CHARACTERISTIC CLASSES
FOR
IRREGULAR SINGULARITIES

Ron Sommeling

CHARACTERISTIC CLASSES FOR IRREGULAR SINGULARITIES

CHARACTERISTIC CLASSES FOR IRREGULAR SINGULARITIES

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van de Wiskunde en Informatica

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Chapter 1

Introduction and notations

1.1 Introduction

For a linear transformation one can define invariants and normal forms such as the determinant, eigenvalues, the characteristic polynomial, the Jordan normal form and the rational Jordan normal form.

In this thesis something analogous is done for differential operators. In this context a differential operator is an additive function $D: V \rightarrow V$ which satisfies the Leibniz rule:

$$D(av) = aDv + \tau(a)v \quad \text{for all } a \in K \text{ and } v \in V.$$

Here V is a finite dimensional vector space over K and K is a differential field of characteristic zero with a non-trivial derivation τ . In many cases K will be the field of the formal Laurent series $k((x))$ and τ will be the derivation $x \frac{d}{dx}$. An example of a differential operator on K^2 with $K = k((x))$ and $\tau = x \frac{d}{dx}$ is:

$$D: \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mapsto \begin{pmatrix} \tau y_1 \\ \tau y_2 \end{pmatrix} + \begin{pmatrix} 1 & 1/x^2 \\ x & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \quad (1.1)$$

The matrix of a differential operator D w.r.t. a K -basis $v = (v_1, \dots, v_n)$ of V will be denoted by $M(D, v)$. Thus for the differential operator in (1.1) one has

$$M(D, e) = \begin{pmatrix} 1 & 1/x^2 \\ x & -\frac{1}{2} \end{pmatrix},$$

where $e = (e_1, e_2)$ denotes the standard basis of K^2 .

A differential operator $D: V \rightarrow V$ gives V the structure of a $K[\tau]$ -module by the action $\tau \cdot v \stackrel{\text{def}}{=} Dv$ for all $v \in V$. Here $K[\tau]$ denotes the K -algebra generated by τ with relation $\tau \cdot a = a\tau + \tau(a)$ for all a in K . For fixed K and τ , the Abelian category of pairs (V, D) where D is a differential operator on V , will be denoted by $\mathcal{V}$.

The formal theory of differential operators with $K = k((x))$ and $\tau = x \frac{d}{dx}$ have been studied by Turrittin [Tur55], Levelt [Lev75], Moser [Mos60], Hilali [HW87], Babbitt-Varadarajan [BV83] and many others. Several invariants related to the nature of the singularity of the differential operator have been defined, e.g., the invariant of Moser (cf. [Mos60]) and Hilali (cf. [HW87]), the order of the singularity of Katz (cf. [Del70, II.1.10]), the irregularity index of Malgrange (cf. [Mal74]), the ρ -invariants of Gérard and Levelt (cf. [GL73]), the Newton-Puiseux polygon and the Newton-Puiseux polynomials (cf. [Die87]). These invariants and their connection with the characteristic class are described in section 3.7 and section 3.8. However these invariants give only partial information about the differential operator. For instance they don't give enough information to decide whether a differential operator is simple or whether two differential operators are isomorphic.

If the differential operator $D: V \rightarrow V$ has a regular singularity then it is well-known that there exists a K -basis $v = (v_1, \dots, v_n)$ of V such that $C \stackrel{\text{def}}{=} M(D, v)$ is a constant matrix and that the monodromy matrix $\exp(2\pi i C)$ determines the isomorphism class of the differential operator, cf. [Man65]. For semisimple differential operators $D: V \rightarrow V$ and $D': V' \rightarrow V'$ with a regular singularity this implies the following. Suppose $M(D, v) = C$ and $M(D', v') = C'$ for some K -bases v of V and v' of V' with C and C' constant matrices. Then the $K[\tau]$ -modules V and V' are isomorphic if and only if $\det(\lambda I - C) \equiv \det(\lambda I - C')$, where the equivalence relation $\equiv$ on $k[\lambda]$ is defined as the transitive closure of

$$\begin{cases} f(\lambda) \equiv f(\lambda + r) & \text{for all } f \in k[\lambda] \text{ and } r \in \mathbb{Z}; \\ f \equiv g \implies fh \equiv gh & \text{for all } f, g, h \in k[\lambda]. \end{cases}$$

If $D: V \rightarrow V$ is a differential operator with an irregular singularity, one can define a Jordan normal form (cf. [Lev75]), i.e., there exists a finite field extension $L = k'(K(t))$ of K (k' is a finite field extension of k and $t^e = x$ for some $e \in \mathbb{N}^*$) and an L -basis $v = (v_1, \dots, v_n)$ of $L \otimes_K V$ such that $M(D, v)$ is a block diagonal matrix with blocks of the form

$$\begin{pmatrix} \alpha_i & 1 & 0 & & 0 \\ 0 & \alpha_i & 1 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & & & 0 & \alpha_i \end{pmatrix}$$

and every $\alpha_i \in k'[1/t]$.

The characteristic polynomial of D is now defined as

$$\prod_i (\lambda - p(\alpha_i))^{m_i},$$

where m_i is the multiplicity of α_i in the Jordan normal form and the function p on $k'[1/t]$ is defined by

$$p(c_\tau t^\tau + \cdots + c_{-1}t^{-1} + c_0) \stackrel{\text{def}}{=} c_\tau t^\tau + \cdots + c_{-1}t^{-1} \quad (\tau \in \mathbb{Z}, c_\tau, \dots, c_0 \in k').$$

The characteristic polynomial is well-defined (independent of the field extension and the chosen basis) and is an element of $k[1/x][\lambda]$. However the characteristic polynomial doesn't determine the isomorphism class of the semisimple part of a differential operator, even if the constant terms of the α_i were taken into account somehow, since differential operators which are not isomorphic over K can become isomorphic over a finite field extension of K . The most simple example is given by the differential operators $x \frac{d}{dx}$ and $x \frac{d}{dx} + \frac{1}{2}$ which are *not* isomorphic over $k((x))$, but become isomorphic over $k((\sqrt{x}))$. The characteristic class of a differential operator which will be defined in chapter 3 doesn't have this deficiency. In chapter 4 a rational Jordan normal form of a differential operator is defined.

For a given differential operator $D: V \rightarrow V$ and a K -basis v of V , one can consider the Dieudonné determinant of the matrix $\tau I + M(D, v)$. This is an element of $\mathcal{L}(K)^{\text{ab}} \stackrel{\text{def}}{=} \mathcal{L}(K)/\mathcal{L}(K)'$ where $\mathcal{L}(K)$ is the (skew) quotient field of $K[\tau]$ and $\mathcal{L}(K)'$ is the commutator subgroup of $\mathcal{L}(K)$. The Dieudonné determinant doesn't depend on the K -basis v and there exists an algorithm to compute the Dieudonné determinant of a given differential operator (cf. [Adj85]). An element representing the Dieudonné determinant of D in (1.1) is $\tau^2 - \frac{1}{2}\tau - 1/x$.

In chapter 2 it is shown that the function $\det: \mathcal{V} \rightarrow \mathcal{L}(K)^{\text{ab}}$ induces an *isomorphism* from the Grothendieck group $K_0(\mathcal{V})$ to $\mathcal{L}^0(K)^{\text{ab}}$, where $\mathcal{L}^0(K)$ denotes the multiplicative subgroup of the monic differential operators in $\mathcal{L}(K)$. This result should be compared with [KS68, p. 327] and [Alm74].

From chapter 3 on, we take $K = k((x))$ (the formal Laurent series) and $\tau = x \frac{d}{dx}$. The Newton diagram and some other well-known invariants of a differential operator follow directly from its Dieudonné determinant, but it is not clear how one can determine whether two given elements in $\mathcal{L}^0(K)^{\text{ab}}$ are equal, nor how to define a multiplication on $\mathcal{L}^0(K)^{\text{ab}}$ corresponding to the multiplication on $K_0(\mathcal{V})$ (from the tensor product). However in section 3.5 we define another invariant which also induces a group isomorphism on the Grothendieck group $K_0(\mathcal{V})$, but for which one can more easily determine whether two given elements are equal and which can be used to describe the multiplication on $K_0(\mathcal{V})$.

We will call this invariant the *characteristic class* of the differential operator, since it strongly resembles the characteristic polynomial of a linear transformation and also the construction of characteristic classes in algebraic topology. We will now sketch how the characteristic class and the multiplication of characteristic classes are defined.

Every differential operator over $K = k((x))$ has an eigenvalue over a finite field extension of K as was shown in [Lev75], i.e., there exists a finite field

extension L of K , an element $a \in L$ and a non-zero vector $v \in L \otimes_K V$ such that $D_L v = av$. In (1.1) for example, $1/\sqrt{x}$ is an eigenvalue of the differential operator D_L with $L \stackrel{\text{def}}{=} k((\sqrt{x}))$, since

$$D_L \left(\frac{1}{\sqrt{x}} \right) = \frac{1}{\sqrt{x}} \left(\frac{1}{\sqrt{x}} \right).$$

The set of eigenvalues over a finite field extension of a differential operator is not finite: if a is an eigenvalue then $a + r$ is also an eigenvalue for all $r \in \mathbb{Z} + xk[[x]]$.

This leads to the definition of a *normalized eigenvalue* in section 3.2; an eigenvalue $a \in L$ of D_L is called *normalized* if $a \in k'[1/t]$ and $D_L v = av$ for some non-zero $v \in K(a) \otimes_K V$. Here k' is the field of constants of $K(a)$ and $t \in K(a)$ satisfies $x/t^e \in k'$ for some $e \in \mathbb{N}^*$. The eigenvalues $1/\sqrt{x}$ and $-1/\sqrt{x}$ of the differential operator D in (1.1) are normalized.

In section 3.3 it is shown that a simple differential operator is determined up to isomorphism by a normalized eigenvalue. Proposition 3.3.6 gives a precise statement on this. The ramification index of the field extension $K(a)/K$ (this is the natural number e in the previous paragraph), where a is a normalized eigenvalue, plays an important rôle.

This leads to the definition of the *characteristic class* of a differential operator in section 3.5. For a simple differential operator the characteristic class is the set of minimal polynomials of all its normalized eigenvalues. In general the characteristic class is defined as the set of all products of the elements in the characteristic classes of the simple components in a Jordan-Hölder decomposition series of the $K[\tau]$ -module V . The characteristic class of D in (1.1) turns out to be

$$[\lambda^2 - 1/x] = \{ (\lambda + a)^2 - 1/x \mid a \in \tfrac{1}{2}\mathbb{Z} \}.$$

The set of all characteristic classes forms an Abelian monoid in which the cancellation law holds. Its Grothendieck group (the Abelian group of the quotients) will be denoted by $\mathcal{C}$. The characteristic class induces a group isomorphism from the Grothendieck group $K_0(\mathcal{V})$ to $\mathcal{C}$.

The tensor product of differential operators gives $K_0(\mathcal{V})$ a structure of a commutative ring. In section 3.4 the corresponding multiplication is defined on the Abelian group $\mathcal{C}$ without reference to this product structure on $K_0(\mathcal{V})$. In contrast to the situation of linear transformations, or more general of endomorphisms of finitely generated projective modules over an Abelian ring (cf. [Alm74]), the multiplication on $\mathcal{C}$ can not be extended by universal polynomials starting with the product on elements of degree one.

In our case the situation is more complicated since, in contrast with the situation of linear transformations, two simple differential operators which are not isomorphic over K can become isomorphic over a finite field extension. One of the consequences is that the ring $\mathcal{C}$ has zero-divisors.

The ramification index of the field extension $K(a)/K$, where a is a normalized eigenvalue, plays again an important rôle in the definition of the multiplication on $\mathcal{C}$. In theorem 3.5.5 it is shown the multiplication defined in section 3.4 is the one induced by the tensor product on $K_0(\mathcal{V})$, i.e., the characteristic class induces a *ring isomorphism* from $K_0(\mathcal{V})$ to the ring $\mathcal{C}$.

In [Lev75] it was shown that every differential operator can be written in exactly one way as the sum of a semisimple differential operator S and a nilpotent linear transformation N with the additional property that S and N commute. One calls S the semisimple part and N the nilpotent part of the differential operator.

In section 3.6 it is shown that mapping a differential operator to its semisimple part is an exact functor. This implies that the Dieudonné determinant and the characteristic class both characterize the semisimple part of a differential operator.

In section 3.7 we investigate up to which order of x the coefficients of an element representing the Dieudonné determinant of a differential operator are independent of the chosen representative. This is done by using an equivalence relation $\sim$ on $K[\tau]$ whose definition uses the Newton-Puiseux polygon. This equivalence relation is also used to compare the coefficients of the characteristic class of a differential operator and its Dieudonné determinant. That result makes it possible to obtain the Newton-Puiseux polygon and polynomials (cf. [Die87]) of a differential operator from its characteristic class.

In section 3.8 we describe some well-known invariants of a differential operator which are all derivable from its Newton-Puiseux polygon and hence from its characteristic class.

In section 3.9 a relation between the coefficients of the characteristic class of a differential operator and of the characteristic polynomial of the matrix of the differential operator w.r.t. a K -basis is given. This generalizes a result of Hilali for super-irreducible matrices stated in [Hil87a, theorem 1].

Every linear transformation has a so called rational Jordan normal form. The nilpotent part, the semisimple part and the simple components of the latter follow immediately from this normal form. The uniqueness of this form can be used to determine whether (the modules corresponding to) two linear transformations are isomorphic.

In chapter 4 of this thesis a rational normal form of differential operators is defined, which resembles the rational Jordan normal form of linear transformations. Theorem 4.1.2 gives a rational normal form of a simple differential operator. This rational normal form is unique up to a very special block diagonal transformation, which is described in the second part of the theorem. In the event that the field of constants k is algebraically closed, one obtains the normal form of [vdEL92]. The differential operator D in (1.1) turns out to be a

simple differential operator and a rational normal form of D is given by

$$\begin{pmatrix} 0 & 1/x \\ 1 & -\frac{1}{2} \end{pmatrix}.$$

Theorem 4.2.3 gives a rational normal form of an arbitrary differential operator. In this case the block structure is completely analogous to the rational Jordan normal form of linear transformations. The characteristic class can be computed directly from a rational normal form.

In section 4.3 it is shown that a rational normal form is super-irreducible, a notion defined in [HW87], and section 4.4 describes the connection with the canonical form of Babbitt and Varadarajan defined in [BV83].

In section 4.5 the rational normal form is used to prove that every holonomic $\mathcal{D}_1$ -module without ∂ -torsion and without x -torsion is a holonomic $\mathcal{D}(K)$ -module.

In section 4.6 the trace of a differential operator is defined and the rational normal form is used to prove a result relating it with the trace of the matrix of the differential operator w.r.t. a K -basis.

Chapter 5 starts with an overview of several known algorithms to bring the matrix of a differential operator in a special form or to compute some invariants.

The existence proofs of the characteristic class and a rational normal form are mainly constructive, so in principle it is possible to compute them. However this requires computations, amongst which factorization of polynomials, in finite field extensions of the field of constants which is in practice often the field of the rational numbers $\mathbb{Q}$. These computations are very laborious and difficult to implement in most existing computer algebra systems. Furthermore it is not clear a priori how many terms of the Laurent series of the matrix are needed to compute its characteristic class and rational normal form.

In section 5.2 a slightly improved version of the algorithm of Babbitt and Varadarajan (cf. [Bab]) is described which leads to a canonical form:

$$x \frac{d}{dx} + A_r x^{r/m} + A_{r+1} x^{(r+1)/m} + \cdots + A_0$$

Here $m \in \mathbb{N}^*$ and all A_i are matrices with coefficients in the field of constants k . Furthermore A_i is semisimple for all $i < 0$ and A_i and A_j commute for all i and j . The characteristic polynomial of this canonical form is an element representing the characteristic class of the differential operator in some finite field extension. If however the so-called “nilpotent case” never occurs during the execution of the algorithm then one can obtain the characteristic class and a rational normal form of the differential operator.

In section 5.3 complete rational algorithms are described to compute the characteristic class and a rational normal form of differential operators over two and three dimensional vector spaces.

The thesis concludes with listing some open problems and items for future research.

1.2 Notations and conventions

The set of natural numbers (including zero) will be denoted by $\mathbb{N}$, while $\mathbb{N}^*$ denotes the set of natural numbers with zero excluded.

For a ring A with unit element and $m, n \in \mathbb{N}^*$, the additive group of $m \times n$ matrices with coefficients in A will be denoted by $\text{Mat}_{m,n}(A)$ and the ring of $n \times n$ matrices with coefficients in A will be denoted by $\text{Mat}_n(A)$. The union of all $\text{Mat}_n(A)$ will be denoted by $\text{Mat}(A)$.

The multiplicative subgroup of $\text{Mat}_n(A)$ consisting of matrices which have a left and a right inverse will be denoted by $\text{Gl}_n(A)$. The union of all $\text{Gl}_n(A)$ will be denoted by $\text{Gl}(A)$.

For a group G with unit element, G^{ab} will denote the commutative group G/G' , where G' is the commutator subgroup of G . For an element $a \in G$, the element $a + G'$ in G^{ab} will often be denoted by $\bar{a}$.

For a commutative monoid M the *Grothendieck group* of M , notation $K(M)$, is defined as the group F/B , where F is the free Abelian group generated by M and B is the subgroup of F generated by all elements of the form

$$[x + y] - [x] - [y],$$

where $[x]$ denotes the element in F corresponding to the element $x \in M$ (cf. [Lan84, Ch. I, § 9]). The canonical monoid homomorphism $\gamma: M \rightarrow K(M)$ given by $x \mapsto [x] + B$ has the universal property with respect to homomorphisms of M into commutative groups. If the cancellation law holds in M , then γ is injective.

In this thesis K will always denote a differential field of characteristic zero with a non-trivial derivation τ . Let k be the field of constants of K , i.e.,

$$k \stackrel{\text{def}}{=} \{x \in K \mid \tau(x) = 0\}.$$

From page 19 onwards K will be the field of formal Laurent series over k , i.e., $K = k((x))$ and τ will be the derivation $x \frac{d}{dx}$.

A *differential operator* (V, D) is a k -linear endomorphism of a finite dimensional vector space V over K satisfying

$$D(av) = \tau(a)v + aDv \quad \text{for all } a \in K \text{ and } v \in V.$$

Let $\mathcal{V}$ denote the category whose objects are the differential operators (V, D) . A morphism φ from (V, D) to (V', D') in this category is a K -linear function $\varphi: V \rightarrow V'$ satisfying $\varphi \circ D = D' \circ \varphi$. One easily verifies that $\mathcal{V}$ is an Abelian category.

