It is well known that the $C^*$-algebra of an ordered pair of qubits is $M_2 \otimes M_2$. What about unordered pairs? We show in detail that $M_3 \oplus \mathbb{C}$ is the $C^*$-algebra of an unordered pair of qubits. Then we use Schur-Weyl duality to characterize the $C^*$-algebra of an unordered $n$-tuple of $d$-level quantum systems. Using some further elementary representation theory and number theory, we characterize the quantum cycles. We finish with a characterization of the von Neumann algebra for unordered words.

Finite dimensional quantum computation is naturally viewed as occurring in the category of finite dimensional $C^*$-algebras together with completely positive unital maps, in the opposite of their usual direction. The $C^*$-algebras are the types (systems). For instance:

- a single qubit: $M_2$
- an ordered pair of qutrits: $M_3 \otimes M_3$
- a single qutrit: $M_3$
- a bit: $\mathbb{C}^2$
- a qubit or a qutrit: $M_2 \oplus M_3$
- a trit or a qubit: $\mathbb{C}^3 \oplus M_2$

More generally, writing $[[\text{type}]]$ for the $C^*$-algebra for the type, we have:

$$
[[\text{d-level quantum system}]] = M_d
d[[\text{ordered pair of } t \text{ and } s]] = [t] \otimes [s]
d[[\text{d-level classical system}]] = \mathbb{C}^d
d[[t \text{ (classical) or } s]] = [t] \oplus [s].
$$

The completely positive unital maps are the programs (operations) in the opposite direction. For example:

1. **Measure a qubit in the standard basis**
   - $m: M_2 \leftrightarrow \mathbb{C}^2$ (qubit $\rightarrow$ bit)
   - $(\lambda, \mu) \mapsto \lambda \ket{0} \bra{0} + \mu \ket{1} \bra{1}$

2. **Apply Hadamard gate to a qubit**
   - $h: M_2 \leftrightarrow M_2$ (qubit $\rightarrow$ qubit)
   - $a \mapsto H^* a H$, where $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

3. **Initialize a qutrit as 0**
   - $i: \mathbb{C} \leftrightarrow M_3$ (empty $\rightarrow$ qutrit)
   - $a \mapsto \ket{0} a \bra{0}$

4. **Forget about a qubit**
   - $d: M_2 \leftrightarrow \mathbb{C}$ (qubit $\rightarrow$ empty)
   - $\lambda \mapsto \lambda 1$

These basic quantum types are well known, but what about an unordered pair of qubits? An unordered pair of bits is simply a trit (00, 01 = 10 or 11). However, we will see an unordered pair of qubits is not a qutrit, but rather its $C^*$-algebra is $M_3 \oplus \mathbb{C}$.

In Section 1, we prove this in detail to get a feel for this surprising result. Then in Section 2, we characterize the $C^*$-algebras of unordered $n$-tuples of $d$-level quantum systems using Schur-Weyl duality, which dates back to the early 20th century. Applying some elementary representation theory, we characterize the $C^*$-algebra for qubit 3-cycles in Section 3. Then, using some number theory, we characterize arbitrary quantum cycles in Section 4. We finish with a characterization of the von Neumann algebra for the quantum unordered words in Section 5.
Unordered quantum types have been considered before. For instance in [5] they are used to give denotational semantics to a quantum lambda calculus.\footnote{The type \([\texttt{qubit}] \otimes 2\) from [5] Example 23] corresponds to an unordered pair of qubits and thus has as \(C^\ast\)-algebra \(M_3 \oplus \mathbb{C}\).} A concrete description, however, has to our knowledge not been published before.

At the end of the paper we will have demonstrated the following.

<table>
<thead>
<tr>
<th>System</th>
<th>Algebra</th>
</tr>
</thead>
<tbody>
<tr>
<td>unordered pair of qubits</td>
<td>(M_3 \oplus \mathbb{C})</td>
</tr>
<tr>
<td>unordered (n)-tuple of (d)-level quantum systems</td>
<td>(\bigoplus_{\lambda \in Y_n} M_{m_{\lambda}})</td>
</tr>
<tr>
<td>words of qubits</td>
<td>(B(\ell^2))</td>
</tr>
<tr>
<td>unordered words</td>
<td>(B(\ell^2) \oplus \bigoplus_{\lambda \in Y^*} M_{m_{\lambda}})</td>
</tr>
<tr>
<td>3-cycle of qubits</td>
<td>(M_4 \oplus M_2 \oplus M_2)</td>
</tr>
<tr>
<td>(n)-cycle of (d)-level quantum systems</td>
<td>(\bigoplus_{0 \leq k &lt; n} M_{c_{k}})</td>
</tr>
</tbody>
</table>

\[Y_n = \left\{ \lambda; \lambda \in \mathbb{N}^n; \left[ \begin{array}{c} \lambda_1 \geq \ldots \geq \lambda_d \geq 0 \\ \lambda_1 + \ldots + \lambda_d = n \end{array} \right] \right\} \quad (n\text{-block Young diagrams of height at most } d)\]

\[Y^* = \bigcup_{n \geq 2} \left\{ \lambda; \lambda \in Y_n; \lambda_2 \neq 0 \right\}\]

\[m_{\lambda} = \prod_{1 \leq i < j \leq d} \frac{\lambda_i - \lambda_j + j - i}{j - i} \quad (\text{Dimension corresponding representation } GL(d))\]

\[c_k = \frac{1}{n} \sum_{l \mid n} d^\frac{\phi(l)}{\phi(l)} \mu\left(\frac{l}{\gcd(l,k)}\right) \phi\left(\frac{l}{\gcd(l,k)}\right) \quad (\text{Ramanujan sum})\]

\(\phi\) : Euler’s totient

\(\mu\) : Möbius function

See appendix A for some decompositions computed using these formulae.

