A Recipe for State-and-Effect Triangles

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Abstract

In the semantics of programming languages one can view programs as state transformers, or as predicate transformers. Recently the author has introduced 'state-and-effect' triangles which captures this situation categorically, involving an adjunction between state- and predicate-transformers. The current paper exploits a classical result in category theory, part of Jon Beck's monadicity theorem, to systematically construct such a state-and-effect triangle from an adjunction. The power of this construction is illustrated in many examples, both for the Boolean and probabilistic (quantitative) case.

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1 Introduction

In program semantics three approaches can be distinguished.

- Interpreting programs themselves as morphisms in certain categories. Composition in the category then corresponds to sequential composition. Parallel composition may be modeled via tensors $\otimes$. Since [26] the categories involved are often Kleisli categories $\mathcal{K}\ell(T)$ of a monad T, where the monad T captures a specific form of computation: deterministic, non-deterministic, probabilistic, *etc.*
- Interpreting programs via their actions on states, as *state transformers*. For instance, in probabilistic programming the states may be probabilistic distributions over certain valuations (mapping variables to values). Execution of a program changes the state, by adapting the probabilities of valuations. The state spaces often have algebraic structure, and take the form of Eilenberg-Moore categories $\mathcal{EM}(T)$ of a monad T.
- Interpreting programs via their actions on predicates, as *predicate transformers*. The predicates involved describe what holds (is true) at a specific point. Execution of a program may then adapt the validity of predicates. A particular form of semantics of this sort is weakest precondition computation [6]. In the context of (coalgebraic) modal logic, these predicate transformers appear as modal operators.

A systematic picture of these three approaches has emerged in categorical language, using triangles of the form described below, see [15], and also [13, 14].

$$
\begin{tikzcd}
\boxed{\textbf{Heisenberg}} & & \boxed{\textbf{Schrödinger}} \\
\mathbf{Log}^{\mathrm{op}} = \begin{pmatrix}\text{predicate}\\ \text{transformers}\end{pmatrix} \arrow[rr, bend left=10] \arrow[rr, phantom, "\top"] & & \begin{pmatrix}\text{state}\\ \text{transformers}\end{pmatrix} \arrow[ll, bend left=10] \\
& \Big(\text{computations}\Big) \arrow[ul, "\text{Pred}"] \arrow[ur, "\text{Stat"'] &
\end{tikzcd}
\tag{1}
$$

The three nodes in this diagram represent categories of which only the morphisms are described. The arrows between these nodes are functors, where the two arrows $\rightleftarrows$ at the top form an adjunction. The two triangles involved should commute. In the case where two up-going ‘predicate’ and ‘state’ functors Pred and Stat in (1) are full and faithful, we have three equivalent ways of describing computations. On morphisms, the predicate functor yields what is called substitution in categorical logic, but what amounts to a weakest precondition operation in program semantics, or a modal operator in programming logic. The upper category on the left is of the form $\mathbf{Log}^{\mathrm{op}}$, where $\mathbf{Log}$ is some category of logical structures. The opposite category $(-)^{\mathrm{op}}$ is needed because predicate transformers operate in the reverse direction, taking a postcondition to a precondition.

In a setting of quantum computation this translation back-and-forth $\rightleftarrows$ in (1) is associated with the different approaches of Heisenberg (logic-based, working backwards) and Schrödinger (state-based, working forwards), see *e.g.* [12]. In quantum foundations one speaks of the duality between states and effects (predicates). Since the above triangles first emerged in the context of semantics of quantum computation [14], they are sometimes referred to as ‘state-and-effect’ triangles.

In certain cases the adjunction $\rightleftarrows$ in (1) forms — or may be restricted to — an equivalence of categories, yielding a duality situation. It shows the importance of duality theory in program semantics and logic; this topic has a long history, going back to [1].

In [14] it is shown that in the presence of relatively weak structure in a category $\mathbf{B}$, a diagram of the form (1) can be formed, with $\mathbf{B}$ as base category of computations, with predicates forming effect modules (see below) and with states forming convex sets. A category with this relatively weak structure is now called an *effectus*, see [20].

The main contribution of this paper is a “new” way of generating state-and-effect triangles, namely from adjunctions. We write the word ‘new’ between quotes, because the underlying category theory uses a famous of result of Jon Beck, and is not new at all. What the paper contributes is mainly a new perspective: it reorganises the work of Beck in such a way that an appropriate triangle appears, see Section 2. The rest of the paper is devoted to illustrations of this recipe for triangles. These examples are either of a Boolean or a probabilistic nature, see Sections 3 and 4 respectively. The Boolean examples are all obtained from an adjunction using “homming into $\{0,1\}$”, whereas the probabilistic (quantitative) examples all arise from “homming into $[0,1]$”, where $[0,1]$ is the unit interval of probabilities.

The series of examples in this paper involves many mathematical structures, ranging from Boolean algebras to compact Hausdorff spaces and C^*-algebras. It is impossible to explain all these notions in detail here. Hence the reader is assumed to be reasonably familiar with these structures. It does not matter so much if some of the examples involve unfamiliar mathematical notions. The structure of these sections 3 and 4 is clear enough, and it does not matter if some of the examples are skipped.

An exception is made for the notions of effect algebra and effect module. They are explicitly explained (briefly) in the beginning of Section 4 because they play such a prominent role in quantitative logic.

The examples involve many adjunctions that are known in the literature. Here they are displayed in triangle form. In several cases monads arise that are familiar in coalgebraic research, like the neighbourhood monad $\mathcal{N}$ in Subsection 3.1, the monotone neighbourhood monad $\mathcal{M}$ in Subsection 3.2, the infinite distribution monad $\mathcal{D}_\infty$ in Subsection 4.4, and the Giry monad $\mathcal{G}$ in Subsection 4.5. Also we will see several examples where we have pushed the recipe to a limit, and where the monad involved is simply the identity.

2 A basic result about monads

We assume that the reader is familiar with the categorical concept of a monad T, and with its double role, describing a form of computation, via the associated Kleisli category $\mathcal{K}\ell(T)$, and describing algebraic structure, via the category $\mathcal{EM}(T)$ of Eilenberg-Moore algebras.

The following result is a basic part of the theory of monads, see *e.g.* [3, Prop. 3.15 and Exercise (KEM)] or [22, Prop. 6.5 and 6.7] or [2, Thm. 20.42], and describes the initiality and finality of the Kleisli category and Eilenberg-Moore category as 'adjunction resolutions' giving rise to a monad.

▶ **Theorem 1.** *Consider an adjunction $F \dashv G$ with induced monad $T = GF$. Then there are 'comparison' functors $\mathcal{K}\ell(T) \to \mathbf{A} \to \mathcal{EM}(T)$ in a diagram:*

$$\begin{tikzcd}
\mathcal{K}\ell(T) \arrow[rr, "L"] \arrow[drr, bend left=10] \arrow[drr, phantom, "\top"] & & \mathbf{A} \arrow[rr, bend left=10, "K"] \arrow[rr, phantom, "\top"] \arrow[d, bend left, "G"] \arrow[d, phantom, "\dashv"] & & \mathcal{EM}(T) \arrow[ll, dotted, bend left=10, "M"] \arrow[dll, bend left=10] \arrow[dll, phantom, "\perp"] \\
& & \mathbf{B} \arrow[ull, bend left=10] \arrow[u, bend left, "F"] \arrow[urr, bend left=10] \arrow[loop below, "T=GF"] & &
\end{tikzcd} \tag{2}$$

where the functor $L\colon \mathcal{K}\ell(T) \to \mathbf{A}$ is full and faithful.