If $n \in \mathbb{N}^*$ and $A \in \text{Mat}_n(K)$, then one often writes $\tau + A$ for the differential operator (K^n, D) , where $D: K^n \rightarrow K^n$ is defined by

$$D \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} \tau b_1 \\ \vdots \\ \tau b_n \end{pmatrix} + A \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \quad \text{for all } \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \in K^n.$$

$\mathcal{D}(K)$ or $K[\tau]$ will denote the subring of $\text{End}_K(K)$ generated by K and τ . This definition of $\mathcal{D}(K)$ is in general not the same as the $\mathcal{D}(K)$ defined by Grothendieck in [Gro67, prop. 16.8.8]. However for $K = k((x))$ and $\tau = x \frac{d}{dx}$ both definitions agree.

Every element of $\mathcal{D}(K)$ can be written in one and only one way as

$$\sum_{i=0}^n a_i \tau^i \quad (n \in \mathbb{N}, a_0, \dots, a_n \in K).$$

$\mathcal{D}(K)$ is a (left) Ore domain, i.e., a ring without zero-divisors such that for all $a, b \in \mathcal{D}(K)$ with $ab \neq 0$, there exist elements $c, d \in \mathcal{D}(K)$ with $cd \neq 0$ such that $ca = db$. Since $\mathcal{D}(K)$ is an Ore domain, one can form its left (skew) quotient field (cf. [Ore31] or [Ste75]), which will be denoted by $\mathcal{L}(K)$. $\mathcal{D}^0(K)$ will denote the multiplicative subset of $\mathcal{D}(K)$ containing the monic elements of $\mathcal{D}(K)$, i.e., elements of the form $\tau^n + a_{n-1}\tau^{n-1} + \dots + a_0$ with $a_0, a_1, \dots, a_{n-1}$ in K . $\mathcal{L}^0(K)$ will denote the multiplicative subgroup of $\mathcal{L}(K)$ of elements of the form $P^{-1}Q$ with $P, Q \in \mathcal{D}^0(K)$.

Definition 1.2.1 A $\mathcal{D}(K)$ -module M is called *holonomic* if it is a $\mathcal{D}(K)$ -module of finite type such that every $m \in M$ is annihilated by some non-zero element of $\mathcal{D}(K)$.

Lemma 1.2.2 Let M be a $\mathcal{D}(K)$ -module. Then the following two statements are equivalent:

1. M is holonomic.
2. M is a finite dimensional vector space over K .

PROOF. Suppose M is holonomic. Since M is a $\mathcal{D}(K)$ -module of finite type, one can write $M = \mathcal{D}(K)m_1 + \dots + \mathcal{D}(K)m_r$ for some $m_1, m_2, \dots, m_r$ in M . It is sufficient to show that $\mathcal{D}(K)m_i$ is a finite dimensional vector space over K for all $i \in \{1, \dots, r\}$. Let $i \in \{1, \dots, r\}$ then $Pm_i = 0$ for some non-zero $P \in \mathcal{D}(K)$. Write $P = a_n\tau^n + a_{n-1}\tau^{n-1} + \dots + a_0$ with $a_0, a_1, \dots, a_n$ in K and $a_n \neq 0$. Then $\tau^n m_i \in \sum_{j < n} K\tau^j m_i$, so $\dim_K \mathcal{D}(K)m_i \leq n$.

Now suppose that M is a finite dimensional vector space over K . Then $\tau^n m \in \sum_{i < n} K\tau^i m$ for some $n \leq \dim_K M$. Write $\tau^n m = a_{n-1}\tau^{n-1}m + \dots + a_0m$ with $a_0, a_1, \dots, a_{n-1}$ in K and define $P \stackrel{\text{def}}{=} \tau^n - a_{n-1}\tau^{n-1} - \dots - a_0$. Then $Pm = 0$, so m has $\mathcal{D}(K)$ -torsion. This shows that M is holonomic. $\square$

Lemma 1.2.2 shows that the category of holonomic $\mathcal{D}(K)$ -modules is naturally equivalent with the category $\mathcal{V}$ of differential operators in the following way. Let V be a holonomic $\mathcal{D}(K)$ -module, then V is a vector space of finite dimension over K and $D: V \rightarrow V$ defined by

$$D(v) \stackrel{\text{def}}{=} \tau \cdot v \quad \text{for all } v \in V$$

is a differential operator.

Let (V, D) and (V', D') be two differential operators. Then one can define (cf. [Del70, § 1.2.7]):

- $(V, D) \oplus (V', D') \stackrel{\text{def}}{=} (V \oplus V', D \oplus D')$ where

$$(D \oplus D')(v \oplus v') = Dv \oplus D'v' \quad \text{for all } v \in V \text{ and } v' \in V'.$$

- $(V, D) \otimes (V', D') \stackrel{\text{def}}{=} (V \otimes V', D \otimes D')$ where

$$(D \otimes D')(v \otimes v') = Dv \otimes v' + v \otimes D'v' \quad \text{for all } v \in V \text{ and } v' \in V'.$$

- $\text{Hom}((V, D), (V', D')) \stackrel{\text{def}}{=} (\text{Hom}(V, V'), \text{Hom}(D, D'))$ where

$$(\text{Hom}(D, D')(f))(v) \stackrel{\text{def}}{=} D'(fv) - f(Dv) \quad \text{for all } f \in \text{Hom}(V, V') \text{ and } v \in V.$$

- $(V, D)^* \stackrel{\text{def}}{=} (V^*, D^*)$ where V^* denotes the dual vector space of V and

$$D^*(f)(v) \stackrel{\text{def}}{=} \tau(fv) - f(Dv) \quad \text{for all } f \in V^* \text{ and } v \in V.$$

If (V, D) is a differential operator and $e = (e_1, \dots, e_n)$ a K -basis of the vector space V then $M(D, e) \in \text{Mat}_n(K)$ is the matrix $(a_{ij})_{i,j \in \{1, \dots, n\}}$ defined by

$$D(e_j) = \sum_{i=1}^n a_{ij} e_i \quad \text{for all } j \in \{1, \dots, n\}.$$

Notice that if $f = (f_1, \dots, f_n)$ is another K -basis of V and T is the transformation matrix from e to f , i.e., $fT = e$, then

$$TM(D, f)T^{-1} - \tau(T)T^{-1} = M(D, e).$$

We therefore introduce the following notation: for a matrix $A \in \text{Mat}_n(K)$ and $g \in \text{Gl}_n(K)$ define $g[A]$ by

$$g[A] \stackrel{\text{def}}{=} gAg^{-1} - \tau(g)g^{-1}.$$

Notice that $(gh)[A] = g[h[A]]$ for all $g, h \in \text{Gl}_n(K)$ and $A \in \text{Mat}_n(K)$.

Chapter 2

The Dieudonné determinant

In this chapter we will show that the Dieudonné determinant induces a group isomorphism from the Grothendieck group $K_0(\mathcal{V})$ to the multiplicative group $\mathcal{L}^0(K)^{\text{ab}}$. This result should be compared with [KS68, p. 327] and [Alm74].

We will first recall the definition of the Grothendieck group of an Abelian category.

Definition 2.1 (Bass) Let $\mathcal{C}$ be an Abelian category with a small skeleton subcategory $\mathcal{E}$, i.e., the isomorphism classes of $\mathcal{C}$ form a set and the objects in $\mathcal{E}$ are a system of representatives for these isomorphism classes and the set of morphisms from A to B is the same in $\mathcal{C}$ and $\mathcal{E}$ for all objects A and B in $\mathcal{E}$. The Grothendieck group $K_0(\mathcal{C})$ of $\mathcal{C}$ is defined as the quotient group $F(\mathcal{E})/R(\mathcal{E})$, where $F(\mathcal{E})$ is the free Abelian group generated by the objects of $\mathcal{E}$ and $R(\mathcal{E})$ is the subgroup of $F(\mathcal{E})$ generated by $T' + T'' - T$ for all objects T, T' and T'' in $\mathcal{E}$ such that there is an exact sequence

$$0 \rightarrow T' \rightarrow T \rightarrow T'' \rightarrow 0 \quad (2.1)$$

in $\mathcal{E}$ (cf. [Bas68, Ch. VIII, § 1] or [Sil81, section 5.1, p. 89]). The image of an object A in $\mathcal{C}$ under the natural map $\mathcal{C} \rightarrow K_0(\mathcal{C})$ will be denoted by $[A]$.

The natural map $\mathcal{C} \rightarrow K_0(\mathcal{C})$ has the universal property with respect to maps γ from $\mathcal{C}$ into commutative groups satisfying $\gamma(T) = \gamma(T') + \gamma(T'')$ for all short exact sequences (2.1) in $\mathcal{C}$.

We will now define a multiplication and an involution on the Grothendieck group $K_0(\mathcal{V})$. Let $(W, \tilde{D})$ be a differential operator and let

$$0 \rightarrow (V', D') \rightarrow (V, D) \rightarrow (V'', D'') \rightarrow 0$$

be a short exact sequence of differential operators. Then

$$0 \leftarrow (V', D')^* \leftarrow (V, D)^* \leftarrow (V'', D'')^* \leftarrow 0$$

and

$$0 \rightarrow (W, \tilde{D}) \otimes (V', D') \rightarrow (W, \tilde{D}) \otimes (V, D) \rightarrow (W, \tilde{D}) \otimes (V'', D'') \rightarrow 0$$

are also exact sequences of differential operators. So one can define a multiplication and an involution on $K_0(\mathcal{V})$ by

$$[(V, D)]^* \stackrel{\text{def}}{=} [(V, D)^*] \quad \text{and} \quad [(W, \tilde{D})] \cdot [(V, D)] \stackrel{\text{def}}{=} [(W, \tilde{D}) \otimes (V, D)].$$

The multiplication gives $K_0(\mathcal{V})$ the structure of a commutative ring.

We will now recall the definition of the Dieudonné determinant. For every $n \in \mathbb{N}^*$ and every skew field L one can define one and only one homomorphism of monoids

$$\det: \text{Gl}_n(L) \longrightarrow (L^*)^{\text{ab}}$$

satisfying

$$\det\left(\begin{pmatrix} a_1 & 0 & & 0 \\ 0 & a_2 & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & & 0 & a_n \end{pmatrix}\right) = \overline{a_1} \cdots \overline{a_n} \quad \text{and} \quad E_n(L) \subseteq \ker(\det),$$

where $a_1, a_2, \dots, a_n$ are elements in L^* and where $E_n(L)$ denotes the subgroup of $\text{Gl}_n(L)$ generated by the elementary matrices. This homomorphism is called the Dieudonné determinant (cf. [Die43]). One of the properties of the Dieudonné determinant is

$$\det\left(\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}\right) = \det(A) \det(C),$$

where A and C are square matrices with coefficients in L .

Let $n \in \mathbb{N}^*$ and $A \in \text{Mat}_n(K)$. Consider the matrix $\tau I + A \subseteq \text{Mat}_n(\mathcal{D}(K))$ where I denotes the $n \times n$ identity matrix. Since $\mathcal{D}(K)$ is a (left) Euclidean ring one can transform $\tau I + A$ into an upper triangular matrix by Gauss elimination (cf. [Adj85]). It is not hard to see that $\tau I + A \in \text{Gl}_n(\mathcal{L}(K))$ and that the Dieudonné determinant can be represented by an element in $\mathcal{D}^0(K)$.

Definition 2.2 (Adjamagbo) Let (V, D) be a differential operator and let $e = (e_1, \dots, e_n)$ be a K -basis of V . Define

$$\det(D) = \det((V, D)) \stackrel{\text{def}}{=} \det(\tau I + M(D, e)).$$

If f is another K -basis of V and T is the transformation matrix from e to f , i.e., $fT = e$, then

$$\tau I + M(D, f) = T(\tau I + M(D, e))T^{-1}.$$

Since $\det$ is a group homomorphism to an Abelian group, this implies

$$\det(\tau I + M(D, f)) = \det(\tau I + M(D, e)).$$

This shows that $\det(D)$ is well-defined.

Example 2.3 Let $K = k(x)$, $\tau = x \frac{d}{dx}$ and

$$D = \tau + \begin{pmatrix} 2x^{-1} & x^{-2} \\ x^{-1} & 2x^{-1} - \frac{1}{2} \end{pmatrix}.$$

Multiplying on the left by the matrix

$$\begin{pmatrix} 1 & -x\tau - 2 - x \\ 0 & 1 \end{pmatrix}$$

gives

$$\begin{pmatrix} 0 & -x\tau^2 - (4 + \frac{3}{2}x)\tau + x^{-2} - 4x^{-1} + 1 + \frac{1}{2}x \\ x^{-1} & \tau + 2x^{-1} - \frac{1}{2} \end{pmatrix}.$$

So an element representing the Dieudonné determinant of D is

$$\tau^2 + (4x^{-1} + \frac{3}{2})\tau - x^{-3} + 4x^{-2} - x^{-1} - \frac{1}{2}.$$

If

$$0 \rightarrow (V', D') \rightarrow (V, D) \rightarrow (V'', D'') \rightarrow 0$$

is a short exact sequence of differential operators, then

$$\det(D) = \det(D') \det(D''),$$

because fixing a K -basis $(e_1, \dots, e_r)$ of V' and a K -basis $(e_{r+1}, \dots, e_n)$ of V'' , one has

$$M(D, (e_1, \dots, e_n)) = \begin{pmatrix} M(D', (e_1, \dots, e_r)) & * \\ 0 & M(D'', (e_{r+1}, \dots, e_n)) \end{pmatrix}.$$

This shows that there is a group homomorphism

$$\det: K_0(\mathcal{V}) \longrightarrow \mathcal{L}^0(K)^{\text{ab}}.$$

Proposition 2.4 *The function $\det: K_0(\mathcal{V}) \rightarrow \mathcal{L}^0(K)^{\text{ab}}$ is a group isomorphism.*

PROOF. Define $\gamma: \mathcal{D}(K) \setminus \{0\} \rightarrow K_0(\mathcal{V})$ by

$$\gamma(P) \stackrel{\text{def}}{=} [(\mathcal{D}(K)/\mathcal{D}(K)P, \tau)] \quad \text{for all } P \in \mathcal{D}(K) \setminus \{0\}.$$

Now let $P, Q \in \mathcal{D}(K) \setminus \{0\}$, then

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{D}(K)/\mathcal{D}(K)P & \xrightarrow{f} & \mathcal{D}(K)/\mathcal{D}(K)PQ & \longrightarrow & \mathcal{D}(K)/\mathcal{D}(K)Q \longrightarrow 0 \\ & & \downarrow \tau & & \downarrow \tau & & \downarrow \tau \\ 0 & \longrightarrow & \mathcal{D}(K)/\mathcal{D}(K)P & \xrightarrow{f} & \mathcal{D}(K)/\mathcal{D}(K)PQ & \longrightarrow & \mathcal{D}(K)/\mathcal{D}(K)Q \longrightarrow 0 \end{array}$$

is an exact commutative diagram, where f is induced by the function $S \mapsto SQ$. This shows that γ is multiplicative, so γ induces a group homomorphism $\gamma: \mathcal{L}^0(K)^{\text{ab}} \rightarrow K_0(\mathcal{V})$. To prove the proposition, it is enough to show that

$$\begin{array}{ccc} K_0(\mathcal{V}) & \xrightarrow{\det} & \mathcal{L}^0(K)^{\text{ab}} \\ & \nwarrow * & \nearrow \gamma \\ & K_0(\mathcal{V}) & \end{array} \quad (2.2)$$

is a commuting triangle of homomorphisms.

- Let $P = \tau^n + a_{n-1}\tau^{n-1} + \cdots + a_0 \in \mathcal{D}^0(K)$ with $a_0, \dots, a_{n-1} \in K$. Then

$$M(\mathcal{D}(K)/\mathcal{D}(K)P, (1, \bar{\tau}, \dots, \bar{\tau}^{n-1})) = \begin{pmatrix} 0 & 0 & -a_0 \\ 1 & \vdots & -a_1 \\ 0 & \ddots & \vdots \\ \vdots & \ddots & 1 & 0 & \vdots \\ 0 & \dots & 0 & 1 & -a_{n-1} \end{pmatrix},$$

where $\bar{\tau} \stackrel{\text{def}}{=} \tau + \mathcal{D}(K)P$. The matrix of $(\mathcal{D}(K)/\mathcal{D}(K)P)^*$ w.r.t. the dual basis of $(1, \tau, \dots, \tau^{n-1})$ is

$$A \stackrel{\text{def}}{=} \begin{pmatrix} 0 & -1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & -1 & 0 \\ 0 & \dots & 0 & -1 & \\ a_0 & a_1 & \dots & a_{n-1} \end{pmatrix}. \quad (2.3)$$

By multiplying $\tau + A$ from the right by the matrix

$$\begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ \tau & 1 & \ddots & & \vdots \\ \tau^2 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ \tau^{n-1} & 0 & \dots & 0 & 1 \end{pmatrix}$$

it is easily seen that P is an element representing the Dieudonné determinant of $\tau + A$. This shows that $\det(\gamma(P)^*) = \bar{P}$.

- Now let (V, D) be a differential operator. There exists a *cyclic vector* $v \in V$ for the differential operator (V, D) (cf. [Del70, lemme II.1.3]), i.e., $v \stackrel{\text{def}}{=} (v, Dv, \dots, D^{n-1}v)$ is a K -basis of V , where $n = \dim_K V$. If

$$D^n v + a_{n-1}D^{n-1}v + \cdots + a_0 = 0 \quad \text{for some } a_0, \dots, a_{n-1} \in K,$$

then

$$M(D, v) = \begin{pmatrix} 0 & \dots & 0 & -a_0 \\ 1 & \ddots & \vdots & -a_1 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & 0 \\ 0 & \dots & 0 & 1 & -a_{n-1} \end{pmatrix}$$

and therefore $M(D^*, v^*)$ is given by (2.3), so

$$\det((V, D)^*) = \overline{\tau^n + a_{n-1}\tau^{n-1} + \dots + a_0}.$$

Let $P \stackrel{\text{def}}{=} \tau^n + a_{n-1}\tau^{n-1} + \dots + a_0$. Then $(\mathcal{D}(K)/\mathcal{D}(K)P, \tau) \simeq (V, D)$. This shows that $\bar{\gamma}(\det([(V, D)]^*)) = [(V, D)]$.

