(A)$-modules with circle action. Notice that P being cofibrant ensures that $A \wedge_{\mathbb{S}^{\mathcal{J}}[M]} \mathbb{S}^{\mathcal{J}}[P]$ is a cofibrant commutative symmetric ring spectrum, which justifies the notation $\mathrm{THH}(A \wedge_{\mathbb{S}^{\mathcal{J}}[M]} \mathbb{S}^{\mathcal{J}}[P])$. □ □

Remark 4.19. Our use of the terminology "THH(A)-module with circle action" in Proposition 4.18 refers to a module for THH(A) thought of as a commutative monoid in the symmetric monoidal category of symmetric spectra with circle action. Thus, if X denotes a THH(A)-module with circle action, it is understood that the action map $\mathrm{THH}(A) \wedge X \to X$ is circle equivariant.

4.20. **Logification.** In order to introduce log ring spectra, we recall the definition of units:

Definition 4.21. The (graded) units $\mathrm{GL}_1^{\mathcal{J}}(A)$ of a positive fibrant commutative symmetric ring spectrum A is the sub commutative $\mathcal{J}$-space monoid of $\Omega^{\mathcal{J}}(A)$ consisting of the path components that map to units in $\pi_0(\Omega^{\mathcal{J}}(A)_{h\mathcal{J}})$.

This notion of units is extensively studied in [SS12] and [Sag13]. The use of $\mathcal{J}$-spaces ensures that the canonical map $\mathrm{GL}_1^{\mathcal{J}}(A) \to \Omega^{\mathcal{J}}(A)$ corresponds to the inclusion $\pi_*(A)^\times \to (\pi_*(A), \cdot)$ of the units in the underlying signed *graded* multiplicative monoid of the graded commutative ring of homotopy groups $\pi_*(A)$.

Definition 4.22. A pre-log ring spectrum (A, M, α) is a *log ring spectrum* if the top horizontal map $\widetilde{\alpha}$ in the following pullback square is a $\mathcal{J}$-equivalence:

$$\begin{tikzcd} \alpha^{-1}\,\mathrm{GL}_1^{\mathcal{J}}(A) \arrow[r, "\widetilde{\alpha}"] \arrow[d] & \mathrm{GL}_1^{\mathcal{J}}(A) \arrow[d] \\ M \arrow[r, "\alpha"] & \Omega^{\mathcal{J}}(A) \end{tikzcd}$$

The map $\mathrm{GL}_1^{\mathcal{J}}(A) \to \Omega^{\mathcal{J}}(A)$ is a fibration because it is an inclusion of path components. Hence the log condition is a homotopy invariant notion as soon as A is positive fibrant.

Example 4.23. (i) The inclusion $\mathrm{GL}_1^{\mathcal{J}}(A) \to \Omega^{\mathcal{J}}(A)$ of commutative $\mathcal{J}$-space monoids defines a log structure, known as the *trivial log structure* on A.
(ii) If (B, N) is a log ring spectrum, then the direct image pre-log structure associated with $A \to B$, introduced in Definition 4.4, is a log structure if $N \to \Omega^{\mathcal{J}}(B)$ is a fibration in $\mathcal{CS}^{\mathcal{J}}$ or $A \to B$ is a positive fibration in $\mathcal{C}\mathrm{Sp}^{\Sigma}$.

We can combine these two examples in order to produce log structures from maps in $\mathcal{C}\mathrm{Sp}^{\Sigma}$:

Definition 4.24. Let $j\colon A \to B$ be a map of commutative symmetric ring spectra with B positive fibrant. The *direct image log structure* $j_*\mathrm{GL}_1^{\mathcal{J}}(B) \to \Omega^{\mathcal{J}}(A)$ is obtained by forming the direct image log structure associated with $j\colon A \to B$ and the trivial log structure $\mathrm{GL}_1^{\mathcal{J}}(B) \to \Omega^{\mathcal{J}}(B)$.

As illustrated in Theorem 1.5 and [Sag14, Theorem 1.4], the direct image log structure is an interesting log structure when $A \to B$ is the connective cover map of a periodic ring spectrum B.

Construction 4.25. If (A, M, α) is a pre-log ring spectrum, then the *associated log ring spectrum* (A^a, M^a, α^a) is defined as follows: We choose a factorization

$$\alpha^{-1}\,\mathrm{GL}_1^{\mathcal{J}}(A) \rightarrowtail G \xrightarrow{\sim} \mathrm{GL}_1^{\mathcal{J}}(A)$$

of $\widetilde{\alpha}$ in the positive $\mathcal{J}$-model structure, and define M^a by the following pushout square in $\mathcal{CS}^{\mathcal{J}}$

$$\begin{tikzcd} \alpha^{-1}\,\mathrm{GL}_1^{\mathcal{J}}(A) \arrow[r, tail] \arrow[d] & G \arrow[d] \\ M \arrow[r] & M^a. \end{tikzcd} \tag{4.2}$$

The universal property of the pushout determines a map $M^a \to \Omega^{\mathcal{J}}(A)$. As proven in [Rog09, Lemma 7.7] or [Sag14, Lemma 3.12], the induced map $M \to M^a$ is a $\mathcal{J}$-equivalence if (A, M) is already a log ring spectrum, and (A, M^a) is always a log ring spectrum. However, in connection with log THH we want a logification that preserves cofibrancy, so we carry on and define A^a by the following functorial factorization

$$\begin{tikzcd} A \wedge_{\mathbb{S}^{\mathcal{J}}[M]} \mathbb{S}^{\mathcal{J}}[M^a] \arrow[r, tail] \arrow[d] & A^a \arrow[d, two heads, "\simeq"] \\ A \wedge_{\mathbb{S}^{\mathcal{J}}[M]} A \arrow[r] & A \end{tikzcd}$$

in the positive stable model structure on $\mathcal{C}\mathrm{Sp}^{\Sigma}$. We let α^a be the adjoint of the map $\mathbb{S}^{\mathcal{J}}[M^a] \to A^a$ given by the factorization. In this way we have defined a functor $(A, M, \alpha) \mapsto (A^a, M^a, \alpha^a)$ from the category of pre-log ring spectra to the full subcategory of log ring spectra that comes with a natural morphism $(A, M, \alpha) \to (A^a, M^a, \alpha^a)$ of pre-log ring spectra such that $A \to A^a$ is a stable equivalence. If (A, M, α) is cofibrant, then also (A^a, M^a, α^a) is cofibrant.

Example 4.26. Let E be a d-periodic commutative symmetric ring spectrum with connective cover $j\colon e \to E$, where $d > 0$. As explained in [Sag14, Construction 4.2], the choice of a representative x of a periodicity element gives rise to a pre-log structure $D(x) \to \Omega^{\mathcal{J}}(e)$. It has the property $D(x)_{h\mathcal{J}} \simeq Q_{\geq 0}S^0$ [Sag14, Lemma 4.6]. There is a canonical map $(e, D(x)) \to (e, j_* \mathrm{GL}_1^{\mathcal{J}}(E))$ that induces a weak equivalence $(e^a, D(x)^a) \to (e, j_* \mathrm{GL}_1^{\mathcal{J}}(E))$. We will return to these pre-log structures in the sequel [RSS14].

4.27. **Log** THH **and logification.** Log THH is invariant under logification:

Theorem 4.28. *Let (A, M, α) be a cofibrant pre-log ring spectrum. Then the logification map $(A, M) \to (A^a, M^a)$ induces a stable equivalence* $\mathrm{THH}(A, M) \to \mathrm{THH}(A^a, M^a)$.

Example 4.29. The theorem implies that for the pre-log structure $(e, D(x))$ of Example 4.26, there is a stable equivalence $\mathrm{THH}(e, D(x)) \to \mathrm{THH}(e, j_*\mathrm{GL}_1^{\mathcal{J}}(E))$, where we suppress a cofibrant replacement of the respective pre-log ring spectra from the notation. This equivalence is useful for calculations, since the homotopy type of $D(x)$ only depends on the degree of the periodicity element of E and the homology of $D(x)_{h\mathcal{J}}$ is well understood.

The homotopy cartesian and cocartesian squares appearing in the next proof and the subsequent lemma refer to the positive $\mathcal{J}$-model structure on $\mathcal{CS}^{\mathcal{J}}$.

of Theorem 4.28. By Proposition 3.19, there is a chain of $\mathcal{J}$-equivalences relating $B^{\mathrm{rep}}(M)$ and $B^{\mathrm{cy}}(M)^{\mathrm{rep}}$. This chain of $\mathcal{J}$-equivalences can be chosen naturally with respect to the map $M \to M^a$ by forming the factorizations and lifts in Lemmas 3.21 and 3.23 in a model category of arrows. Using that the $\mathcal{J}$-equivalences in the chain are augmented over a cofibrant object, it follows from Corollary A.8 that the map $\mathrm{THH}(A, M) \to \mathrm{THH}(A^a, M^a)$ is stably equivalent to the map

$$\mathrm{THH}(A) \wedge_{\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)]} \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)^{\mathrm{rep}}] \to \mathrm{THH}(A^a) \wedge_{\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M^a)]} \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M^a)^{\mathrm{rep}}]\,.$$

Let P be a cofibrant replacement of $\alpha^{-1}\,\mathrm{GL}_1^{\mathcal{J}}(A)$ in the positive $\mathcal{J}$-model structure, let $\rho\colon B^{\mathrm{cy}}(P) \to B^{\mathrm{cy}}(P)^{\mathrm{rep}}$ be the repletion map, and let $\gamma\colon B^{\mathrm{cy}}(P) \to B^{\mathrm{cy}}(P^{\mathrm{gp}})$ be the map induced by the group completion. We consider the pushout square

$$\begin{tikzcd} B^{\mathrm{cy}}(P) \arrow[r, "\gamma"] \arrow[d, "\rho"'] & B^{\mathrm{cy}}(P^{\mathrm{gp}}) \arrow[d, "\tau"] \\ B^{\mathrm{cy}}(P)^{\mathrm{rep}} \arrow[r] & B^{\mathrm{cy}}(P)^{\mathrm{rep}} \boxtimes_{B^{\mathrm{cy}}(P)} B^{\mathrm{cy}}(P^{\mathrm{gp}}). \end{tikzcd}$$

Since ρ is an acyclic cofibration in the group completion model structure, so is τ. The domain of τ is grouplike. Since ρ is surjective on $\pi_0(-)_{h\mathcal{J}}$, the codomain of τ is also grouplike and it follows that τ is a $\mathcal{J}$-equivalence.

Next we notice that the evident map $f\colon \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(P)] \to \mathrm{THH}(A)$ factors as a composite $\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(P)] \to \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(P^{\mathrm{gp}})] \to \mathrm{THH}(A)$ because $P \to \Omega^{\mathcal{J}}(A)$ factors through $\mathrm{GL}_1^{\mathcal{J}}(A)$ and therefore extends over $P \to P^{\mathrm{gp}}$. It follows that the cobase change of $\mathbb{S}^{\mathcal{J}}[\rho]$ along f is a stable equivalence, because it can be computed as the cobase change of the acyclic cofibration $\mathbb{S}^{\mathcal{J}}[\tau]$ along $\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(P^{\mathrm{gp}})] \to \mathrm{THH}(A)$.

Let $P \rightarrowtail G \overset{\sim}{\twoheadrightarrow} \mathrm{GL}_1^{\mathcal{J}}(A)$ be a factorization in the positive $\mathcal{J}$-model structure. Using the gluing lemma in $\mathcal{CS}^{\mathcal{J}}$, the pushout of $M \leftarrow P \to G$ is $\mathcal{J}$-equivalent to the logification M^a from Construction 4.25. Applying the cyclic and repleted bar constructions and the functor $\mathbb{S}^{\mathcal{J}}$ provides the following commutative cube in $\mathcal{C}\mathrm{Sp}^{\Sigma}$:

$$\begin{tikzcd}
 & \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(G)] \arrow[rr] \arrow[dd] & & \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M^a)] \arrow[dd] \\
\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(P)] \arrow[ur] \arrow[rr] \arrow[dd, "\mathbb{S}^{\mathcal{J}}[\rho]"'] & & \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)] \arrow[ur] \arrow[dd] & \\
 & \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(G)^{\mathrm{rep}}] \arrow[rr] & & \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M^a)^{\mathrm{rep}}] . \\
\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(P)^{\mathrm{rep}}] \arrow[ur] \arrow[rr] & & \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)^{\mathrm{rep}}] \arrow[ur] &
\end{tikzcd}$$

The map $\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(G)] \to \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(G)^{\mathrm{rep}}]$ is a stable equivalence by Lemma 3.16. Combining this with the above statement about $\mathbb{S}^{\mathcal{J}}[\rho]$, we conclude that the left hand face becomes homotopy cocartesian after cobase change along the canonical map $\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(G)] \to \mathrm{THH}(A)$, in the sense that the cobase change of

$$\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(G)] \wedge_{\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(P)]} \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(P)^{\mathrm{rep}}] \to \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(G)^{\mathrm{rep}}]$$

along $\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(G)] \to \mathrm{THH}(A)$ is a stable equivalence. The top face is homotopy cocartesian because $\mathbb{S}^{\mathcal{J}}$ and $-\otimes S^1$ are left Quillen. The bottom face is seen to be homotopy cocartesian by combining Lemma 4.30 below with the fact that $\mathbb{S}^{\mathcal{J}}$ is left Quillen. By commuting homotopy pushouts, the statements about the top and bottom face in the above cube imply that the diagram

$$\begin{tikzcd}
\mathrm{THH}(A) \wedge_{\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(P)]} \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(P)^{\mathrm{rep}}] \arrow[r] \arrow[d] & \mathrm{THH}(A) \wedge_{\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)]} \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)^{\mathrm{rep}}] \arrow[d] \\
\mathrm{THH}(A) \wedge_{\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(G)]} \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(G)^{\mathrm{rep}}] \arrow[r] & \mathrm{THH}(A^a) \wedge_{\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M^a)]} \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M^a)^{\mathrm{rep}}]
\end{tikzcd}$$

induced by the logification construction is homotopy cocartesian. This gives the result since the vertical map on the left is a stable equivalence. □ □

The next lemma was used in the previous proof.