1 An Unordered Pair of Qubits

The Hilbert space of a pair of qubits is \(\mathbb{C}^2 \otimes \mathbb{C}^2\). Write \(\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2\). Let \(\sigma: \mathcal{H} \to \mathcal{H}\) denote the unitary map that exchanges the two qubits:

\[
\sigma: \begin{array}{c|c|c|c}
|00\rangle & |00\rangle & |01\rangle & |10\rangle \\
|10\rangle & |01\rangle & |11\rangle & |11\rangle \\
\end{array}
\]

An important category to study semantics of finite-dimensional quantum computation is \(\text{Star}_{\text{CPU}}^{\text{op}}\), which we will define in a moment. It is important as every object corresponds to a type of finite-dimensional quantum system and every arrow to a program of the corresponding type. Conversely, every physical
finite dimensional quantum type and program corresponds to an object and arrow (respectively) in this category.

The objects are finite-dimensional C*-algebras. As the norm plays no rôle in this paper, these are equivalently semisimple *-algebras over C, and are also equivalently finite dimensional W*-algebras. Accordingly, we refer to them as *-algebras in the rest of the paper. We remark, however, that there are finite-dimensional *-algebras in the axiomatic sense that are not C*-algebras and these are excluded.

The arrows in Star_{cPU} are completely positive unital linear maps, and in Star_{op} they are in the opposite direction.

The *-algebra of a pair of qubits is $B(H) \cong M_4$. The map that exchanges the two qubits is given by:

$$B(\sigma): M_4 \leftarrow M_4, \quad a \mapsto \sigma^{-1} a \sigma = \sigma a \sigma.$$  

We claim the *-algebra of an unordered pair of qubits must be the coequalizer of $B(\sigma)$ and id. This is the equalizer in Star_{cPU}, which is the following subalgebra of $M_4$:

$$E = \{ a; \ a \in M_4; \ \sigma a \sigma = a \} \subseteq M_4.$$  

First note the analogy with the classical case: to form an unordered pair of bits, one takes the quotient with respect to the equivalence relation defined by permuting the bits, which identifies 01 and 10. This is a coequalizer in the category Set. Why is the coequalizer used? The definition gives the following rule: for every program $f: M_4 \leftarrow \mathcal{A}$ invariant under swapping ($\sigma \circ f = f$) there is a unique lift $f': E \leftarrow \mathcal{A}$ such that $e \circ f' = f$, where $e: E \subseteq M_4$ is the coequalizer map.

What *-algebra is $E$? We write $\mathcal{S}$ for the symmetric part of $H$:

$$\mathcal{S} = \{ v; \ v \in H; \ \sigma v = v \}.$$  

One might expect $E = B(\mathcal{S})$, but this is not the case. There is another summand of $E$. First, we must take a small detour. It is easy to verify that the projection onto $\mathcal{S}$ is given by

$$P_{\mathcal{S}}: v \mapsto \frac{v + \sigma v}{2},$$  

which is called the symmetrizer. The complementary projection $P_{\mathcal{A}} = I - P_{\mathcal{S}}$

$$P_{\mathcal{A}}: v \mapsto \frac{v - \sigma v}{2}$$

projects onto the antisymmetric subspace of $H$, which is given by

$$\mathcal{A} = \{ v; \ v \in H; \ \sigma v = -v \}.$$  

By considering the images of the standard basis vectors under $P_{\mathcal{A}}$ and $P_{\mathcal{S}}$, it is easy to determine that

$$\{ |00\rangle, |11\rangle, \frac{1}{\sqrt{2}} |01\rangle + \frac{1}{\sqrt{2}} |10\rangle \} \quad \text{and} \quad \{ \frac{1}{\sqrt{2}} |01\rangle - \frac{1}{\sqrt{2}} |10\rangle \}$$

are orthonormal bases for $\mathcal{S}$ and respectively $\mathcal{A}$.

---

2 Which are automatically unital.

3 The category Star_{op} and its variations occur under different names in the literature. The category Star_{op} of finite dimensional C*-algebras with c.p. maps in the opposite direction is equivalent to the category CP*[FHilb] from [1]. If we restrict to subunital maps, we call the category Star_{op}^{sub}, which is equivalent to the category Q from [7] and the category CPM_{e} from [5].
There is a map \( i: B(\mathcal{F}) \oplus B(\mathcal{A}) \to B(\mathcal{F} \oplus \mathcal{A}) \cong B(\mathcal{H}) \), given by

\[
(s, a) \mapsto s \oplus a = \begin{pmatrix} s & 0 \\ 0 & a \end{pmatrix}.
\]

Its image, \( \text{Im} i \), is actually the equalizer \( E \). We have to show both inclusions.

First, suppose \( a \in \text{Im} i \). Then \( a = P_\mathcal{F} aP_\mathcal{F} + P_\mathcal{A} aP_\mathcal{A} \). Note that \( \sigma P_\mathcal{F} = P_\mathcal{F} \sigma = P_\mathcal{F} \) and \( \sigma P_\mathcal{A} = P_\mathcal{A} \sigma = -P_\mathcal{A} \). Thus:

\[
\sigma a \sigma = \sigma P_\mathcal{F} aP_\mathcal{F} \sigma + \sigma P_\mathcal{A} aP_\mathcal{A} \sigma = P_\mathcal{F} aP_\mathcal{F} + P_\mathcal{A} aP_\mathcal{A} = a.
\]

Hence \( a \in E \).