In case the category $\mathbf{A}$ has coequalisers (of reflexive pairs), then K has a left adjoint M, as indicated via the dotted arrow, satisfying $MKL \cong L$.

The famous monadicity theorem of Jon Beck gives conditions that guarantee that the functor $K\colon \mathbf{A} \to \mathcal{EM}(T)$ is an equivalence of categories, so that objects of $\mathbf{A}$ are algebras. The existence of the left adjoint M is the part of this theorem that we use in the current setting. Other (unused) parts of Beck's theorem require that the functor G preserves and reflects coequalisers of reflexive pairs. For convenience we include a proof sketch.

Proof. Define $L(X) = F(X)$ and $L\big(X \xrightarrow{f} GF(Y)\big) = \varepsilon_{F(Y)} \circ F(f)\colon F(X) \to F(Y)$. This functor L is full and faithful because there is a bijective adjoint correspondence:

$$\begin{array}{c} F(X) \longrightarrow F(Y) \\ \hline\hline X \longrightarrow GF(Y) = T(Y) \end{array}$$

The functor $K\colon \mathbf{A} \to \mathcal{EM}(T)$ is defined as:

$$K(A) = \begin{pmatrix} GFG(A) \\ \big\downarrow G(\varepsilon_A) \\ G(A) \end{pmatrix} \qquad \text{and } K\big(A \xrightarrow{f} B\big) = G(f).$$

We leave it to the reader to see that K is well-defined. For a Kleisli map $f\colon X \to T(Y)$ the map $KL(f)$ is Kleisli extension:

$$KL(f) \;=\; G(\varepsilon_{F(Y)} \circ F(f)) \;=\; \mu_Y \circ T(f) : \; T(X) \longrightarrow T(Y).$$

Assume that the category $\mathbf{A}$ has coequalisers. For an algebra $a\colon T(X) \to X$ let $M(X,a)$ be the (codomain of the) coequaliser in:

$$\begin{tikzcd} FGF(X) \arrow[r, shift left, "F(a)"] \arrow[r, shift right, "\varepsilon_{F(X)}"'] & F(X) \arrow[r, two heads, "c"] & M(X,a) \end{tikzcd}$$

It is not hard to see that there is a bijective correspondence:

$$\frac{M(X,a) \xrightarrow{\;f\;} A \qquad\qquad \text{in } \mathbf{A}}{\left(\begin{array}{c} T(X) \\ \downarrow a \\ X \end{array}\right) \xrightarrow[\;g\;]{} \left(\begin{array}{c} TG(A) \\ \downarrow G(\varepsilon_A) \\ G(A) \end{array}\right) = K(A) \qquad \text{in } \mathcal{EM}(T)}$$

What remains is to show $MKL \cong L$. This follows because for each $X \in \mathbf{B}$, the following diagram is a coequaliser in $\mathbf{A}$.

$$\begin{tikzcd} FGFGF(X) \arrow[r, shift left, "F(\mu_X)=FG(\varepsilon_{F(X)})"] \arrow[r, shift right, "\varepsilon_{FGF(X)}"'] & FGF(X) \arrow[r, two heads, "\varepsilon_{F(X)}"] & F(X) \end{tikzcd}$$

Hence the codomain $MKL(X)$ of the coequaliser of $FKL(X) = FG(\varepsilon_{F(X)})$ and the counit map $\varepsilon_{FGF(X)}$ is isomorphic to $F(X) = L(X)$. Proving naturality of $MKL \cong L$ (wrt. Kleisli maps) is a bit of work, but is essentially straightforward. ◀

An essential 'aha moment' underlying this paper is that the above result can be massaged into triangle form. This is what happens in the next result, to which we will refer as the 'triangle corollary'. It is the 'recipe' that occurs in the title of this paper.

▶ **Corollary 2.** *Consider an adjunction $F \dashv G$, where F is a functor $\mathbf{B} \to \mathbf{A}$, the category $\mathbf{A}$ has coequalisers, and the induced monad on $\mathbf{B}$ is written as $T = GF$. Diagram* (2) *then gives rise to a triangle as below, where both up-going functors are full and faithful.*

$$\begin{tikzcd} \mathbf{A} \arrow[rr, bend left, "K"] \arrow[rr, phantom, "\top"] & & \mathcal{EM}(T) \arrow[ll, bend left, "M"] \\ & \mathcal{K}\ell(T) \arrow[ul, "\mathrm{Pred}=L"] \arrow[ur, "KL=\mathrm{Stat}"'] & \end{tikzcd} \tag{3}$$

This triangle commutes, trivially from left to right, and up-to-isomorphism from right to left, since $MKL \cong L$. In this context we refer to the functor L as the 'predicate' functor Pred, *and to the functor KL as the 'states' functor* Stat.

The remainder of the paper is devoted to instances of this triangle corollary. In each of these examples the category $\mathbf{A}$ will be of the form $\mathbf{P}^{\mathrm{op}}$, where $\mathbf{P}$ is a category of predicates (with equalisers). The full and faithfulness of the functors $\mathrm{Pred}\colon \mathcal{K}\ell(T) \to \mathbf{P}^{\mathrm{op}}$ and $\mathrm{Stat}\colon \mathcal{K}\ell(T) \to \mathcal{EM}(T)$ means that there are bijective correspondences between:

$$\frac{X \xrightarrow{\text{computations}} T(Y)}{\mathrm{Pred}(Y) \xrightarrow[\text{predicate transformers}]{} \mathrm{Pred}(X)} \qquad \frac{X \xrightarrow{\text{computations}} T(Y)}{\mathrm{Stat}(X) \xrightarrow[\text{state transformers}]{} \mathrm{Stat}(Y)} \tag{4}$$

Since $\mathrm{Stat}(X) = T(X)$, the correspondence on the right is given by Kleisli extension, sending a map $f\colon X \to T(Y)$ to $\mu \circ T(f)\colon T(X) \to T(Y)$. This bijective correspondence on the right is a categorical formality. But the correspondence on the left is much more interesting, since it precisely describes to which kind of predicate transformers (preserving which structure) computations correspond. This will be illustrated below.

Aside: as discussed in [14], the predicate functor $\mathrm{Pred}\colon \mathcal{K}\ell(T) \to \mathbf{A}$ is in some cases an *enriched* functor, preserving additional structure that is of semantical/logical relevance. For instance, operations on programs, like $\cup$ for non-deterministic sum, may be expressed as structure on Kleisli homsets. Preservation of this structure by the functor Pred gives the logical rules for dealing with such structure in weakest precondition computations. These enriched aspects will not be elaborated in the current context.

3 Boolean examples

We split our series of examples in two parts, namely into Boolean and probabilistic examples. The Boolean ones are obtained via adjunctions that involve 'homming into 2', where $2 = \{0, 1\}$ is the 2-element set of Booleans. The probabilistic (aka. quantitative) examples in the next section are obtained via 'homming into $[0, 1]$', where $[0, 1] \subseteq \mathbb{R}$ is the unit interval of probabilities.