Since $K_0(\mathcal{V}) \xrightarrow{*} K_0(V)$ is a group automorphism (even a ring automorphism) and $\mathcal{L}^0(K)^{\text{ab}}$ is generated by the image of $\mathcal{D}^0(K)$ in $\mathcal{L}^0(K)^{\text{ab}}$ and $K_0(\mathcal{V})$ is generated by the elements $[(V, D)]$ where (V, D) is a differential operator, it follows from the above that (2.2) is a commutative triangle of group homomorphisms. $\square$

We will now show that one can define a function $\mathcal{D}(K) \xrightarrow{*} \mathcal{D}(K)$ with the properties $(PQ)^* = Q^*P^*$ and $P^{**} = P$.

Definition 2.5 Let $P \in \mathcal{D}(K)$. The *adjoint operator* $P^\circledast$ is defined by the following properties:

$$\begin{aligned} f^\circledast &= f \quad \text{for all } f \in K, \\ \tau^\circledast &= -\tau, \\ (Q + R)^\circledast &= Q^\circledast + R^\circledast \quad \text{for all } Q, R \in \mathcal{D}(K), \\ (QR)^\circledast &= R^\circledast Q^\circledast \quad \text{for all } Q, R \in \mathcal{D}(K). \end{aligned}$$

Now define $P^* \stackrel{\text{def}}{=} (-1)^{\deg P} P^\circledast$. Then one readily verifies that $(QR)^* = R^*Q^*$ and $Q^{**} = Q$ for all $Q, R \in \mathcal{D}(K)$.

So the function $\mathcal{D}(K) \xrightarrow{*} \mathcal{D}(K)$ induces a group automorphism of $\mathcal{L}^0(K)^{\text{ab}}$.

Proposition 2.6 *The following diagram commutes:*

$$\begin{array}{ccc} K_0(\mathcal{V}) & \xrightarrow{\det} & \mathcal{L}^0(K)^{\text{ab}} \\ \downarrow * & & \downarrow * \\ K_0(V) & \xrightarrow{\det} & \mathcal{L}^0(K)^{\text{ab}} \end{array}$$

PROOF. Let A be the matrix in (2.3). It is enough to show that $\det((\tau + A)^*) = (\det(\tau + A))^*$. Since $(\tau + A)^* = \tau - A^t$ and (see the proof of proposition 2.4)

$$\det(\tau + A) = \tau^n + \overline{a_{n-1}}\tau^{n-1} + \dots + a_0$$

it is enough to show that $\tau^n - a_{n-1}\tau^{n-1} + \dots + (-1)^n a_0$ is an element representing the Dieudonné determinant of the matrix

$$\begin{pmatrix} \tau & 0 & 0 & -a_0 \\ 1 & \tau & \vdots & -a_1 \\ 0 & \ddots & \ddots & 0 \\ \vdots & & 1 & \tau \\ 0 & & 0 & 1 \end{pmatrix}.$$

This is immediately clear by multiplying the matrix from the left by the matrix

$$\begin{pmatrix} 1 & -\tau & \tau^2 & \dots & (-\tau)^{n-1} \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & 0 \\ 0 & & & 0 & 1 \end{pmatrix}.$$

□

Chapter 3

Normalized eigenvalues and characteristic classes

3.1 Field extensions and complete local fields

Let L be a finite field extension of K . The derivation τ on K can be extended in one and only one way to a derivation on L , which will also be denoted by τ . A differential operator (V, D) can be extended in one and only one way to a differential operator (V_L, D_L) where $V_L \stackrel{\text{def}}{=} L \otimes_K V$ by

$$D_L(a \otimes v) \stackrel{\text{def}}{=} \tau(a) \otimes v + a \otimes Dv \quad \text{for all } a \in K \text{ and } v \in V.$$

We will often drop the subscript L when there is no confusion possible.

Let L' be another finite field extension of K and let σ be a ring homomorphism from L to L' leaving K elementwise fixed. The uniqueness of the extension of τ to L and σL implies that $\sigma\tau = \tau\sigma$ on L . One can define σ from V_L to $V_{L'}$ by

$$\sigma(a \otimes v) \stackrel{\text{def}}{=} \sigma(a) \otimes v \quad \text{for all } a \in L \text{ and } v \in V.$$

Then $\sigma: V_L \rightarrow V_{L'}$ is a K -linear function satisfying $\sigma D_L = D_{L'} \sigma$.

Lemma 3.1.1 *Let L be a finite field extension of K and let c be an element in L . Then c is algebraic over k if and only if c is a constant in L .*

PROOF. Suppose c is algebraic over k . Let

$$f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_0 \in k[X]$$

be the minimal polynomial of c over k . Applying τ to $f(c)$ gives

$$(nc^{n-1} + (n-1)a_{n-1}c^{n-2} + \cdots + a_1)\tau(c) = 0.$$

Since $f(X)$ is the minimal polynomial of c over k , this implies that $\tau(c) = 0$, so c is a constant in L .

Now assume c is a constant in L . Let

$$f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_0 \in K[X]$$

be the minimal polynomial of c over K . Applying τ to $f(c)$ gives

$$\tau(a_{n-1})c^{n-1} + \cdots + \tau(a_0) = 0.$$

Since $f(X)$ is the minimal polynomial of c over K , this implies that $\tau(a_{n-1}) = \cdots = \tau(a_0) = 0$, so $f(X) \in k[X]$. $\square$

Now let K be a complete local field of equal characteristic zero, i.e., the quotient field of a complete discrete valuation ring with a residue field of characteristic zero. We will denote the valuation ring by $\mathcal{O}_K$, its maximal ideal by $\mathfrak{m}_K$ and the valuation on K by val_K . The subscript will often be omitted if there is no confusion possible.

We will further assume that

$$\tau(\mathcal{O}) \subseteq \mathfrak{m} \quad \text{and} \quad \tau(y)\mathcal{O} = \mathfrak{m} \quad \text{for some } y \in \mathfrak{m}. \quad (3.1)$$

Notice that this implies that the derivation τ is a continuous function w.r.t. the $\mathfrak{m}$ -adic topology and that $\mathfrak{m} = y\mathcal{O}$.

Proposition 3.1.2 *Let K be a complete local field of equal characteristic zero and let τ be a derivation on K satisfying condition (3.1). Let k be the field of constants in K . Then there is a ring isomorphism of the differential k -algebras $(\mathcal{O}, \tau)$ and $(k[[x]], cx \frac{d}{dx})$ for some $c \in k^*$. Here $k[[x]]$ denotes the ring of formal power series in one indeterminate over the field k .*

PROOF. By [Del70, lemme II.1.5] $(\mathcal{O}, \tau)$ and $(k[[t]], \tau)$ are isomorphic, where τ is a derivation on $k[[t]]$ with constant field k . Property (3.1) implies that $\tau(t)/t$ is a unit in $k[[t]]$, so $\tau(t)/t = c - b\tau(t)$ for some $c \in k^*$ and $b \in k[[t]]$. Let $B \in k[[t]]$ with $dB/dt = b$ and define $x \stackrel{\text{def}}{=} \exp(B)t$. Since $\tau = \tau(t) \frac{d}{dt}$, one easily verifies that $\tau(x)/x = c \in k^*$. The fact that $\exp(B)$ is a unit in $k[[t]]$ now implies that $(\mathcal{O}, \tau)$ and $(k[[x]], cx \frac{d}{dx})$ are isomorphic. $\square$

Proposition 3.1.3 *Let K be a complete local field of equal characteristic zero and let τ be a derivation on K satisfying condition (3.1). Let L be a finite field extension of K . Then L is also a complete local field of equal characteristic zero, $\mathcal{O}_L$ is the integral closure of $\mathcal{O}_K$ in L and the field of constants in L is a system of representatives of $\mathcal{O}_L/\mathfrak{m}_L$.*

PROOF. The first two statements follow from [Ser62, Ch. II, prop. 3].

Let k' be the field of constants in L . By lemma 3.1.1 k' is algebraic over k , so in particular $k' \subseteq \mathcal{O}_L$. Let k'' be a maximal field in $\mathcal{O}_L$ containing k' . The

proof of [Ser62, Ch. II, prop. 6] implies that k'' is a system of representatives of $\mathcal{O}_L/\mathfrak{m}_L$. We have to show that $k' = k''$. One has $\mathcal{O}_L/\mathfrak{m}_L \simeq k''$ and, by proposition 3.1.2, $\mathcal{O}_K/\mathfrak{m}_K \simeq k$. Since L is a finite field extension of K , k'' is also a finite field extension of k . Applying lemma 3.1.1 shows that $k' = k''$. $\square$

Definition 3.1.4 Let K and L be as in proposition 3.1.3. Define the *ramification index of L over K* , notation $\epsilon_{L/K}$, as the ramification index of $\mathfrak{m}_L$ over $\mathfrak{m}_K$, i.e.,

$$\mathcal{O}_L \mathfrak{m}_K = \mathfrak{m}_L^{\epsilon_{L/K}}.$$

Define the *inertial degree of L over K* , notation $f_{L/K}$, as the inertial degree of $\mathfrak{m}_L$ over $\mathfrak{m}_K$, i.e., $f_{L/K} = [\mathcal{O}_L/\mathfrak{m}_L : \mathcal{O}_K/\mathfrak{m}_K]$. It is well-known that $[L : K] = \epsilon_{L/K} f_{L/K}$. If L' is a finite field extension of L , then $\epsilon_{L'/K} = \epsilon_{L'/L} \epsilon_{L/K}$ and $f_{L'/K} = f_{L'/L} f_{L/K}$.

From this point on we will fix a complete local field $K = k((x))$ of equal characteristic zero. Define $\partial \stackrel{\text{def}}{=} \frac{d}{dx}$. To avoid the constant $\tau(x)/x$ in numerous places we will assume that the derivation τ on K is $x\partial$.

Proposition 3.1.5 *Let L be a finite field extension of K and let k' be the field of constants in L . Then $[k' : k] = f_{L/K}$ and $L = k'K(t)$ for some non-zero $t \in \mathcal{O}_L$ with $\tau(t) = t/\epsilon_{L/K}$.*

PROOF. The statement $[k' : k] = f_{L/K}$ follows immediately from the definition of the inertial degree and the proof of the last statement of proposition 3.1.3.

Let $t \in \mathfrak{m}_L$ be a uniformizing parameter in $\mathcal{O}_L$, i.e., $t\mathcal{O}_L = \mathfrak{m}_L$. Then $x = ut^e$ for some $u \in \mathcal{O}_L^*$ and $e = \epsilon_{L/K}$. Write $u = u_0 + m$ with $u_0 \in k'$ and $m \in \mathfrak{m}_L$. Since $\mathcal{O}_L$ is complete there is an element $w \in \mathcal{O}_L^*$ such that $u/u_0 = w^e$. Define $t' \stackrel{\text{def}}{=} wt$. Then $x = u_0(t')^e$, so $\tau(t') = t'/e$. One has $\mathcal{O}_L = k'[[t']]$ and therefore $L = k'K(t')$. $\square$

Let L , k' and t as in proposition 3.1.5. Then one has the following field extensions

$$\begin{array}{c} L = k'K(t) \\ \left| \epsilon_{L/K} \right. \\ k'K \\ \left| f_{L/K} \right. \\ K \end{array}$$

If L/K is a Galois extension, then $\text{Gal}(L/k'K)$ is the inertia group of $\mathfrak{m}_L$ over K . It is a cyclic group of order $\epsilon_{L/K}$ generated by any automorphism sending t to ζt , where ζ is a primitive $\epsilon_{L/K}$ -th root of unity.

Proposition 3.1.6 *Let L be a finite field extension of K and let K_1 and K_2 be two subfields of L containing K . Let $K' \stackrel{\text{def}}{=} K_1 K_2 \subseteq L$. Without loss of generality one may assume that L is a normal closure of the field extension K'/K . Then*

$$\epsilon_{L/K} = \epsilon_{K'/K} = \text{lcm}(\epsilon_{K_1/K}, \epsilon_{K_2/K}).$$

PROOF. Denote the field of constants in K_1 , K_2 and K' by respectively k_1 , k_2 and k' . Notice that $k_1 k_2 \subseteq k'$. By proposition 3.1.5 there exists a non-zero $t_1 \in \mathcal{O}_{K_1}$ and a non-zero $t_2 \in \mathcal{O}_{K_2}$ satisfying $\tau(t_1) = t_1/e_1$ and $\tau(t_2) = t_2/e_2$, where $e_1 \stackrel{\text{def}}{=} \epsilon_{K_1/K}$ and $e_2 \stackrel{\text{def}}{=} \epsilon_{K_2/K}$. Put $e \stackrel{\text{def}}{=} \text{lcm}(e_1, e_2)$ and write

$$\frac{e_1 e_2}{e} = r e_1 + s e_2 \quad \text{for some } r, s \in \mathbb{Z}.$$

Put $t \stackrel{\text{def}}{=} t_1^r t_2^s$. Then $\tau(t^{e/e_1} t_1^{-1}) = \tau(t^{e/e_2} t_2^{-1}) = 0$, so $t_1, t_2 \in k'[t]$. This shows that

$$K' = k_1 k_2 K(t_1, t_2) = k' K(t).$$

Furthermore $\tau(x t^{-e}) = 0$ implies that $x \mathcal{O}_{K'} = t^e \mathcal{O}_{K'}$, hence $\epsilon_{K'/K} = e$.

Let G be the set of K -homomorphisms from K' to L . Then L is the smallest field in L containing $\sigma K'$ for all $\sigma \in G$. Since K' and $\sigma K'$ are isomorphic, one has $\epsilon_{\sigma K'/K} = \epsilon_{K'/K}$ for all $\sigma \in G$. Applying the previous result now yields $\epsilon_{L/K} = \epsilon_{K'/K}$. $\square$

Definition 3.1.7 Let L be a finite field extension of K . Define

$$\dot{\mathcal{O}}_L \stackrel{\text{def}}{=} \left\{ \frac{\tau(a)}{a} \mid a \in L^* \right\}.$$

Proposition 3.1.8 Let L be a finite field extension of K . Put $e \stackrel{\text{def}}{=} \epsilon_{L/K}$. Then

$$\dot{\mathcal{O}}_L = \frac{1}{e} \mathbb{Z} + \mathfrak{m}_L.$$

PROOF. Let k' be the field of constants in L . By proposition 3.1.5 there is a non-zero $t \in \mathcal{O}_L$ satisfying $\tau(t) = t/e$.

Let $b \in L^*$, then one can write $b = u t^r$ with $r \in \mathbb{Z}$ and $u \in \mathcal{O}_L^*$. Applying the derivation τ yields

$$\frac{\tau(b)}{b} = \frac{r}{e} + \frac{\tau(u)}{u}.$$

Since u is a unit in $\mathcal{O}_L$, it is enough to show that $\tau(u) \in \mathfrak{m}_L$. This follows from the fact that $\mathcal{O}_L$ and $k'[[t]]$ are isomorphic.

Since $\tau(t) = t/e$ and $\dot{\mathcal{O}}_L$ is a $\mathbb{Z}$ -module, one has $(1/e)\mathbb{Z} \subseteq \dot{\mathcal{O}}_L$. Remains to show that $\mathfrak{m}_L \subseteq \dot{\mathcal{O}}_L$.

Let $b \in \mathfrak{m}_L$ then $b/t \in \mathcal{O}_L$. Since $\mathcal{O}_L$ and $k'[[t]]$ are isomorphic, one has $b/t = db'/dt$ for some $b' \in \mathcal{O}_L$. Now define $B \stackrel{\text{def}}{=} \exp(eb')$ then $dB/dt = ebB/t$, so $b = \tau(B)/B \in \dot{\mathcal{O}}_L$. $\square$

Definition 3.1.9 Let L be a finite field extension of K and let e , k' and t as in proposition 3.1.5. Define

$$\mathcal{P}_L \stackrel{\text{def}}{=} k' \left[\frac{1}{t} \right].$$

Notice that $k'[1/t]$ is independent of the chosen t and that $\mathcal{P}_L \cap \mathcal{O}_L = k'$ and $\mathcal{P}_L \cap \dot{\mathcal{O}}_L = (1/e)\mathbb{Z}$ by proposition 3.1.8.

Proposition 3.1.10 *Let L and L' be finite field extensions of K and let $\sigma: L \rightarrow L'$ be a K -homomorphism. Then $\sigma(\mathcal{O}_L) = \mathcal{O}_{\sigma L}$ and $\sigma(\mathcal{P}_L) = \mathcal{P}_{\sigma L}$.*

PROOF. It follows from proposition 3.1.3 that $\mathcal{O}_L$ is the integral closure of $\mathcal{O}_K$ in L and that $\mathcal{O}_{\sigma L}$ is the integral closure of $\mathcal{O}_K$ in σL . From this it is clear that $\sigma(\mathcal{O}_L) = \mathcal{O}_{\sigma L}$. Let e, k' and t as in proposition 3.1.5. Then using $\tau(t) = t/e$ and $\sigma(\tau(c)) = \tau(\sigma(c))$ for all $c \in L$, one deduces that $\sigma(k')$ is the field of constants in $\sigma(L)$ and $\tau(\sigma(t)) = \sigma(t)/e$. Furthermore $\sigma(t) \in \sigma(\mathcal{O}_L) = \mathcal{O}_{\sigma L}$. Therefore $\sigma(t)$ satisfies the condition in proposition 3.1.5 for the field extension $\sigma(L)$ of K . Hence $\sigma(\mathcal{P}_L) = \mathcal{P}_{\sigma L}$ by definition 3.1.9. $\square$

It is occasionally convenient to know that in proposition 3.1.5 one can choose an element $t \in L^*$ with $\tau(t) = t/\epsilon_{L/K}$ and $L = K(t)$. To prove this, we first need a lemma.

Lemma 3.1.11 *Let $e \in \mathbb{N}^*$, k' a finite field extension of k (a field of characteristic zero) and γ a non-zero element in k' . Then there exists an element $\alpha \in k'$ such that $k' = k(\alpha^e \gamma)$.*

PROOF. We will first show that $k' = k(\alpha_0^e \gamma, \dots, \alpha_m^e \gamma)$ for some $m \in \mathbb{N}$ and $\alpha_0, \alpha_1, \dots, \alpha_m$ in k' .