Conversely, suppose \( a \in E \). First note that \( a = P_\mathcal{F} aP_\mathcal{F} + P_\mathcal{A} aP_\mathcal{A} + P_\mathcal{F} aP_\mathcal{A} + P_\mathcal{A} aP_\mathcal{F} \). Now since \( \sigma a \sigma = a \), we have:

\[
P_\mathcal{F} aP_\mathcal{F} + P_\mathcal{A} aP_\mathcal{A} + P_\mathcal{F} aP_\mathcal{A} + P_\mathcal{A} aP_\mathcal{F} = P_\mathcal{F} aP_\mathcal{F} + P_\mathcal{A} aP_\mathcal{A} - P_\mathcal{F} aP_\mathcal{A} - P_\mathcal{A} aP_\mathcal{F}.
\]

Thus \( P_\mathcal{F} aP_\mathcal{F} = -P_\mathcal{A} aP_\mathcal{A} \). Their images are orthogonal, hence \( P_\mathcal{F} aP_\mathcal{A} = P_\mathcal{A} aP_\mathcal{F} = 0 \). So \( a = P_\mathcal{F} aP_\mathcal{F} + P_\mathcal{A} aP_\mathcal{A} \), and hence \( a \in \text{Im} i \).

Thus \( E \cong B(\mathcal{F}) \oplus B(\mathcal{A}) \cong M_3 \oplus \mathbb{C} \). At first one might be surprised that the antisymmetric vector \( \frac{1}{\sqrt{2}} |01 \rangle - \frac{1}{\sqrt{2}} |10 \rangle \) of \( \mathcal{H} \) is a possible state of an unordered pair of qubits, since \( \sigma \) changes its sign. The explanation is simple: in \( \ast \)-algebras, two states that differ only by global phase are identified. Thus the antisymmetric vector is symmetric up to global phase \(-1 \).

An astute reader might note that we have proven a bit more: the \( \ast \)-algebra associated to an unordered pair of \( d \)-level quantum systems is given by \( B(\mathcal{A}) \oplus B(\mathcal{F}) \) as well, where \( \mathcal{A}, \mathcal{F} \subseteq \mathbb{C}^d \otimes \mathbb{C}^d \) are defined similarly.

## 2 Unordered Tuples

In the previous section, we have shown how to characterize the \( \ast \)-algebra for a pair of qubits. In this section, we will generalize to arbitrary tuples. We define an unordered \( n \)-tuple of \( d \)-level quantum systems as follows. Consider the Hilbert space \((\mathbb{C}^d)^{\otimes n}\). A permutation of \( n \) elements \( \pi \in S_n \) acts on it in an obvious way, by permuting the basis vectors as follows:

\[
\pi: \ |i_1i_2 \ldots i_n \rangle \mapsto |i_{\pi^{-1}(1)} \ldots i_{\pi^{-1}(n)} \rangle.
\]

The equalizer of all \( \pi \in S_n \) in \( \text{Star}_{\mathbb{C}PU} \) is the \( \ast \)-algebra for unordered \( n \)-tuples of \( d \)-level quantum systems. It is given by the following subalgebra of \( B((\mathbb{C}^d)^{\otimes n}) \)

\[
E = \{ a; \ \pi^{-1} a \pi = a \text{ for all } \pi \in S_n \} \subseteq B((\mathbb{C}^d)^{\otimes n}).
\]

The final result is:

\[
E \cong \bigoplus_{\lambda} M_{m_\lambda} \text{ where } m_\lambda = \prod_{1 \leq i < j \leq d} \frac{\lambda_i - \lambda_j + j - i}{j - i}.
\]

To prove this, we will first review some of the basics of representation theory of finite groups. Then we will introduce Schur-Weyl duality to prove the result.
A representation of a group is a pair \((V, \rho)\), where \(V\) is a vector space and \(\rho : G \to GL(V)\) is a group homomorphism. Often, one refers to the vector space \(V\) as the representation instead of the group homomorphism. When considering the action of \(g \in G\) on vectors \(v \in V\) it is common to leave out the \(\rho\) and write \(gv\) instead of \(\rho(g)v\).

We now give some examples of representations. The vector space \((\mathbb{C}^d)^\otimes n\) is a representation of \(S_n\), by the action given in equation (1). Another one is that for any group \(G\), we can consider \(\rho_{\text{trivial}} : G \to GL(\mathbb{C})\) given by \(\rho_{\text{trivial}}(g) = I\). This is called the trivial representation.

Given two representations \(\rho : G \to GL(V)\) and \(\sigma : G \to GL(W)\) a morphism \(f\) from \(\rho\) to \(\sigma\) is a linear map \(f : U \to V\) such that \(\sigma(g)f = f\rho(g)\) for every \(g \in G\). That is: linear maps that commute with the group actions of the representations.

We can relate morphisms of representations to the equalizer that we want to calculate as follows.

\[
\text{Rep}(S_n)((\mathbb{C}^d)^\otimes n, (\mathbb{C}^d)^\otimes n) = \{a; a \in B((\mathbb{C}^d)^\otimes n); \pi a = a\pi \text{ for all } \pi \in S_n\}
\]
\[
= \{a; a \in B((\mathbb{C}^d)^\otimes n); \pi^{-1}a\pi = a \text{ for all } \pi \in S_n\}
\]
\[
= E.
\]

Given two representations \(\rho : G \to GL(V), \sigma : G \to GL(W)\), one can define the direct sum representation on \(V \oplus W\) by \((\rho, \sigma)(g)(v, w) = (\rho(g)(v), \sigma(g)(w))\). A representation is called indecomposable if it is not the direct sum in this way of two other representations.