Lemma 4.30. *Consider the following commutative diagram of cofibrant objects in $\mathcal{CS}^{\mathcal{J}}$ in which the N_i^{rep} are the repletions of the horizontal composites $\epsilon_i\colon N_i \to M_i$:*

$$\begin{tikzcd}
 & N_2 \arrow[rr] \arrow[dd] & & N_2^{\mathrm{rep}} \arrow[rr] \arrow[dd] & & M_2 \arrow[dd] \\
N_1 \arrow[ur, tail] \arrow[rr] \arrow[dd] & & N_1^{\mathrm{rep}} \arrow[ur] \arrow[rr] \arrow[dd] & & M_1 \arrow[ur, tail] \arrow[dd] & \\
 & N_4 \arrow[rr] & & N_4^{\mathrm{rep}} \arrow[rr] & & M_4 . \\
N_3 \arrow[ur, tail] \arrow[rr] & & N_3^{\mathrm{rep}} \arrow[ur] \arrow[rr] & & M_3 \arrow[ur, tail] &
\end{tikzcd}$$

If the left and right hand faces are homotopy cocartesian squares and all ϵ_i are virtually surjective, then the middle square of repletions is homotopy cocartesian.

Proof. We use the characterization of the repletion from Corollary 3.22. Group completion preserves homotopy cocartesian squares because the homotopy pushout of grouplike objects in $\mathcal{CS}^{\mathcal{J}}$ is grouplike. Since the two-sided bar construction over $\boxtimes$ can be used to compute homotopy pushouts, this reduces the statement to showing that

$$\begin{array}{ccc} B^{\boxtimes}(N_3^{\mathrm{rep}}, N_1^{\mathrm{rep}}, N_2^{\mathrm{rep}}) & \longrightarrow & B^{\boxtimes}(N_3^{\mathrm{gp}}, N_1^{\mathrm{gp}}, N_2^{\mathrm{gp}}) \\ \downarrow & & \downarrow \\ B^{\boxtimes}(M_3, M_1, M_2) & \longrightarrow & B^{\boxtimes}(M_3^{\mathrm{gp}}, M_1^{\mathrm{gp}}, M_2^{\mathrm{gp}}) \end{array}$$

is homotopy cartesian. By [SS12, Corollary 11.4], a square of $\mathcal{J}$-spaces is homotopy cartesian if and only if it becomes a homotopy cartesian diagram of simplicial sets after applying $(-)_{h\mathcal{J}}$. Using that the monoidal structure map of $(-)_{h\mathcal{J}}$ is a $\mathcal{J}$-equivalence for cofibrant objects (see e.g. [Sag14, Lemma 2.11]), it remains to show that the following square of simplicial sets is homotopy cartesian:

$$\begin{array}{ccc} B((N_3^{\mathrm{rep}})_{h\mathcal{J}}, (N_1^{\mathrm{rep}})_{h\mathcal{J}}, (N_2^{\mathrm{rep}})_{h\mathcal{J}}) & \longrightarrow & B((N_3^{\mathrm{gp}})_{h\mathcal{J}}, (N_1^{\mathrm{gp}})_{h\mathcal{J}}, (N_2^{\mathrm{gp}})_{h\mathcal{J}}) \\ \downarrow & & \downarrow \\ B((M_3)_{h\mathcal{J}}, (M_1)_{h\mathcal{J}}, (M_2)_{h\mathcal{J}}) & \longrightarrow & B((M_3^{\mathrm{gp}})_{h\mathcal{J}}, (M_1^{\mathrm{gp}})_{h\mathcal{J}}, (M_2^{\mathrm{gp}})_{h\mathcal{J}}). \end{array}$$

Using the hypothesis of virtual surjectivity, this follows from the Bousfield–Friedlander Theorem [BF78, Theorem B.4] about the realization of pointwise homotopy cartesian squares of bisimplicial sets: The square of bisimplicial sets obtained by applying $B_\bullet$ in the situation above is pointwise homotopy cartesian,

$$B_\bullet((N_3^{\mathrm{gp}})_{h\mathcal{J}}, (N_1^{\mathrm{gp}})_{h\mathcal{J}}, (N_2^{\mathrm{gp}})_{h\mathcal{J}}) \quad \text{and} \quad B_\bullet((M_3^{\mathrm{gp}})_{h\mathcal{J}}, (M_1^{\mathrm{gp}})_{h\mathcal{J}}, (M_2^{\mathrm{gp}})_{h\mathcal{J}})$$

satisfy the π_*-Kan condition because all simplicial monoids involved are grouplike, and the assumed virtual surjectivity implies that the map between these two bisimplicial sets induces a Kan fibration on vertical path components. □ □

5. The case of discrete pre-log rings

A *discrete pre-log ring* (B, N, β) is a commutative ring B together with a commutative monoid N and a monoid homomorphism $\beta\colon N \to (B, \cdot)$. These objects are usually just called *pre-log rings*, but we use the additional term *discrete* to distinguish them from pre-log ring spectra. In this section we explain how discrete pre-log rings can be viewed as pre-log ring spectra, construct a homotopy cofiber sequence for the log THH of (B, N, β) if N is a free commutative monoid $\langle x \rangle$ on a single generator x, and compute the mod p homotopy of $\mathrm{THH}(\mathbb{Z}, \langle p \rangle)$. This is interesting in its own right and serves as an example of the general principle in Proposition 4.18 for setting up homotopy cofiber sequences.

Let (B, N, β) be a discrete pre-log ring. The standard model for the Eilenberg–Mac Lane spectrum associated with B is the commutative symmetric ring spectrum HB with $(HB)_n = \widetilde{B}[S^n]$, the reduced B-linearization of the n-sphere. Since the underlying multiplicative monoid of 0-simplices of $(HB)_0$ is the multiplicative monoid of B, the structure map β provides a morphism of discrete simplicial monoids $N \to (HB)_0 = (\Omega^{\mathcal{J}}(HB))(\mathbf{0}, \mathbf{0})$. In the following we write $F = F^{\mathcal{J}}_{(\mathbf{0},\mathbf{0})}$ for the strong symmetric monoidal functor $\mathcal{S} \to \mathcal{S}^{\mathcal{J}}$ that is left adjoint to the evaluation functor at $(\mathbf{0}, \mathbf{0})$. With this notation, we obtain a map $\beta\colon FN \to \Omega^{\mathcal{J}}(HB)$ in $\mathcal{CS}^{\mathcal{J}}$ that defines the pre-log ring spectrum (HB, FN) associated with (B, N). Here the monoid structure of FN is provided by the paring

$$FN \boxtimes FN = F^{\mathcal{J}}_{(\mathbf{0},\mathbf{0})}(N) \boxtimes F^{\mathcal{J}}_{(\mathbf{0},\mathbf{0})}(N) \cong F^{\mathcal{J}}_{(\mathbf{0},\mathbf{0})}(N \times N) \to F^{\mathcal{J}}_{(\mathbf{0},\mathbf{0})}(N) = FN,$$

and $\mathbb{S}^{\mathcal{J}}[FN] \cong \mathbb{S}[N]$, where $\mathbb{S}[-] = \Sigma^{\infty}(-)_{+}$ denotes the unreduced symmetric suspension spectrum of an ordinary simplicial set.

5.1. **The cyclic and the replete bar construction for discrete monoids.** If N is a commutative monoid, we can view it as a discrete simplicial commutative monoid and form the cyclic bar construction $B^{\mathrm{cy}}(N)$ in simplicial commutative monoids. Using the algebraic group completion $N \to N^{\mathrm{gp}}$ of N, we define the replete bar construction $B^{\mathrm{rep}}(N)$ as the pullback of $N \to N^{\mathrm{gp}} \leftarrow B^{\mathrm{cy}}(N^{\mathrm{gp}})$ in simplicial commutative monoids. To relate this to the replete bar construction in commutative $\mathcal{J}$-space monoids, we first note that the strong symmetric monoidal functor F induces a natural isomorphism $F(B^{\mathrm{cy}}(N)) \cong B^{\mathrm{cy}}(FN)$. Secondly, the map $FN \to F(N^{\mathrm{gp}})$ has grouplike codomain and induces a weak equivalence when applying $B((-)_{h\mathcal{J}})$ by [FM94, Proposition Q.1]. So after composing with a fibrant replacement $(-)^{\mathrm{fib}}$ in the positive $\mathcal{J}$-model structure, it provides a group completion $FN \to (F(N^{\mathrm{gp}}))^{\mathrm{fib}}$ of commutative $\mathcal{J}$-space monoids. We also note that even though the underlying $\mathcal{J}$-space is cofibrant, FN is usually not cofibrant in the positive $\mathcal{J}$-model structure on $\mathcal{CS}^{\mathcal{J}}$. Thus, for the purpose of defining log THH, we should first pass to a cofibrant replacement $(FN)^{\mathrm{cof}}$.

Lemma 5.2. *Let N be a commutative monoid, and let $(FN)^{\mathrm{cof}} \to FN$ be a cofibrant replacement in the positive $\mathcal{J}$-model structure.*

(i) The induced map $B^{\mathrm{cy}}((FN)^{\mathrm{cof}}) \to B^{\mathrm{cy}}(FN)$ is a $\mathcal{J}$-equivalence and gives rise to a stable equivalence

$$\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}((FN)^{\mathrm{cof}})] \xrightarrow{\simeq} \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(FN)] \cong \mathbb{S}[B^{\mathrm{cy}}(N)]$$

of commutative $\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}((FN)^{\mathrm{cof}})]$-algebras with circle action.

(ii) There is a chain of $\mathcal{J}$-equivalences of commutative $B^{\mathrm{cy}}((FN)^{\mathrm{cof}})$-algebras, connecting $B^{\mathrm{rep}}((FN)^{\mathrm{cof}})$ to $F(B^{\mathrm{rep}}(N))$, that induces a chain of stable equivalences

$$\mathbb{S}^{\mathcal{J}}[B^{\mathrm{rep}}((FN)^{\mathrm{cof}})] \simeq \mathbb{S}^{\mathcal{J}}[F(B^{\mathrm{rep}}(N))] \cong \mathbb{S}[B^{\mathrm{rep}}(N)]$$

of commutative $\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}((FN)^{\mathrm{cof}})]$-algebras with circle action.

Proof. The commutative $\mathcal{J}$-space monoid FN is flat by the definition of the generating cofibrations for the flat model structure on $\mathcal{J}$-spaces [SS12, Corollary 5.10 and Proposition 6.16]. Hence it follows from Lemma 2.8 that the $\mathcal{J}$-equivalence $(FN)^{\mathrm{cof}} \to FN$ induces a $\mathcal{J}$-equivalence of cyclic bar constructions. Since $B^{\mathrm{cy}}(FN)$ is augmented over the monoidal unit $U^{\mathcal{J}}$, it follows from Corollary A.8 that $\mathbb{S}^{\mathcal{J}}$ takes the latter $\mathcal{J}$-equivalence to a stable equivalence. This implies (i).

For part (ii) we observe that there is a commutative diagram

$$\begin{tikzcd}
FN \arrow[d] & (FN)^{\mathrm{cof}} \arrow[l] \arrow[r, "="] \arrow[d] & (FN)^{\mathrm{cof}} \arrow[d] & (FN)^{\mathrm{cof}} \arrow[l, "="'] \arrow[d] \\
F(N^{\mathrm{gp}}) & (F(N^{\mathrm{gp}}))^{\mathrm{cof}} \arrow[l] \arrow[r] & (F(N^{\mathrm{gp}})^{\mathrm{cof}})^{\mathrm{fib}} & ((FN)^{\mathrm{cof}})^{\mathrm{gp}}. \arrow[l]
\end{tikzcd}$$

Here the left hand square and the middle square are induced by the cofibrant and fibrant replacements in the positive $\mathcal{J}$-model structure, respectively, and the lower horizontal map in the right hand square results from the lifting property in the group completion model structure by the remark before the lemma. Each vertical map in the diagram induces a weak equivalence when applying $B((-)_{h\mathcal{J}})$, and all terms in the lower row are grouplike. Since $(FN)^{\mathrm{cof}} \to FN$ is a $\mathcal{J}$-equivalence, it follows that all horizontal maps in the diagram are $\mathcal{J}$-equivalences.

Next we apply the cyclic bar construction to the terms in the lower row of the diagram and form the homotopy pullbacks of the vertical maps and the respective augmentations of the cyclic bar constructions to obtain a chain of $\mathcal{J}$-equivalences between the homotopy pullbacks. Since the map of discrete simplicial sets $N \to N^{\mathrm{gp}}$ is a Kan fibration, the pullback defining $B^{\mathrm{rep}}(N)$ is already a homotopy pullback. By [SS12, Corollary 11.4], the functor F maps it to a homotopy cartesian square in $\mathcal{CS}^{\mathcal{J}}$. So we have constructed the desired chain of $\mathcal{J}$-equivalences, and arguing as in part (i), we see that $\mathbb{S}^{\mathcal{J}}$ takes this to a chain of stable equivalences. □ □

Let (A, M, α) be a pre-log ring spectrum with $M = FN$ for a discrete commutative monoid N. Choosing a factorization of the adjoint structure map $\mathbb{S}[N] \to A$ into a cofibration $\bar{\alpha}^c\colon \mathbb{S}[N] \to A^c$ followed by an acyclic fibration $A^c \to A$ in commutative symmetric ring spectra, we can give the following simpler description of $\mathrm{THH}(A, M)$ that does not involve a cofibrant replacement of (A, M). This shows in particular that for a discrete pre-log ring (B, N), the definition of log THH via the pre-log ring spectrum (HB, FN) is equivalent to the definition given by the first author in [Rog09].