Let ℓ be a normal closure of the field extension k'/k and let $\sigma \in \text{Gal}(\ell/k)$. Suppose that $\sigma(\alpha^e \gamma) = \alpha^e \gamma$ for all $\alpha \in k'$. Then $\sigma(\gamma) = \gamma$, so $\sigma(\alpha^e) = \alpha^e$ for all $\alpha \in k'$. Hence $(\lambda + \sigma(\alpha))^e = \sigma((\lambda + \alpha)^e) = (\lambda + \alpha)^e$ for all $\lambda \in k$ and all $\alpha \in k'$. Since k is a field of characteristic zero, this implies that $\sigma(\alpha) = \alpha$ for all $\alpha \in k'$, so $\sigma \in \text{Gal}(\ell/k')$. This shows that $k' = k(\alpha_0^e \gamma, \dots, \alpha_m^e \gamma)$ for some $m \in \mathbb{N}$ and $\alpha_0, \alpha_1, \dots, \alpha_m$ in k' .

Let $Z = \{z_0, \dots, z_m\}$ be a subset of k and let $f(X) \in k'[X]$ be a polynomial of degree m or less such that

$$f(z_i) = \alpha_i \quad \text{for all } i \in \{0, \dots, m\}.$$

Since k has infinitely many elements and there are only finitely many subfields of k' containing k , there is a subfield k'' of k' containing k and an infinite subset W of k such that

$$k'' = k(f(\lambda)^e \gamma) \quad \text{for all } \lambda \in W.$$

Since $f(\lambda)^e \gamma \in k''$ for at least $me + 1$ different values of λ , one must have $f(X)^e \gamma \in k''[X]$. The fact that

$$k' = k(f(z_0)^e \gamma, \dots, f(z_m)^e \gamma)$$

now implies that $k'' = k'$, hence $k' = k(\alpha^e \gamma)$ for some $\alpha \in k'$. $\square$

Proposition 3.1.12 *Let L be a finite field extension of K . Put $e \stackrel{\text{def}}{=} \epsilon_{L/K}$ and let k' be the field of constants in L . Then there exists a non-zero element $t \in L$ such that $\tau(t) = t/e$ and $L = K(t)$.*

Furthermore for any $s \in L$ satisfying $\tau(s) = s/e$ and $L = K(s)$ one has $k' = k(s^e/x)$, so in particular $k' = k(t^e/x)$.

PROOF. By proposition 3.1.5 there exists a non-zero element $t' \in O_L$ such that $\tau(t') = t'/e$ and $L = k'K(t')$. Define $\gamma \stackrel{\text{def}}{=} (t')^e/x$, then $\tau(\gamma) = 0$, so $\gamma \in k'$. By lemma 3.1.11 there exists an element $\alpha \in k'$ such that $k' = k(\alpha^e\gamma)$. We may assume that $\alpha \neq 0$. Now define $t \stackrel{\text{def}}{=} \alpha t'$. It is clear that $\tau(t) = t/e$. Since k' is generated by $\alpha^e\gamma = t^e/x$ as a k -algebra, one has the inclusion $k'K \subseteq K(t)$. Also $t' = t/\alpha$, so $L = k'K(t') \subseteq K(t) \subseteq L$, hence $L = K(t)$.

To prove the second part of the proposition suppose $L = K(s)$ with $\tau(s) = s/e$ for some $s \in L$. We want to show that this implies $k' = k(s^e/x)$. $L = K(s)$ implies that s is of degree $[L : K] = e_{L/K} \cdot f_{L/K}$ over K and therefore s^e/x is of degree $f_{L/K}$ or more over K . The assumption that $\tau(s) = s/e$ implies that $s^e/x \in k'$, so

$$f_{L/K} \leq [k(s^e/x) : k] \leq [k' : k] = f_{L/K}.$$

Conclusion: $[k(s^e/x) : k] = f_{L/K}$ and therefore $k' = k(s^e/x)$. $\square$

3.2 Normalized eigenvalues

Let us recall an important result from [Lev75].

Proposition 3.2.1 (Levelt) *Let (V, D) be a differential operator. Then there exists a finite field extension L of K , a non-zero element $v \in V_L$ and an element $a \in L$ such that $D_L v = av$.*

Example 3.2.2 Let $k = \mathbb{Q}$, $K = \mathbb{Q}((x))$, $L = K(\sqrt{2}, t)$ with $x = 2t^4$ and

$$D = x \frac{d}{dx} + \begin{pmatrix} 2/x & 6/x \\ 3 & 2/x - \frac{1}{2} \end{pmatrix}.$$

Then

$$\begin{aligned} \begin{pmatrix} 1 \\ t^2 \end{pmatrix} & \text{ is an eigenvector of } D_L \text{ with eigenvalue } \frac{2}{x} + \frac{3}{t^2}, \\ \begin{pmatrix} t \\ t^3 \end{pmatrix} & \text{ is an eigenvector of } D_L \text{ with eigenvalue } \frac{2}{x} + \frac{3}{t^2} + \frac{1}{4}, \\ \begin{pmatrix} t^2 \\ t^4 \end{pmatrix} & \text{ is an eigenvector of } D_L \text{ with eigenvalue } \frac{2}{x} + \frac{3}{t^2} + \frac{1}{2}, \\ \begin{pmatrix} e^{\sqrt{2}x} \\ t^2 e^{\sqrt{2}x} \end{pmatrix} & \text{ is an eigenvector of } D_L \text{ with eigenvalue } \frac{2}{x} + \frac{3}{t^2} + \sqrt{2}x. \end{aligned}$$

Example 3.2.2 shows that a differential operator can have many related different eigenvalues. In general, if $a \in L$ is an eigenvalue of the differential operator D_L with corresponding eigenvector $v \in V_L$ and $f \in L^*$, then $D_L(fv) = (a + \tau(f)/f)(fv)$ so every element of $a + \mathcal{O}_L$ is also an eigenvalue of D_L .

We will therefore first introduce the notion of a “normalized” eigenvalue. In the next section we will show that normalized eigenvalues determine simple differential operators up to isomorphism.

Definition 3.2.3 Let (V, D) be a differential operator and let L be a finite field extension of K . An element $a \in L$ is called a *normalized eigenvalue* of the differential operator D_L if

$$a \in \mathcal{P}_L \quad \text{and} \quad D_L v = av \quad \text{for some non-zero } v \in V_{K(a)}.$$

In example 3.2.2

$$\frac{2}{x} + \frac{3}{t^2} \quad \text{and} \quad \frac{2}{x} + \frac{3}{t^2} + \frac{1}{2}$$

are normalized eigenvalues of D_L , while

$$\frac{2}{x} + \frac{3}{t^2} + \frac{1}{4} \quad \text{and} \quad \frac{2}{x} + \frac{3}{t^2} + \sqrt{2}x$$

are not normalized.

We will now show that every differential operator has a normalized eigenvalue.

Proposition 3.2.4 *Let (V, D) be a differential operator. Then there exists a finite field extension L of K , a non-zero element $v \in V_L$ and an element $a \in \mathcal{P}_L$ such that $D_L v = av$.*

PROOF. By proposition 3.2.1 there exists a finite field extension L of K , a non-zero element $v \in V_L$ and an element $a \in L$ such that $D_L v = av$. One has $L = \mathcal{P}_L \oplus \mathfrak{m}_L$. Write $a = a' + g$ with $a' \in \mathcal{P}_L$ and $g \in \mathfrak{m}_L$. By proposition 3.1.8, $g \in \mathcal{O}_L$. Let $f \in L^*$ with $\tau(f)/f = -g$ and define $w \in V_L$ by $w \stackrel{\text{def}}{=} fv$. One easily verifies that $D_L w = a'w$. $\square$

Proposition 3.2.5 *Let (V, D) be a differential operator. Then D_L has a normalized eigenvalue over some finite field extension L of K .*

PROOF. By proposition 3.2.4 there exists a finite field extension L of K , a non-zero element $v \in V_L$ and an element $a \in \mathcal{P}_L$ such that $D_L v = av$. We may assume that $L/K(a)$ is a Galois extension. Let G be its Galois group. By proposition 3.1.12 there exists an element $t \in L^*$ with $L = K(t)$ and $\tau(t) = t/e$ where $e \stackrel{\text{def}}{=} \epsilon_{L/K}$. Put $n \stackrel{\text{def}}{=} [L : K(a)]$ and define

$$w_i \stackrel{\text{def}}{=} \sum_{\sigma \in G} \sigma(t^i v) \quad \text{for all } i \in \{0, \dots, n-1\}.$$

Then $\sigma(w_i) = w_i$ for all $\sigma \in G$, so $w_i \in V_{K(a)}$ for all $i \in \{0, \dots, n-1\}$. Let $\sigma \in G$ and $i \in \{0, \dots, n-1\}$, then

$$D_L \sigma(t^i v) = \sigma D_L(t^i v) = \sigma((a + i/e)t^i v) = (a + i/e)\sigma(t^i v).$$

This implies that $D_L w_i = (a + i/e)w_i$ for all $i \in \{0, \dots, n-1\}$. Remains to show that at least one of the w_i is non-zero. To show this, notice that

$$w_i = \sum_{\sigma \in G} \sigma(t)^i \sigma(v) \quad \text{for all } i \in \{0, \dots, n-1\}.$$

Since G is the Galois group of $K(t)/K(a)$, we have

$$\sigma_1(t) \neq \sigma_2(t) \quad \text{for all } \sigma_1, \sigma_2 \in G \text{ with } \sigma_1 \neq \sigma_2.$$

This shows that the Vandermonde determinant $\det(\sigma(t)^i)$ is non-zero and therefore at least one of the w_i is non-zero, as desired. $\square$

Proposition 3.2.6 *Let (V, D) be a differential operator. Let L be a Galois extension of K and let $a \in L$ be a normalized eigenvalue of D_L with corresponding non-zero eigenvector $v \in V_{K(a)}$. Let G be the set of K -homomorphisms from $K(a)$ to L . Then:*

1. *If $\sigma(a) - a \in K$ for some $\sigma \in G$, then $\sigma = 1$.*
2. *The set $\{\sigma(v) \mid \sigma \in G\}$ is linearly independent over L .*

PROOF. Let $\sigma \in G$ and assume that $c \stackrel{\text{def}}{=} \sigma(a) - a \in K$. Then $\sigma(K(a)) \subseteq K(a)$, so $\sigma^n \in G$ for all $n \in \mathbb{N}$. Since G is finite, one has $\sigma^n = 1$ for some $n \in \mathbb{N}$. So $a = \sigma^n(a) = a + nc$, hence $\sigma(a) - a = c = 0$. This implies that $\sigma = 1$.

Let $H \subseteq G$ be a maximal subset of G for which $\{\sigma(v) \mid \sigma \in H\}$ is linearly independent over L . Suppose $H \neq G$. Let $\sigma_0 \in G \setminus H$, then

$$\sigma_0(v) = \sum_{\sigma \in H} \lambda_\sigma \sigma(v) \quad \text{for some } \lambda_\sigma \in L. \quad (3.2)$$

Applying D_L gives

$$\sigma_0(a)\sigma_0(v) = \sum_{\sigma \in H} (\tau(\lambda_\sigma) + \lambda_\sigma \sigma(a))\sigma(v).$$

Since $\{\sigma(v) \mid \sigma \in H\}$ is linearly independent over L , this implies that

$$\lambda_\sigma \sigma_0(a) = \tau(\lambda_\sigma) + \lambda_\sigma \sigma(a) \quad \text{for all } \sigma \in H.$$

Equation (3.2) implies that at least one of the λ_σ is non-zero. Assume $\lambda_\sigma \neq 0$, then $\sigma_0(a) - \sigma(a) = \tau(\lambda_\sigma)/\lambda_\sigma \in \mathcal{O}_L$ and also $\sigma_0(a) - \sigma(a) \in \mathcal{P}_L$ by proposition 3.1.10, hence $\sigma_0(a) - \sigma(a) \in K$. By the first part of the proposition this implies $\sigma_0 = \sigma \in H$, a contradiction. This shows that $H = G$, i.e., $\{\sigma(v) \mid \sigma \in G\}$ is linearly independent over L . $\square$

3.3 Simple differential operators

A differential operator (V, D) is called *simple* if its corresponding $\mathcal{D}(K)$ -module V is simple, i.e., $V \neq 0$ and there is no D -invariant K -linear subspace W of V with $0 \neq W \neq V$.

A differential operator (V, D) is called *semisimple* if its corresponding $\mathcal{D}(K)$ -module is semisimple, i.e. (cf. [Bou58, § 3, Déf. 3]), if the following equivalent conditions hold:

1. V is a sum of simple $\mathcal{D}(K)$ -submodules.
2. V is a direct sum of simple $\mathcal{D}(K)$ -submodules.
3. Every $\mathcal{D}(K)$ -submodule of V is a direct factor.

We will now investigate normalized eigenvalues of simple differential operators.

Proposition 3.3.1 *Let (V, D) be a differential operator. Let L be a finite field extension of K and let $a \in L$ be a normalized eigenvalue of D_L with corresponding non-zero eigenvector $v \in V_{K(a)}$. Then there is a simple $\mathcal{D}(K)$ -submodule W of V such that $v \in W_{K(a)}$ and $\dim_K W = [K(a) : K]$.*

PROOF. Put $n \stackrel{\text{def}}{=} [K(a) : K]$ and write

$$v = \sum_{i=0}^{n-1} a^i v_i \quad (v_0, \dots, v_{n-1} \in V).$$

Without loss of generality one may assume that L/K is a Galois extension. Let G be the set of K -homomorphisms from $K(a)$ to L . One has

$$\sigma(v) = \sum_{i=0}^{n-1} \sigma(a)^i v_i \quad \text{for all } \sigma \in G. \quad (3.3)$$

The Vandermonde determinant $\det(\sigma(a)^i)$ is non-zero and by proposition 3.2.6 the set $\{\sigma(v) \mid \sigma \in G\}$ is linearly independent over L , hence $v_0, \dots, v_{n-1}$ are linearly independent over K . Now define

$$W \stackrel{\text{def}}{=} Kv_0 + \dots + Kv_{n-1}.$$

Then $v \in W_{K(a)}$ and $\dim_K W = n$. We will now show that W is a simple $\mathcal{D}(K)$ -module.

By equation (3.3)

$$W_L = \sum_{\sigma \in G} L\sigma(v).$$

Notice that $D_L \sigma(v) = \sigma D_L(v) = \sigma(a)\sigma(v)$ so $D_L(L\sigma(v)) = L\sigma(v)$ for all $\sigma \in G$. This shows that $D(W) \subseteq W_L \cap V = W$.

Let S be a simple $\mathcal{D}(L)$ -submodule of W_L . Then S is isomorphic to some $L\sigma_0(v)$, so $S = Lw$ for some non-zero $w \in S$ and $D_L w = \sigma_0(a)w$. Since $w \in W_L$, w can be written as $\sum \lambda_\sigma \sigma(v)$ for some $\lambda_\sigma \in L$. The relation $D_L w = \sigma_0(a)w$ implies

$$\sum_{\sigma \in G} (\tau(\lambda_\sigma) + \lambda_\sigma \sigma(a)) \sigma(v) = \sum_{\sigma \in G} \sigma_0(a) \lambda_\sigma \sigma(v),$$

and, since the $\sigma(v)$ are linearly independent over L , we conclude

$$\tau(\lambda_\sigma) + \lambda_\sigma \sigma(a) = \sigma_0(a) \lambda_\sigma \quad \text{for all } \sigma \in G.$$

By the same argument as in the proof of proposition 3.2.6 this implies $\lambda_\sigma = 0$ for all $\sigma \neq \sigma_0$. Conclusion: $S = L\sigma_0(v)$. So the only simple $\mathcal{D}(L)$ -submodules of W_L are the $L\sigma(v)$.

Now let W' be a non-zero $\mathcal{D}(K)$ -submodule of W . Then W'_L is a non-zero $\mathcal{D}(L)$ -submodule of W_L . By definition W_L is a semisimple $\mathcal{D}(L)$ -module, so W'_L is also a semisimple $\mathcal{D}(L)$ -module. We have just shown that the only simple $\mathcal{D}(L)$ -submodules of W_L are the $L\sigma(v)$, hence W'_L must be a direct sum of some $L\sigma(v)$. Since W'_L is invariant under the action of $\text{Gal}(L/K)$ and $\text{Gal}(L/K)$ acts transitively on the set of all $L\sigma(v)$, we conclude that

$$W'_L = \sum_{\sigma \in G} L\sigma(v) = W_L.$$

Consequently $W' = W$. This shows that W is a simple $\mathcal{D}(K)$ -module. $\square$

Corollary 3.3.2 *Let (V, D) be a differential operator. Let L be a finite field extension of K and let $a \in \mathcal{P}_L$ be an eigenvalue of D_L . Then:*

$$D \text{ is simple} \quad \text{iff} \quad [K(a) : K] = \dim_K V.$$

PROOF. The proof of proposition 3.2.5 shows that D_L has a normalized eigenvalue $a' \in L$ with $a' - a \in k$. Since $a' - a \in k$, one has $K(a') = K(a)$. Now use proposition 3.3.1. $\square$

Notice that we can't replace the condition $a \in \mathcal{P}_L$ by $a \in L$ in the previous corollary. A counterexample is

$$D = x \frac{d}{dx} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad a = \sqrt{x}, \quad v = \begin{pmatrix} 0 \\ e^{2\sqrt{x}} \end{pmatrix}.$$

The differential operator D is not simple since $K \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a subspace of K^2 which is invariant under D , but $[K(a) : K] = 2$.