Given a representation on a vector space \(V\) and a subspace \(U\), one calls \(U\) invariant (under \(G\)) if for every \(u \in U\) and \(g \in G\) we have \(gu \in U\). A representation on \(V\) is called irreducible if the only invariant subspaces are \(\{0\}\) and \(V\) itself. This intentionally implies that the unique representation on the zero-dimensional vector space is not irreducible, for the same reason that \(1\) is not prime and \(0\) is not connected as a topological space.

A slightly surprising, but welcome, theorem is that a representation of a finite group is indecomposable if and only if it is irreducible. Furthermore, every representation is uniquely the direct sum of irreducible representations (up to isomorphism). See [2, Proposition 1.5].

Thus there are distinct irreducible representations \(U_\lambda\) and natural numbers \(m_\lambda\), called multiplicities, such that \((\mathbb{C}^d)^\otimes n \cong \bigoplus U_\lambda^\otimes m_\lambda\) and hence

\[
E \cong \bigoplus_{\lambda, \mu} \text{Rep}(S_n)(U_\lambda^{m_\lambda}, U_\mu^{m_\mu}).
\]

Now, given a morphism between representations, it is easy to see that its kernel and image are invariant. Thus, the only morphisms between irreducible representations are invertible or zero maps. This is the first part of Schur’s lemma. Consequently the maps between non-isomorphic irreducible representations are 0 and do not contribute to the direct sum, giving

\[
E \cong \bigoplus_{\lambda} \text{Rep}(S_n)(U_\lambda^{m_\lambda}, U_\lambda^{m_\lambda}).
\]

The second part of Schur’s lemma is the following observation. Suppose we have an endomorphism \(f\) of an irreducible representation \(V\). Since the base field \(\mathbb{C}\) is algebraically closed, \(f\) must have an eigenvalue \(\lambda\), which is to say that \(f - \lambda I\) has non-trivial kernel. The map \(f - \lambda I\) is itself a morphism of representations, and since \(V\) is irreducible, \(\ker(f - \lambda I) = V\) and so \(f - \lambda I = 0\). That is to say: \(f = \lambda I\). Thus endomorphisms of irreducible representations are scalar multiples of the identity. We deduce

\[
E \cong \bigoplus_{\lambda} M_{m_\lambda}.
\]
Thus, if we know the irreducible representations of $S_n$ and their multiplicities in $\left(\mathbb{C}^d\right)^\otimes n$, then we know $E$. Schur-Weyl duality solves this problem for us. It gives a correspondence between the irreducible representations of $S_n$ in $\left(\mathbb{C}^d\right)^\otimes n$ and of $\text{GL}(d)$ in $\left(\mathbb{C}^d\right)^\otimes n$. The space $\left(\mathbb{C}^d\right)^\otimes n$ is a representation of $\text{GL}(d)$, via the following action

$$gv_1 \otimes \ldots \otimes v_n = (gv_1) \otimes \ldots \otimes (gv_d).$$

Schur-Weyl duality asserts

$$\left(\mathbb{C}^d\right)^\otimes n \cong \bigoplus_{\lambda: \lambda_1 \geq \ldots \geq \lambda_d \geq 0, \lambda_1 + \ldots + \lambda_d = n} U_\lambda \otimes V_\lambda \equiv \bigoplus_{\lambda: \lambda_1 \geq \ldots \geq \lambda_d \geq 0, \lambda_1 \ldots \lambda_d \in \mathbb{N}} U_\lambda \otimes \dim V_\lambda,$$

where $U_\lambda$ are irreducible representations of $S_n$ and $V_\lambda$ are irreducible representations of $\text{GL}(d)$. See [2, Exercise 6.30]. Thus $m_\lambda = \dim V_\lambda$. Together with the duality statement, we are given explicit constructions for $U_\lambda$ and $V_\lambda$. See [2, Theorem 4.3] and [2, §6.1]. From this one can derive [2, Theorem 6.3 (1)] that

$$\dim V_\lambda = \prod_{1 \leq i < j \leq d} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

In particular, in the case of unordered $n$-tuples of qubits, we see $\dim V_\lambda = \lambda_i - \lambda_j + j - i$ and hence

$$E \cong \begin{cases} \bigoplus_{1 \leq i \leq \frac{n+1}{2}} M_{2i-1} & n \text{ even} \\ \bigoplus_{1 \leq i \leq \frac{n}{2}} M_{2i} & n \text{ odd.} \end{cases}$$

### 3 A 3-cycle of Qubits

Unordered tuples are defined by quotienting out the action of the symmetric group. Similarly, we can define other types by quotienting out the action of a subgroup of the symmetric group. The methods of the previous section can be adapted to this situation as well. We will consider cycles, which are not as interesting a type as unordered tuples, but they serve as an example easily related to regular combinatorics.

A 3-cycle of qubits is given by the equalizer

$$E = \{a; a \in M_8; \pi^{-1}a\pi = a; \pi \in C_3 \leq S_3\} \subseteq M_8.$$

The cyclic subgroup $C_3$ of $S_3$ contains $\{(\ ), (1\ 2\ 3), (1\ 3\ 2)\}$. We can use the same argument as before to derive that $E \cong \bigoplus_i M_{m_i}$, where $m_i$ is the multiplicity of the $i$th irreducible representation of $C_3$ in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. However, Schur-Weyl duality will not help this time. We need to determine the multiplicities $m_i$ in another way.