Proposition 5.3. *Let (A, M) be a pre-log ring spectrum with $M = FN$ for a discrete commutative monoid N, and let $(A^{\mathrm{cof}}, M^{\mathrm{cof}})$ be a cofibrant replacement. Then there is a chain of stable equivalences*

$$\mathrm{THH}(A^{\mathrm{cof}}, M^{\mathrm{cof}}) \simeq B^{\mathrm{cy}}(A^c) \wedge_{\mathbb{S}[B^{\mathrm{cy}}(N)]} \mathbb{S}[B^{\mathrm{rep}}(N)]$$

of $\mathrm{THH}(A^{\mathrm{cof}})$*-modules with circle action.*

Proof. We first use the lifting axiom for the positive stable model structure on $\mathcal{C}\mathrm{Sp}^{\Sigma}$ to get a stable equivalence $A^{\mathrm{cof}} \to A^c$ of commutative $\mathbb{S}^{\mathcal{J}}[M^{\mathrm{cof}}]$-algebras, which in turn induces a stable equivalence $B^{\mathrm{cy}}(A^{\mathrm{cof}}) \to B^{\mathrm{cy}}(A^c)$ of commutative $\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M^{\mathrm{cof}})]$-algebras. It now follows from left properness of the positive stable model structure that the stable equivalences in Lemma 5.2 give rise to a chain of stable equivalences as stated in the proposition. □ □

5.4. **Free monoids and log THH localization sequences.** In order to prepare for subsequent computations of log THH, we describe the homotopy type and the homology of $B^{\mathrm{cy}}(N)$ and $B^{\mathrm{rep}}(N)$ (or rather of their geometric realizations) for a free commutative monoid N on one generator. We write $H_*X = H_*(X; \mathbb{F}_p)$ for (mod p) homology, and denote polynomial, height h truncated polynomial, exterior and divided power algebras by $P(x)$, $P_h(x)$, $E(x)$ and $\Gamma(x) = \mathbb{F}_p\{\gamma_k x \,|\, k \geq 0\}$, respectively.

Let $N = \langle x \rangle = \{x^k \mid k \geq 0\}$ be the free commutative monoid on the generator x. Its cyclic bar construction decomposes as a disjoint union

$$B^{\mathrm{cy}}(N) = \coprod_{k \geq 0} B^{\mathrm{cy}}(N; k) \simeq * \sqcup \coprod_{k \geq 1} S^1(k)\,, \tag{5.1}$$

where $B^{\mathrm{cy}}(N; k)$ denotes the geometric realization of the cyclic subset consisting of the simplices $(x^{i_0}, \dots, x^{i_q})$ with $i_0 + \cdots + i_q = k$. Here $B^{\mathrm{cy}}(N; 0)$ is a point, while $B^{\mathrm{cy}}(N; k)$ for $k \geq 1$ is equivariantly homotopy equivalent to S^1 with the degree k circle action, which we write as $S^1(k)$. Hence

$$H_* B^{\mathrm{cy}}(N) \cong P(x) \otimes E(dx)\,, \tag{5.2}$$

with $x \in H_0 S^1(1)$ and $dx \in H_1 S^1(1)$ represented by the cycles (x) and $(1, x)$, respectively. See [Rog09, Proposition 3.20] or [Hes96, Lemma 2.2.3] for further details.

The group completion $N^{\mathrm{gp}} = \langle x^{\pm 1} \rangle = \{x^k \mid k \in \mathbb{Z}\}$ has cyclic bar construction $B^{\mathrm{cy}}(N^{\mathrm{gp}}) = \coprod_{k \in \mathbb{Z}} B^{\mathrm{cy}}(N^{\mathrm{gp}}; k)$, containing the replete bar construction

$$B^{\mathrm{rep}}(N) = \coprod_{k \geq 0} B^{\mathrm{cy}}(N^{\mathrm{gp}}; k) \simeq \coprod_{k \geq 0} S^1(k) \tag{5.3}$$

as the non-negatively indexed summands. The repletion map restricts to equivalences $B^{\mathrm{cy}}(N;k) \simeq B^{\mathrm{cy}}(N^{\mathrm{gp}};k)$ for $k \geq 1$, while $B^{\mathrm{cy}}(N^{\mathrm{gp}};0) \cong B(N^{\mathrm{gp}})$ is related to $S^1(0)$ (that is, S^1 equipped with the trivial circle action) by a chain of weak equivalences that are circle equivariant. Hence

$$H_* B^{\mathrm{rep}}(N) \cong P(x) \otimes E(d\log x)\,,$$

with $x \in H_0 S^1(1)$ and $d\log x \in H_1 S^1(0)$ represented by the cycles (x) and (x^{-1}, x), respectively. In homology, $\rho_*(x) = x$ and $\rho_*(dx) = x \cdot d\log x$. All this can be deduced from [Rog09, Proposition 3.21]. The next lemma is an immediate consequence of the equivalences in (5.1) and (5.3).

Lemma 5.5. *For $N = \langle x \rangle$ there is a homotopy cofiber sequence*

$$\mathbb{S}[B^{\mathrm{cy}}(N)] \xrightarrow{\rho} \mathbb{S}[B^{\mathrm{rep}}(N)] \xrightarrow{\partial} \Sigma\mathbb{S}$$

of $\mathbb{S}[B^{\mathrm{cy}}(N)]$-modules with circle action, in which the action on $\Sigma\mathbb{S}$ is trivial. □

Notice that the commutative $\mathcal{J}$-space monoid $F\langle x \rangle$, obtained by applying the strong symmetric monoidal functor F to $\langle x \rangle$, can be identified with the free commutative $\mathcal{J}$-space monoid $\mathbb{C}\langle x \rangle$ on a generator in bidegree $(\mathbf{0}, \mathbf{0})$, cf. Example 2.5. Given a commutative symmetric ring spectrum A, every 0-simplex $x \in A_0$ thus gives rise to a pre-log ring spectrum $(A, \mathbb{C}\langle x \rangle)$. This will usually not be cofibrant, so in order to define log THH we should first pass to a cofibrant replacement $(A^{\mathrm{cof}}, \mathbb{C}\langle x \rangle^{\mathrm{cof}})$. In order to form the balanced smash product $A^{\mathrm{cof}} \wedge_{\mathbb{S}^{\mathcal{J}}[\mathbb{C}\langle x \rangle^{\mathrm{cof}}]} \mathbb{S}$, we view $\mathbb{S}$ as a commutative $\mathbb{S}^{\mathcal{J}}[\mathbb{C}\langle x \rangle^{\mathrm{cof}}]$-algebra via the composition $\mathbb{C}\langle x \rangle^{\mathrm{cof}} \to \mathbb{C}\langle x \rangle \to U^{\mathcal{J}}$, using the isomorphism $\mathbb{S}^{\mathcal{J}}[U^{\mathcal{J}}] \cong \mathbb{S}$. The next lemma shows that we may view $A^{\mathrm{cof}} \wedge_{\mathbb{S}^{\mathcal{J}}[\mathbb{C}\langle x \rangle^{\mathrm{cof}}]} \mathbb{S}$ as a homotopy invariant model for the quotient of A by the ideal generated by x.

Lemma 5.6. *For a cofibrant replacement $(A^{\mathrm{cof}}, \mathbb{C}\langle x \rangle^{\mathrm{cof}}) \to (A, \mathbb{C}\langle x \rangle)$ there is a homotopy cofiber sequence $A \xrightarrow{x} A \to A^{\mathrm{cof}} \wedge_{\mathbb{S}^{\mathcal{J}}[\mathbb{C}\langle x \rangle^{\mathrm{cof}}]} \mathbb{S}$ of A^{cof}-modules.*

Proof. The conditions for $(A^{\mathrm{cof}}, \mathbb{C}\langle x \rangle^{\mathrm{cof}})$ to be a cofibrant replacement imply that the induced map $A^{\mathrm{cof}} \wedge_{\mathbb{S}^{\mathcal{J}}[\mathbb{C}\langle x \rangle^{\mathrm{cof}}]} \mathbb{S}[\langle x \rangle] \to A$ is a stable equivalence. Using this observation and Lemma 4.8, the homotopy cofiber sequence in the lemma is obtained by applying the functor $A^{\mathrm{cof}} \wedge_{\mathbb{S}^{\mathcal{J}}[\mathbb{C}\langle x \rangle^{\mathrm{cof}}]} (-)$ to the obvious homotopy cofiber sequence $\mathbb{S}[\langle x \rangle] \xrightarrow{x} \mathbb{S}[\langle x \rangle] \to \mathbb{S}$ of $\mathbb{S}[\langle x \rangle]$-modules. □ □

Specializing the general principle in Proposition 4.18 to the case of the commutative $\mathcal{J}$-space monoid $\mathbb{C}\langle x \rangle$, we get the localization sequence in the next theorem.

Theorem 5.7. *Let A be a commutative symmetric ring spectrum, and let $(A, \mathbb{C}\langle x \rangle)$ be the pre-log ring spectrum determined by a 0-simplex $x \in A_0$. Then there is a homotopy cofiber sequence*

$$\mathrm{THH}(A^{\mathrm{cof}}) \xrightarrow{\rho} \mathrm{THH}(A^{\mathrm{cof}}, \mathbb{C}\langle x \rangle^{\mathrm{cof}}) \xrightarrow{\partial} \Sigma\,\mathrm{THH}(A^{\mathrm{cof}} \wedge_{\mathbb{S}^{\mathcal{J}}[\mathbb{C}\langle x \rangle^{\mathrm{cof}}]} \mathbb{S})$$

of $\mathrm{THH}(A^{\mathrm{cof}})$-modules with circle action.

Proof. With the notation from Proposition 4.18, let $P = U^{\mathcal{J}}$, and consider the map $\mathbb{C}\langle x \rangle^{\mathrm{cof}} \to \mathbb{C}\langle x \rangle \to U^{\mathcal{J}}$. Using the stable equivalences in Lemma 5.2, the homotopy cofiber sequence from Lemma 5.5 translates into a homotopy cofiber sequence

$$\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(\mathbb{C}\langle x \rangle^{\mathrm{cof}})] \xrightarrow{\rho} \mathbb{S}^{\mathcal{J}}[B^{\mathrm{rep}}(\mathbb{C}\langle x \rangle^{\mathrm{cof}})] \xrightarrow{\partial} \Sigma\mathbb{S}$$

of $\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(\mathbb{C}\langle x \rangle^{\mathrm{cof}})]$-modules with circle action. Because of the canonical isomorphism $\mathbb{S} \cong \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(U^{\mathcal{J}})]$, this provides the necessary input for Proposition 4.18. □ □

Remark 5.8. Using the description in Proposition 5.3, one can argue as in the above theorem to get a homotopy cofiber sequence

$$B^{\mathrm{cy}}(A^c) \xrightarrow{\rho} B^{\mathrm{cy}}(A^c) \wedge_{\mathbb{S}[B^{\mathrm{cy}}(\langle x\rangle)]} \mathbb{S}[B^{\mathrm{rep}}(\langle x\rangle)] \xrightarrow{\partial} \Sigma B^{\mathrm{cy}}(A^c \wedge_{\mathbb{S}[\langle x\rangle]} \mathbb{S}),$$

which is stably equivalent to the one in the theorem.

If (B, N) is a discrete pre-log ring, we shall simplify the notation by writing $\mathrm{THH}(B, N)$ for $\mathrm{THH}((HB)^{\mathrm{cof}}, (FN)^{\mathrm{cof}})$ and $\mathrm{THH}(B)$ for $\mathrm{THH}((HB)^{\mathrm{cof}})$. We remark that since the underlying symmetric spectrum of HB is flat [Sch12, Proposition I.7.14(ii) and Example I.7.33], we could equally well use $B^{\mathrm{cy}}(HB)$ as a model for $\mathrm{THH}(B)$.

Example 5.9. Let B be a commutative ring and $x \in B$ an element that does not divide zero. Then $(HB)^{\mathrm{cof}} \wedge_{\mathbb{S}^{\mathcal{J}}[\mathbb{C}\langle x\rangle^{\mathrm{cof}}]} \mathbb{S} \simeq H(B/(x))$, and there is a homotopy cofiber sequence

$$\mathrm{THH}(B) \xrightarrow{\rho} \mathrm{THH}(B, \langle x\rangle) \longrightarrow \Sigma\, \mathrm{THH}(B/(x))$$

of $\mathrm{THH}(B)$-modules with circle action. In particular, when B is a discrete valuation ring with uniformizer π, residue field $\ell = B/(\pi)$ and fraction field $L = B[\pi^{-1}]$, there is a homotopy cofiber sequence

$$\mathrm{THH}(B) \xrightarrow{\rho} \mathrm{THH}(B, \langle \pi\rangle) \longrightarrow \Sigma\, \mathrm{THH}(\ell)$$

of $\mathrm{THH}(B)$-modules with circle action. We expect that it agrees with the homotopy cofiber sequence

$$\mathrm{THH}(B) \xrightarrow{j^*} \mathrm{THH}(B|L) \xrightarrow{\partial} \Sigma\, \mathrm{THH}(\ell)$$

of Hesselholt and Madsen [HM03, §1.5], which is only defined in this more restricted setting.

Remark 5.10. A commutative symmetric ring spectrum A that does not have the homotopy type of an Eilenberg–Mac Lane spectrum will often not admit an interesting pre-log structure with $M = \mathbb{C}\langle x\rangle$. One reason is that for this to exist, the higher multiplicative Dyer–Lashof operations on the image of x in $H_0(\Omega^\infty A; \mathbb{F}_p)$ must vanish. For example, this is not the case for the element $[p]$ in $H_0(\Omega^\infty ku; \mathbb{F}_p)$; see [Rog09, Lemma 9.6] and the discussion in [Rog09, Remark 9.17].