Proposition 3.3.3 *Let (V, D) be a simple differential operator. Let L be a Galois extension of K and let $a \in L$ be a normalized eigenvalue of D_L with corresponding non-zero eigenvector $v \in V_{K(a)}$. Let G be the set of K -homomorphisms from $K(a)$ to L . Then $\{\sigma(v) \mid \sigma \in G\}$ is an L -basis of the vector space V_L .*

PROOF. By proposition 3.2.6 the set $\{\sigma(v) \mid \sigma \in G\}$ is linearly independent over L . By corollary 3.3.2 one has $\dim_K V = [K(a) : K]$, so $|G| = [K(a) : K] = \dim_K V = \dim_L V_L$. This finishes the proof. $\square$

Proposition 3.3.4 *Let (V, D) be a simple differential operator, L a finite field extension of K and $a \in L$ a normalized eigenvalue of D_L . Let W be the vector space $K(a)$ over K on which an action of τ is defined by $\tau \cdot w = \tau(w) + aw$ for all $w \in K(a)$. Then the $\mathcal{D}(K)$ -modules V and W are isomorphic.*

PROOF. Without loss of generality one may assume that L/K is a Galois extension. Let $v \in V_{K(a)}$ be a non-zero eigenvector of D_L by the eigenvalue a and let $G = \{\sigma_1, \dots, \sigma_n\}$ be the set of K -homomorphisms from $K(a)$ to L with $\sigma_1 = 1$ and $n = \dim_K V$. By proposition 3.3.3 one has

$$V_L = L\sigma_1(v) \oplus \dots \oplus L\sigma_n(v).$$

Let $\varphi: V \rightarrow L$ be the composition of the inclusion $V \rightarrow V_L$, the projection $V_L \rightarrow L$ and the function $L \rightarrow L$ defined by $b \mapsto b$ for all $b \in L$. Let $\sigma \in \text{Gal}(L/K(a))$ then $\sigma(Lv) = Lv$ and σ permutes the other $L\sigma_i(v)$. This shows that $\sigma(\varphi w) = \varphi w$ for all $w \in V$, so $\varphi(V) \subseteq K(a)$. Let $w \in \ker \varphi$. Write

$$1 \otimes w = \lambda_2 \sigma_2(v) + \dots + \lambda_n \sigma_n(v) \quad \text{for some } \lambda_2, \dots, \lambda_n \in L. \quad (3.4)$$

Let $i \in \{2, \dots, n\}$. Let $\sigma \in \text{Gal}(L/K)$ be an automorphism such that $\sigma\sigma_i$ is the identity on $K(a)$. Then $\sigma\sigma_j = \sigma_{\pi(j)}$ for all $j \in \{1, \dots, n\}$ where π is a permutation on $\{1, \dots, n\}$ with $\pi(i) = 1$. Applying σ to equation (3.4) gives

$$1 \otimes w = \sigma(\lambda_2)\sigma_{\pi(2)}(v) + \dots + \sigma(\lambda_i)v + \dots + \sigma(\lambda_n)\sigma_{\pi(n)}(v).$$

This shows that $\sigma(\lambda_i) = 0$ and hence $\lambda_i = 0$. This is true for all $i \in \{2, \dots, n\}$, so $w = 0$ and therefore φ is injective. Corollary 3.3.2 now implies that φ is a K -linear isomorphism from V to $K(a)$ and one easily verifies that the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & K(a) \\ \downarrow D & & \downarrow \tau+a \\ V & \xrightarrow{\varphi} & K(a) \end{array}$$

$\square$

Proposition 3.3.5 *Let (V, D) be a semisimple differential operator. Then*

$$V = \ker D \oplus \text{im } D$$

as vector spaces over k .

PROOF. It is enough to prove the statement for simple differential operators, so assume that (V, D) is a simple differential operator. By [Mal74, thm. 2.3b] the index of the differential operator D is zero, i.e., $\dim_k \ker D$ and $\dim_k \operatorname{coker} D$ are finite and $\dim_k \ker D - \dim_k \operatorname{coker} D = 0$. So it is enough to show that $\ker D \cap \operatorname{im} D = 0$.

If $\ker D = 0$ this is clear, so assume that $v \in \ker D$ for some $v \in V$ with $v \neq 0$. Then Kv is a subspace of V which is invariant under D . Since we assumed that D is a simple differential operator, we must have $V = Kv$. It is clear that $\ker D = kv$. Let $c \in k$ and suppose that $cv \in \operatorname{im} D$, i.e., $D(av) = cv$ for some $a \in K$. Then $\tau(a) = c$. By writing a as a Laurent series in x , one immediately verifies that this implies that $c = 0$. This shows that $\ker D \cap \operatorname{im} D = 0$ and finishes the proof of the proposition. $\square$

Proposition 3.3.6 *Let (V, D) and (V', D') be two simple differential operators. Let L be a finite field extension of K and let $a \in L$ be a normalized eigenvalue of D_L and $a' \in L$ a normalized eigenvalue of D'_L . Let $f(X) \in K[X]$ be the minimal polynomial of a over K and let $g(X) \in K[X]$ be the minimal polynomial of a' over K . Put $e \stackrel{\text{def}}{=} \epsilon_{K(a)/K}$. Then:*

$$(V, D) \simeq (V', D') \quad \text{iff} \quad g(X) = f(X + \frac{\tau}{e}) \quad \text{for some } \tau \in \mathbb{Z}.$$

PROOF. Without loss of generality one may assume that L is a normal closure of the field extension $K(a, a')/K$. Let $v \in V_{K(a)}$ a non-zero eigenvector of D_L with eigenvalue a and let $v' \in V'_{K(a')}$ a non-zero eigenvector of D'_L with eigenvalue a' . Let G be the set of K -homomorphisms from $K(a)$ to L . By proposition 3.3.3 the set $\{\sigma(v) \mid \sigma \in G\}$ is an L -basis of V_L .

Suppose φ from V to V' is an isomorphism of $\mathcal{D}(K)$ -modules. Then the set $\{\varphi_L(\sigma(v)) \mid \sigma \in G\}$ is an L -basis of V'_L , where $\varphi_L \stackrel{\text{def}}{=} 1 \otimes \varphi$. Write

$$v' = \sum_{\sigma \in G} \lambda_\sigma \varphi_L(\sigma(v)) \quad (\text{all } \lambda_\sigma \text{ in } L).$$

Applying D'_L gives

$$\lambda_\sigma a' = \tau(\lambda_\sigma) + \lambda_\sigma \sigma(a) \quad \text{for all } \sigma \in G.$$

Since $v' \neq 0$, at least one of the λ_σ is non-zero. Assume $\lambda_\sigma \neq 0$. Then $\sigma(a) \in \mathcal{P}_L$ by proposition 3.1.10, so

$$a' - \sigma(a) = \frac{\tau(\lambda_\sigma)}{\lambda_\sigma} \in \mathcal{O}_L \cap \mathcal{P}_L \subseteq K.$$

This shows that L is also a normal closure of the field extension $K(a)/K$, so $\epsilon_{L/K} = e$ by proposition 3.1.6. By proposition 3.1.8, $\mathcal{O}_L \cap \mathcal{P}_L = (1/e)\mathbb{Z}$, so $a' - \sigma_0(a) \in (1/e)\mathbb{Z}$, hence $g(X) = f(X + \tau/e)$ for some $\tau \in \mathbb{Z}$.

Now suppose $g(X) = f(X + r/e)$ for some $r \in \mathbb{Z}$. This implies that $a' = \sigma(a) - r/e$ for some $\sigma \in G$. By proposition 3.1.8, there is an element $\alpha \in K(a)$ with $\tau(\alpha)/\alpha = r/e$. Define

$$w \stackrel{\text{def}}{=} \alpha \sigma^{-1}(v').$$

Then $w \in V'_{K(a)}$ and $D'_L w = aw$. By proposition 3.3.3 the set $\{\sigma(w) \mid \sigma \in G\}$ is an L -basis of V'_L . Define $\varphi: V_L \rightarrow V'_L$ by

$$\varphi\left(\sum_{\sigma \in G} \lambda_\sigma \sigma(v)\right) \stackrel{\text{def}}{=} \sum_{\sigma \in G} \lambda_\sigma \sigma(w) \quad \text{for all } \lambda_\sigma \in L.$$

It is clear that $\varphi: V_L \rightarrow V'_L$ is an isomorphism of vector spaces and for all $\sigma \in G$

$$\begin{aligned} (\varphi \circ D_L)(\lambda_\sigma \sigma(v)) &= \varphi\left((\tau(\lambda_\sigma) + \lambda_\sigma \sigma(a))\sigma(v)\right) \\ &= (\tau(\lambda_\sigma) + \lambda_\sigma \sigma(a))\sigma(w) \\ &= D'_L(\lambda_\sigma \sigma(w)) \\ &= (D'_L \circ \varphi)(\lambda_\sigma \sigma(v)), \end{aligned}$$

so φ is an isomorphism of the $\mathcal{D}(L)$ -modules V_L and V'_L . If one shows that $\varphi(V) \subseteq V'$, then φ is an isomorphism of the $\mathcal{D}(K)$ -modules V and V' . Let $u \in V$ and $\sigma' \in \text{Gal}(L/K)$. We have to show that $\sigma'(\varphi(1 \otimes u)) = \varphi(1 \otimes u)$. Write $1 \otimes u = \sum_{\sigma \in G} \lambda_\sigma \sigma(v)$ with all λ_σ in L . Applying σ' to this equation gives $\sigma'(\lambda_\sigma) = \lambda_{\sigma'\sigma}$ for all $\sigma \in G$. This implies

$$\sigma'(\varphi(1 \otimes u)) = \sigma'\left(\sum_{\sigma \in G} \lambda_\sigma \sigma(w)\right) = \sum_{\sigma \in G} \lambda_{\sigma'\sigma}(\sigma'\sigma)(w) = \varphi(1 \otimes u).$$

□

Notice that if (V, D) and (V', D') are simple differential operators and L is a finite field extension of K such that the $\mathcal{D}(L)$ -modules V_L and V'_L are isomorphic, then the $\mathcal{D}(K)$ -modules V and V' need *not* to be isomorphic. An example is

$$V = V' = K, \quad D = x \frac{d}{dx}, \quad D' = x \frac{d}{dx} + \frac{1}{2}, \quad L = K(\sqrt{x}).$$

Proposition 3.3.6 implies that the $\mathcal{D}(L)$ -modules V_L and V'_L are isomorphic, but that the $\mathcal{D}(K)$ -modules V and V' are not isomorphic. So the situation is different from simple linear transformations.

Definition 3.3.7 Let (V, D) be a simple differential operator. Let L be finite field extension of K and $a \in L$ a normalized eigenvalue of D_L . Define the *ramification index* of D , notation $\epsilon(D)$, and the *inertial degree* of D , notation $f(D)$, by

$$\epsilon(D) \stackrel{\text{def}}{=} \epsilon_{K(a)/K} \quad \text{and} \quad f(D) \stackrel{\text{def}}{=} f_{K(a)/K}.$$

If a' is another normalized eigenvalue of D_L then the fields $K(a)$ and $K(a')$ are isomorphic by proposition 3.3.6, hence $\epsilon_{K(a)/K}$ and $f_{K(a)/K}$ are independent of the chosen normalized eigenvalue a .

Corollary 3.3.8 *Let (V, D) be a simple differential operator. Then*

$$\dim V = \epsilon(D)f(D).$$

PROOF. Let L be finite field extension of K and $a \in L$ a normalized eigenvalue of D_L . From definition 3.3.7 it is clear that $\epsilon(D)f(D) = [K(a) : K]$. Now use corollary 3.3.2. $\square$

Proposition 3.3.9 *Let (V, D) be a simple differential operator. Let R be the ring of $\mathcal{D}(K)$ -endomorphisms of V . Then R is a finite field extension of k and $[R : k] = f(D)$.*

PROOF. Let $f \in R \setminus \{0\}$. Then $\ker(f)$ and $\text{im}(f)$ are $\mathcal{D}(K)$ -submodules of V . Since V is a simple $\mathcal{D}(K)$ -module and $f \neq 0$, one has $\ker(f) = 0$ and $\text{im}(f) = V$, so f is a $\mathcal{D}(K)$ -automorphism of V .

Now let $f, g \in R$. Let L be a finite field extension of K and $a \in L$ a normalized eigenvalue of D_L with corresponding non-zero eigenvector $v \in V_{K(a)}$. We may assume that L/K is a Galois extension. Let $\{\sigma_1, \dots, \sigma_n\}$ be the set of K -homomorphisms from $K(a)$ to L with $n = \dim_K V$. By proposition 3.3.3 the set $\{\sigma_1 v, \dots, \sigma_n v\}$ is an L -basis of V_L . The $\mathcal{D}(K)$ -endomorphisms f and g can be uniquely extended to $\mathcal{D}(L)$ -endomorphisms f_L and g_L of V_L . Let $C = (c_{ij})$ be the matrix of f_L w.r.t. the L -basis $(\sigma_1 v, \dots, \sigma_n v)$ of V_L . Since $fD = Df$ one has $f_L D_L = D_L f_L$, so

$$\tau(c_{ij}) + (\sigma_i(a) - \sigma_j(a))c_{ij} = 0 \quad \text{for all } i, j \in \{1, \dots, n\}.$$

Let $i, j \in \{1, \dots, n\}$ with $i \neq j$ and suppose that $c_{ij} \neq 0$. Then $\tau(c_{ij})/c_{ij} = \sigma_j(a) - \sigma_i(a) \in \mathcal{P}_L \cap \dot{O}_L \subseteq k$ by proposition 3.1.8 and proposition 3.1.10, so $\sigma_i = \sigma_j$ by proposition 3.2.6, a contradiction. So we have

$$c_{ij} = \begin{cases} 0, & \text{if } i \neq j; \\ \text{a constant in } L, & \text{if } i = j. \end{cases}$$

The same holds for g_L , so f_L and g_L commute and therefore f and g commute.

We have now shown that the multiplication in R is commutative and that every non-zero element in R has an inverse. Now it is straightforward to check that R is a (commutative) field and that $k \subseteq R$.

Let k' be the constant field of $K(a)$. We will show that R and k' are isomorphic. Let $f \in R$ and let $C = (c_{ij})$ be the matrix of f_L w.r.t. the L -basis $(\sigma_1 v, \dots, \sigma_n v)$ of V_L . We have already shown that C is a constant diagonal matrix. Without loss of generality we may assume that $\sigma_1 = 1$. Let $\sigma \in \text{Gal}(L/K)$. Then $\sigma|_{K(a)} = \sigma_i$ for some $i \in \{1, \dots, n\}$. One has

$$c_{ii}\sigma_i(v) = f_L(\sigma_i(v)) = \sigma(f_L(v)) = \sigma(c_{11}v) = \sigma(c_{11})\sigma_i(v). \quad (3.5)$$

Let $G \stackrel{\text{def}}{=} \text{Gal}(L/K(a))$. Then $\sigma|_{K(a)} = 1$ for all $\sigma \in G$, so $\sigma(c_{11}) = c_{11}$ for all $\sigma \in G$ by equation (3.5), hence $c_{11} \in K(a)$. Since c_{11} is also a constant in L , one has $c_{11} \in k'$. In other words

$$M(f_L, (\sigma_1 v, \dots, \sigma_n v)) = \begin{pmatrix} \sigma_1(c) & 0 & \dots & 0 \\ 0 & \sigma_2(c) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & & 0 & \sigma_n(c) \end{pmatrix} \quad \text{for some } c \in k'. \quad (3.6)$$

Conversely, let g be an L -linear transformation of V_L such that equation (3.6) holds with f_L replaced by g . We will show that there exists an element $f \in R$ such that $g|_V = f$. It is clear that g and D_L commute, so g is a $\mathcal{D}(L)$ -endomorphism of V . Remains to show that $\sigma g = g \sigma$ for all $\sigma \in \text{Gal}(L/K)$. Let $\sigma \in \text{Gal}(L/K)$ and let $i \in \{1, \dots, n\}$. Then $\sigma \sigma_i = \sigma_j$ for some $j \in \{1, \dots, n\}$ and

$$(\sigma g)(\sigma_i(v)) = \sigma(\sigma_i(c)\sigma_i(v)) = \sigma_j(c)\sigma_j(v) = g(\sigma_j(v)) = (g\sigma)(\sigma_i(v)).$$

This shows that $\sigma g = g \sigma$ for all $\sigma \in \text{Gal}(L/K)$.

Now it is straightforward to check that R and k' are isomorphic, so R is a finite field extension of k and $[R : k] = [k' : k] = f(D)$. $\square$

3.4 The characteristic ring

In this section we will construct a ring $\mathcal{C}$ which is isomorphic to $K_0(\mathcal{V})$. The construction of this ring is completely formal. The reader can forget all about differential operators.

The set $\mathcal{C}$ will be defined as the set of equivalence classes of some subset of $k[1/x](\lambda)$ under a certain equivalence relation $\equiv$. The main idea is to construct an isomorphism c from $K_0(\mathcal{V})$ to $\mathcal{C}$ such that if (V, D) is a simple differential operator, then $\bar{c}([(V, D)])$ will be the equivalence class in $\mathcal{C}$ of the minimal polynomial of a normalized eigenvalue of D_L where L is some finite field extension of K . Proposition 3.3.6 will therefore guide the definition of the equivalence relation $\equiv$. The addition in $\mathcal{C}$ comes from the multiplication in the rational polynomial ring $k[1/x](\lambda)$. The multiplication $\star$ in $\mathcal{C}$ has to correspond with the tensor product in $K_0(\mathcal{V})$. Looking at one-dimensional differential operators, this implies that one has to define $\star$ in such a way that

$$(\lambda - a) \star (\lambda - b) \equiv (\lambda - (a + b)) \quad \text{for all } a, b \in k[1/x].$$

However this multiplication can not be extended by universal polynomials. The main reason being that two differential operators which are not isomorphic over K can become isomorphic over a finite field extension of K .

Define $\mathcal{F}$ as the multiplicative monoid of the monic polynomials in one variable over the field K and $\mathcal{I}$ as the subset of $\mathcal{F}$ consisting of the irreducible polynomials.

Let $m \in \mathbb{N}^*$ and L a finite field extension of K . Define equivalence relations $\equiv_m$ and $\equiv$ on L by

$$\alpha \equiv_m \beta \quad \text{iff} \quad \alpha - \beta \in \frac{1}{m} \mathbb{Z}$$

and

$$\alpha \equiv \beta \quad \text{iff} \quad \alpha - \beta \in \frac{1}{e_{K(\alpha)/K}} \mathbb{Z}$$

for all α and β in L .

Now define the equivalence relations $\equiv_m$ and $\equiv$ on $\mathcal{F}$ as follows. Let $f, g \in \mathcal{F}$ and let L be a splitting field of fg over K . Write

$$f(\lambda) = \prod_{i=1}^n (\lambda - \alpha_i) \quad \text{and} \quad g(\lambda) = \prod_{j=1}^p (\lambda - \beta_j),$$

with $n, p \in \mathbb{N}$ and all α_i and β_j in L . Then $f \equiv_m g$ resp. $f \equiv g$ if and only if $n = p$ and there is a permutation σ on $\{1, \dots, n\}$ such that

$$\alpha_{\sigma(i)} \equiv_m \beta_i \quad \text{for all } i \in \{1, \dots, n\},$$

respectively

$$\alpha_{\sigma(i)} \equiv \beta_i \quad \text{for all } i \in \{1, \dots, n\}.$$

If $f \equiv g$, then f and g are called *similar*, otherwise they are called *non-similar*. Denote the equivalence class in $\mathcal{F}/\equiv$ of an element $f \in \mathcal{F}$ by $[f]$.