To this end, we recall the theory of characters. Given a representation $\rho: G \to \text{GL}(V)$. For each $g \in G$ we can consider the trace $\text{Tr} \rho(g)$. This yields a map $\chi_V = \text{Tr} \circ \rho : G \to \mathbb{C}$, which is called the character of $\rho$.

By the cyclic property of the trace, we have for any character $\chi$ that $\chi(h^{-1}gh) = \chi(ggh^{-1}) = \chi(g)$. Thus on the same conjugacy class, a character will give the same value. Such a function is called a class function.

Using Schur’s lemma one can work out that

$$\dim \text{Rep}(G)(V, W) = \frac{1}{\#G} \sum_{g \in G} \chi_V(g) \chi_W(g) = \begin{cases} 1 & V \cong W \\ 0 & V \not\cong W. \end{cases}$$
Also, using spectral decomposition, we can derive \( \chi_{V \oplus W} = \chi_V + \chi_W \). Thus, for two such class functions \( \alpha, \beta : G \to \mathbb{C} \), one is lead to define

\[
(\alpha, \beta) = \frac{1}{\#G} \sum_{g \in G} \alpha(g) \overline{\beta(g)}.
\]

This is an Hermitian inner product on the class functions. In fact, with respect to this inner product

1. the characters of irreducible representations are an orthonormal basis of the class functions;
2. a representation \( V \) is irreducible if and only if \((\chi_V, \chi_V) = 1\);
3. there are as many irreducible representations as conjugacy classes and
4. the multiplicity of \( V \) in \( W \) is \((\chi_V, \chi_W)\).

See \[2, \S 2.2 and Proposition 2.30\].

Thus, to determine the multiplicities of the irreducible representations of \( C_3 \) in \( \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \), it is sufficient to determine the character of \( \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \) and the characters of the irreducible representations of \( C_3 \).

We determine the irreducible representations of \( C_3 \) as follows. As \( C_3 \) is Abelian, its conjugacy classes are trivial. Write \( \pi \) for the generator of \( C_3 \) such that \( C_3 = \{1, \pi, \pi^2\} \). Thus, we are looking for \#C_3 = 3 irreducible representations. The trivial representation maps every group element to the identity matrix. It has character \((1, 1, 1)\). Then we have two 1-dimensional representations given by \( \pi \mapsto \omega \) and \( \pi \mapsto \omega^2 \), where \( \omega = e^{\frac{2\pi i}{3}} \). Using the inner product, we can compute that these are distinct irreducible representations. We summarize these results in a character table:

<table>
<thead>
<tr>
<th>( C_3 ) ( \leq ) ( S_3 )</th>
<th>1</th>
<th>( \pi )</th>
<th>( \pi^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>trivial</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>first</td>
<td>1</td>
<td>( \omega )</td>
<td>( \omega^2 )</td>
</tr>
<tr>
<td>second</td>
<td>1</td>
<td>( \omega^2 )</td>
<td>( \omega )</td>
</tr>
</tbody>
</table>

Now we compute the character \( \chi \) of \( \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \). This is particularly easy because of the way the action is defined: the value of the character on \( g \) is the number of basis vectors fixed by \( g \). Thus:

<table>
<thead>
<tr>
<th>( C_3 ) ( \leq ) ( S_3 )</th>
<th>1</th>
<th>( \pi )</th>
<th>( \pi^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 )</td>
<td>8</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

We compute

\[
(\chi_{\text{trivial}}, \chi) = 4 \quad (\chi_{\text{first}}, \chi) = 2 \quad (\chi_{\text{second}}, \chi) = 2.
\]

Thus the *-algebra for a 3-cycle of qubits is given by \( E = M_4 \oplus M_2 \oplus M_2 \).

## 4 Cycles

Now we characterize arbitrary cycles: a \( n \)-cycle of \( d \)-level quantum systems is given by the equalizer

\[
E = \{ a; a \in B((\mathbb{C}^d)^{\otimes n}); g^{-1}ag = a; g \in C_n \leq S_n \} \subseteq B((\mathbb{C}^d)^{\otimes n}).
\]

First, we compute the irreducible representation of \( C_n \). Let \( \pi \in C_n \) be such that \( C_n = \{1, \pi, \pi^2, \ldots, \pi^n\} \). Note that by commutativity, the conjugacy classes are trivial. For any \( 0 \leq k \leq n \), define a 1-dimensional representation \( \rho_k \) by

\[
\rho_k : C_n \to \text{GL}(\mathbb{C}) \quad \pi^i \mapsto (\omega^k),
\]
In [6], Ramanujan introduced (what are now called) Ramanujan sums $ω_j$ where

$$\omega = e^{2\pi i/n}.$$  

Note that $\rho_0$ is the trivial representation. Now, observe

$$(\rho_i, \rho_j) = \frac{1}{n} \sum_{0 \leq k < n} |\omega_j|^2 = 1$$

and $\text{Tr}\rho_j(\pi) \neq \text{Tr}\rho_i(\pi)$ whenever $i \neq j$, so these are $k$ distinct irreducible representations. The character table is given by