Example 5.11. In the case of the canonical pre-log ring spectrum structure on $A = \mathbb{S}^{\mathcal{J}}[\mathbb{C}\langle x\rangle] \cong \mathbb{S}[\langle x\rangle]$, with $M = \mathbb{C}\langle x\rangle$, we have stable equivalences

$$\mathrm{THH}(\mathbb{S}[\langle x\rangle]^{\mathrm{cof}}) \simeq \mathbb{S}[B^{\mathrm{cy}}\langle x\rangle] \simeq \mathbb{S}[* \sqcup \coprod_{k\geq 1} S^1(k)]$$

and

$$\mathrm{THH}(\mathbb{S}[\langle x\rangle]^{\mathrm{cof}}, \mathbb{C}\langle x\rangle^{\mathrm{cof}}) \simeq \mathbb{S}[B^{\mathrm{rep}}\langle x\rangle] \simeq \mathbb{S}[\coprod_{k\geq 0} S^1(k)] \simeq \mathbb{S}[\Lambda_{\geq 0} S^1]\,,$$

of commutative symmetric ring spectra with circle action, by Proposition 5.3 and equations (5.1) and (5.3). Here $\Lambda_{\geq 0} S^1$ is the subspace of the free loop space ΛS^1 that consists of maps $S^1 \to S^1$ of non-negative degree. By the discussion before Lemma 5.2, the commutative $\mathcal{J}$-space monoid $F\langle x^{\pm 1}\rangle$ is $\mathcal{J}$-equivalent to $\mathbb{C}\langle x\rangle^{\mathrm{gp}}$. Hence $\mathbb{S}^{\mathcal{J}}[\mathbb{C}\langle x\rangle^{\mathrm{gp}}]$ is stably equivalent to $\mathbb{S}[\langle x^{\pm 1}\rangle]$, and the natural map to the localization $A[M^{-1}] = \mathbb{S}^{\mathcal{J}}[\mathbb{C}\langle x\rangle^{\mathrm{gp}}] \simeq \mathbb{S}[\langle x^{\pm 1}\rangle]$ induces the evident inclusions to

$$\mathrm{THH}(\mathbb{S}[\langle x^{\pm 1}\rangle]^{\mathrm{cof}}) \simeq \mathbb{S}[B^{\mathrm{cy}}\langle x^{\pm 1}\rangle] \simeq \mathbb{S}[\coprod_{k\in\mathbb{Z}} S^1(k)] \simeq \mathbb{S}[\Lambda S^1]\,.$$

5.12. **Logarithmic** THH **of** $(\mathbb{Z}, \langle p \rangle)$**.** Let us write $\bar{\pi}_* X = \pi_*(X; \mathbb{Z}/p)$ for mod p homotopy, where p is a prime, and let $\doteq$ denote equality up to a unit in $\mathbb{F}_p$. In this section all balanced smash products of symmetric spectra should be understood in the homotopy invariant left derived sense. We begin by determining the structure of a Künneth spectral sequence obtained by identifying $\mathrm{THH}(\mathbb{Z}/p)$ with the left derived balanced smash product $\mathrm{THH}(\mathbb{Z}) \wedge_{\mathbb{S}[B^{\mathrm{cy}}\langle p \rangle]} \mathbb{S}$, where we use $B^{\mathrm{cy}}(H\mathbb{Z})$ as a model for $\mathrm{THH}(\mathbb{Z})$, cf. the remarks preceding Example 5.9.

Proposition 5.13. *Consider the case $B = \mathbb{Z}$ and $N = \langle p \rangle$. There is an algebra spectral sequence*

$$\begin{aligned} E^2_{**} &= \mathrm{Tor}_{**}^{H_* B^{\mathrm{cy}}\langle p \rangle}(\bar{\pi}_* \mathrm{THH}(\mathbb{Z}), \mathbb{F}_p) \\ &\Longrightarrow \bar{\pi}_* \mathrm{THH}(\mathbb{Z}/p)\,, \end{aligned}$$

where $H_* B^{\mathrm{cy}}\langle p \rangle = P(p) \otimes E(dp)$ by (5.2), $\bar{\pi}_* \mathrm{THH}(\mathbb{Z}) = E(\lambda_1) \otimes P(\mu_1)$ and $\bar{\pi}_* \mathrm{THH}(\mathbb{Z}/p) = E(\epsilon_0) \otimes P(\mu_0)$, with $|p| = 0$, $|dp| = 1$, $|\lambda_1| = 2p-1$, $|\mu_1| = 2p$, $|\epsilon_0| = 1$ and $|\mu_0| = 2$. Here

$$E^2_{**} = E(\lambda_1) \otimes P(\mu_1) \otimes E([p]) \otimes \Gamma([dp])\,,$$

where $[p]$ has bidegree $(1,0)$ and $[dp]$ has bidegree $(1,1)$. There are non-trivial d^p-differentials

$$d^p(\gamma_k[dp]) \doteq \lambda_1 \cdot \gamma_{k-p}[dp]$$

for all $k \geq p$, leaving

$$E^\infty_{**} = P(\mu_1) \otimes E([p]) \otimes P_p([dp])\,.$$

Hence $[p]$ represents ϵ_0, $[dp]$ represents μ_0, and μ_1 represents μ_0^p (up to units in $\mathbb{F}_p$) in the abutment, and there is a multiplicative extension $[dp]^p \doteq \mu_1$.

Proof. Applying cobase change along $\mathbb{S} \to H\mathbb{Z} \to H = H\mathbb{F}_p$ to the left derived balanced smash product mentioned above, we get a homotopy cocartesian square

$$\begin{array}{ccc} H \wedge B^{\mathrm{cy}}\langle p \rangle_+ & \longrightarrow & H \\ \downarrow & & \downarrow \\ H \wedge_{H\mathbb{Z}} \mathrm{THH}(\mathbb{Z}) & \longrightarrow & H \wedge_{H\mathbb{Z}} \mathrm{THH}(\mathbb{Z}/p) \end{array} \tag{5.4}$$

of commutative H-algebras. The spectral sequence in question is the associated Tor spectral sequence [EKMM97, Theorem IV.4.1], in view of the identification $\pi_*(H \wedge_{H\mathbb{Z}} X) \cong \bar{\pi}_* X$ for $H\mathbb{Z}$-modules.

Bökstedt computed that $\pi_* \mathrm{THH}(\mathbb{Z}/p) = P(\mu_0)$ with $|\mu_0| = 2$, so $\bar{\pi}_* \mathrm{THH}(\mathbb{Z}/p) = E(\epsilon_0) \otimes P(\mu_0)$ with $|\epsilon_0| = 1$ (the mod p Bockstein element). Bökstedt also computed that $\bar{\pi}_* \mathrm{THH}(\mathbb{Z}) = E(\lambda_1) \otimes P(\mu_1)$. For published proofs, see e.g. the cases $m = 0$ and $m = 1$ of [AR05, Theorem 5.12], or [AR12, §3, §4]. This leads to the stated E^2-term and abutment.

The algebra generators λ_1, μ_1, $[p]$ and $[dp]$ must be infinite cycles for filtration reasons. To determine the differentials on the remaining algebra generators, i.e., the divided powers $\gamma_{p^i}[dp]$ for $i \geq 1$, we note that the abutment $E(\epsilon_0) \otimes P(\mu_0)$ has exactly one generator in each non-negative degree. In total degree $2p-1$ the E^2-term is generated by λ_1 and $[p] \cdot \gamma_{p-1}[dp]$. Hence one of these must be hit by a differential, and for filtration reasons the only possibility is $d^p(\gamma_p[dp]) \doteq \lambda_1$.

Taking this into account, it follows that $[p]$, $[dp]$ and μ_1 survive to E^∞, where they must represent ϵ_0, μ_0 and μ_0^p, respectively, up to units in $\mathbb{F}_p$. Furthermore, in degree $2p^2 - 1$ the only remaining generators are

$$\mu_1^{p-1} \cdot [p] \cdot \gamma_{p-1}[dp] \qquad \text{and} \qquad \lambda_1 \cdot \gamma_{p^2-p}[dp]\,.$$

The first of these must represent a unit in $\mathbb{F}_p$ times $(\mu_0^p)^{p-1} \cdot \epsilon_0 \cdot \mu_0^{p-1}$ in the abutment. Since this product is nonzero, the first generator cannot be a boundary, and must survive to E^∞. Hence the second generator must be hit by a differential, which must come from $\gamma_{p^2}[dp]$. This explains the second differential, $d^p(\gamma_{p^2}[dp]) \doteq \lambda_1 \cdot \gamma_{p^2-p}[dp]$. The cases $i \geq 3$ are very similar. □ □

Having determined the differentials and multiplicative extensions above, we can now analyze the Künneth spectral sequence computing $\bar{\pi}_* \operatorname{THH}(\mathbb{Z}, \langle p \rangle)$.

Proposition 5.14. *Consider the case $B = \mathbb{Z}$ and $N = \langle p \rangle$. There is an algebra spectral sequence*

$$\begin{aligned} E^2_{**} &= \operatorname{Tor}^{H_* B^{\mathrm{cy}}\langle p \rangle}_{**}(\bar{\pi}_* \operatorname{THH}(\mathbb{Z}), H_* B^{\mathrm{rep}}\langle p \rangle) \\ &\Longrightarrow \bar{\pi}_* \operatorname{THH}(\mathbb{Z}, \langle p \rangle)\,, \end{aligned}$$

where $H_* B^{\mathrm{cy}}\langle p \rangle = P(p) \otimes E(dp)$ by (5.2), $\bar{\pi}_* \operatorname{THH}(\mathbb{Z}) = E(\lambda_1) \otimes P(\mu_1)$ and $H_* B^{\mathrm{rep}}\langle p \rangle = P(p) \otimes E(d \log p)$, with $|d \log p| = 1$ and the remaining degrees as above. Here

$$E^2_{**} = E(\lambda_1) \otimes P(\mu_1) \otimes E(d \log p) \otimes \Gamma([dp])$$

where $[dp]$ has bidegree $(1, 1)$. There are non-trivial differentials

$$d^p(\gamma_k[dp]) \doteq \lambda_1 \cdot \gamma_{k-p}[dp]$$

for all $k \geq p$, leaving

$$E^\infty_{**} = P(\mu_1) \otimes E(d \log p) \otimes P_p([dp])\,.$$

There is a multiplicative extension $[dp]^p \doteq \mu_1$, so the abutment is

$$\bar{\pi}_* \operatorname{THH}(\mathbb{Z}, \langle p \rangle) = E(d \log p) \otimes P(\kappa_0)$$

where κ_0 is represented by $[dp]$, with $|\kappa_0| = 2$.

Proof. Applying cobase change along $\mathbb{S} \to H\mathbb{Z} \to H = H\mathbb{F}_p$ to the balanced smash product defining $\operatorname{THH}(\mathbb{Z}, \langle p \rangle)$, we get a homotopy cocartesian square

$$\begin{array}{ccc} H \wedge B^{\mathrm{cy}}\langle p \rangle_+ & \xrightarrow{\rho} & H \wedge B^{\mathrm{rep}}\langle p \rangle_+ \\ \downarrow & & \downarrow \\ H \wedge_{H\mathbb{Z}} \operatorname{THH}(\mathbb{Z}) & \xrightarrow{\rho} & H \wedge_{H\mathbb{Z}} \operatorname{THH}(\mathbb{Z}, \langle p \rangle) \end{array} \tag{5.5}$$

of commutative H-algebras, and an associated Tor spectral sequence, as asserted. Here

$$\begin{aligned} E^2_{**} &= \operatorname{Tor}^{P(p)\otimes E(dp)}_{**}(E(\lambda_1) \otimes P(\mu_1), P(p) \otimes E(d \log p)) \\ &\cong \operatorname{Tor}^{E(dp)}_{**}(E(\lambda_1) \otimes P(\mu_1), E(d \log p)) \\ &\cong E(\lambda_1) \otimes P(\mu_1) \otimes E(d \log p) \otimes \Gamma([dp]) \end{aligned}$$

by change-of-rings and the observation that $E(dp)$ acts trivially on $E(\lambda_1) \otimes P(\mu_1)$ and $E(d \log p)$.

To determine the differentials and multiplicative extensions in this spectral sequence, we use naturality of the Künneth spectral sequence with respect to the map of homotopy cocartesian squares from (5.5) to (5.4), induced by the augmentation

$$\mathbb{S}[B^{\mathrm{rep}}\langle p \rangle] \longrightarrow \mathbb{S}$$

viewed as a map of $\mathbb{S}[B^{\mathrm{cy}}\langle p \rangle]$-algebras. The induced homomorphism of E^2-terms

$$E(\lambda_1) \otimes P(\mu_1) \otimes E(d \log p) \otimes \Gamma([dp]) \longrightarrow E(\lambda_1) \otimes P(\mu_1) \otimes E([p]) \otimes \Gamma([dp])$$

takes $d \log p$ to 0 and preserves the other generators. The class $d \log p$ is an infinite cycle, for filtration reasons, so the differentials $d^p(\gamma_k[dp]) \doteq \lambda_1 \cdot \gamma_{k-p}[dp]$ for $k \geq p$ on the right hand side lift to the left hand side. This leaves the E^{p+1}-term

$$E^{p+1}_{**} = P(\mu_1) \otimes E(d \log p) \otimes P_p([dp])\,,$$

which must equal the E^∞-term for filtration reasons. The multiplicative extension $[dp]^p \doteq \mu_1$ on the right hand side must also lift to the left hand side, completing the proof. □ □

Remark 5.15. The completion map $\mathbb{Z} \to \mathbb{Z}_p$ induces an isomorphism from this spectral sequence to the one discussed in [Rog09, Example 8.13], converging to $\bar{\pi}_* \operatorname{THH}(\mathbb{Z}_p, \langle p \rangle)$. The argument just given justifies the assertions about differentials and multiplicative extensions that were made without proof in the cited example.