3.1 Sets and sets

We will present examples in the following manner, in three stages.

$$\begin{tikzcd}
\mathbf{Sets}^{\mathrm{op}} \arrow[d, bend left, "\mathcal{P}=\mathrm{Hom}(-,2)"] \arrow[d, phantom, "\dashv"] \\
\mathbf{Sets} \arrow[u, bend left, "\mathcal{P}=\mathrm{Hom}(-,2)"] \arrow[loop below, "\mathcal{N}=\mathcal{P}\mathcal{P}"]
\end{tikzcd}
\qquad
\begin{array}{c}
\mathcal{P}(X) \xrightarrow{\mathbf{Sets}^{\mathrm{op}}} Y \\
\hline\hline
Y \xrightarrow{\mathbf{Sets}} \mathcal{P}(X) \\
\hline\hline
X \xrightarrow[\mathbf{Sets}]{} \mathcal{P}(Y)
\end{array}
\qquad
\begin{tikzcd}
\mathbf{Sets}^{\mathrm{op}} \arrow[rr, bend left] \arrow[rr, phantom, "\top"] & & \mathcal{E}\mathcal{M}(\mathcal{N}) = \mathbf{CABA} \arrow[ll, bend left] \\
& \mathcal{K}\ell(\mathcal{N}) \arrow[ul, "\mathrm{Pred}"] \arrow[ur, "\mathrm{Stat}"'] &
\end{tikzcd}$$

On the left we describe the adjunction that forms the basis for the example at hand, together with the induced monad. In this case we have the familiar fact that the powerset functor $\mathcal{P}$ is adjoint to itself, as indicated. The induced double-powerset monad $\mathcal{P}\mathcal{P}$ is known in the coalgebra/modal logic community as the neighbourhood monad $\mathcal{N}$, because its coalgebras are related to neighbourhood frames in modal logic.

In the middle the bijective correspondence is described that forms the basis of the adjunction. In this case there is the obvious correspondence between functions $Y \to \mathcal{P}(X)$ and functions $X \to \mathcal{P}(Y)$ — which are all relations on $X \times Y$.

On the right the result is shown of applying the triangle corollary 2 to the adjunction on the left. The full and faithfulness of the predicate functor $\mathrm{Pred}\colon \mathcal{K}\ell(\mathcal{N}) \to \mathbf{Sets}^{\mathrm{op}}$ plays an important role in the approach to coalgebraic dynamic logic in [11], relating coalgebras $X \to \mathcal{N}(X)$ to predicate transformer functions $\mathcal{P}(X) \to \mathcal{P}(X)$, going in the opposite direction. The category $\mathcal{E}\mathcal{M}(\mathcal{N})$ of Eilenberg-Moore algebras of the neighbourhood monad $\mathcal{N}$ is the category **CABA** of complete atomic Boolean algebras (see *e.g.* [28]). The adjunction $\mathbf{Sets}^{\mathrm{op}} \rightleftarrows \mathcal{E}\mathcal{M}(\mathcal{N})$ is thus an equivalence.

3.2 Sets and posets

We now restrict the adjunction in the previous subsection to posets.

$$\begin{tikzcd} \mathbf{PoSets}^{\mathrm{op}} \arrow[d, bend left, "\mathrm{Up}=\mathrm{Hom}(-,2)"] \arrow[d, phantom, "\dashv"] \\ \mathbf{Sets} \arrow[u, bend left, "\mathcal{P}=\mathrm{Hom}(-,2)"] \arrow[loop below, "\mathcal{M}=\mathrm{Up}\mathcal{P}"] \end{tikzcd} \qquad \frac{Y \xrightarrow{\mathbf{PoSets}} \mathcal{P}(X)}{X \xrightarrow[\mathbf{Sets}]{} \mathrm{Up}(Y)} \qquad \begin{tikzcd} \mathbf{PoSets}^{\mathrm{op}} \arrow[rr, bend left] \arrow[rr, phantom, "\top"] & & \mathcal{EM}(\mathcal{M}) = \mathbf{CDL} \arrow[ll, bend left] \\ & \mathcal{K}\ell(\mathcal{M}) \arrow[ul, "\mathrm{Pred}"] \arrow[ur, "\mathrm{Stat}"'] & \end{tikzcd}$$

The functor $\mathrm{Up}\colon \mathbf{PoSets}^{\mathrm{op}} \to \mathbf{Sets}$ sends a poset Y to the collection of upsets $U \subseteq Y$, satisfying $y \geq x \in U$ implies $y \in U$. These upsets can be identified with monotone maps $p\colon Y \to 2$, namely as $p^{-1}(1)$.

Notice that this time there is a bijective correspondence between computations $X \to \mathcal{M}(Y) = \mathrm{Up}\mathcal{P}(Y)$ and *monotone* predicate transformers $\mathcal{P}(Y) \to \mathcal{P}(X)$. This fact is used in [11]. The algebras of the monad $\mathcal{M}$ are completely distributive lattices, see [24] and [21, I, Prop. 3.8].

3.3 Sets and meet-semilattices

We now restrict the adjunction further to meet semilattices, that is, to posets with finite meets $\wedge, \top$.

$$\begin{tikzcd} \mathbf{MSL}^{\mathrm{op}} \arrow[d, bend left, "\mathrm{Hom}(-,2)"] \arrow[d, phantom, "\dashv"] \\ \mathbf{Sets} \arrow[u, bend left, "\mathcal{P}=\mathrm{Hom}(-,2)"] \arrow[loop below, "\mathcal{F}=\mathbf{MSL}(\mathcal{P}(-),2)"] \end{tikzcd} \qquad \frac{Y \xrightarrow{\mathbf{MSL}} \mathcal{P}(X)}{X \xrightarrow[\mathbf{Sets}]{} \mathbf{MSL}(Y,2)} \qquad \begin{tikzcd} \mathbf{MSL}^{\mathrm{op}} \arrow[rr, bend left] \arrow[rr, phantom, "\top"] & & \mathcal{EM}(\mathcal{F}) = \mathbf{CCL} \arrow[ll, bend left] \\ & \mathcal{K}\ell(\mathcal{F}) \arrow[ul, "\mathrm{Pred}"] \arrow[ur, "\mathrm{Stat}"'] & \end{tikzcd}$$

Morphisms in the category **MSL** of meet semilattices preserve the meet $\wedge$ and the top element $\top$ (and hence the order too). For $Y \in \mathbf{MSL}$ one can identify a map $Y \to 2$ with a *filter* of Y, that is, with an upset $U \subseteq Y$ closed under $\wedge, \top$.

The resulting monad $\mathcal{F}(X) = \mathbf{MSL}(\mathcal{P}(X), 2)$ gives the filters in $\mathcal{P}(X)$. This monad is thus called the *filter monad*. In [29] it is shown that its category of algebras $\mathcal{EM}(\mathcal{F})$ is the category **CCL** of continuous complete lattices, that is, of complete lattices in which each element x is the (directed) join $x = \bigvee\{y \mid y \ll x\}$ of the elements way below it.