<table>
<thead>
<tr>
<th>$C_n \leq S_m$</th>
<th>1</th>
<th>$\pi$</th>
<th>$\pi^2$</th>
<th>...</th>
<th>$\pi^{n-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_0$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>...</td>
<td>1</td>
</tr>
<tr>
<td>$\rho_1$</td>
<td>1</td>
<td>$\omega$</td>
<td>$\omega^2$</td>
<td>...</td>
<td>$\omega^{n-1}$</td>
</tr>
<tr>
<td>$\rho_2$</td>
<td>1</td>
<td>$\omega^2$</td>
<td>$\omega^4$</td>
<td>...</td>
<td>$\omega^{2(n-1)}$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$\rho_{n-1}$</td>
<td>1</td>
<td>$\omega^{n-1}$</td>
<td>$\omega^{2(n-1)}$</td>
<td>...</td>
<td>$\omega^{(n-1)^2}$</td>
</tr>
</tbody>
</table>

Now we will compute that character $\chi$ of the representation on $(\mathbb{C}^d)^\otimes n$. The value of $\chi(\pi^i)$ is the number of basis vectors that are fixed by $\pi^i$. All of the basis vectors are fixed by $1 = \pi^0$, so $\chi(1) = d^n$. The only basis vectors fixed by $\pi$ are of the form $|vv...v)$. The general case is more subtle. For instance, suppose $n = 4$ and $d = 2$. Then $|0101\rangle$ is fixed by $\pi^2$.

Given $0 \leq i < n$. If a basis vector $|v_1...v_n\rangle$ is fixed by $\pi^i$, then we must have $v_j = v_{\pi^i(j)} = v_{\pi^2(j)} = \ldots$ for any $0 \leq j < n$. If $i$ is coprime to $n$, then $\{0, \pi^i(0), \pi^i(2), \ldots\}$ (the orbit of the subgroup generated by $\pi^i$) ranges over all indices and thus the basis vector must be of the form $|vv...v)$. If $j$ is not coprime to $n$, then $\{1, 2, ..., n\}$ splits into several equally sized orbits. The size of each of them is the order of $\pi^i$, which equals $\frac{n}{\gcd(i, n)}$. Thus the number of orbits is $\gcd(i, n)$. On each of the orbits, the basis vector has the same value, but is otherwise unrestricted. Thus there are $d^{\gcd(i, n)}$ basis vectors fixed by $\pi^i$. Thus $\chi(\pi^i) = d^{\gcd(i, n)}$.

Now, we will compute the multiplicity of the $k$th irreducible representation in $\rho$, which is given by $(\chi_k, \chi)$:

$$\langle \chi_k, \chi \rangle = \frac{1}{n} \sum_{0 \leq j < n} \omega_{jk} d^{\gcd(j, n)}$$

$$= \frac{1}{n} \sum_{j | n} \sum_{1 \leq j \leq n} \omega_{jk} d^{l} \quad (\gcd(j, n) = l)$$

$$= \frac{1}{n} \sum_{j | n} d^{l} \sum_{1 \leq j \leq n} \omega_{jk}. \quad (2)$$

As $l$ divides $j$, we may substitute $jl$ for $j$ and get:

$$\langle \chi_k, \chi \rangle = \frac{1}{n} \sum_{j | n} d^{l} \sum_{1 \leq jl \leq n} \omega_{jl} = \frac{1}{n} \sum_{l | n} d^{l} \sum_{1 \leq j \leq \frac{n}{l}} \omega_{jl}.$$  

In [6], Ramanujan introduced (what are now called) Ramanujan sums:

$$c_n(m) = \sum_{1 \leq h \leq n \atop \gcd(h, n) = 1} e\left(\frac{hm}{n}\right),$$
where \( e(x) = e^{2\pi ix} \). Note that \( \omega_{jk} = e^{\frac{ijk}{n}} \). Consequently

\[
(\chi_k, \chi) = \frac{1}{n} \sum_{l|n} d_l^\nu c_l^\nu(k) = \frac{1}{n} \sum_{l|n} d_l^\nu c_l(\chi).
\]

Hölder gave a simple expression for \( c_l(\chi) \), see \cite{4}, Theorem 272:

\[
c_l(\chi) = \mu \left( \frac{l}{\gcd(l,k)} \right) \frac{\phi(l)}{\phi\left( \frac{l}{\gcd(l,k)} \right)},
\]

where \( \mu \) is the Möbius function and \( \phi \) is Euler’s totient. Therefore:

\[
(\chi_k, \chi) = \frac{1}{n} \sum_{l|n} d_l^\nu \mu \left( \frac{l}{\gcd(l,k)} \right) \frac{\phi(l)}{\phi\left( \frac{l}{\gcd(l,k)} \right)}.
\]

There are two cases of particular interest, which can be proven directly from (2):

- If \( n \) is a prime number, then:

\[
(\chi_k, \chi) = \begin{cases} 
  d^n \frac{n}{\phi(n)} & k = 0 \\
  d^n \frac{n}{\phi(n)} & k > 0.
\end{cases}
\]

- The multiplicity corresponding to the trivial representation is

\[
(\chi_0, \chi) = \frac{1}{n} \sum_{l|n} d_l^\nu \mu(1) \phi(l) = \frac{1}{n} \sum_{l|n} d_l^\nu \phi\left( \frac{n}{l} \right).
\]

This is MacMahon’s formula for counting the number of possible necklaces with \( n \) beads, where we may choose from \( d \) different colors of beads. See \cite{3} 4.63.

## 5 Unordered Words

Classically, a word is just a \( n \)-tuple for some \( n \). To work out what should be an unordered word, we simply work out what is an unordered \( n \)-tuple. In the quantum analogue, such a reduction does not work. Again, we need to tune our methods to work out a suitable equalizer.