Corollary 5.16. *There is an isomorphism*

$$\bar{\pi}_* \operatorname{THH}(\mathbb{Z}_p, \langle p \rangle) \cong \bar{\pi}_* \operatorname{THH}(\mathbb{Z}_p | \mathbb{Q}_p)$$

of $\bar{\pi}_* \operatorname{THH}(\mathbb{Z}_p)$-algebras, where the right hand side is as defined by Hesselholt-Madsen [HM03]. □

6. Localization sequences for log THH of periodic ring spectra

In this section we construct homotopy cofiber sequences relating $\operatorname{THH}(A, M)$ and $\operatorname{THH}(A)$ for certain pre-log ring spectra (A, M). We begin by describing the commutative $\mathcal{J}$-space monoids M that participate in these pre-log ring spectra.

6.1. **Repetitive commutative $\mathcal{J}$-space monoids.** The category $\mathcal{J}$ has a $\mathbb{Z}$-grading defined by letting the degree of an object $(\mathbf{n_1}, \mathbf{n_2})$ be the difference $n = n_2 - n_1 \in \mathbb{Z}$. Morphisms in $\mathcal{J}$ preserve the degree. We let $\mathcal{J}_n$ be the full subcategory of $\mathcal{J}$ whose objects have degree n. If X is a $\mathcal{J}$-space, then the restriction along the inclusion $\iota_n \colon \mathcal{J}_n \to \mathcal{J}$ defines a $\mathcal{J}_n$-space $\iota_n^* X$, which we call the *$\mathcal{J}$-degree n part* of X. The decomposition of $X_{h\mathcal{J}}$ into the components $(\iota_n^* X)_{h\mathcal{J}_n}$ defines a $\mathbb{Z}$-grading on $X_{h\mathcal{J}}$. We can view this grading as a map $X_{h\mathcal{J}} \to \mathbb{Z}$. If M is a $\mathcal{J}$-space monoid, then $M_{h\mathcal{J}}$ is a simplicial monoid because $(-)_{h\mathcal{J}}$ is monoidal, and $M_{h\mathcal{J}} \to \mathbb{Z}$ is a monoid homomorphism to the (discrete simplicial) monoid $(\mathbb{Z}, +)$.

Definition 6.2. For each $\mathcal{J}$-space X and each subset $S \subseteq \mathbb{Z}$ let $X_S \subseteq X$ be the sub $\mathcal{J}$-space with

$$X_S(\mathbf{n}_1, \mathbf{n}_2) = \begin{cases} X(\mathbf{n}_1, \mathbf{n}_2) & \text{if } n_2 - n_1 \in S, \\ \emptyset & \text{otherwise.} \end{cases}$$

In particular we have the part $X_{\{0\}}$ in $\mathcal{J}$-degree zero, the part $X_{\neq 0} = X_{\mathbb{Z}\setminus\{0\}}$ in non-zero $\mathcal{J}$-degrees, the part $X_{>0} = X_{\mathbb{N}}$ in positive $\mathcal{J}$-degrees, and the part $X_{\geq 0} = X_{\mathbb{N}_0}$ in non-negative $\mathcal{J}$-degrees. If X is the cyclic or replete bar construction on a commutative $\mathcal{J}$-space monoid M, we often denote X_S by $B^{\mathrm{cy}}_S(M)$ or $B^{\mathrm{rep}}_S(M)$, respectively.

Let T be the terminal $\mathcal{J}$-space, i.e., the constant functor $T \colon \mathcal{J} \to \mathcal{S}$ with value a chosen one-point space. We can obtain the inclusions $X_{\geq 0} \to X$ and $X_{\{0\}} \to X$ as the base changes along $X \to T$ of $T_{\geq 0} \to T$ and $T_{\{0\}} \to T$, respectively. The last two maps are maps of commutative $\mathcal{J}$-space monoids. This verifies:

Lemma 6.3. *If M is a (commutative) $\mathcal{J}$-space monoid, then so are $M_{\{0\}}$ and $M_{\geq 0}$, and the inclusions $M_{\{0\}} \to M_{\geq 0} \to M$ are monoid maps.* □

Lemma 6.4. *If M is a cofibrant commutative $\mathcal{J}$-space monoid such that $M_{<0} = \emptyset$, then $M_{\{0\}}$ is also cofibrant.*

Proof. We may assume that M is a cell complex obtained by attaching generating cofibrations of the positive $\mathcal{J}$-model structure. Since $\mathcal{J}$-spaces X and Y with $X_{<0} = \emptyset$ and $Y_{<0} = \emptyset$ satisfy $(X \boxtimes Y)_{\{0\}} \cong X_{\{0\}} \boxtimes Y_{\{0\}}$, it follows from the analysis of cell attachments in [SS12, Proposition 10.1] that $M_{\{0\}}$ is cofibrant. □ □

We are now ready to introduce the condition that will ensure the existence of the homotopy cofiber sequence for logarithmic THH that we are after.

Definition 6.5. A commutative $\mathcal{J}$-space monoid M is *repetitive* if it is not concentrated in $\mathcal{J}$-degree zero and if the group completion map $M \to M^{\mathrm{gp}}$ induces a $\mathcal{J}$-equivalence $M \to (M^{\mathrm{gp}})_{\geq 0}$.

It follows that a repetitive commutative $\mathcal{J}$-space monoid M satisfies $M_{<0} = \emptyset$ and $M_{>0} \neq \emptyset$. If M is a grouplike commutative $\mathcal{J}$-space monoid that is not concentrated in $\mathcal{J}$-degree zero, then we say that M is *d-periodic* if d is the minimal positive element in the image of the grading map $M_{h\mathcal{J}} \to \mathbb{Z}$. In this case, $M_{\{n\}} = \emptyset$ if and only if $d \nmid n$.

Definition 6.6. A repetitive commutative $\mathcal{J}$-space monoid M is *repetitive of period d* if its group completion M^{gp} is d-periodic.

Thus, if M is repetitive, then $M_{\{n\}}$ is non-empty if and only if $d \mid n$ and $n \geq 0$. The next lemma shows that every grouplike commutative $\mathcal{J}$-space monoid that is not concentrated in $\mathcal{J}$-degree zero gives rise to a repetitive commutative $\mathcal{J}$-space monoid.

Lemma 6.7. *If M is a grouplike commutative $\mathcal{J}$-space monoid of period $d > 0$, then $M_{\geq 0}$ is repetitive of period d.*

Proof. Since M is grouplike, it follows that the grading map $M_{h\mathcal{J}} \to \mathbb{Z}$ induces surjective monoid maps $(M_{\geq 0})_{h\mathcal{J}} \to d\mathbb{N}_0$ and $M_{h\mathcal{J}} \to d\mathbb{Z}$. Applying the simplicial bar construction, we get the following square of bisimplicial sets:

$$\begin{array}{ccc} B_\bullet((M_{\geq 0})_{h\mathcal{J}}) & \longrightarrow & B_\bullet(M_{h\mathcal{J}}) \\ \downarrow & & \downarrow \\ B_\bullet(d\mathbb{N}_0) & \longrightarrow & B_\bullet(d\mathbb{Z})\,. \end{array}$$

By the Bousfield–Friedlander Theorem [BF78, Theorem B.4], the square induces a homotopy cartesian square after realization. Hence $B((M_{\geq 0})_{h\mathcal{J}}) \to B(M_{h\mathcal{J}})$ is a weak equivalence because $B(d\mathbb{N}_0) \to B(d\mathbb{Z})$ is one, and consequently $M_{\geq 0} \to M$ is $\mathcal{J}$-equivalent to the group completion $M_{\geq 0} \to (M_{\geq 0})^{\mathrm{gp}}$. □ □

Let E be a commutative symmetric ring spectrum with $0 \neq 1$ in $\pi_0(E)$. We say that E is d-periodic if $\pi_*(E)$ has a unit of non-zero degree and d is the minimal positive degree of such a unit.

Corollary 6.8. *Let E be a positive fibrant commutative symmetric ring spectrum that is d-periodic for some $d > 0$, and let $j\colon e \to E$ be a model for the connective cover of E with e positive fibrant. Then the commutative $\mathcal{J}$-space monoid $j_*\mathrm{GL}_1^{\mathcal{J}}(E)$ is repetitive of period d.*

Proof. In this situation $\mathrm{GL}_1^{\mathcal{J}}(E)$ is grouplike of period d and the map $\Omega^{\mathcal{J}}(e \to E)_{\geq 0}$ is a $\mathcal{J}$-equivalence. Hence the map $j_*\mathrm{GL}_1^{\mathcal{J}}(E) \to \mathrm{GL}_1^{\mathcal{J}}(E)$ is $\mathcal{J}$-equivalent to the inclusion of $\mathrm{GL}_1^{\mathcal{J}}(E)_{\geq 0}$. □ □

Example 6.9. The commutative $\mathcal{J}$-space monoid $D(x)$ described in Example 4.26 is repetitive of period d.

6.10. **Homotopy cofiber sequences.** Let M be a commutative $\mathcal{J}$-space monoid that is concentrated in non-negative $\mathcal{J}$-degrees, i.e., $M = M_{\geq 0}$. We then have a collapse map of commutative symmetric ring spectra

$$\pi\colon \mathbb{S}^{\mathcal{J}}[M] \cong \bigvee_{n \geq 0} \mathbb{S}^{\mathcal{J}}[M_{\{n\}}] \to \mathbb{S}^{\mathcal{J}}[M_{\{0\}}]$$

which is the identity on $\mathbb{S}^{\mathcal{J}}[M_{\{0\}}]$ and takes each wedge summand $\mathbb{S}^{\mathcal{J}}[M_{\{n\}}]$ for $n > 0$ to the base point.

Definition 6.11. Let (A, M) be a pre-log ring spectrum with M concentrated in non-negative $\mathcal{J}$-degrees. The *quotient of A along the map* $\pi\colon \mathbb{S}^{\mathcal{J}}[M] \to \mathbb{S}^{\mathcal{J}}[M_{\{0\}}]$ is the commutative A-algebra spectrum

$$A/(M_{>0}) = A \wedge_{\mathbb{S}^{\mathcal{J}}[M]} \mathbb{S}^{\mathcal{J}}[M_{\{0\}}]$$

given by the pushout of $A \leftarrow \mathbb{S}^{\mathcal{J}}[M] \to \mathbb{S}^{\mathcal{J}}[M_{\{0\}}]$ in $\mathcal{C}\mathrm{Sp}^{\Sigma}$.

We observe that if (A, M) is a cofibrant pre-log ring spectrum, then the adjoint structure map $\bar{\alpha}\colon \mathbb{S}^{\mathcal{J}}[M] \to A$ is a cofibration, and the pushout square defining $A/(M_{>0})$ is homotopy cocartesian since $\mathcal{C}\mathrm{Sp}^{\Sigma}$ is left proper. Furthermore, $A/(M_{>0})$ is then cofibrant by Lemma 6.4.

The statement of the next theorem is as in Theorem 1.4 from the introduction, except that we make the cofibrancy condition on (A, M) explicit.

Theorem 6.12. *Let (A, M) be a cofibrant pre-log ring spectrum with M repetitive. Then there is a natural homotopy cofiber sequence*

$$\mathrm{THH}(A) \xrightarrow{\rho} \mathrm{THH}(A, M) \xrightarrow{\partial} \Sigma\,\mathrm{THH}(A/(M_{>0}))$$

of $\mathrm{THH}(A)$*-module spectra with circle action.*

We proceed to explain how to derive this theorem from the general principle in Proposition 4.18. Evaluating the cyclic bar construction on $\pi\colon \mathbb{S}^{\mathcal{J}}[M] \to \mathbb{S}^{\mathcal{J}}[M_{\{0\}}]$ we get a map of commutative symmetric ring spectra that can be identified with the projection $\pi\colon \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)] \to \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M_{\{0\}})]$, collapsing the positive part of $B^{\mathrm{cy}}(M)$ and mapping $\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}_{\{0\}}(M)]$ isomorphically onto $\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M_{\{0\}})]$. This applies in particular if M is repetitive, and we shall view $\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M_{\{0\}})]$ as an $\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)]$-module via π.

Proposition 6.13. *Let M be a repetitive and cofibrant commutative $\mathcal{J}$-space monoid. Then there is a homotopy cofiber sequence*

$$\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)] \xrightarrow{\rho} \mathbb{S}^{\mathcal{J}}[B^{\mathrm{rep}}(M)] \xrightarrow{\partial} \Sigma\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M_{\{0\}})]$$

of $\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)]$-modules with circle action.

The proof of this proposition requires a thorough analysis of the repletion map ρ and is postponed until Section 7.

of Theorem 6.12. With the notation from Proposition 4.18, we consider the cofibrant commutative $\mathcal{J}$-space monoid $P = M_{\{0\}}$ and the map of commutative symmetric ring spectra $\pi\colon \mathbb{S}^{\mathcal{J}}[M] \to \mathbb{S}^{\mathcal{J}}[M_{\{0\}}]$. The homotopy cofiber sequence in Proposition 6.13 now gives the necessary input for Proposition 4.18, and the result follows. □ □

6.14. **Identification of the homotopy cofiber.** Let M be a commutative $\mathcal{J}$-space monoid with multiplication $\mu\colon M \boxtimes M \to M$. We noticed in Example 2.5 that a 0-simplex $x \in M(\mathbf{d_1}, \mathbf{d_2})_0$ corresponds to a map $\bar{x}\colon F^{\mathcal{J}}_{(\mathbf{d_1},\mathbf{d_2})}(*) \to M$ from the free $\mathcal{J}$-space on a point in bidegree $(\mathbf{d_1}, \mathbf{d_2})$. In the following, we will refer to the composite map of $\mathcal{J}$-spaces

$$F^{\mathcal{J}}_{(\mathbf{d_1},\mathbf{d_2})}(*) \boxtimes M \xrightarrow{\bar{x}\boxtimes \mathrm{id}} M \boxtimes M \xrightarrow{\mu} M \tag{6.1}$$

as the *multiplication with x*.