3.4 Sets and Boolean algebras

We further restrict the adjunction to the category **BA** of Boolean algebras.

$$\begin{tikzcd} \mathbf{BA}^{\mathrm{op}} \arrow[d, bend left, "\mathrm{Hom}(-,2)"] \arrow[d, phantom, "\dashv"] \\ \mathbf{Sets} \arrow[u, bend left, "\mathcal{P}=\mathrm{Hom}(-,2)"] \arrow[loop below, "\mathcal{U}=\mathbf{BA}(\mathcal{P}(-),2)"] \end{tikzcd} \qquad \frac{Y \xrightarrow{\mathbf{BA}} \mathcal{P}(X)}{X \xrightarrow[\mathbf{Sets}]{} \mathbf{BA}(Y,2)} \qquad \begin{tikzcd} \mathbf{BA}^{\mathrm{op}} \arrow[rr, bend left] \arrow[rr, phantom, "\top"] & & \mathcal{EM}(\mathcal{U}) = \mathbf{CH} \arrow[ll, bend left] \\ & \mathcal{K}\ell(\mathcal{U}) \arrow[ul, "\mathrm{Pred}"] \arrow[ur, "\mathrm{Stat}"'] & \end{tikzcd}$$

The functor $\mathrm{Hom}(-,2)\colon \mathbf{BA}^{\mathrm{op}} \to \mathbf{Sets}$ sends a Boolean algebra Y to the set $\mathbf{BA}(Y, 2)$ of Boolean algebra maps $Y \to 2$. They can be identified with *ultrafilters* of Y. The resulting monad $\mathcal{U} = \mathbf{BA}(\mathcal{P}(-), 2)$ is the *ultrafilter monad*, sending a set X to the BA-maps $\mathcal{P}(X) \to 2$, or equivalently, the ultrafilters of $\mathcal{P}(X)$.

An important result of Manes (see [23], and also [21, III, 2.4]) says that the category of Eilenberg-Moore algebras of the ultrafilter monad $\mathcal{U}$ is the category **CH** of compact Hausdorff spaces. This adjunction $\mathbf{BA}^{\mathrm{op}} \rightleftarrows \mathbf{CH}$ restricts to an equivalence $\mathbf{BA}^{\mathrm{op}} \simeq \mathbf{Stone}$ called Stone duality, where $\mathbf{Stone} \hookrightarrow \mathbf{CH}$ is the full subcategory of Stone spaces — in which each open subset is the union of the clopens contained in it.

3.5 Sets and complete Boolean algebras

We can restrict the adjunction $\mathbf{BA}^{\mathrm{op}} \rightleftarrows \mathbf{Sets}$ from the previous subsection to an adjunction $\mathbf{CBA}^{\mathrm{op}} \rightleftarrows \mathbf{Sets}$ between *complete* Boolean algebras and sets. The resulting monad on **Sets** is of the form $X \mapsto \mathbf{CBA}(\mathcal{P}(X), 2)$. But here we hit a wall, since this monad is the identity.

▶ **Lemma 3.** *For each set X the unit map $\eta\colon X \to \mathbf{CBA}(\mathcal{P}(X), 2)$, given by $\eta(x)(U) = 1$ iff $x \in U$, is an isomorphism.*

Proof. Let $h\colon \mathcal{P}(X) \to 2$ be a map of complete Boolean algebras, preserving the BA-structure and all joins (unions). Since each subset $U \in \mathcal{P}(X)$ can be described as union of singletons, the function h is determined by its values $h(\{x\})$ for $x \in X$. We have $1 = h(X) = \bigcup_{x\in X} h(\{x\})$. Hence $h(\{x\}) = 1$ for some $x \in X$. But then $h(X - \{x\}) = h(\neg\{x\}) = \neg h(\{x\}) = \neg 1 = 0$. This implies $h(\{x'\}) = 0$ for each $x' \neq x$. But then $h = \eta(x)$. ◀

4 Probabilistic examples

The next series of examples starts from adjunctions that are obtained by homming into the unit interval $[0, 1]$. The quantitative logic that belongs to these examples is given in terms of effect modules. These can be seen as "probabilistic vector spaces", involving scalar multiplication with scalars from the unit interval $[0, 1]$, instead of from $\mathbb{R}$ or $\mathbb{C}$. We provide a crash course for these structures, and refer to [17, 15] or [7] for more information.

A partial commutative monoid (PCM) consists of a set M with a partial binary operation $\ovee$ and a zero element $0 \in M$. The operation $\ovee$ is commutative and associative, in an appropriate partial sense. One writes $x \perp y$ if $x \ovee y$ is defined.

An *effect algebra* is a PCM with an orthocomplement $(-)^{\perp}$, so that $x \ovee x^{\perp} = 1$, where $1 = 0^{\perp}$, and $x \perp 1$ implies $x = 0$. An effect algebra is automatically a poset, via the definition $x \leq y$ iff $x \ovee z = y$ for some z. The main example is the unit interval $[0, 1]$, with $x \perp y$ iff $x + y \leq 1$, and in that case $x \ovee y = x + y$; the orthocomplement is $x^{\perp} = 1 - x$. A map of effect algebras $f\colon E \to D$ is a function that preserves 1 and $\ovee$, if defined. We write **EA** for the resulting category. Each Boolean algebra is an effect algebra, with $x \perp y$ iff $x \wedge y = 0$, and in that case $x \ovee y = x \vee y$. This yields a functor $\mathbf{BA} \to \mathbf{EA}$, which is full and faithful.

An *effect module* is an effect algebra E with an action $[0, 1] \times E \to E$ that preserves $\ovee, 0$ in each argument separately. A map of effect modules f is a map of effect algebras that preserves scalar multiplication: $f(r \cdot x) = r \cdot f(x)$. We thus get a subcategory $\mathbf{EMod} \hookrightarrow \mathbf{EA}$. For each set X, the set $[0, 1]^X$ of fuzzy predicates on X is an effect module, with $p \perp q$ iff $p(x)+q(x) \leq 1$ for all $x \in X$, and in that case $(p \ovee q)(x) = p(x) + q(x)$. Orthocomplement is given by $p^{\perp}(x) = 1 - p(x)$ and scalar multiplication by $r \cdot p \in [0, 1]^X$, for $r \in [0, 1]$ and $p \in [0, 1]^X$, by $(r \cdot p)(x) = r \cdot p(x)$. This assignment $X \mapsto [0, 1]^X$ yields a functor $\mathbf{Sets} \to \mathbf{EMod}^{\mathrm{op}}$ that will be used below. Important examples of effect modules arise in quantum logic. For instance, for each Hilbert space $\mathcal{H}$, the set $\mathcal{E}f(\mathcal{H}) = \{A\colon \mathcal{H} \to \mathcal{H} \mid 0 \leq A \leq \mathrm{id}\}$ of effects is an effect module. More generally, for a (unital) C^*-algebra A, the set of effects $[0, 1]_A = \{a \in A \mid 0 \leq a \leq 1\}$ is an effect module. In [8] it is shown that taking effects yields a full and faithful functor:

$$\mathbf{Cstar}_{\mathrm{PU}} \xrightarrow{[0,1]_{(-)}} \mathbf{EMod} \tag{5}$$

Here we write $\mathbf{Cstar}_{\mathrm{PU}}$ for the category of C^*-algebras with positive unital maps.

An *MV-algebra* [5] can be understood as a 'commutative' effect algebra. It is an effect algebra with a join $\vee$, and thus also a meet $\wedge$, via De Morgan, in which the equation $(x \vee y)^\perp \ovee x = y^\perp \ovee (x \wedge y)$ holds. There is a subcategory $\mathbf{MVA} \hookrightarrow \mathbf{EA}$ with maps additionally preserving joins $\vee$ (and hence also $\wedge$). Within an MV-algebra one can define (total) addition and subtraction operations as $x + y = x \ovee (x^\perp \wedge y)$ and $x - y = (x^\perp + y)^\perp$. The unit interval $[0,1]$ is an MV-algebra, in which $+$ and $-$ are truncated (to 1 or 0), if needed.