The Hilbert space for quantum words over a \( d \)-level quantum system is the infinite dimensional Hilbert space

\[
\mathcal{H} := \bigoplus_{n \in \mathbb{N}} (\mathbb{C}^d)^\otimes n.
\]

Note that it only contains sequences that are square summable. The corresponding von Neumann algebra is the set of all bounded operators \( B(\mathcal{H}) \).

We will define an action \( \rho_{\mathcal{H}} \) of \( \prod_{n \in \mathbb{N}} S_n \) on \( \mathcal{H} \) as follows.

\[
\rho_{\mathcal{H}}(\pi_1, \pi_2, \ldots)(|i_1 \ldots i_n\rangle) = |i_{\pi_1^{-1}(1)} \ldots i_{\pi_n^{-1}(n)}\rangle
\]

We wish to compute the equalizer of the actions, which is simply given by

\[
E = \{ a; a \in B(\mathcal{H}); \pi^{-1}a\pi = a \text{ for all } \pi \in \prod_{n \in \mathbb{N}} S_n \}
\]

\[
= \text{BRep}(\prod_{n \in \mathbb{N}} S_n)(\mathcal{H}, \mathcal{H}),
\]
where BRep(ΠnSn)(H, n) denotes the morphisms of representations that are bounded (as linear maps between Hilbert spaces).

We cannot simply apply the same techniques as in Section 2. There are various difficulties. First, H is infinite dimensional and the group ΠnSn is not finite so it does not follow from the theory we used previously that H splits into irreducible representations of ΠnSn. Secondly, the infinite product ⊕ is not a coproduct anymore. We will work around these issues ad hoc. It is possible to give ΠnSn a compact topology using Tychonoff’s theorem and use the representation theory of compact groups, but we do not pursue that direction.

Let iω: Sn → Πn∈N Sn denote the obvious inclusion and pω: B(H) → B((Cd)⊗n) the obvious projection. Then pω ◦ ρ ◦ iω is the action we considered in (1). Recall that

\[(Cd)^⊗n = \bigoplus_{λ∈Yn} U_λ^{⊗m_λ} \text{ where } m_λ = \prod_{1≤i<j≤d} \frac{λ_i - λ_j + j - i}{j - i}\]

and Uλ are distinct irreducible representations for Sn indexed by

\[Yn = \{ λ; λ ∈ \mathbb{N}^d, \left[ \begin{array}{c} λ_1 ≥ ... ≥ λ_d ≥ 0 \\ λ_1 + ... + λ_d = n \end{array} \right] \},\]

which are called n-block Young diagrams of height at most d. The diagram λ is often depicted as a row of λ1 blocks, then a row of λ2 blocks beneath it and so on. All blocks are left justified. For instance, (4, 2, 0) is written as

\[
\begin{array}{c}
\square \\
\square \\
\square \\
\square \\
\end{array}
\]

Note that Uλ for any λ ∈ Yn is an irreducible representation for Πn∈N Sn as well, since Sm acts trivially on Uλ if m ≠ n. However, not all Uλ are distinct.

For each n ∈ N, there is the trivial representation of Sn. They correspond to the Young diagrams of height 1 (□ □ □ □ ...). They are all isomorphic as representations of Πn∈N Sn. The representation isomorphism between any two, is the unique non-zero map between the 1-dimensional subspaces. We will show all other representations are distinct.

The kernel of a representation (V, ρ), is the subgroup of elements that map to the identity operator, equivalently the kernel of ρ as a group homomorphism. If two representations are isomorphic then their kernels and dimensions are the same.

Given n, m ∈ N and λ ∈ Yn and μ ∈ Ym with λ ≠ μ such that, without loss of generality, Uλ is not a trivial representation. Suppose n = m and Uλ is isomorphic to Uμ as representation of Πn∈N Sn. Then it is also isomorphic via the same isomorphism as representation of Sn = Sm, which is a contradiction. Thus Uλ and Uμ are distinct.

For the remaining case, suppose n ≠ m. Because Uλ is not a trivial representation, there is an element π ∈ Sn that is not in its kernel. If Uμ is a trivial representation, then Uλ and Uμ must be distinct as they have different kernels. If Uμ is not a trivial representation, then there is an element π′ ∈ Sm that is not in its kernel. By definition of the action on H, every element of Sn is in the kernel of Uμ. Thus Uλ and Uμ have different kernel. Hence they are distinct.

We have a direct sum decomposition of H into irreducible representations of Πn∈N Sn:

\[H = \bigoplus_{n∈N} (Cd)^⊗n ⊕ U_{trivial} ⊕ \bigoplus_{λ∈Yn} U_λ^{⊗m_λ},\]

where Utrivial = U□ = U□ = ... is the trivial representation. Write

\[Y^* = \bigcup_{n∈N} \{ λ; λ ∈ Yn; h(λ) ≠ 1 \}.\]
Now recall (3):

\[
E = B\text{Rep}(\prod_n S_n)(\mathcal{H}, \mathcal{H})
\approx B\text{Rep}(\prod_n S_n)(U^{\oplus \omega}_{\text{trivial}} \oplus \bigoplus_{\lambda \in Y^*} U^{m_{\lambda}}_{\lambda} \oplus \bigoplus_{\lambda \in Y^*} U^{\oplus m_{\lambda}}_{\lambda}).
\]