Lemma 6.15. *If M is a grouplike commutative $\mathcal{J}$-space monoid, then the multiplication with any 0-simplex x is a $\mathcal{J}$-equivalence.*

Proof. The 0-simplex in $(F^{\mathcal{J}}_{(\mathbf{d_1},\mathbf{d_2})}(*))_{h\mathcal{J}}$ specified by $(\mathbf{d_1},\mathbf{d_2})$, the monoidal structure map of $(-)_{h\mathcal{J}}$ and the multiplication with x induce a sequence of maps

$$M_{h\mathcal{J}} \to \Big(F^{\mathcal{J}}_{(\mathbf{d_1},\mathbf{d_2})}(*)\Big)_{h\mathcal{J}} \times M_{h\mathcal{J}} \to \Big(F^{\mathcal{J}}_{(\mathbf{d_1},\mathbf{d_2})}(*) \boxtimes M\Big)_{h\mathcal{J}} \to M_{h\mathcal{J}}. \tag{6.2}$$

The space $(F^{\mathcal{J}}_{(\mathbf{d_1},\mathbf{d_2})}(*))_{h\mathcal{J}}$ is contractible because it is isomorphic to the classifying space of the comma category $((\mathbf{d_1},\mathbf{d_2})\downarrow\mathcal{J})$. So the first map in (6.2) is a weak equivalence. The second map is a weak equivalence by [Sag14, Lemma 2.11]. Hence the multiplication with x is a weak equivalence if and only if the composite map in (6.2) is one. The latter condition holds since M is grouplike. □ □

Corollary 6.16. *If M is repetitive of period d, then the multiplication with x induces a $\mathcal{J}$-equivalence*

$$F^{\mathcal{J}}_{(\mathbf{d_1},\mathbf{d_2})}(*) \boxtimes M \to M_{>0}$$

for any 0-simplex $x \in M(\mathbf{d_1},\mathbf{d_2})$ of $\mathcal{J}$-degree d.

Proof. This is an immediate consequence of Lemma 6.15. □ □

We now explain how to identify $A/(M_{>0})$ in examples. A first step is:

Lemma 6.17. *Let M be a cofibrant commutative $\mathcal{J}$-space monoid that is repetitive of period d, and let $x \in M(\mathbf{d_1},\mathbf{d_2})$ be a 0-simplex of $\mathcal{J}$-degree d. Then there is a homotopy cocartesian square of $\mathbb{S}^{\mathcal{J}}[M]$-modules*

$$\begin{array}{ccc} \mathbb{S}^{\mathcal{J}}[F^{\mathcal{J}}_{(\mathbf{d_1},\mathbf{d_2})}(*) \boxtimes M] & \longrightarrow & \mathbb{S}^{\mathcal{J}}[M] \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ * & \longrightarrow & \mathbb{S}^{\mathcal{J}}[M_{\{0\}}], \end{array}$$

where the top horizontal map is induced by the multiplication with x.

Proof. It suffices to show that the square is homotopy cocartesian as a diagram of symmetric spectra, and as such the top horizontal map factors as the composition

$$\mathbb{S}^{\mathcal{J}}[F^{\mathcal{J}}_{(\mathbf{d_1},\mathbf{d_2})}(*) \boxtimes M] \xrightarrow{\simeq} \mathbb{S}^{\mathcal{J}}[M_{>0}] \to \mathbb{S}^{\mathcal{J}}[M_{\{0\}}] \vee \mathbb{S}^{\mathcal{J}}[M_{>0}] \cong \mathbb{S}^{\mathcal{J}}[M]\,.$$

The result easily follows from the fact that the first map is a stable equivalence, by Corollaries 6.16 and A.8. □ □

Let (A, M) be a pre-log ring spectrum. The choice of a 0-simplex $x \in M(\mathbf{d_1},\mathbf{d_2})_0$ and the structure map $\alpha\colon M \to \Omega^{\mathcal{J}}(A)$ determine a 0-simplex in

$$\Omega^{\mathcal{J}}(A)(\mathbf{d_1},\mathbf{d_2}) = \Omega^{d_2}(A_{d_1}) \cong \mathrm{Map}_{\mathrm{Sp}^{\Sigma}}(F_{d_1}S^{d_2}, A),$$

i.e., a map $F_{d_1}S^{d_2} \to A$. Here $F_{d_1}S^{d_2} \simeq \Sigma^{d_2-d_1}(\mathbb{S})$ is the free symmetric spectrum on the space S^{d_2} in degree d_1. Extension of scalars provides an A-module map $\widetilde{x}\colon F_{d_1}S^{d_2} \wedge A \to A$ from the free A-module spectrum on S^{d_2} in degree d_1, which depends on x and α.

Lemma 6.18. *Let (A, M) be a cofibrant pre-log ring spectrum with M repetitive of period d. Then there is a homotopy cocartesian square of A-module spectra*

$$\begin{array}{ccc} F_{d_1}S^{d_2} \wedge A & \xrightarrow{\widetilde{x}} & A \\ \big\downarrow & & \big\downarrow \\ * & \longrightarrow & A/(M_{>0}), \end{array}$$

where the right hand vertical map is the canonical map to the quotient of A along the collapse map $\pi\colon \mathbb{S}^{\mathcal{J}}[M] \to \mathbb{S}^{\mathcal{J}}[M_{\{0\}}]$.

Proof. Notice first that the diagram in the lemma is isomorphic to that obtained by applying the functor $A \wedge_{\mathbb{S}^{\mathcal{J}}[M]} (-)$ to the diagram in Lemma 6.17. Writing $C(\tilde{x})$ for the mapping cone of $\tilde{x}$, it follows from Lemmas 6.17 and 4.8 that the canonical map

$$C(\tilde{x}) \cong A \wedge_{\mathbb{S}^{\mathcal{J}}[M]} C\left(\mathbb{S}^{\mathcal{J}}[F^{\mathcal{J}}_{(\mathbf{d_1},\mathbf{d_2})}(*) \boxtimes M] \to \mathbb{S}^{\mathcal{J}}[M]\right) \to A \wedge_{\mathbb{S}^{\mathcal{J}}[M]} \mathbb{S}^{\mathcal{J}}[M_{\{0\}}]$$

is a stable equivalence. This is equivalent to the statement in the lemma. □ □

Lemma 6.19. *Let (A, M) be a cofibrant pre-log ring spectrum with A connective and with M repetitive of period d. Suppose that there is a 0-simplex $x \in M(\mathbf{d_1}, \mathbf{d_2})_0$ of $\mathcal{J}$-degree d such that the homotopy class $[x] \in \pi_d(A)$ represented by the image of x in $\Omega^{\mathcal{J}}(A)$ has the property that multiplication with $[x]$ induces an isomorphism $\pi_i(A) \to \pi_{i+d}(A)$ for each $i \geq 0$.*

Then the map $\pi_i(A) \to \pi_i(A/(M_{>0}))$ is an isomorphism for $0 \leq i < d$, and $\pi_i(A/(M_{>0}))$ is trivial for all other i.

Proof. In the long exact sequence of homotopy groups associated with the homotopy cofiber sequence arising from Lemma 6.18, the map $\widetilde{x}$ induces multiplication with $[x]$. The claim follows by inspection of the long exact sequence. □ □

Now let E be a d-periodic and positive fibrant commutative symmetric ring spectrum with connective cover $e \to E$, where $d > 0$. In the following we shall consider the corresponding direct image log ring spectrum $(e, j_*\mathrm{GL}^{\mathcal{J}}_1(E))$ from Definition 4.24. Notice that if $(e^{\mathrm{cof}}, j_*\mathrm{GL}^{\mathcal{J}}_1(E)^{\mathrm{cof}})$ denotes a cofibrant replacement of the latter, then the composition $e^{\mathrm{cof}} \to E$ is again a connective cover and $j_*\mathrm{GL}^{\mathcal{J}}_1(E)^{\mathrm{cof}} \to j_*\mathrm{GL}^{\mathcal{J}}_1(E)$ is a $\mathcal{J}$-equivalence. It will be convenient to simplify notation by writing $(e, j_*\mathrm{GL}^{\mathcal{J}}_1(E))$ also for the cofibrant replacement when the meaning is clear from the context.

Corollary 6.20. *Let E be a d-periodic positive fibrant commutative symmetric ring spectrum with connective cover $j\colon e \to E$, and let $(e, j_*\mathrm{GL}^{\mathcal{J}}_1(E))$ denote a cofibrant replacement of the corresponding direct image log ring spectrum. Then the commutative symmetric ring spectrum $e/(j_*\mathrm{GL}^{\mathcal{J}}_1(E)_{>0})$ associated with $(e, j_*\mathrm{GL}^{\mathcal{J}}_1(E))$ is stably equivalent to the $(d-1)$-th Postnikov section $e[0, d\rangle$ of e.*

Proof. The commutative $\mathcal{J}$-space monoid $j_*\mathrm{GL}^{\mathcal{J}}_1(E)$ is repetitive by Corollary 6.8. By construction, the composite $j_*\mathrm{GL}^{\mathcal{J}}_1(E) \to \Omega^{\mathcal{J}}(e) \to \Omega^{\mathcal{J}}(E)$ maps any 0-simplex $x \in (j_*\mathrm{GL}^{\mathcal{J}}_1(E))(\mathbf{d_1}, \mathbf{d_2})_0$ of $\mathcal{J}$-degree d to a representative of a unit in $\pi_*(E)$. So the image of x in $\Omega^{\mathcal{J}}(e)$ represents a homotopy class satisfying the assumptions of the previous lemma. □ □

Combining this corollary with Theorem 6.12 gives the following result, which is similar to the statement in Theorem 1.5 from the introduction, except that we here make the fibrancy and cofibrancy conditions on E and $(e, j_*\mathrm{GL}^{\mathcal{J}}_1(E))$ explicit. We require E to be positive fibrant to ensure that $\Omega^{\mathcal{J}}(E)$ and $\mathrm{GL}^{\mathcal{J}}_1(E)$ capture the desired homotopy type, so that $j_*\mathrm{GL}^{\mathcal{J}}_1(E)$ is repetitive.

Theorem 6.21. *Let E be a d-periodic positive fibrant commutative symmetric ring spectrum with connective cover $j\colon e \to E$, and let $(e, j_*\mathrm{GL}^{\mathcal{J}}_1(E))$ denote a cofibrant replacement of the corresponding direct image log ring spectrum. Then there is a natural homotopy cofiber sequence*

$$\mathrm{THH}(e) \xrightarrow{\rho} \mathrm{THH}(e, j_*\mathrm{GL}^{\mathcal{J}}_1(E)) \xrightarrow{\partial} \Sigma\,\mathrm{THH}(e[0, d\rangle)$$

of $\mathrm{THH}(e)$-module spectra with circle action. □

Remark 6.22. Currently we do not know if it is possible to find repetitive pre-log structures on ring spectra A that do not arise as the connective covers of periodic ring spectra. Such pre-log structures would lead to new examples of homotopy cofiber sequences for log THH, where the term $A/(M_{>0})$ might be more difficult to describe.

7. The proof of Proposition 6.13

In this section we prove Proposition 6.13, thereby completing the last step in setting up the homotopy cofiber sequence in Theorem 6.12. As a beginning, we observe that if M is a commutative $\mathcal{J}$-space monoid, then the $\mathbb{Z}$-grading of $\mathcal{J}$ induces an augmentation $B^{\mathrm{cy}}(M)_{h\mathcal{J}} \to B^{\mathrm{cy}}(\mathbb{Z})$ of the cyclic bar construction. Here $B^{\mathrm{cy}}(\mathbb{Z})$ denotes the cyclic bar construction of the discrete monoid $\mathbb{Z}$ as considered in Section 5.1. Using that $B^{\mathrm{cy}}(M)_{h\mathcal{J}}$ is isomorphic to the realization of the bisimplicial set $[q] \mapsto B^{\mathrm{cy}}_q(M)_{h\mathcal{J}}$, the augmentation is defined as the realization of the map

$$B^{\mathrm{cy}}_q(M)_{h\mathcal{J}} \cong \coprod_{(x_0,\dots,x_q)\in B^{\mathrm{cy}}_q(\mathbb{Z})} (M_{\{x_0\}} \boxtimes \cdots \boxtimes M_{\{x_q\}})_{h\mathcal{J}} \to B^{\mathrm{cy}}_q(\mathbb{Z})$$

that collapses $(M_{\{x_0\}} \boxtimes \cdots \boxtimes M_{\{x_q\}})_{h\mathcal{J}}$ to the point $(x_0,\dots,x_q) \in B^{\mathrm{cy}}_q(\mathbb{Z})$. Notice that if we view $B^{\mathrm{cy}}(\mathbb{Z})$ as a constant commutative $\mathcal{J}$-space monoid, then there is an analogous augmentation $B^{\mathrm{cy}}(M) \to B^{\mathrm{cy}}(\mathbb{Z})$ before passing to the homotopy colimit. Using these augmentations, we can relate the repletion map for $B^{\mathrm{cy}}(M)$ to the repletion map for $B^{\mathrm{cy}}(\mathbb{Z})$ analyzed in Section 5.4.

Proposition 7.1. *For a repetitive and cofibrant commutative $\mathcal{J}$-space monoid M, the repletion map $\rho\colon B^{\mathrm{cy}}(M) \to B^{\mathrm{rep}}(M)$ restricts to a $\mathcal{J}$-equivalence*

$$\rho_{>0}\colon B^{\mathrm{cy}}_{>0}(M) \xrightarrow{\sim} B^{\mathrm{rep}}_{>0}(M)$$

in positive $\mathcal{J}$-degrees.