There is a category $\mathbf{MVMod}$ of *MV-modules*, which are MV-algebras with $[0,1]$-scalar multiplication. Thus $\mathbf{MVMod}$ is twice a subcategory in: $\mathbf{MVA} \hookleftarrow \mathbf{MVMod} \hookrightarrow \mathbf{EMod}$. The effect module $[0,1]^X$ of fuzzy predicates is an MV-module. For a commutative C^*-algebra A the set of effects $[0,1]_A$ is an MV-module. In fact there is a full and faithful functor:

$$\mathbf{CCstar}_{\mathrm{MIU}} \xrightarrow{[0,1]_{(-)}} \mathbf{MVMod} \tag{6}$$

where $\mathbf{CCstar}_{\mathrm{MIU}}$ is the category of commutative C^*-algebras, with MIU-maps, preserving multiplication, involution and unit (aka. $*$-homomorphisms).

Having seen this background information we continue our series of examples.

4.1 Sets and effect modules

As noted above, fuzzy predicates yield a functor $\mathbf{Sets} \to \mathbf{EMod}^{\mathrm{op}}$. This functor involves homming into $[0,1]$, and has an adjoint that is used as starting point for several variations.

$$\begin{tikzcd}
\mathbf{EMod}^{\mathrm{op}} \arrow[d, bend left, "\mathrm{Hom}(-,[0,1])"] \arrow[d, phantom, "\dashv"] \\
\mathbf{Sets} \arrow[u, bend left, "\mathrm{Hom}(-,[0,1])"] \arrow[loop below, "\mathcal{E}=\mathbf{EMod}([0,1]^{(-)},[0,1])"]
\end{tikzcd}
\qquad
\frac{Y \xrightarrow[\mathbf{EMod}]{} [0,1]^X}{X \xrightarrow[\mathbf{Sets}]{} \mathbf{EMod}(Y,[0,1])}
\qquad
\begin{tikzcd}
\mathbf{EMod}^{\mathrm{op}} \arrow[rr, bend left, ""] \arrow[rr, phantom, "\top"] & & \mathcal{EM}(\mathcal{E}) = \mathbf{CCH}_{\mathrm{sep}} \arrow[ll, bend left] \\
& \mathcal{K}\ell(\mathcal{E}) \arrow[ul, "\mathrm{Pred}"] \arrow[ur, "\mathrm{Stat}"'] &
\end{tikzcd}$$

The induced monad $\mathcal{E}$ is the *expectation* monad introduced in [16]. It can be understood as an extension of the (finite probability) distribution monad $\mathcal{D}$, since $\mathcal{E}(X) \cong \mathcal{D}(X)$ if X is a finite set. The triangle corollary on the right says in particular that Kleisli maps $X \to \mathcal{E}(Y)$ are in bijective correspondence with effect module maps $[0,1]^Y \to [0,1]^X$ acting as predicate transformers, on fuzzy predicates.

The category of algebras $\mathcal{EM}(\mathcal{E})$ of the expectation monad is the category $\mathbf{CCH}_{\mathrm{sep}}$ of convex compact Hausdorff spaces, with a separation condition (see [16, 18] for details). State spaces in quantum computing are typically such convex compact Hausdorff spaces.

Using the full and faithfulness of the functor $[0,1]_{(-)} : \mathbf{Cstar}_{\mathrm{PU}} \to \mathbf{EMod}$ from (5), the expectation monad can alternatively be described in terms of the states of the commutative C^*-algebra $\ell^\infty(X)$ of bounded functions $X \to \mathbb{C}$, via:

$$\begin{aligned}
\mathrm{Stat}(\ell^\infty(X)) \stackrel{\mathrm{def}}{=} \mathbf{Cstar}_{\mathrm{PU}}\big(\ell^\infty(X), \mathbb{C}\big) &\stackrel{(5)}{\cong} \mathbf{EMod}\big([0,1]_{\ell^\infty(X)}, [0,1]_{\mathbb{C}}\big) \\
&= \mathbf{EMod}\big([0,1]^X, [0,1]\big) \;=\; \mathcal{E}(X).
\end{aligned} \tag{7}$$

In this way one obtains the result from [8] that there is a full & faithful functor:

$$\mathcal{K}\ell(\mathcal{E}) \longrightarrow \big(\mathbf{CCstar}_{\mathrm{PU}}\big)^{\mathrm{op}} \tag{8}$$

embedding the Kleisli category $\mathcal{K}\ell(\mathcal{E})$ of the expectation monad into commutative C^*-algebras with positive unital maps. On objects this functor (8) is given by $X \mapsto \ell^\infty(X)$.

4.2 Compact Hausdorff spaces and effect modules

In the previous example we have used the *set* $\mathbf{EMod}(E,[0,1])$ of effect module maps $E \to [0,1]$, for an effect module E. It turns out that this homset has much more structure: it is a compact Hausdorff space. The reason is that the unit interval $[0,1]$ is compact Hausdorff, and so the function space $[0,1]^E$ too, by Tychonoff. The homset $\mathbf{EMod}(E,[0,1]) \hookrightarrow [0,1]^E$ can be described via a closed subset of maps satisfying the effect module map requirements. Hence $\mathbf{EMod}(E,[0,1])$ is compact Hausdorff itself. We thus obtain the following situation.

$$\begin{tikzcd}
\mathbf{EMod}^{\mathrm{op}} \arrow[d, bend left, "\mathrm{Hom}(-,[0,1])"] \arrow[d, phantom, "\dashv"] \\
\mathbf{CH} \arrow[u, bend left, "\mathrm{Hom}(-,[0,1])"] \arrow[loop below, "\mathcal{R}=\mathbf{EMod}(\mathrm{C}(-,[0,1]),[0,1])"]
\end{tikzcd}
\qquad
\begin{array}{c}
Y \xrightarrow{\mathbf{EMod}} \mathrm{C}(X,[0,1]) \\
\hline\hline
X \xrightarrow[\mathbf{CH}]{} \mathbf{EMod}(Y,[0,1])
\end{array}
\qquad
\begin{tikzcd}
\mathbf{EMod}^{\mathrm{op}} \arrow[rr, bend left] \arrow[rr, phantom, "\top"] & & \mathcal{EM}(\mathcal{R}) = \mathbf{CCH}_{\mathrm{sep}} \arrow[ll, bend left] \\
& \mathcal{K}\ell(\mathcal{R}) \arrow[ul, "\mathrm{Pred}"] \arrow[ur, "\mathrm{Stat}"'] &
\end{tikzcd}$$

For a compact Hausdorff space X, the subset $\mathrm{C}(X,[0,1]) \hookrightarrow [0,1]^X$ of continuous maps $X \to [0,1]$ is a (sub) effect module. The induced monad $\mathcal{R}(X) = \mathbf{EMod}\big(\mathrm{C}(X,[0,1]),[0,1]\big)$ is the *Radon* monad. Using the full & faithful functor (5) the monad can equivalently be described as $X \mapsto \mathrm{Stat}(\mathrm{C}(X))$, where $\mathrm{C}(X)$ is the commutative C^*-algebra of functions $X \to \mathbb{C}$. The monad occurs in [25] as part of a topological and domain-theoretic approach to information theory. The main result of [8] is the equivalence of categories

$$\mathcal{K}\ell(\mathcal{R}) \simeq \big(\mathbf{CCstar}_{\mathrm{PU}}\big)^{\mathrm{op}}$$

between the Kleisli category of this Radon monad $\mathcal{R}$ and the category of commutative C^*-algebras and positive unital maps. This shows how (commutative) C^*-algebras appear in state-and-effect triangles (see also [15]).