Using Schur’s lemma and the fact that \(\oplus\) is a biproduct, we derive

\[
E \approx B\text{Rep}(\prod_n S_n)(U^{\oplus \omega}_{\text{trivial}} \oplus U^{\oplus \omega}_{\text{trivial}}) \\
\oplus B\text{Rep}(\prod_n S_n)(\bigoplus_{\lambda \in Y^*} U^{m_{\lambda}}_{\lambda}) \\
\approx B(l^2) \oplus B\text{Rep}(\prod_n S_n)(\bigoplus_{\lambda \in Y^*} U^{m_{\lambda}}_{\lambda}).
\]

We have to be a bit more careful for the right-hand summand, since \(\oplus\) is not a countable biproduct.

\[
B\text{Rep}(\prod_n S_n)(\bigoplus_{\lambda \in Y^*} U^{m_{\lambda}}_{\lambda}) = \left\{ \begin{array}{l}
\{ (a_{\lambda \mu})_{\lambda \in Y^*}; \\
(\lambda, \mu \in Y^*; a_{\lambda \mu} \in B(\bigoplus_{\lambda \in Y^*} U^{m_{\lambda}}_{\lambda}); a_{\lambda \mu} \in B(\bigoplus_{\lambda \in Y^*} U^{m_{\lambda}}_{\lambda}) \}
\end{array} \right. \\
(\text{dfn.})
\]

\[
= \left\{ \begin{array}{l}
\{ (a_{\lambda \lambda})_{\lambda \in Y^*}; \\
(\lambda \in Y^*; a_{\lambda \lambda} \in B(\bigoplus_{\lambda \in Y^*} U^{m_{\lambda}}_{\lambda}); a_{\lambda \lambda} \in B(\bigoplus_{\lambda \in Y^*} U^{m_{\lambda}}_{\lambda}) \}
\end{array} \right. \\
(\text{Schur’s lemma})
\]

\[
= \left\{ \begin{array}{l}
\{ (a_{\lambda \lambda})_{\lambda \in Y^*}; \\
(\lambda \in Y^*; \sup_{\lambda} \|a_{\lambda \lambda}\| < \infty; a_{\lambda \lambda} \in M_{m_{\lambda}} \}
\end{array} \right. \\
(\ast, \text{see below})
\]

\[
= \left\{ \begin{array}{l}
\{ (a_{\lambda})_{\lambda \in Y^*}; \\
(\lambda \in Y^*; \sup_{\lambda} \|a_{\lambda}\| < \infty; a_{\lambda} \in M_{m_{\lambda}} \}
\end{array} \right. \\
(\text{reindexing})
\]

Consequently

\[
E \approx B(l^2) \oplus \prod_{\lambda \in Y^*} M_{m_{\lambda}}.
\]

For step \(\ast\), note that the inclusion \(\subseteq\) is easy, and the other inclusion is can be carefully checked using the definition of the direct sum and noting the cross terms are zero. We also emphasize that the infinite product should be interpreted for \(C^*\) or \(W^*\)-algebras, with the norm bounded (the \(C^*\)-sum). This is, in general, a strict subalgebra of the infinite product in \(C\)-algebras or rings.

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References


A Computed decompositions

For easy reference, we have computed the decompositions into matrix algebras of the C*-algebras for unordered pairs, triples and quads for various types.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|c|}
\hline
\text{d} & \text{M}_3 & \text{M}_6 & \text{M}_{10} & \text{M}_{15} & \text{M}_{21} & \text{M}_{28} & \text{M}_{36} & \text{M}_{55} \\
\hline
\text{2} & \text{M}_3 & \text{M}_3 & \text{M}_6 & \text{M}_{10} & \text{M}_{15} & \text{M}_{21} & \text{M}_{28} & \text{M}_{36} & \text{M}_{55} \\
\hline
\text{3} & \text{M}_6 & \text{M}_3 & \text{M}_6 & \text{M}_{10} & \text{M}_{15} & \text{M}_{21} & \text{M}_{28} & \text{M}_{36} & \text{M}_{55} \\
\hline
\text{4} & \text{M}_{10} & \text{M}_6 & \text{M}_{20} & \text{M}_{20} & \text{M}_4 & \text{M}_{35} & \text{M}_{40} & \text{M}_{10} & \text{M}_{35} \\
\hline
\text{5} & \text{M}_{15} & \text{M}_{10} & \text{M}_{20} & \text{M}_{20} & \text{M}_4 & \text{M}_{35} & \text{M}_{40} & \text{M}_{10} & \text{M}_{35} \\
\hline
\text{6} & \text{M}_{21} & \text{M}_{15} & \text{M}_{35} & \text{M}_{40} & \text{M}_{10} & \text{M}_{35} & \text{M}_{40} & \text{M}_{10} & \text{M}_{35} \\
\hline
\text{7} & \text{M}_{28} & \text{M}_{21} & \text{M}_{68} & \text{M}_{112} & \text{M}_{35} & \text{M}_{120} & \text{M}_{240} & \text{M}_{240} & \text{M}_{240} \\
\hline
\text{8} & \text{M}_{36} & \text{M}_{28} & \text{M}_{68} & \text{M}_{112} & \text{M}_{35} & \text{M}_{120} & \text{M}_{240} & \text{M}_{240} & \text{M}_{240} \\
\hline
\text{9} & \text{M}_{55} & \text{M}_{45} & \text{M}_{720} & \text{M}_{330} & \text{M}_{120} & \text{M}_{715} & \text{M}_{1485} & \text{M}_{825} & \text{M}_{990} \\
\hline
\end{tabular}
\caption{Decompositions into matrix algebras of unordered pairs, triples and quads of various types.}
\end{table}

\footnote{The script used for the computation can be found here: \url{https://westerbaan.name/~bas/math/bags.py}}