Now define a function $e_K: \mathcal{F} \rightarrow \mathbb{N}$. Let $f \in \mathcal{F}$ and define $e_K(f) \stackrel{\text{def}}{=} e_{L/K}$, where L is a splitting field of f over K . We will often write $e(f)$ instead of $e_K(f)$ when there is no confusion possible.

Corollary 3.4.1 *Let $f \in \mathcal{F}$. Write $f = \prod_{i=1}^m f_i$ with $m \in \mathbb{N}$ and all f_i in $\mathcal{I}$. For every $i \in \{1, \dots, m\}$ let α_i be a root of f_i in some common splitting field of f over K . Then*

$$e(f) = \text{lcm}(e(f_1), \dots, e(f_m)) = \text{lcm}(e_{K(\alpha_1)/K}, \dots, e_{K(\alpha_m)/K}) = e_{K(\alpha_1, \dots, \alpha_m)/K}.$$

PROOF. This follows immediately from proposition 3.1.6. $\square$

Notice that for all $m \in \mathbb{N}^*$ and all f and g in $\mathcal{I}$ one has

$$f \equiv_m g \quad \text{iff} \quad f(\lambda + \frac{r}{m}) = g(\lambda) \quad \text{for some } r \in \mathbb{Z}.$$

and

$$f \equiv g \quad \text{iff} \quad f(\lambda + \frac{r}{e(f)}) = g(\lambda) \quad \text{for some } r \in \mathbb{Z}.$$

Also notice that for all $m \in \mathbb{N}^*$ and all f and g in $\mathcal{F}$ with either $f \equiv g$ or $f \equiv_m g$ one has $\epsilon(f) = \epsilon(g)$.

We will now define an addition $\cdot$ and a multiplication $\star$ on the set $\mathcal{F}/\equiv$. Define

$$[f] \cdot [g] \stackrel{\text{def}}{=} [fg] \quad \text{for all } f, g \in \mathcal{F}.$$

The definition of the equivalence relation $\equiv$ immediately implies that this is a good definition. The definition of the multiplication on $\mathcal{F}/\equiv$ is more complicated. This multiplication should correspond with the tensor product of differential operators.

To motivate the definition of this multiplication, let (V, D) and (V', D') be two simple differential operators with $e \stackrel{\text{def}}{=} \epsilon(D)$ and $e' \stackrel{\text{def}}{=} \epsilon(D')$. Then

$$(V, D) \otimes (K, \tau + \frac{\tau}{e}) \simeq (V, D) \quad \text{and} \quad (V', D') \otimes (K, \tau + \frac{\tau'}{e'}) \simeq (V', D')$$

for all $\tau, \tau' \in \mathbb{Z}$, hence

$$(V \otimes V', D \otimes D') \otimes (K, \tau + \frac{1}{\text{lcm}(e, e')}) \simeq (V \otimes V', D \otimes D'). \quad (3.7)$$

Let L be a finite field extension of K such that

$$\begin{aligned} (V_L, D_L) &\simeq (L, \tau + a_1) \oplus \cdots \oplus (L, \tau + a_m), \\ (V'_L, D'_L) &\simeq (L, \tau + b_1) \oplus \cdots \oplus (L, \tau + b_n), \end{aligned}$$

where $m, n \in \mathbb{N}^*$ and all a_i and b_j are elements in L . One can choose a field L with $\epsilon_{L/K} = \text{lcm}(e, e')$. Then

$$(V_L \otimes V'_L, D_L \otimes D'_L) \simeq \bigoplus_{i,j} (L, \tau + a_i + b_j). \quad (3.8)$$

With equations (3.7) and (3.8) in mind we will define a multiplication $\star$ on $\mathcal{F}/\equiv$ such that if $f, g \in \mathcal{I}$ and $h \in \mathcal{F}$ with $[f] \star [g] = [h]$, then

$$h(\lambda) \equiv h(\lambda + \frac{1}{e}) \quad \text{and} \quad h \equiv_e \mathcal{R}(f, g),$$

where $e \stackrel{\text{def}}{=} \epsilon(fg) = \text{lcm}(\epsilon(f), \epsilon(g))$ and the function $\mathcal{R}: \mathcal{F} \times \mathcal{F} \rightarrow \mathcal{F}$ is defined by

$$\mathcal{R}(f, g)(\lambda) \stackrel{\text{def}}{=} \text{Res}_y(f(y), g(\lambda - y)) \quad \text{for all } f, g \in \mathcal{F},$$

where Res_y denotes the resultant w.r.t. the variable y . Notice that if

$$f(\lambda) = \prod_{i=1}^m (\lambda - \alpha_i) \quad \text{and} \quad g(\lambda) = \prod_{j=1}^n (\lambda - \beta_j)$$

with all α_i and β_j in some finite field extension of K , then

$$\mathcal{R}(f, g)(\lambda) = \prod_{i=1}^m \prod_{j=1}^n (\lambda - (\alpha_i + \beta_j)) \in K[\lambda]. \quad (3.9)$$

Proposition 3.4.2 *Let $m \in \mathbb{N}^*$ and $f \in \mathcal{F}$. Let*

$$\prod_{i=1}^n f_i^{r_i} \quad (n \in \mathbb{N}, r_1, \dots, r_n \in \mathbb{N}^*, f_1, \dots, f_n \in \mathcal{I})$$

be the prime factor decomposition of f . Suppose that $\epsilon(f_i) \mid m \mid r_i \epsilon(f_i)$ for all $i \in \{1, \dots, n\}$. Then there exists an element $h \in \mathcal{F}$ satisfying

$$h \equiv_m f \quad \text{and} \quad h(\lambda) \equiv h(\lambda + \frac{1}{m}). \quad (3.10)$$

Furthermore, for every $h' \in \mathcal{F}$:

$$h' \equiv h \quad \text{iff} \quad h' \text{ satisfies (3.10).}$$

PROOF. Define

$$h(\lambda) \stackrel{\text{def}}{=} \prod_{i=1}^n \prod_{j=0}^{n_i-1} f_i(\lambda + \frac{j}{m})^{r_i/n_i},$$

where $n_i \stackrel{\text{def}}{=} m/\epsilon(f_i)$ for all $i \in \{1, \dots, n\}$. It is immediately clear that $h \equiv_m f$. To prove $h(\lambda) \equiv h(\lambda + 1/m)$, it is enough to show that for all $i \in \{1, \dots, n\}$

$$\prod_{j=0}^{n_i-1} f_i(\lambda + \frac{j}{m}) \equiv \prod_{j=0}^{n_i-1} f_i(\lambda + \frac{j+1}{m}). \quad (3.11)$$

Let $i \in \{1, \dots, n\}$. All factors but one in the left hand side and right hand side of equation (3.11) are equal. Remains to show

$$f_i(\lambda) \equiv f_i(\lambda + \frac{n_i}{m}) = f_i(\lambda + \frac{1}{\epsilon(f_i)}).$$

This is immediately clear.

To prove the second statement of the proposition, let $h' \in \mathcal{F}$ and suppose that $h' \equiv h$. Then $h' \equiv_m h \equiv_m f$ and

$$h'(\lambda + \frac{1}{m}) \equiv h(\lambda + \frac{1}{m}) \equiv h(\lambda) \equiv h'(\lambda).$$

Now suppose that h' satisfies (3.10). We have to show that $h' \equiv h$. Notice that

$$f_i(\lambda + \frac{j_1}{m}) \equiv f_i(\lambda + \frac{j_2}{m}) \iff n_i \mid j_1 - j_2 \quad (3.12)$$

for all $i \in \{1, \dots, n\}$ and $j_1, j_2 \in \mathbb{Z}$. Since $h' \equiv_m f$ one can write

$$h'(\lambda) \equiv \prod_{i=1}^n \prod_{j=0}^{n_i-1} f_i(\lambda + \frac{j}{m})^{r_{i,j}}$$

with all $r_{i,j}$ in $\mathbb{N}$ (possibly zero) and $r_i = \sum_j r_{i,j}$ for all $i \in \{1, \dots, n\}$. The fact that $h'(\lambda) \equiv h'(\lambda + 1/m)$ implies that

$$\prod_{j=0}^{n_i-1} f_i(\lambda + \frac{j}{m})^{r_{i,j}} \equiv \prod_{j=0}^{n_i-1} f_i(\lambda + \frac{j+1}{m})^{r_{i,j}}$$

for all $i \in \{1, \dots, n\}$. Using (3.12), this implies $r_{i,0} = r_{i,1} = \dots = r_{i,n_i-1}$, hence $r_{i,j} = r_i/n_i$ for all i and j . This shows that

$$h'(\lambda) \equiv \prod_{i=1}^n \prod_{j=0}^{n_i-1} f_i(\lambda + \frac{j}{m})^{r_i/n_i} = h(\lambda).$$

□

Lemma 3.4.3 *Let $f, g, h \in \mathcal{I}$. Let $r \in \mathbb{N}$ with $h^r \mid \mathcal{R}(f, g)$ and $h^{r+1} \nmid \mathcal{R}(f, g)$. Then $\epsilon(h) \mid \epsilon(fg) \mid r \cdot \epsilon(h)$.*

PROOF. Let L be a splitting field of fg over K . Let $\alpha \in L$ be a root of f and $\beta \in L$ be a root of g , such that $h(\alpha + \beta) = 0$. Then by corollary 3.4.1 one has $\epsilon(fg) = \epsilon_{K(\alpha, \beta)/K}$ and $\epsilon(h) = \epsilon_{K(\alpha + \beta)/K}$. Since $K(\alpha + \beta) \subseteq K(\alpha, \beta)$, we have $\epsilon(h) \mid \epsilon(fg)$.

Put $q \stackrel{\text{def}}{=} \epsilon_{K(\alpha, \beta)/K(\alpha + \beta)}$, then $q = \epsilon(fg)/\epsilon(h)$. We have to show that $q \mid r$. Let k' be the field of constants in L . Since $\epsilon_{L/K} = \epsilon_{K(\alpha, \beta)/K}$ by proposition 3.1.6, one has $\epsilon_{L/K(\alpha, \beta)} = 1$ and therefore $L = k'K(\alpha, \beta)$. So we have the following field extensions

$$\begin{array}{c} L = k'K(\alpha, \beta) \\ \mid q \\ k'K(\alpha + \beta) \\ \mid \epsilon(h) \\ k'K \end{array}$$

Let H be the Galois group of the field extension $L/k'K(\alpha + \beta)$, G_1 the set of K -homomorphisms from $K(\alpha)$ to L and G_2 the set of K -homomorphisms from $K(\beta)$ to L . We have

$$\mathcal{R}(f, g)(\lambda) = \prod_{\sigma_1 \in G_1} \prod_{\sigma_2 \in G_2} \left(\lambda - (\sigma_1(\alpha) + \sigma_2(\beta)) \right).$$

Therefore define

$$J \stackrel{\text{def}}{=} \{ (\sigma_1, \sigma_2) \in G_1 \times G_2 \mid \sigma_1(\alpha) + \sigma_2(\beta) = \alpha + \beta \}.$$

It is clear that $|J| = r$. H acts on J in the following way:

$$\langle \sigma_1, \sigma_2 \rangle \xrightarrow{\sigma} \langle \sigma \sigma_1, \sigma \sigma_2 \rangle.$$

Suppose that $\sigma \sigma_1 = \sigma_1$ on $K(\alpha)$ and $\sigma \sigma_2 = \sigma_2$ on $K(\beta)$. Then σ fixes $\sigma_1(\alpha)$ and $\sigma_2(\beta)$, so σ elementwise fixes $k'K(\sigma_1\alpha, \sigma_2\beta) = L$. Hence $\sigma = 1$. This shows that $|H|$ divides $|J|$. $\square$

We are now ready to define a multiplication $\star$ on the set $\mathcal{F}/\equiv$. Let $f, g \in \mathcal{I}$. By proposition 3.4.2 and lemma 3.4.3 there is an element $h \in \mathcal{F}$ satisfying

$$h \equiv_m \mathcal{R}(f, g) \quad \text{and} \quad h(\lambda) \equiv h(\lambda + \frac{1}{m}), \quad (3.13)$$

where $m \stackrel{\text{def}}{=} \epsilon(fg)$. Define

$$[f] \star [g] \stackrel{\text{def}}{=} [h].$$

By the second statement of proposition 3.4.2, $[h]$ is independent of the choice of h satisfying (3.13). Remains to prove that $[h]$ only depends on the equivalence classes of f and g in $\mathcal{F}/\equiv$. Let $f', g' \in \mathcal{F}$ with $f' \equiv f$ and $g' \equiv g$. Then $\epsilon(f') = \epsilon(f)$ and $\epsilon(g') = \epsilon(g)$, so by proposition 3.4.2 it is enough to show that $h \equiv_m \mathcal{R}(f', g')$. We know that $f' \equiv_{\epsilon(f)} f$ and $g' \equiv_{\epsilon(g)} g$, so using equation (3.9), $\mathcal{R}(f', g') \equiv_m \mathcal{R}(f, g)$ and therefore $h \equiv_m \mathcal{R}(f', g')$.

Example 3.4.4 In this example we will compute $[\lambda^2 - 1/x] \star [\lambda^2 - 1/x]$ and $[\lambda^2 - 2] \star [\lambda^2 - 2]$.

The roots of $\lambda^2 - 1/x$ in $K(\sqrt{x})$ are $1/\sqrt{x}$ and $-1/\sqrt{x}$. Therefore

$$\mathcal{R}(\lambda^2 - \frac{1}{x}, \lambda^2 - \frac{1}{x}) = \lambda^2(\lambda - \frac{2}{\sqrt{x}})(\lambda + \frac{2}{\sqrt{x}}) = \lambda^2(\lambda^2 - \frac{4}{x}).$$

Since $\epsilon_{K(\sqrt{x})/K} = 2$ one has $\epsilon(\lambda^2 - 1/x) = 2$. So we are looking for an element $f \in K[\lambda]$ satisfying

$$f(\lambda) \equiv_2 \lambda^2(\lambda^2 - \frac{4}{x}) \quad \text{and} \quad f(\lambda + \frac{1}{2}) \equiv f(\lambda).$$

An f satisfying the first relation is similar to $\lambda^2(\lambda^2 - 4/x)$, $(\lambda + \frac{1}{2})^2(\lambda^2 - 4/x)$ or $\lambda(\lambda + \frac{1}{2})(\lambda^2 - 4/x)$. Only the last one satisfies the last relation, so

$$[\lambda^2 - \frac{1}{x}] \star [\lambda^2 - \frac{1}{x}] = [\lambda(\lambda + \frac{1}{2})(\lambda^2 - \frac{4}{x})].$$

The roots of $\lambda^2 - 2$ in $K(\sqrt{2})$ are $\sqrt{2}$ and $-\sqrt{2}$. Therefore

$$\mathcal{R}(\lambda^2 - 2, \lambda^2 - 2) = \lambda^2(\lambda - 2\sqrt{2})(\lambda + 2\sqrt{2}) = \lambda^2(\lambda^2 - 8).$$

Since $\epsilon_{K(\sqrt{2})/K} = 1$ one has $\epsilon(\lambda^2 - 2) = 1$. So we are looking for an element $f \in K[\lambda]$ satisfying

$$f(\lambda) \equiv_1 \lambda^2(\lambda^2 - 8) \quad \text{and} \quad f(\lambda + 1) \equiv f(\lambda).$$

The last relation is always satisfied, so

$$[\lambda^2 - 2] \star [\lambda^2 - 2] = [\lambda^2(\lambda^2 - 8)].$$

Now let $f, g \in \mathcal{F}$. Write $f = \prod f_i$ and $g = \prod g_j$ with all f_i and g_j in $\mathcal{I}$ and define

$$[f] \star [g] \stackrel{\text{def}}{=} \prod_{i,j} [f_i] \star [g_j]. \quad (3.14)$$

To prove the distributivity and associativity of the addition $\cdot$ and multiplication $\star$ on $\mathcal{F}/\equiv$, we first need a new definition and a few lemmas.

Let $m \in \mathbb{N}^*$ and $f, g \in \mathcal{F}$. Define

$$[f] \equiv_m [g] \quad \text{iff} \quad f' \equiv_m g' \quad \text{for some } f', g' \in \mathcal{F} \text{ with } f' \equiv f \text{ and } g' \equiv g.$$

Lemma 3.4.5 $\equiv_m$ is an equivalence relation for all $m \in \mathbb{N}^*$.

PROOF. We will only the transitivity as the rest is clear. Let $m \in \mathbb{N}^*$ and $f, g, h \in \mathcal{F}$ with $[f] \equiv_m [g]$ and $[g] \equiv_m [h]$. Then there exist elements $f', g', g'', h'' \in \mathcal{F}$ with $f' \equiv f$, $g' \equiv g'' \equiv g$, $h' \equiv h$ and $f' \equiv_m g'$ and $g'' \equiv_m h''$. It is enough to show that there exists an $h' \in \mathcal{F}$ with $h' \equiv h''$ and $g' \equiv_m h'$. We will show this if $h'' \in \mathcal{I}$. The general case follows immediately from this. Put $e = \epsilon(h'')$. Since we assumed that $h'' \in \mathcal{I}$, we have $g', g'' \in \mathcal{I}$ and $\epsilon(g') = \epsilon(g'') = e$. Since $g' \equiv g''$, there is an integer $r \in \mathbb{Z}$ with $g'(\lambda) = g''(\lambda + r/e)$. Now define $h'(\lambda) \stackrel{\text{def}}{=} h''(\lambda + r/e)$. Then $h' \equiv h''$ and $g' \equiv_m h'$. $\square$

Lemma 3.4.6 Let $f, g \in \mathcal{F}$ and $m \in \mathbb{N}^*$.

1. If $[f] \equiv_m [g]$ then $[f] \cdot [h] \equiv_m [g] \cdot [h]$ and $[f] \star [h] \equiv_m [g] \star [h]$ for all $h \in \mathcal{F}$.
2. $[f] \star [g] \equiv_{\epsilon(fg)} \mathcal{R}(f, g)$.