The algebras of the Radon monad are convex compact Hausdorff spaces (with separation), like for the expectation monad $\mathcal{E}$, see [9] for details.

4.3 Compact Hausdorff spaces and MV-modules

The adjunction $\mathbf{EMod}^{\mathrm{op}} \rightleftarrows \mathbf{CH}$ can be restricted to an adjunction $\mathbf{MVMod}^{\mathrm{op}} \rightleftarrows \mathbf{CH}$, involving MV-modules instead of effect modules. This can be done since continuous functions $X \to [0,1]$ are appropriately closed under joins $\vee$, and thus form an MV-module. Additionally, for an MV-module E, the MV-module maps $E \to [0,1]$ form a compact Hausdorff space (using the same argument as in the previous subsection).

Via this restriction to an adjunction $\mathbf{MVMod}^{\mathrm{op}} \rightleftarrows \mathbf{CH}$ we hit a wall again.

▶ **Lemma 4.** *For a compact Hausdorff space X, the unit $\eta\colon X \to \mathbf{MVMod}\big(\mathrm{C}(X,[0,1]),[0,1]\big)$, given by $\eta(x)(p) = p(x)$, is an isomorphism in $\mathbf{CH}$.*

This result can be understood as part of the Yosida duality for Riesz spaces. It is well-known in the MV-algebra community, but possibly not precisely in this form. For convenience, we include a proof.

Proof. We only show that the unit η is an isomorphism, not that it is also a homeomorphism. Injectivity is immediate by Urysohn. For surjectivity, we first establish the following two auxiliary results.

1. For each $p \in \mathrm{C}(X,[0,1])$ and $\omega \in \mathbf{MVMod}\big(\mathrm{C}(X,[0,1]),[0,1]\big)$, if $\omega(p) = 0$, then there is an $x \in X$ with $p(x) = 0$.
   If not, then $p(x) > 0$ for all $x \in X$. Hence there is an inclusion $X \subseteq \bigcup_{r>0} p^{-1}\big((r,1]\big)$. By compactness there are finitely many r_i with $X \subseteq \bigcup_i p^{-1}\big((r_i,1]\big)$. Thus for $r = \bigwedge_i r_i > 0$ we have $p(x) > r$ for all $x \in X$. Find an $n \in \mathbb{N}$ with $n \cdot r \geq 1$. The n-fold sum $n \cdot p$ in the MV-module $\mathrm{C}(X,[0,1])$ then satisfies $p(x) = 1$ for all x, so that $n \cdot p = 1$ in $\mathrm{C}(X,[0,1])$. But now we get a contradiction: $1 = \omega(1) = \omega(n \cdot p) = n \cdot \omega(p) = 0$.
2. For each finite collection of maps $p_1, \ldots, p_n \in \mathrm{C}(X,[0,1])$ and for each function $\omega \in \mathbf{MVMod}\big(\mathrm{C}(X,[0,1]),[0,1]\big)$ there is an $x \in X$ with $\omega(p_i) = p_i(x)$ for all $1 \leq i \leq n$.
   For the proof, define $p \in \mathrm{C}(X,[0,1])$ using the MV-structure of $\mathrm{C}(X,[0,1])$ as:
   $$p = \bigvee\nolimits_i \big(p_i - \omega(p_i) \cdot 1\big) \vee \big(\omega(p_i) \cdot 1 - p_i\big).$$
   Since the state $\omega\colon \mathrm{C}(X,[0,1]) \to [0,1]$ preserves the MV-structure we get in $[0,1]$:
   $$\omega(p) \;=\; \bigvee\nolimits_i \big(\omega(p_i) - \omega(p_i) \cdot 1\big) \vee \big(\omega(p_i) \cdot 1 - \omega(p_i)\big) \;=\; 0.$$
   Hence by the previous point there is an $x \in X$ with $p(x) = 0$. But then $p_i(x) = \omega(p_i)$, as required.

Now we can prove surjectivity of the unit map $\eta\colon X \to \mathbf{MVMod}\big(\mathrm{C}(X,[0,1]),[0,1]\big)$. Let $\omega\colon \mathrm{C}(X,[0,1]) \to [0,1]$ be an MV-module map. Define for each $p \in \mathrm{C}(X,[0,1])$ the subset $U_p = \{x \in X \mid \omega(p) \neq p(x)\}$. This subset $U_p \subseteq X$ is open since it can be written as $f^{-1}(\mathbb{R} - \{0\})$, for the continuous function $f(x) = p(x) - \omega(p)$.

Suppose towards a contradiction that $\omega \neq \eta(x)$ for all $x \in X$. Thus, for each $x \in X$ there is a $p \in \mathrm{C}(X,[0,1])$ with $\omega(p) \neq \eta(x)(p) = p(x)$. This means $X \subseteq \bigcup_p U_p$. By compactness of X there are finitely many $p_i \in \mathrm{C}(X,[0,1])$ with $X \subseteq \bigcup_i U_{p_i}$. The above second point however gives an $x \in X$ with $\omega(p_i) = p_i(x)$ for all i. But then $x \notin \bigcup_i U_{p_i}$. ◀

4.4 Sets and directed complete effect modules

In the remainder of this paper we shall consider effect modules with additional completeness properties (wrt. its standard order). Specifically, we consider ω-complete, and directed-complete effect modules. In the first case each ascending ω-chain $x_0 \leq x_1 \leq \cdots$ has a least upperbound $\bigvee_n x_n$; and in the second case each directed subset D has a join $\bigvee D$. We write the resulting subcategories as:

$$\mathbf{DcEMod} \hookrightarrow \omega\text{-}\mathbf{EMod} \hookrightarrow \mathbf{EMod}$$

where maps are required to preserve the relevant joins $\bigvee$.