Proof. Because M is repetitive, it follows from the definition of $B^{\mathrm{rep}}(M)$ that it is enough to show that the group completion $M \to M^{\mathrm{gp}}$ induces a $\mathcal{J}$-equivalence $B^{\mathrm{cy}}_{>0}(M) \to B^{\mathrm{cy}}_{>0}(M^{\mathrm{gp}})$. Assuming that M is repetitive of period d, the augmentation introduced above induces a commutative square of bisimplicial sets

$$\begin{array}{ccc} B^{\mathrm{cy}}_\bullet(M)_{h\mathcal{J}} & \longrightarrow & B^{\mathrm{cy}}_\bullet(M^{\mathrm{gp}})_{h\mathcal{J}} \\ \downarrow & & \downarrow \\ B^{\mathrm{cy}}_\bullet(d\mathbb{N}_0) & \longrightarrow & B^{\mathrm{cy}}_\bullet(d\mathbb{Z}). \end{array} \tag{7.1}$$

Since M is cofibrant, it follows from Lemma 2.8 and [Sag14, Lemma 2.11] that there is a weak equivalence $B^{\mathrm{cy}}_q(M_{h\mathcal{J}}) \to B^{\mathrm{cy}}_q(M)_{h\mathcal{J}}$, and similarly for M^{gp}. This implies that the square is a homotopy cartesian square of simplicial sets in every simplicial degree. To see that it is homotopy cartesian after realization, we use the Bousfield–Friedlander theorem [BF78, Theorem B.4]: The bisimplicial sets $B^{\mathrm{cy}}_\bullet(M^{\mathrm{gp}})_{h\mathcal{J}}$ and $B^{\mathrm{cy}}_\bullet(d\mathbb{Z})$ satisfy the π_*-Kan condition because M^{gp} and $d\mathbb{Z}$ are grouplike. Moreover, the map $B^{\mathrm{cy}}_\bullet(M^{\mathrm{gp}})_{h\mathcal{J}} \to B^{\mathrm{cy}}_\bullet(d\mathbb{Z})$ is a Kan fibration on vertical path components because $B^{\mathrm{cy}}_\bullet(\pi_0(M^{\mathrm{gp}}_{h\mathcal{J}})) \to B^{\mathrm{cy}}_\bullet(d\mathbb{Z})$ is a degree-wise surjective homomorphism of simplicial abelian groups. This implies that [BF78, Theorem B.4] applies.

As we observed in Section 5.1, $B^{\mathrm{cy}}(d\mathbb{N}_0) \to B^{\mathrm{cy}}(d\mathbb{Z})$ restricts to an equivalence on the positive components in the $\mathbb{Z}$-grading. Hence $B^{\mathrm{cy}}_{>0}(M)_{h\mathcal{J}} \to B^{\mathrm{cy}}_{>0}(M^{\mathrm{gp}})_{h\mathcal{J}}$ is a weak equivalence because the realization of (7.1) is homotopy cartesian. □ □

Next we observe that the group completion map $M \to M^{\mathrm{gp}}$ and the collapse map π applied to the cyclic and replete bar constructions give rise to the following diagram of commutative symmetric ring spectra with circle action

$$\begin{tikzcd}
\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)] \arrow[r, "\pi"] \arrow[d, "\rho"] & \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}_{\{0\}}(M)] \arrow[r, equal] \arrow[d, "\rho_{\{0\}}"] & \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M_{\{0\}})] \arrow[r, "\simeq"] & \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M^{\mathrm{gp}}_{\{0\}})] \arrow[d, "\sigma"] \\
\mathbb{S}^{\mathcal{J}}[B^{\mathrm{rep}}(M)] \arrow[r, "\pi"] & \mathbb{S}^{\mathcal{J}}[B^{\mathrm{rep}}_{\{0\}}(M)] \arrow[rr, "\simeq"] & & \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}_{\{0\}}(M^{\mathrm{gp}})],
\end{tikzcd}$$

where σ is induced by the obvious inclusion $B^{\mathrm{cy}}(M^{\mathrm{gp}}_{\{0\}}) \to B^{\mathrm{cy}}_{\{0\}}(M^{\mathrm{gp}})$. Here we are implicitly using Corollary A.8 to ensure that the horizontal maps on the right hand side are stable equivalences as indicated. We shall view this as a commutative diagram of $\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)]$-modules with circle action.

Proposition 7.2. *The left hand square above is homotopy cocartesian in the category of $\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)]$-modules with circle action, hence induces a stable equivalence of the vertical mapping cones $C(\rho) \xrightarrow{\simeq} C(\sigma)$ as $\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)]$-modules with circle action.*

Proof. It suffices to show that the left hand square is homotopy cocartesian as a diagram of symmetric spectra. Clearly the map ρ decomposes as the wedge sum of the restrictions $\rho_{\{0\}}$ and $\rho_{>0}$. The result therefore follows from Proposition 7.1 and Corollary A.8 which together show that the latter map is a stable equivalence. □ □

It remains to analyze the mapping cone $C(\sigma)$. For this we momentarily return to the cyclic bar construction $B^{\mathrm{cy}}(d\mathbb{Z})$ and write $B^{\mathrm{cy}}_{\{0\}}(d\mathbb{Z})$ for the cyclic subobject with q-simplices all tuples $(x_0, x_1, \dots, x_q)$ of elements in $d\mathbb{Z}$, subject to the condition that $x_0 + x_1 + \cdots + x_q = 0$. Here the notation is supposed to emphasize the fact that the augmentation of $B^{\mathrm{cy}}(M^{\mathrm{gp}})$ discussed before Proposition 7.1 restricts to a map $B^{\mathrm{cy}}_{\{0\}}(M^{\mathrm{gp}}) \to B^{\mathrm{cy}}_{\{0\}}(d\mathbb{Z})$ if M is repetitive of period d.

Writing $\mathcal{B}(d\mathbb{Z})$ for the groupoid with one object $*$ and automorphism group $d\mathbb{Z}$, we may identify $B^{\mathrm{cy}}(d\mathbb{Z})$ with the cyclic nerve of $\mathcal{B}(d\mathbb{Z})$. Let $\mathbb{Z}/2 = \{0, 1\}$ be the cyclic group of order two, and let $\mathcal{E}(\mathbb{Z}/2)$ be the groupoid with objects 0 and 1, and a unique morphism from each object to each object. We use the notation $E\mathbb{Z}/2$ for the nerve of $\mathcal{E}(\mathbb{Z}/2)$. This is isomorphic to the cyclic nerve of $\mathcal{E}(\mathbb{Z}/2)$, since the zeroth component of a simplex in the cyclic nerve of a category is redundant if every object is initial.

Let $\phi\colon \mathcal{E}(\mathbb{Z}/2) \to \mathcal{B}(d\mathbb{Z})$ be the functor that takes the morphism $0 \to 1$ to $d\colon * \to *$, the morphism $1 \to 0$ to $-d\colon * \to *$, and the identity morphisms $0 \to 0$ and $1 \to 1$ to the identity morphism $0\colon * \to *$. With the notation introduced above, the cyclic nerve of ϕ is a map $E\mathbb{Z}/2 \to B^{\mathrm{cy}}(d\mathbb{Z})$. Its image C is the simplicial subset of $B^{\mathrm{cy}}(d\mathbb{Z})$ whose q-simplices are tuples $(x_0, x_1, \dots, x_q)$ of elements in $\{d, 0, -d\}$ satisfying the condition that the signs of the nonzero terms x_i alternate cyclically. This is a cyclic subobject of $B^{\mathrm{cy}}_{\{0\}}(d\mathbb{Z})$ that fits in the diagram of cyclic sets

(7.2)
$$\begin{tikzcd}
& \mathbb{Z}/2 \arrow[r, two heads] \arrow[d, tail] & \{(0)\} \arrow[d, tail] & \\
\{0\} \arrow[r, tail] & E\mathbb{Z}/2 \arrow[r, two heads] & C \arrow[r, tail] & B^{\mathrm{cy}}_{\{0\}}(d\mathbb{Z}),
\end{tikzcd}$$

where $\mathbb{Z}/2 = \{0, 1\}$ is viewed as the discrete cyclic set on the set of 0-simplices in $E\mathbb{Z}/2$ and $\{(0)\}$ is the set of 0-simplices of C. It is immediate from the definition that the maps are degreewise injective and surjective as indicated.

Lemma 7.3. *The square in diagram* (7.2) *is a pushout, and the two remaining maps, $\{0\} \to E\mathbb{Z}/2$ and $C \to B^{\mathrm{cy}}_{\{0\}}(d\mathbb{Z})$, are weak equivalences.*

Proof. It is easy to see that the square is a pushout in every simplicial degree and hence a pushout of simplicial sets. Since $\mathcal{E}(\mathbb{Z}/2)$ has an initial object its nerve $E\mathbb{Z}/2$ is contractible and hence C has the homotopy type of $\Sigma\{0,1\} \cong S^1$. The map $C \to B^{\mathrm{cy}}_{\{0\}}(d\mathbb{Z}) \simeq S^1$ is a weak equivalence since it maps the generator $[(-d,d)]$ of $\pi_1(C)$ to a generator of $\pi_1(B^{\mathrm{cy}}_{\{0\}}(d\mathbb{Z}))$. □ □

Consider in general a d-periodic grouplike and cofibrant commutative $\mathcal{J}$-space monoid N with $d > 0$. The idea for the next step is to pull $B^{\mathrm{cy}}_{\{0\}}(N)$ back over the diagram (7.2) using the structure map to $B^{\mathrm{cy}}_{\{0\}}(d\mathbb{Z})$. It will be convenient to use the following ad hoc notation. For a map of cyclic sets $\theta\colon K \to B^{\mathrm{cy}}(d\mathbb{Z})$, we write $\theta(k) = (\theta(k)_0, \dots, \theta(k)_q)$ for the image of a q-simplex $k \in K_q$ and let $K \odot N$ denote the cyclic $\mathcal{J}$-space with

$$(K \odot N)_q = \coprod_{k \in K_q} N_{\{\theta(k)_0\}} \boxtimes \cdots \boxtimes N_{\{\theta(k)_q\}}$$

and the obvious structure maps.

Lemma 7.4. *If $K \to L \to B^{\mathrm{cy}}(d\mathbb{Z})$ are maps of cyclic sets such that $K \to L$ is a weak equivalence on underlying simplicial sets, then the induced map of $\mathcal{J}$-spaces $K \odot N \to L \odot N$ is a $\mathcal{J}$-equivalence.*

Proof. We consider the induced commutative diagram of bisimplicial sets

$$\begin{tikzcd} (K \odot N)_{h\mathcal{J}} \arrow[r] \arrow[d] & (L \odot N)_{h\mathcal{J}} \arrow[r] \arrow[d] & B^{\mathrm{cy}}(N)_{h\mathcal{J}} \arrow[d] \\ K \arrow[r] & L \arrow[r] & B^{\mathrm{cy}}(d\mathbb{Z}). \end{tikzcd}$$

The right hand square and the outer square are homotopy cartesian in every simplicial degree q because they are actual pullbacks and $K_q \to L_q \to B^{\mathrm{cy}}_q(d\mathbb{Z})$ are maps of discrete simplicial sets. As in the proof of Proposition 7.1, it follows that these two squares are homotopy cartesian after realization. Hence the left hand square is also homotopy cartesian after realization, and $K \odot N \to L \odot N$ is a $\mathcal{J}$-equivalence since $K \to L$ is a weak equivalence. □ □

With notation from diagram (7.2) we have the isomorphisms

$$\{0\} \odot N \cong B^{\mathrm{cy}}(N_{\{0\}}), \quad \mathbb{Z}/2 \odot N \cong \mathbb{Z}/2 \times B^{\mathrm{cy}}(N_{\{0\}}), \quad B^{\mathrm{cy}}_{\{0\}}(d\mathbb{Z}) \odot N \cong B^{\mathrm{cy}}_{\{0\}}(N)$$

of cyclic $\mathcal{J}$-spaces. Pulling $B^{\mathrm{cy}}_{\{0\}}(N)$ back over diagram (7.2) we therefore get the following commutative diagram of cyclic $\mathcal{J}$-spaces

$$\begin{tikzcd} & \mathbb{Z}/2 \times B^{\mathrm{cy}}(N_{\{0\}}) \arrow[r] \arrow[d] & B^{\mathrm{cy}}(N_{\{0\}}) \arrow[d] & \\ B^{\mathrm{cy}}(N_{\{0\}}) \arrow[r, "\simeq"] & E\mathbb{Z}/2 \odot N \arrow[r] & C \odot N \arrow[r, "\simeq"] & B^{\mathrm{cy}}_{\{0\}}(N), \end{tikzcd}$$

where the square is a pushout by construction and the remaining horizontal maps are $\mathcal{J}$-equivalences by Lemmas 7.3 and 7.4. Applying the functor $\mathbb{S}^{\mathcal{J}}$ and passing to the geometric realizations this in turn gives a commutative diagram

(7.3)

$$\begin{tikzcd} & \mathbb{S}^{\mathcal{J}}[\mathbb{Z}/2 \times B^{\mathrm{cy}}(N_{\{0\}})] \arrow[r] \arrow[d] & \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(N_{\{0\}})] \arrow[d] & \\ \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(N_{\{0\}})] \arrow[r, "\simeq"] & \mathbb{S}^{\mathcal{J}}[E\mathbb{Z}/2 \odot N] \arrow[r] & \mathbb{S}^{\mathcal{J}}[C \odot N] \arrow[r, "\simeq"] & \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}_{\{0\}}(N)] \end{tikzcd}$$

of $\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(N_{\{0\}})]$-modules with circle action. The square is again a pushout square and the remaining vertical maps are stable equivalences by Corollary A.8. We claim that the square is in fact homotopy cocartesian and for this it suffices to observe that the vertical map on the left is a levelwise cofibration. Indeed, before geometric realization it is clearly a levelwise cofibration (in fact the inclusion of a wedge summand) in each simplicial degree and the geometric realization is therefore also a levelwise cofibration. Using this we can analyze the mapping cone $C(\sigma)$ of the composite map $\sigma\colon \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(N_{\{0\}})] \to \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}_{\{0\}}(N)]$ in the diagram.