PROOF. Let $h \in \mathcal{F}$. Suppose $[f] \equiv_m [g]$. Without loss of generality one may assume that $f \equiv_m g$. This implies that $fh \equiv_m gh$ and therefore $[f] \cdot [h] = [fh] \equiv_m [gh] = [g] \cdot [h]$. Write $f = f_1 \dots f_p$ with $p \in \mathbb{N}$ and $f_1, \dots, f_p \in \mathcal{I}$. Then one can also write $g = g_1 \dots g_p$ with $g_1, \dots, g_p \in \mathcal{I}$ and $f_i \equiv_m g_i$ for all $i \in \{1, \dots, p\}$. Write $h = h_1 \dots h_q$ with $q \in \mathbb{N}$ and $h_1, \dots, h_q \in \mathcal{I}$. One has

$$[f] \star [h] = \prod_{i,j} [f_i] \star [h_j] \quad \text{and} \quad [g] \star [h] = \prod_{i,j} [g_i] \star [h_j].$$

If we show that $[f_i] \star [h_j] \equiv_m [g_i] \star [h_j]$ for all i and j , then, by repeatedly applying the first part of the statement, $[f] \star [h] \equiv_m [g] \star [h]$. So, let $i \in \{1, \dots, p\}$ and

$j \in \{1, \dots, q\}$. Let $[f_i] \star [h_j] = [F]$ for some $F \in \mathcal{F}$. Then $F \equiv_e \mathcal{R}(f_i, h_j)$ and $F(\lambda) \equiv F(\lambda + 1/e)$ by definition, where $e \stackrel{\text{def}}{=} \epsilon(f_i, h_j)$. Since $f_i \equiv_m g_i$, one has $g_i(\lambda) = f_i(\lambda + r/m)$ for some $r \in \mathbb{Z}$. Define $G \in \mathcal{F}$ by $G(\lambda) \stackrel{\text{def}}{=} F(\lambda + r/m)$. Then $G \equiv_e \mathcal{R}(g_i, h_j)$ and $G(\lambda) \equiv G(\lambda + 1/e)$, so $[g_i] \star [h_j] = [G]$ by definition. This shows that $[f_i] \star [h_j] \equiv_m [g_i] \star [h_j]$.

To prove the second statement, write $f = f_1 \dots f_p$ and $g = g_1 \dots g_q$ with $p, q \in \mathbb{N}$ and all f_i and g_j in $\mathcal{I}$. Then

$$[f] \star [g] = \prod_{i,j} [f_i] \star [g_j] \quad \text{and} \quad [\mathcal{R}(f, g)] = \prod_{i,j} [\mathcal{R}(f_i, g_j)].$$

By the first part of the first statement it is therefore sufficient to prove that $[f_i] \star [g_j] \equiv_e [\mathcal{R}(f_i, g_j)]$ for all i and j , where $e \stackrel{\text{def}}{=} \epsilon(fg)$. So, let $i \in \{1, \dots, p\}$ and $j \in \{1, \dots, q\}$. Let $[f_i] \star [g_j] = [h]$ for some $h \in \mathcal{F}$. Then by definition

$$h \equiv_{\epsilon(f, g_j)} \mathcal{R}(f_i, g_j).$$

Since $\epsilon(f_i, g_j) \mid \epsilon(fg)$, this implies $h \equiv_e \mathcal{R}(f_i, g_j)$, hence $[f_i] \star [g_j] = [h] \equiv_e [\mathcal{R}(f_i, g_j)]$. $\square$

Proposition 3.4.7 *Let $f, g, h \in \mathcal{F}$. Then:*

$$\begin{aligned} [f] \star ([g] \cdot [h]) &= ([f] \star [g])([f] \star [h]), \\ [f] \star [g] &= [g] \star [f], \\ [\lambda] \star [f(\lambda)] &= [f(\lambda)], \\ ([f] \star [g]) \star [h] &= [f] \star ([g] \star [h]). \end{aligned}$$

PROOF. The first equality follows immediately from the definition of the $\star$ operator in (3.14). Also by (3.14), it is enough to show the rest of the equalities for f, g and h in $\mathcal{I}$. Equation (3.9) immediately implies that $\mathcal{R}(f, g) = \mathcal{R}(g, f)$ and $\mathcal{R}(\lambda, f(\lambda)) = f(\lambda)$. This implies that $[f] \star [g] = [g] \star [f]$ and $[\lambda] \star [f(\lambda)] = [f(\lambda)]$.

Remains to show the associativity of $\star$. Let $f, g, h \in \mathcal{I}$. Define

$$\mathcal{R}(f, g, h) \stackrel{\text{def}}{=} \mathcal{R}(\mathcal{R}(f, g), h) = \mathcal{R}(f, \mathcal{R}(g, h)).$$

Lemma 3.4.3 implies that $\epsilon(\mathcal{R}(f, g, h)) \mid \epsilon(\mathcal{R}(f, g)h)$ and $\epsilon(\mathcal{R}(f, g)) \mid \epsilon(fg)$, hence

$$\epsilon(\mathcal{R}(f, g, h)) \mid \epsilon(\mathcal{R}(f, g)h) \mid \epsilon(fgh). \quad (3.15)$$

Using lemma 3.4.6, one finds

$$([f] \star [g]) \star [h] \equiv_{\epsilon(fg)} [\mathcal{R}(f, g)] \star [h] \equiv_{\epsilon(\mathcal{R}(f, g)h)} [\mathcal{R}(f, g, h)].$$

Using (3.15), this implies $[\mathcal{R}(f, g, h)] \equiv_{\epsilon(fgh)} ([f] \star [g]) \star [h]$. Now write

$$\frac{1}{\epsilon(fgh)} = \frac{a}{\epsilon(f)} + \frac{b}{\epsilon(g)} + \frac{c}{\epsilon(h)},$$

with a, b and c integers. Then

$$\begin{aligned} ([f] \star [g]) \star [h] &= \left([f(\lambda + \frac{a}{e(f)})] \star [g(\lambda + \frac{b}{e(g)})] \right) \star [h(\lambda + \frac{c}{e(h)})] \\ &= \left(([f] \star [g]) \star [h] \right) (\lambda + \frac{1}{e(fgh)}). \end{aligned}$$

The same can be done for $[f] \star ([g] \star [h])$.

In the same way as lemma 3.4.3, one can prove that $e(H) \mid e(fgh) \mid r \cdot e(H)$ for every $H \in \mathcal{I}$ and $r \in \mathbb{N}^*$ satisfying $H^r \mid \mathcal{R}(f, g, h)$ and $H^{r+1} \nmid \mathcal{R}(f, g, h)$. Hence, by the second statement of proposition 3.4.2, one finds $([f] \star [g]) \star [h] = [f] \star ([g] \star [h])$. $\square$

Now define R as the multiplicative subgroup of $k[1/x](\lambda)$ consisting of the elements of the form f/g with $f, g \in \mathcal{F}$ and all roots of f and g are in P_L (see definition 3.1.9), where L is a splitting field of fg . Define a relation $\equiv$ on R by

$$\frac{f}{g} \equiv \frac{f'}{g'} \quad \text{iff} \quad fg' \equiv f'g$$

for all f, f', g and g' in $\mathcal{F} \cap R$. Using the properties of $\equiv$ on $\mathcal{F}$, one easily verifies that this is a good definition and that $\equiv$ is an *equivalence* relation on R . Now define the *characteristic ring* $\mathcal{C}$ by $\mathcal{C} \stackrel{\text{def}}{=} R / \equiv$. We will denote the equivalence class in $\mathcal{C}$ of $f \in R$ by $[f]$. Let $f, g, h, j \in \mathcal{F} \cap R$ and define

$$\begin{bmatrix} f \\ g \end{bmatrix} \cdot \begin{bmatrix} h \\ j \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} fh \\ gj \end{bmatrix}.$$

One easily verifies that this is a good definition and that the binary operation $\cdot$ gives $\mathcal{C}$ the structure of an Abelian group. Finally define

$$\begin{bmatrix} f \\ g \end{bmatrix} \star \begin{bmatrix} h \\ j \end{bmatrix} \stackrel{\text{def}}{=} \frac{([f] \star [h])([g] \star [j])}{([f] \star [j])([g] \star [h])}.$$

Using the properties of proposition 3.4.7, one can verify that $\star$ is well-defined on R and that it gives $\mathcal{C}$ the structure of a commutative ring with identity with $+$ as addition and $\star$ as multiplication.

Notice that $\mathcal{C}$ has zero-divisors, e.g.,

$$[\lambda^2 - \frac{1}{x}] \star \begin{bmatrix} \lambda + \frac{1}{2} \\ \lambda \end{bmatrix} = [1] = 0 \quad \text{and} \quad [\lambda(\lambda + \frac{1}{2})] \star \begin{bmatrix} \lambda + \frac{1}{2} \\ \lambda \end{bmatrix} = [1] = 0.$$

In the next section we will show that the rings $K_0(\mathcal{V})$ and $\mathcal{C}$ are isomorphic.

3.5 The characteristic class

Let (V, D) be a simple differential operator. By proposition 3.2.5 the differential operator D_L has a normalized eigenvalue a in some finite field extension L of K . Define $c((V, D))$, or short $c(D)$, as the equivalence class in $\mathcal{C}$ of the minimal polynomial of a over K . By proposition 3.3.6 this equivalence class is independent of the chosen normalized eigenvalue.

For a general differential operator (V, D) , let

$$0 = V_0 \subset V_1 \subset \cdots \subset V_r = V$$

be a (finite) composition series of the $\mathcal{D}(K)$ -module V , in other words

- $0 = V_0 \subset V_1 \subset \cdots \subset V_r = V$ are D -invariant vector spaces over K .
- The differential operator D_i on V_i/V_{i-1} , which is induced by D , is simple for all $i \in \{1, \dots, r\}$.

Define $c((V, D))$, or short $c(D)$, by

$$c(D) = c((V, D)) \stackrel{\text{def}}{=} \prod_{i=1}^r c((V_i/V_{i-1}, D_i)).$$

By the Jordan-Hölder theorem, the right hand side doesn't depend on the composition series of the $\mathcal{D}(K)$ -module V . We will call $c(D)$ the *characteristic class* of the $\mathcal{D}(K)$ -module V . This is a generalization of the characteristic polynomial defined by Levelt in [Lev75].

Corollary 3.5.1 *Let (V, D) be a differential operator and let $h \in \mathcal{F} \cap R$ with $c(D) = [h]$. Then D is simple if and only if h is irreducible.*

Example 3.5.2 Let

$$D = x \frac{d}{dx} + \begin{pmatrix} 0 & 1/x \\ 1 & -\frac{1}{2} \end{pmatrix}.$$

Then

$$D \begin{pmatrix} 1 \\ \sqrt{x} \end{pmatrix} = \frac{1}{\sqrt{x}} \begin{pmatrix} 1 \\ \sqrt{x} \end{pmatrix},$$

so $1/\sqrt{x}$ is a normalized eigenvalue of D_L , where $L \stackrel{\text{def}}{=} K(\sqrt{x})$. The differential operator D is simple by corollary 3.3.2, hence

$$c(D)(\lambda) = [\lambda^2 - 1/x].$$

Proposition 3.5.3 *Let (V, D) be a differential operator and let $h \in \mathcal{F} \cap R$ with $c(D) = [h]$. Let L be a finite field extension of K , then h is also an element representing the characteristic class of the differential operator (V_L, D_L) .*

PROOF. Let

$$0 = V_0 \subset V_1 \subset \cdots \subset V_r = V$$

be a composition series of the $\mathcal{D}(K)$ -module V . Then

$$0 = (V_0)_L \subset (V_1)_L \subset \cdots \subset (V_r)_L = V_L$$

can be refined to a composition series of the $\mathcal{D}(L)$ -module V_L . We may therefore assume that (V, D) is a simple differential operator. Write

$$h = \prod_{i=1}^m f_i \quad (m \in \mathbb{N}, \text{ all } f_i \in L[\lambda] \text{ irreducible over } L).$$

Let L' be a splitting field of h over L . For every $i \in \{1, \dots, m\}$, let $a_i \in L'$ be some root of f_i . Let $v_1 \in K(a_1) \otimes V$ be a non-zero vector with $D_L v_1 = a_1 v_1$. For every $i \in \{2, \dots, m\}$, define $v_i \stackrel{\text{def}}{=} \sigma_i(v_1)$, where $\sigma_i \in \text{Gal}(L'/K)$ is a homomorphism sending a_1 to a_i . Then $D_{L'} v_i = a_i v_i$ for every $i \in \{1, \dots, m\}$. For every $i \in \{1, \dots, m\}$, let G_i be the set of L -homomorphisms from $L(a_i)$ to L' . By proposition 3.3.1 there is a simple $\mathcal{D}(L)$ -submodule W_i of V_L such that $\dim_L W_i = [L(a_i) : L] = \deg f_i$ and $v_i \in L(a_i) \otimes_L W_i$ for every $i \in \{1, \dots, m\}$. Since $\{\sigma(v_i) \mid i \in \{1, \dots, m\}, \sigma \in G_i\} = \{\sigma(v_1) \mid \sigma \in \text{Gal}(L'/K)\}$ is an L' -basis of $L' \otimes V$ by proposition 3.3.3, one has

$$L' \otimes V = \sum_{i=1}^m L' \otimes_L W_i,$$

so $V_L = \sum_{i=1}^m W_i$. Comparing dimensions shows that V_L is a direct sum of the W_i . One has $c(D_L|_{W_i}) = [f_i]$ for every $i \in \{1, \dots, m\}$, so $c(D_L) = \prod_i [f_i] = [h]$. $\square$

Proposition 3.5.4 *Let (V, D) be a differential operator, L a finite field extension of K and let $h \in \mathcal{F} \cap R$ be a polynomial with $c(D_L) = [h]$. Suppose that $\epsilon_{L/K} \mid \epsilon(f)$ for all $f \in \mathcal{I}$ dividing h . Then $c(D) = [h]$.*

PROOF. Let $g \in \mathcal{F} \cap R$ with $c(D) = [g]$. We have to show that $g \equiv h$. By proposition 3.5.3 we have $c(D_L) = [g]$, so g and h are similar over the field L . Let L' be a splitting field of h over L . Let $\alpha_1, \alpha_2, \dots, \alpha_m$ be the roots of h in L' and let $\beta_1, \beta_2, \dots, \beta_m$ be the roots of g in L' . The fact that g and h are similar over L implies that (after relabeling)

$$\alpha_i - \beta_i \in \frac{1}{\epsilon_{L(\alpha_i)/K}} \mathbb{Z} \quad \text{for all } i \in \{1, \dots, m\}.$$

The fact that $\epsilon_{L/K} \mid \epsilon(f)$ for all $f \in \mathcal{I}$ dividing h implies $\epsilon_{L/K} \mid \epsilon_{K(\alpha_i)/K}$ for all $i \in \{1, \dots, m\}$, so

$$\epsilon_{L(\alpha_i)/K} = \text{lcm}(\epsilon_{K(\alpha_i)/K}, \epsilon_{L/K}) = \epsilon_{K(\alpha_i)/K}.$$

Hence $g \equiv h$. $\square$

If $0 \rightarrow (V', D') \xrightarrow{f} (V, D) \xrightarrow{g} (V'', D'') \rightarrow 0$ is an exact sequence in $\mathcal{V}$ and

$$0 = (V'_0, D'_0) \subset (V'_1, D'_1) \subset \cdots \subset (V'_n, D'_n) = (V', D')$$

is a composition series of (V', D') and

$$0 = (V''_0, D''_0) \subset (V''_1, D''_1) \subset \cdots \subset (V''_m, D''_m) = (V'', D'')$$

is a composition series of (V'', D'') , then

$$\begin{aligned} 0 &= f((V'_0, D'_0)) \subset \cdots \subset f((V'_n, D'_n)) \\ &= g^{-1}((V''_0, D''_0)) \subset \cdots \subset g^{-1}((V''_m, D''_m)) = (V, D) \end{aligned}$$

is a composition series of (V, D) , so $c(D')c(D'') = c(D)$. This implies that $c: \mathcal{V} \rightarrow \mathcal{C}$ induces a group homomorphism $\bar{c}: K_0(\mathcal{V}) \rightarrow \mathcal{C}$.

Theorem 3.5.5 *The function $\bar{c}: K_0(\mathcal{V}) \rightarrow \mathcal{C}$ is a ring isomorphism.*

PROOF.

Injective: If (V, D) is a differential operator and

$$0 = (V_0, D_0) \subset (V_1, D_1) \subset \cdots \subset (V_n, D_n) = (V, D)$$

is a composition series of (V, D) , then

$$[(V, D)] = [(V_n/V_{n-1}, \bar{D}_n)] + \cdots + [(V_1/V_0, \bar{D}_1)]$$

in $K_0(\mathcal{V})$, so every element in $K_0(\mathcal{V})$ can be written as

$$\sum_{i=0}^m [A_i] - \sum_{j=0}^n [B_j] \tag{3.16}$$

with $m, n \in \mathbb{N}$ and all A_i and B_j simple differential operators. Now suppose $c(E) = [1]$ where $E \in K_0(\mathcal{V})$ is as in equation (3.16). Then

$$\prod_{i=1}^m c(A_i) = \prod_{j=1}^n c(B_j).$$

This implies that $m = n$ and that there is a permutation π on $\{1, \dots, m\}$ such that $c(A_i) = c(B_{\pi(i)})$ for all $i \in \{1, \dots, m\}$. Applying proposition 3.3.6 gives that A_i and $B_{\pi(i)}$ are isomorphic, so $[A_i] = [B_{\pi(i)}]$ in $K_0(\mathcal{V})$ for all $i \in \{1, \dots, m\}$. This shows that $E = 0$ in $K_0(\mathcal{V})$.

Surjective: Let L be a finite field extension of K and let $a \in \mathcal{P}_L$. It is enough that there exists a differential operator (V, D) with $\dim_K V = [K(a) : K]$ and a non-zero $v \in V_{K(a)}$ such that $D_L v = av$. Put $n \stackrel{\text{def}}{=} [K(a) : K]$. If $n = 1$ then the

differential operator $\tau + a: K \rightarrow K$ satisfies the condition with eigenvector (1), so assume that $n > 1$. Write

$$a^{i+1} - i\tau(a)a^{i-1} = \sum_{j=0}^{n-1} \lambda_{ij} a^j \quad \text{for all } i \in \{0, \dots, n-1\},$$

with $\lambda_{ij} \in K$ for all i and j . Let $(e_1, \dots, e_n)$ denote the natural basis of the vector space K^n . Define the differential operator D on K^n by

$$De_{j+1} \stackrel{\text{def}}{=} \sum_{i=0}^{n-1} \lambda_{ij} e_{i+1} \quad \text{for all } j \in \{0, \dots, n-1\}.$$

Now define $v \in K(a)^n$ by

$$v \stackrel{\text{def}}{=} \sum_{i=0}^{n-1} a^i e_{i+1}.$$

One can easily check that $D_L v = av$.