We start with the directed-complete case. The adjunction $\mathbf{EMod}^{\mathrm{op}} \rightleftarrows \mathbf{Sets}$ from Subsection 4.1 can be restricted to an adjunction as on the left below.

$$\begin{tikzcd}
\mathbf{DcEMod}^{\mathrm{op}} \arrow[d, bend left, "\mathrm{Hom}(-,[0,1])"] \arrow[d, phantom, "\dashv"] \\
\mathbf{Sets} \arrow[u, bend left, "\mathrm{Hom}(-,[0,1])"] \arrow[loop below, "\mathcal{E}_\infty = \mathbf{DcEMod}([0,1]^{(-)},[0,1])"]
\end{tikzcd}
\qquad
\dfrac{Y \xrightarrow{\mathbf{DcEMod}} [0,1]^X}{X \xrightarrow[\mathbf{Sets}]{} \mathbf{DcEMod}(Y,[0,1])}
\qquad
\begin{tikzcd}
\mathbf{DcEMod}^{\mathrm{op}} \arrow[rr, bend left] \arrow[rr, phantom, "\top"] & & \mathcal{EM}(\mathcal{E}_\infty) = \mathbf{Conv}_\infty \arrow[ll, bend left] \\
& \mathcal{K}\ell(\mathcal{E}_\infty) \arrow[ul, "\mathrm{Pred}"] \arrow[ur, "\mathrm{Stat}"'] &
\end{tikzcd}$$

The resulting monad $\mathcal{E}_\infty = \mathbf{DcEMod}\big([0,1]^{(-)}, [0,1]\big)$ on **Sets** is in fact isomorphic[1] to the infinite (discrete probability) distribution monad $\mathcal{D}_\infty$. We recall, for a set X,

$$\mathcal{D}_\infty(X) \;=\; \{\omega\colon X \to [0,1] \mid \mathrm{supp}(\omega) \text{ is countable, and } \textstyle\sum_x \omega(x) = 1\}.$$

The subset $\mathrm{supp}(\omega) \subseteq X$ contains the elements $x \in X$ with $\omega(x) \neq 0$. The requirement in the definition of $\mathcal{D}_\infty(X)$ that $\mathrm{supp}(\omega)$ be countable is superfluous, since it follows from the requirement $\sum_x \omega(x) = 1$. Briefly, $\mathrm{supp}(\omega) \subseteq \bigcup_{n>0} X_n$, where $X_n = \{x \in X \mid \omega(x) > \frac{1}{n}\}$ contains at most $n-1$ elements (see *e.g.* [27, Prop. 2.1.2]).

▶ Proposition 5. There is an isomorphism of monads $\mathcal{D}_\infty \cong \mathcal{E}_\infty$, where $\mathcal{E}_\infty$ is the monad induced by the above adjunction $\mathbf{DcEMod}^{\mathrm{op}} \rightleftarrows \mathbf{Sets}$.

Proof. For a subset $U \subseteq X$ we write $\mathbf{1}_U\colon X \to [0,1]$ for the 'indicator' function, defined by $\mathbf{1}_U(x) = 1$ if $x \in U$ and $\mathbf{1}_U(x) = 0$ if $x \notin U$. We write $\mathbf{1}_x$ for $\mathbf{1}_{\{x\}}$. This function $\mathbf{1}_{(-)}\colon \mathcal{P}(X) \to [0,1]^X$ is a map of effect algebras that preserves all joins.

Let $h \in \mathcal{E}_\infty(X)$, so h is a Scott-continuous map of effect modules $h\colon [0,1]^X \to [0,1]$. Define $\overline{h}\colon X \to [0,1]$ as $\overline{h}(x) = h(\mathbf{1}_x)$. Notice that if $U \subseteq X$ is a finite subset, then:

$$1 \;=\; h(1) \;=\; h(\mathbf{1}_X) \;\geq\; h(\mathbf{1}_U) \;=\; h(\textstyle\bigovee_{x\in U} \mathbf{1}_x) \;=\; \textstyle\bigovee_{x\in U} h(\mathbf{1}_x) \;=\; \textstyle\bigovee_{x\in U} \overline{h}(x).$$

We can write X as directed union of its finite subsets, and thus also $\mathbf{1}_X = \bigvee\{\mathbf{1}_U \mid U \subseteq X \text{ finite}\}$. But then $\overline{h} \in \mathcal{D}_\infty(X)$, because h preserves directed joins:

$$1 \;=\; h(\mathbf{1}_X) \;=\; \textstyle\bigvee\{h(\mathbf{1}_U) \mid U \subseteq X \text{ finite}\} \;=\; \bigvee\{\sum_{x\in U} \overline{h}(x) \mid U \subseteq X \text{ finite}\} \;=\; \sum_{x\in X} \overline{h}(x).$$

Conversely, given $\omega \in \mathcal{D}_\infty(X)$ we define $\overline{\omega}\colon [0,1]^X \to [0,1]$ as $\overline{\omega}(p) = \sum_{x\in X} p(x)\cdot\omega(x)$. It is easy to see that $\overline{\omega}$ is a map of effect modules. It is a bit more challenging to see that it preserves directed joins $\bigvee_i p_i$, for $p_i \in [0,1]^X$.

First we write the countable support of ω as $\mathrm{supp}(\omega) = \{x_0, x_1, x_2, \ldots\} \subseteq X$ in such a way that $\omega(x_0) \geq \omega(x_1) \geq \omega(x_2) \geq \cdots$. We have $1 = \sum_{x\in X} \omega(x) = \sum_{n\in\mathbb{N}} \omega(x_n)$. Hence, for each $N \in \mathbb{N}$ we get:

$$\textstyle\sum_{n>N} \omega(x_n) \;=\; 1 - \sum_{n\leq N} \omega(x_n).$$

By taking the limit $N \to \infty$ on both sides we get:

$$\lim_{N\to\infty} \textstyle\sum_{n>N} \omega(x_n) \;=\; 1 - \displaystyle\lim_{N\to\infty} \textstyle\sum_{n\leq N} \omega(x_n) \;=\; 1 - \sum_{n\in\mathbb{N}} \omega(x_n) \;=\; 1 - 1 \;=\; 0.$$

We have to prove $\overline{\omega}(\bigvee_i p_i) = \bigvee_i \overline{\omega}(p_i)$. The non-trivial part is $(\leq)$. For each $N \in \mathbb{N}$ we have:

$$\begin{array}{rcl}
\overline{\omega}(\bigvee_i p_i) & = & \sum_{n\in\mathbb{N}} (\bigvee_i p_i)(x_n)\cdot\omega(x_n) \\
& = & \sum_{n\in\mathbb{N}} (\bigvee_i p_i(x_n))\cdot\omega(x_n) \\
& = & \sum_{n\in\mathbb{N}} \bigvee_i p_i(x_n)\cdot\omega(x_n) \\
& = & \Big(\sum_{n\leq N} \bigvee_i p_i(x_n)\cdot\omega(x_n)\Big) + \Big(\sum_{n>N} \bigvee_i p_i(x_n)\cdot\omega(x_n)\Big) \\
& = & \Big(\bigvee_i \sum_{n\leq N} p_i(x_n)\cdot\omega(x_n)\Big) + \Big(\sum_{n>N} \bigvee_i p_i(x_n)\cdot\omega(x_n)\Big) \\
& \leq & \Big(\bigvee_i \sum_{n\leq N} p_i(x_n)\cdot\omega(x_n)\Big) + \Big(\sum_{n>N} \omega(x_n)\Big) \qquad \text{since } p_i(x) \in [0,1].
\end{array}$$

[1] This isomorphism $\mathcal{E}_\infty \cong \mathcal{D}_\infty$ in Proposition 5 is inspired by work of Robert Furber (PhD Thesis, forthcoming): he noticed the isomorphism $\mathrm{NStat}(\ell^\infty(X)) \cong \mathcal{D}_\infty(X)$ in (11), which is obtained here as a corollary to Proposition 5.

Hence we are done by taking the limit $N \to \infty$. Notice that we use that the join $\bigvee$ can be moved outside a finite sum. This works precisely because the join is taken over a directed set.