Proposition 7.5. *Let N be a grouplike and cofibrant commutative $\mathcal{J}$-space monoid with period $d > 0$. Then the mapping cone $C(\sigma)$ is related to $\Sigma\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(N_{\{0\}})]$ by a chain of stable equivalences of $\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(N_{\{0\}})]$-modules with circle action.*

Proof. Diagram (7.3) gives rise to a commutative diagram of mapping cones

$$\begin{array}{ccc} & & C\big(\mathbb{S}^{\mathcal{J}}[\mathbb{Z}/2 \times B^{\mathrm{cy}}(N_{\{0\}})] \to \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(N_{\{0\}})]\big) \\ & & \downarrow\simeq \\ C\big(\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(N_{\{0\}})] \to \mathbb{S}^{\mathcal{J}}[C \odot N]\big) & \xrightarrow{\simeq} & C\big(\mathbb{S}^{\mathcal{J}}[E\mathbb{Z}/2 \odot N] \to \mathbb{S}^{\mathcal{J}}[C \odot N]\big) \\ \downarrow\simeq & & \downarrow\simeq \\ C\big(\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(N_{\{0\}})] \to \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}_{\{0\}}(N)]\big) & \xrightarrow{\simeq} & C\big(\mathbb{S}^{\mathcal{J}}[E\mathbb{Z}/2 \odot N] \to \mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}_{\{0\}}(N)]\big) \end{array}$$

where each arrow between mapping cones is a stable equivalence of $\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(N_{\{0\}})]$-modules with circle action. The result follows by identifying $C(\sigma)$ with the lower left hand term and $\Sigma\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(N_{\{0\}})]$ with the upper right hand term. □ □

of Proposition 6.13. Let $N = M^{\mathrm{gp}}$. Pulling the stable equivalences in Proposition 7.5 back to stable equivalences of $\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)]$-modules, we get the desired chain of stable equivalences

$$C(\rho) \xrightarrow{\simeq} C(\sigma) \simeq \Sigma\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(N_{\{0\}})] \xleftarrow{\simeq} \Sigma\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M_{\{0\}})]$$

of $\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}(M)]$-modules with circle action. □ □

Appendix A. Homotopy invariance of $\mathbb{S}^{\mathcal{J}}$

Being a left Quillen functor, $\mathbb{S}^{\mathcal{J}}$ takes $\mathcal{J}$-equivalences between cofibrant $\mathcal{J}$-spaces to stable equivalences. It is useful to know that $\mathbb{S}^{\mathcal{J}}$ is homotopically well-behaved on a larger class of $\mathcal{J}$-spaces that includes cofibrant $\mathcal{J}$-spaces and the underlying $\mathcal{J}$-spaces of cofibrant commutative $\mathcal{J}$-space monoids. For this purpose it is not sufficient to work with flat $\mathcal{J}$-spaces, because $\mathbb{S}^{\mathcal{J}}$ is not left Quillen with respect to the flat model structure [SS12, Remark 4.29].

Definition A.1. A $\mathcal{J}$-space X is *$\mathbb{S}^{\mathcal{J}}$-good* if there exists a cofibrant $\mathcal{J}$-space X' and a $\mathcal{J}$-equivalence $X' \to X$ such that $\mathbb{S}^{\mathcal{J}}[X'] \to \mathbb{S}^{\mathcal{J}}[X]$ is a stable equivalence.

It is clear from the definition that cofibrant $\mathcal{J}$-spaces are $\mathbb{S}^{\mathcal{J}}$-good. The terminal $\mathcal{J}$-space T is an example of a $\mathcal{J}$-space that is not $\mathbb{S}^{\mathcal{J}}$-good. Using that $\mathbb{S}^{\mathcal{J}}$ is a left Quillen functor we see that if X is $\mathbb{S}^{\mathcal{J}}$-good and $Y \to X$ is any $\mathcal{J}$-equivalence with Y cofibrant, then the induced map $\mathbb{S}^{\mathcal{J}}[Y] \to \mathbb{S}^{\mathcal{J}}[X]$ is a stable equivalence. This in turn has the following consequence.

Proposition A.2. *The functor $\mathbb{S}^{\mathcal{J}}$ takes $\mathcal{J}$-equivalences between $\mathbb{S}^{\mathcal{J}}$-good $\mathcal{J}$-spaces to stable equivalences.* □

The automorphism group of an object $(\mathbf{n}_1, \mathbf{n}_2)$ in $\mathcal{J}$ may evidently be identified with $\Sigma_{n_1} \times \Sigma_{n_2}$.

Definition A.3. A $\mathcal{J}$-space X is *Σ-free in the second variable* if Σ_{n_2} acts freely on $X(\mathbf{n}_1, \mathbf{n}_2)$ for each object $(\mathbf{n}_1, \mathbf{n}_2)$ in $\mathcal{J}$.

Lemma A.4. *If a $\mathcal{J}$-space is Σ-free in the second variable, then it is $\mathbb{S}^{\mathcal{J}}$-good.*

Proof. Let X be Σ-free in the second variable and let $X' \to X$ be a cofibrant replacement, which we may assume is a level equivalence. Then X' is also Σ-free in the second variable. The freeness assumptions imply that the quotients by the Σ_k-actions arising in the explicit description of $\mathbb{S}^{\mathcal{J}}$ given in (2.2) preserve weak equivalences. Hence $\mathbb{S}^{\mathcal{J}}[X'] \to \mathbb{S}^{\mathcal{J}}[X]$ is a level equivalence of symmetric spectra and therefore a stable equivalence. □ □

The following condition for $\mathbb{S}^{\mathcal{J}}$-goodness can often be checked in practice.

Corollary A.5. *Let $X \to Y$ be a map of $\mathcal{J}$-spaces such that Y is Σ-free in the second variable. Then X is $\mathbb{S}^{\mathcal{J}}$-good.*

Proof. If Y is Σ-free in the second variable then it is automatic that also X is Σ-free in the second variable. □ □

Lemma A.6. *Let M be a cofibrant commutative $\mathcal{J}$-space monoid. Then M is Σ-free in the second variable and hence $\mathbb{S}^{\mathcal{J}}$-good.*

Proof. Let us say that a $\mathcal{J}$-space X is *strongly free in the second variable* if for every subgroup $G \subseteq \Sigma_{m_1} \times \Sigma_{m_2}$ such that the composite $G \to \Sigma_{m_1} \times \Sigma_{m_2} \to \Sigma_{m_1}$ with the projection is injective, the group Σ_{n_2} acts freely on

$$(X \boxtimes (F^{\mathcal{J}}_{(\mathbf{m_1},\mathbf{m_2})}(*)/G))(\mathbf{n_1},\mathbf{n_2})\,. \tag{A.1}$$

We will prove the lemma by showing the stronger statement that the underlying $\mathcal{J}$-space of M is strongly free in the second variable.

We first show that $U^{\mathcal{J}} = \mathcal{J}((\mathbf{0},\mathbf{0}),-)$ is strongly free in the second variable. Let $G \subseteq \Sigma_{m_1} \times \Sigma_{m_2}$ be a subgroup with $G \to \Sigma_{m_1} \times \Sigma_{m_2} \to \Sigma_{m_1}$ injective. If a morphism $(\alpha_1,\alpha_2,\rho)\colon (\mathbf{m_1},\mathbf{m_2}) \to (\mathbf{n_1},\mathbf{n_2})$ represents an element

$$[(\alpha_1,\alpha_2,\rho)] \in \mathcal{J}((\mathbf{m_1},\mathbf{m_2}),(\mathbf{n_1},\mathbf{n_2}))/G \cong (U^{\mathcal{J}} \boxtimes (F^{\mathcal{J}}_{(\mathbf{m_1},\mathbf{m_2})}(*)/G))(\mathbf{n_1},\mathbf{n_2}),$$

then $\sigma[(\alpha_1,\alpha_2,\rho)] = [(\alpha_1,\alpha_2,\rho)]$ for a $\sigma \in \Sigma_{n_2}$ implies that there is a $(\gamma_1,\gamma_2) \in G$ with

$$(\mathrm{id}_{\mathbf{n_1}},\sigma,\mathrm{id}_\emptyset)(\alpha_1,\alpha_2,\rho) = (\alpha_1,\alpha_2,\rho)(\gamma_1,\gamma_2,\mathrm{id}_\emptyset).$$

By definition of the composition in $\mathcal{J}$ (see [SS12, Definition 4.2]), this implies $\alpha_1 = \alpha_1\gamma_1$. Since α_1 is injective, $\gamma_1 = \mathrm{id}_{\mathbf{m_1}}$. Hence $\gamma_2 = \mathrm{id}_{\mathbf{m_2}}$ because $G \to \Sigma_{m_1}$ is injective. So we have $(\mathrm{id}_{\mathbf{n_1}},\sigma,\mathrm{id}_\emptyset)(\alpha_1,\alpha_2,\rho) = (\alpha_1,\alpha_2,\rho)$. This implies $\sigma(i) = i$ for $i \in \alpha_2(\mathbf{m_2})$. In the third variable, we have $\left(\sigma|_{\mathbf{n_2}\setminus\alpha_2}\right)\rho = \rho$ and hence $\sigma(i) = i$ for every $i \in \mathbf{n_2} \setminus \alpha_2$. Hence the Σ_{n_2}-action on $(U^{\mathcal{J}} \boxtimes (F^{\mathcal{J}}_{(\mathbf{m_1},\mathbf{m_2})}(*)/G))(\mathbf{n_1},\mathbf{n_2})$ is free.

Now we assume that $f\colon X \to Y$ is a generating cofibration for the positive $\mathcal{J}$-model structure on $\mathcal{S}^{\mathcal{J}}$ and that the square

$$\begin{tikzcd} \mathbb{C}(X) \arrow[r] \arrow[d] & \mathbb{C}(Y) \arrow[d] \\ A \arrow[r] & B \end{tikzcd}$$

is a pushout in $\mathcal{C}\mathcal{S}^{\mathcal{J}}$. We want to show that the underlying $\mathcal{J}$-space of B is strongly free in the second variable if that of A is. For this we use [SS12, Proposition 10.1]. It provides a filtration $A = F_0(B) \to F_1(B) \to \dots$ of $A \to B$ with $\mathrm{colim}_i\, F_i(B) = B$ such that there are pushout squares of $\mathcal{J}$-spaces

$$\begin{tikzcd} A \boxtimes (Q^i_{i-1}(f)/\Sigma_i) \arrow[r] \arrow[d] & A \boxtimes (Y^{\boxtimes i}/\Sigma_i) \arrow[d] \\ F_{i-1}(B) \arrow[r] & F_i(B), \end{tikzcd} \tag{A.2}$$

where $Q^i_{i-1}(f) \to Y^{\boxtimes i}$ is the iterated pushout product map associated with f. Here we use that since we only consider the commutative case, the functors $U_i^{\mathbb{C}}$ appearing in [SS12, Proposition 10.1] are the forgetful functors (see [SS12, Example 10.2]). Since the top horizontal map in (A.2) is a monomorphism of $\mathcal{J}$-spaces by [SS12, Proposition 7.1(vii)], it is enough to show that $A \boxtimes (Y^{\boxtimes i}/\Sigma_i)$ is strongly free in the second variable. Using that Y is of the form $F^{\mathcal{J}}_{(\mathbf{k_1},\mathbf{k_2})}(K)$ with $k_1 \geq 1$, it is therefore enough to show that $A \boxtimes ((F^{\mathcal{J}}_{(\mathbf{k_1},\mathbf{k_2})}(*))^{\boxtimes i}/\Sigma_i)$ is strongly free in the second variable if $k_1 \geq 1$. This follows from the hypothesis on A because

$$(F^{\mathcal{J}}_{(\mathbf{k_1},\mathbf{k_2})}(*)^{\boxtimes i}/\Sigma_i) \boxtimes (F^{\mathcal{J}}_{(\mathbf{m_1},\mathbf{m_2})}(*)/G) \cong F^{\mathcal{J}}_{(\mathbf{k_1},\mathbf{k_2})^{\sqcup i}\sqcup(\mathbf{m_1},\mathbf{m_2})}(*)/(\Sigma_i \times G)$$

and $\Sigma_i \times G \to \Sigma_{ik_1+m_1}$ is injective if $k_1 \geq 1$ and $G \to \Sigma_{m_1}$ is injective.

For a general cofibrant commutative $\mathcal{J}$-space monoid M, we may without loss of generality assume that M is a cell complex constructed from the generating cofibrations. This means that there is a λ-sequence $\{M_\alpha \colon \alpha < \lambda\}$ in $\mathcal{CS}^{\mathcal{J}}$ for some ordinal λ such that $M_0 = U^{\mathcal{J}}$ and $M_\alpha \to M_{\alpha+1}$ is the cobase change of a generating cofibration in $\mathcal{CS}^{\mathcal{J}}$. In this situation, the above arguments imply that M is strongly free in the second variable. □ □

The following consequence of Lemma A.6 can also easily be verified directly.

Corollary A.7. *Positive cofibrant $\mathcal{J}$-spaces are Σ-free in the second variable.*

Proof. If X is a positive cofibrant $\mathcal{J}$-space, then $\mathbb{C}(X)$ is a cofibrant commutative $\mathcal{J}$-space monoid. Since there is a canonical map of $\mathcal{J}$-spaces $X \to \mathbb{C}(X)$, the result follows by Lemma A.6 and the proof of Corollary A.5. □ □

Combining Proposition A.2, Corollary A.5 and Lemma A.6 provides the following result.

Corollary A.8. *If M is cofibrant in $\mathcal{CS}^{\mathcal{J}}$ and $X \to Y \to M$ is a sequence of maps of $\mathcal{J}$-spaces with $X \to Y$ a $\mathcal{J}$-equivalence, then $\mathbb{S}^{\mathcal{J}}[X] \to \mathbb{S}^{\mathcal{J}}[Y]$ is a stable equivalence.* □

Remark A.9. When working with the topological version of $\mathcal{J}$-spaces, the Σ-freeness condition should be replaced by a suitable equivariant cofibrancy condition. The analogue of Lemma A.6 then continues to hold, but we do not have a direct topological analogue of Corollary A.5.

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