What remains is to show that these mappings $h \mapsto \overline{h}$ and $\omega \mapsto \overline{\omega}$ yield an isomorphism $\mathcal{D}_\infty(X) \cong \mathcal{E}_\infty(X)$, which is natural in X, and forms an isomorphism of monads. This is left to the interested reader. ◀

As a result, the Eilenberg-Moore category $\mathcal{EM}(\mathcal{E}_\infty)$ is isomorphic to $\mathcal{EM}(\mathcal{D}_\infty) = \mathbf{Conv}_\infty$, where $\mathbf{Conv}_\infty$ is the category of countably-convex sets X, in which convex sums $\sum_{n\in\mathbb{N}} r_n x_n$ exist, where $x_n \in X$ and $r_n \in [0,1]$ with $\sum_n r_n = 1$.

We briefly look at the relation with C^*-algebras (actually W^*-algebras), like in Subsection 4.1. We write $\mathbf{Wstar}_{\mathrm{NPU}}$ for the category of W^*-algebras with normal positive unital maps. The term 'normal' is used in the operator algebra community for what is called 'Scott-continuity' (preservation of directed joins) in the domain theory community. This means that taking effects yields a full and faithful functor:

$$\mathbf{Wstar}_{\mathrm{NPU}} \xrightarrow{\;[0,1]_{(-)}\;} \mathbf{DcEMod} \tag{9}$$

This is similar to the situation in (5) and (6). One could also use AW^*-algebras here. Next, there is now a full and faithful functor to the category of commutative W^*-algebras:

$$\mathcal{K}\ell(\mathcal{D}_\infty) \cong \mathcal{K}\ell(\mathcal{E}_\infty) \longrightarrow \mathbf{CWstar}_{\mathrm{NPU}} \tag{10}$$

On objects it is given by $X \mapsto \ell^\infty(X)$. This functor is full and faithful since there is a bijective correspondence:

$$\begin{array}{ll} \ell^\infty(X) \longrightarrow \ell^\infty(Y) & \text{in } \mathbf{CWstar}_{\mathrm{NPU}} \\ \hline\hline Y \longrightarrow \mathrm{NStat}(\ell^\infty(X)) \cong \mathcal{E}_\infty(X) \cong \mathcal{D}_\infty(X) & \text{in } \mathbf{Sets} \end{array}$$

where the isomorphism $\cong$ describing normal states is given, like in (7), by:

$$\begin{array}{rcl} \mathrm{NStat}(\ell^\infty(X)) \stackrel{\mathrm{def}}{=} \mathbf{Wstar}_{\mathrm{NPU}}\big(\ell^\infty(X), \mathbb{C}\big) & \stackrel{(9)}{\cong} & \mathbf{DcEMod}\big([0,1]_{\ell^\infty(X)}, [0,1]_{\mathbb{C}}\big) \\ & = & \mathbf{DcEMod}\big([0,1]^X, [0,1]\big) \\ & = & \mathcal{E}_\infty(X) \\ & \cong & \mathcal{D}_\infty(X). \end{array} \tag{11}$$

4.5 Measurable spaces and ω-complete effect modules

In our final example we use an adjunction between effect modules and measurable spaces (instead of sets or compact Hausdorff spaces). We write **Meas** for the category of measurable spaces (X, Σ_X), where $\Sigma_X \subseteq \mathcal{P}(X)$ is the σ-algebra of measurable subsets, with measurable functions between them (whose inverse image maps measurable subsets to measurable subsets). We use the unit interval $[0,1]$ with its standard Borel σ-algebra (the least one that contains all the usual opens). A basic fact in this situation is that for a measurable space X, the set $\mathbf{Meas}(X, [0,1])$ of measurable functions $X \to [0,1]$ is an ω-effect module. The effect module structure is inherited via the inclusion $\mathbf{Meas}(X, [0,1]) \hookrightarrow [0,1]^X$. Joins of ascending ω-chains $p_0 \leq p_1 \leq \cdots$ exists, because the (pointwise) join $\bigvee_n p_n$ is a measurable function again. In this way we obtain a functor $\mathbf{Meas}(-, [0,1]) \colon \mathbf{Meas} \to \omega\text{-}\mathbf{EMod}^{\mathrm{op}}$.

In the other direction there is also a hom-functor $\omega\text{-}\mathbf{EMod}(-, [0,1]) \colon \omega\text{-}\mathbf{EMod}^{\mathrm{op}} \to \mathbf{Meas}$. For an ω-effect module E we can provide the set of maps $\omega\text{-}\mathbf{EMod}(E, [0,1])$ with a σ-algebra, namely the least one that makes all the evaluation maps $\mathrm{ev}_x \colon \omega\text{-}\mathbf{EMod}(E, [0,1]) \to$

$[0,1]$ measurable, for $x \in E$. This function ev_x is given by $\mathrm{ev}_x(p) = p(x)$. This gives the following situation.

$$\begin{tikzcd}
\omega\text{-}\mathbf{EMod}^{\mathrm{op}} \arrow[d, bend left, "\mathrm{Hom}(-,[0,1])"] \arrow[d, phantom, "\dashv"] \\
\mathbf{Meas} \arrow[u, bend left, "\mathrm{Hom}(-,[0,1])"] \arrow[loop below, "\mathcal{G}=\omega\text{-}\mathbf{EMod}(\mathbf{Meas}(-,[0,1]),[0,1])"]
\end{tikzcd}
\qquad
\frac{Y \xrightarrow{\omega\text{-}\mathbf{EMod}} \mathbf{Meas}(X,[0,1])}{X \xrightarrow[\mathbf{Meas}]{} \omega\text{-}\mathbf{EMod}(Y,[0,1])}
\qquad
\begin{tikzcd}
\omega\text{-}\mathbf{EMod}^{\mathrm{op}} \arrow[rr, bend left] \arrow[rr, phantom, "\top"] & & \mathcal{EM}(\mathcal{G}) \arrow[ll, bend left] \\
& \mathcal{K}\ell(\mathcal{G}) \arrow[ul, "\mathrm{Pred}"] \arrow[ur, "\mathrm{Stat}"'] &
\end{tikzcd}$$

We use the symbol $\mathcal{G}$ for the induced monad because of the following result.

▶ **Proposition 6.** The monad $\mathcal{G} = \omega\text{-}\mathbf{EMod}\big(\mathbf{Meas}(-,[0,1]),[0,1]\big)$ on **Meas** in the above situation is (isomorphic to) the Giry monad [10], given by probability measures:

$$\mathrm{Giry}(X) \stackrel{\text{def}}{=} \{\phi\colon \Sigma_X \to [0,1] \mid \phi \text{ is a probability measure}\} = \omega\text{-}\mathbf{EA}(\Sigma_X,[0,1]).$$

Proof. The isomorphism involves Lebesgue integration:

$$\begin{tikzcd}[column sep=huge]
\mathcal{G}(X) = \omega\text{-}\mathbf{EMod}\big(\mathbf{Meas}(X,[0,1]),[0,1]\big) \arrow[r, bend left, "I\mapsto(M\mapsto I(\mathbf{1}_M))"] \arrow[r, phantom, "\cong"] & \omega\text{-}\mathbf{EA}(\Sigma_X,[0,1]) = \mathrm{Giry}(X) \arrow[l, bend left, "\phi\mapsto(p\mapsto\int p\,\mathrm{d}\phi)"]
\end{tikzcd}$$

See [13] or [19] for more details. ◀

The above triangle is further investigated in [13]. It resembles the situation described in [4] for Markov kernels (the ordinary, not the abstract, ones).

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