Multiplicative: Let (V, D) and (V', D') be simple differential operators. It is enough to show that $c((V, D) \otimes (V', D')) = c(D) \star c(D')$. Let L be a finite field extension of K and $a \in L$ a normalized eigenvalue of D_L with corresponding non-zero eigenvector $v \in V_{K(a)}$ and $a' \in L$ a normalized eigenvalue of D'_L with corresponding non-zero eigenvector $v' \in V'_{K(a')}$. We may assume that L is a normal closure of the field extension $K(a, a')/K$. Let f be the minimal polynomial of a over K and let g be the minimal polynomial of a' over K , then $c(D) = [f]$ and $c(D') = [g]$. Let $h \in \mathcal{F} \cap R$ with $c((V, D) \otimes (V', D')) = [h]$. To show that $[f] \star [g] = [h]$, we have to check that

$$h \equiv_e \mathcal{R}(f, g) \quad \text{and} \quad h(\lambda) \equiv h\left(\lambda + \frac{1}{e}\right),$$

where $e \stackrel{\text{def}}{=} \epsilon(fg) = \epsilon_{L/K}$.

We will first show the second statement. Using a composition series of $(V, D) \otimes (V', D')$, one can write

$$[(V, D) \otimes (V', D')] = [(V_1, D_1)] + \dots + [(V_n, D_n)]$$

in $K_0(\mathcal{V})$, where $n \in \mathbb{N}$ and $(V_1, D_1), \dots, (V_n, D_n)$ are simple differential operators. Let $i \in \{1, \dots, n\}$. Then there exists a finite field extension L' of K such that $(D_i)_{L'}$ has a normalized eigenvalue $a_i \in L'$ with corresponding non-zero eigenvector $v_i \in K(a_i) \otimes V_i$. One easily checks that

$$\left(\left(\tau - \frac{1}{e}\right) \otimes D_i\right)(1 \otimes v_i) = (a_i - \frac{1}{e})(1 \otimes v_i)$$

Hence, by corollary 3.3.2, $(\tau - 1/e) \otimes D_i$ is a simple differential operator and its characteristic class is $c(D_i)(\lambda + 1/e) = c(D_i)(\lambda) \star [\lambda + 1/e]$. This is true for all $i \in \{1, \dots, n\}$, hence

$$\begin{aligned}
 c((\tau - \frac{1}{e}) \otimes D \otimes D')(\lambda) &= \prod_{i=1}^n c((\tau - \frac{1}{e}) \otimes D_i)(\lambda) \\
 &= \prod_{i=1}^n (c(D_i)(\lambda) \star [\lambda + \frac{1}{e}]) \\
 &= \left(\prod_{i=1}^n c(D_i)(\lambda) \right) \star [\lambda + \frac{1}{e}] \\
 &= c(D \otimes D')(\lambda) \star [\lambda + \frac{1}{e}] \\
 &= [h(\lambda)] \star [\lambda + \frac{1}{e}] \\
 &= [h(\lambda + \frac{1}{e})]. \tag{3.17}
 \end{aligned}$$

On the other hand, write $1/e = r/\epsilon(f) + s/\epsilon(g)$ with $r, s \in \mathbb{Z}$. By the same argument as before one finds that $(\tau - r/\epsilon(f)) \otimes D$ is a simple differential operator with characteristic class $[f(\lambda + r/\epsilon(f))] = [f(\lambda)]$, hence

$$(K, \tau - \frac{r}{\epsilon(f)}) \otimes (V, D) \simeq (V, D) \quad \text{and} \quad (K, \tau - \frac{s}{\epsilon(g)}) \otimes (V', D') \simeq (V', D').$$

Notice that

$$(K, \tau - \frac{1}{e}) \simeq (K, \tau - \frac{r}{\epsilon(f)}) \otimes (K, \tau - \frac{s}{\epsilon(g)}).$$

This implies

$$\begin{aligned}
 c((\tau - \frac{1}{e}) \otimes D \otimes D')(\lambda) &= c((\tau - \frac{r}{\epsilon(f)}) \otimes D \otimes (\tau - \frac{s}{\epsilon(g)}) \otimes D')(\lambda) \\
 &= c(D \otimes D')(\lambda) \\
 &= [h(\lambda)]. \tag{3.18}
 \end{aligned}$$

Combining (3.17) and (3.18) gives $[h(\lambda)] = [h(\lambda + 1/e)]$, hence $h(\lambda) \equiv h(\lambda + 1/e)$.

Remains to show $h \equiv_{\epsilon} \mathcal{R}(f, g)$. Let G_1 be the set of K -homomorphisms from $K(a)$ to L and let G_2 be the set of K -homomorphisms from $K(a')$ to L . By proposition 3.3.3 the set $\{\sigma(v) \mid \sigma \in G_1\}$ is an L -basis of the vector space V_L and $\{\sigma(v') \mid \sigma \in G_2\}$ is an L -basis of the vector space V'_L , so $\{\sigma_1(v) \otimes \sigma_2(v') \mid \sigma_1 \in G_1, \sigma_2 \in G_2\}$ is an L -basis of the vector space $V_L \otimes_L V'_L$. We have

$$(D_L \otimes D'_L)(\sigma_1(v) \otimes \sigma_2(v')) = (\sigma_1(a) + \sigma_2(a'))(\sigma_1(v) \otimes \sigma_2(v')) \tag{3.19}$$

for all $\sigma_1 \in G_1$ and $\sigma_2 \in G_2$. Equation (3.19) implies that

$$\prod_{\sigma_1 \in G_1} \prod_{\sigma_2 \in G_2} \left(\lambda - (\sigma_1(a) + \sigma_2(a')) \right) = \mathcal{R}(f, g)(\lambda)$$

is an element representing the characteristic class of the differential operator $D_L \otimes D'_L$.

On the other hand, by proposition 3.5.3, h is also an element representing the characteristic class of the differential operator $D_L \otimes D'_L$. Since $\mathcal{R}(f, g)$ is the product of linear factors over L , this implies that $h \equiv_e \mathcal{R}(f, g)$. $\square$

3.6 The semisimple part of a differential operator

The main result of this section is proposition 3.6.6 which shows that the map sending a differential operator to its semisimple part is an exact functor. The semisimple and nilpotent part of a differential operator are defined by [Lev75, theorem I].

Proposition 3.6.1 (Levelt) *Let (V, D) be a differential operator. Then there is exactly one pair of maps $S, N: V \rightarrow V$ satisfying:*

1. S is a semisimple differential operator.
2. N is a nilpotent K -linear map.
3. $D = S + N$.
4. S and N commute.

One calls S the *semisimple part* of the differential operator, notation $s((V, D))$ or $s(D)$, and N the *nilpotent part* of the differential operator, notation $n((V, D))$ or $n(D)$.

Another important result is [Lev75, theorem II].

Proposition 3.6.2 (Levelt) *Let (V, D) be a differential operator. Then (V, D) is semisimple if and only if (V_L, D_L) is diagonalizable over a finite field extension L of K .*

Corollary 3.6.3 *Let V and V' be two semisimple $\mathcal{D}(K)$ -modules. Then:*

1. $V \oplus V'$ is a semisimple $\mathcal{D}(K)$ -module.
2. V^* is a semisimple $\mathcal{D}(K)$ -module.
3. $V \otimes V'$ is a semisimple $\mathcal{D}(K)$ -module.

4. $\text{Hom}(V, V')$ is a semisimple $\mathcal{D}(K)$ -module.

PROOF. (1) is trivial and (2) follows immediately from (4), while (3) follows from (2) and (4), since $V \otimes V'$ and $\text{Hom}(V^*, V')$ are isomorphic. Remains to show (4). This follows from proposition 3.6.2 and the fact the $\mathcal{D}(L)$ -modules $\text{Hom}_K(V, V')_L$ and $\text{Hom}_L(V_L, V'_L)$ are isomorphic. $\square$

Proposition 3.6.4 *Let (V, D) and (V', D') be two differential operators. Then:*

- $s(D \oplus D') = s(D) \oplus s(D')$;
- $s(D^*) = s(D)^*$;
- $s(D \otimes D') = s(D) \otimes s(D')$;
- $s(\text{Hom}(D, D')) = \text{Hom}(s(D), s(D'))$.

PROOF. By corollary 3.6.3 the differential operators on the right hand side are semisimple, so it is enough to show that there is a corresponding nilpotent K -linear map satisfying the last three conditions of proposition 3.6.1. It is not difficult to verify that one should take:

- $n(D) \oplus n(D')$;
- $-n(D)^*$;
- $n(D) \otimes 1 + 1 \otimes n(D')$;
- $f \mapsto n(D') \circ f - f \circ n(D)$.

$\square$

Lemma 3.6.5 *Let (V, D) be a differential operator and $v \in V$ with $D(v) = 0$. Then $s(D)(v) = 0$.*

PROOF. Write $V' = Kv$, $S = s(D)$, $N = n(D)$ and let $W = V' + N(V') + N^2(V') + \dots$. Let $i \in \mathbb{N}$ then $D(N^i(v)) = N^i(D(v)) = 0$. This shows that W is a D -invariant subspace of V and that $D|_W$ is semisimple. Since W is also N -invariant, it is also S -invariant, so both $(S|_W, N|_W)$ and $(D|_W, 0)$ satisfy the conditions of proposition 3.6.1. By uniqueness $D|_W = S|_W$. In particular $S(v) = D(v) = 0$. $\square$

Proposition 3.6.6 *The map $s: \mathcal{V} \rightarrow \mathcal{V}$ defined by $s: (V, D) \mapsto (V, s(D))$ is an exact functor.*

PROOF. Let (V, D) and (V', D') be two differential operators and let $f: V \rightarrow V'$ be a $\mathcal{D}(K)$ -homomorphism. We have to show that $s(D') \circ f = f \circ s(D)$. It is equivalent to prove that $\text{Hom}(D, D')(f) = 0$ implies $\text{Hom}(s(D), s(D'))(f) = 0$. By proposition 3.6.4, $\text{Hom}(s(D), s(D')) = s(\text{Hom}(D, D'))$. The statement now follows from lemma 3.6.5. $\square$

Corollary 3.6.7 *Let (V, D) be a differential operator. Then $c(s(D)) = c(D)$.*

PROOF. Since $s: \mathcal{V} \rightarrow \mathcal{V}$ is an exact functor, one can define s on $K_0(\mathcal{V})$. Write $[(V, D)] = [(V_1, D_1)] + \cdots + [(V_n, D_n)]$ in $K_0(\mathcal{V})$, where $n \in \mathbb{N}$ and $(V_1, D_1), (V_2, D_2), \dots, (V_n, D_n)$ are simple differential operators. Then:

$$\begin{aligned} [(V, s(D))] &= s([(V, D)]) \\ &= s([(V_1, D_1)]) + \cdots + s([(V_n, D_n)]) \\ &= [(V_1, D_1)] + \cdots + [(V_n, D_n)] \\ &= [(V, D)]. \end{aligned}$$

Hence $s(D)$ and D have the same characteristic class. $\square$

3.7 Newton-Puiseux polygon and polynomials

In this section we will compare the characteristic class of a differential operator with its Dieudonné determinant. We will start by showing up to which order of x the coefficients of an element representing the Dieudonné determinant of a differential operator are independent of the chosen representative. This is done by using an equivalence relation $\sim$ whose definition uses the Newton-Puiseux polygon.

Then we will show that these “representation independent parts” of the Dieudonné determinant and the characteristic class are the same up to a sign. This result is then used to show how one can compute from the characteristic class invariants like the Newton-Puiseux polygon and the Newton-Puiseux polynomials defined in [Die87].

Let us start by recalling the definition and properties of the Newton-Puiseux polygon of elements in $\mathcal{D}(K)$ and in $K[\lambda]$.

Definition 3.7.1 For a finite subset $T \subset \mathbb{Z}^2$ define the *Newton-Puiseux polygon* of T , notation $N(T)$, as the convex hull in $\mathbb{R}^2$ of the set

$$\bigcup_{(i,j) \in T} \{(x, y) \in \mathbb{R}^2 \mid x \leq i, y \geq j\}.$$

Let $\mathcal{N}$ be the set of all Newton-Puiseux polygons. $\mathcal{N}$ becomes a commutative monoid by defining:

$$N_1 + N_2 \stackrel{\text{def}}{=} \{a + b \mid a \in N_1, b \in N_2\} \quad \text{for all } N_1, N_2 \in \mathcal{N}.$$

Let $\mathcal{N}_0$ be the submonoid $\mathcal{N} \setminus \{\emptyset\}$. Define a map $N: \mathcal{D}(K) \rightarrow \mathcal{N}$ by

$$N(a_n \tau^n + \cdots + a_1 \tau + a_0) \stackrel{\text{def}}{=} N\left(\{(i, \text{val}(a_i)) \mid 0 \leq i \leq n, a_i \neq 0\}\right)$$

and a map $N: K[\lambda] \rightarrow \mathcal{N}$ by

$$N(a_n \lambda^n + \cdots + a_1 \lambda + a_0) \stackrel{\text{def}}{=} N\left(\{(i, \text{val}(a_i)) \mid 0 \leq i \leq n, a_i \neq 0\}\right).$$

Example 3.7.2 Figure 3.1 is a picture of the Newton-Puiseux polygon of

$$\tau^4 + \left(\frac{2}{x^2} - \frac{3}{x}\right)\tau^3 - \frac{1}{x^2}\tau^2 + \left(\frac{1}{x^3} + \frac{5}{x^2} + 1\right)\tau + \frac{6}{x^2}$$

Figure 3.1: a Newton-Puiseux polygon.

Lemma 3.7.3 $N(PQ) = N(P) + N(Q)$ for all $P, Q \in \mathcal{D}(K)$ and also for all $P, Q \in K[\lambda]$.

PROOF. A proof for $\mathcal{D}(K)$ can be found in [Mal]. The result for $K[\lambda]$ is proved in the same way. $\square$

Lemma 3.7.4 $N(P^*) = N(P)$ for all $P \in \mathcal{D}(K)$.

PROOF. Let $P \in \mathcal{D}(K)$ and write $P = a_n \tau^n + \cdots + a_0$ with $a_0, a_1, \dots, a_n$ in K . By lemma 3.7.3, $N(\tau^i a_i) = N(a_i \tau^i)$ for all $i \in \{0, \dots, n\}$. Hence

$$N(P^*) \subseteq \text{convex hull of } \bigcup_{i=0}^n N(\tau^i a_i) = \text{convex hull of } \bigcup_{i=0}^n N(a_i \tau^i) = N(P).$$

Since $P^{**} = P$, one also has $N(P) \subseteq N(P^*)$. Hence $N(P^*) = N(P)$. $\square$

The invertible elements of $\mathcal{N}$ are the Newton-Puiseux polygons $N(\{(x_0, y_0)\})$ where $(x_0, y_0) \in \mathbb{Z}^2$. Every element of $\mathcal{N}_0$ can be written in one and only one way as sum of an invertible element and elements of the form

$$N(\{(0, 0), (x_i, y_i)\}) \quad (i = 1, \dots, r)$$

with $x_i > 0$ for all $i \in \{1, \dots, r\}$ and $y_1/x_1 > \dots > y_r/x_r > 0$. The rational integers y_i/x_i are usually called the *slopes* of the Newton-Puiseux polygon N . Using this decomposition one easily shows that the cancellation law holds in $\mathcal{N}_0$.

Let $\tilde{\mathcal{N}}$ be the Grothendieck group (see page 7) of the monoid $\mathcal{N}_0$. Since the cancellation law holds in $\mathcal{N}_0$, the natural map of $\mathcal{N}_0$ into its Grothendieck group $\tilde{\mathcal{N}}$ is injective. Now we can extend $N: \mathcal{D}(K) \setminus \{0\} \rightarrow \mathcal{N}_0$ to $N: \mathcal{L}(K)^* \rightarrow \tilde{\mathcal{N}}$. This is a homomorphism, so N induces a group homomorphism

$$N: \mathcal{L}^0(K)^{\text{ab}} \longrightarrow \tilde{\mathcal{N}}.$$

Definition 3.7.5 Let (V, D) be a differential operator. Then D has a *cyclic vector* (cf. [Del70, lemme II.1.3]) or in other words there exists an element $P \in \mathcal{D}^0(K)$ such that the $\mathcal{D}(K)$ -modules V and $\mathcal{D}(K)/\mathcal{D}(K)P$ are isomorphic.

The *Newton-Puiseux polygon* of D , notation $N(D)$, is defined by

$$N(D) \stackrel{\text{def}}{=} N(P).$$

It is well-known that this is independent of the chosen cyclic vector, cf. [Adj85], [Die87] or [Lev91, appendix A]. In [Adj85] the Dieudonné determinant is used to show this result.

We will now compare the coefficients of elements representing the characteristic class and the Dieudonné determinant of a differential operator. First define a relation $\sim$ on $\mathcal{D}(K)$ by

$$P \sim Q \quad \text{iff} \quad N(P - Q) + (1, 0) \subseteq N(P)$$

for all $P, Q \in \mathcal{D}(K)$.

Lemma 3.7.6 *Let $P, Q \in \mathcal{D}(K)$ with $P \sim Q$. Then $N(P) = N(Q)$.*

PROOF. Suppose $N(P) \not\subseteq N(Q)$. Let $a\tau^m$ be a term of P such that $(m, \text{val}(a)) \notin N(Q)$ and $(m+1, \text{val}(a)) \notin N(P)$ then $(m, \text{val}(a)) \in N(P-Q)$ so $(m+1, \text{val}(a)) \in N(P)$, which gives a contradiction. Hence $N(P) \subseteq N(Q)$. For the same reason $N(Q) \subseteq N(P)$, so $N(P) = N(Q)$. $\square$

Lemma 3.7.7 *$\sim$ is an equivalence relation.*

PROOF. Since $N(0) = \emptyset$, one has $P \sim P$ for all $P \in \mathcal{D}(K)$. Lemma 3.7.6 implies that $P \sim Q$ if and only if $Q \sim P$. Remains to show the transitivity of the relation $\sim$.

Let $P, Q, R \in \mathcal{D}(K)$ with $P \sim Q$ and $Q \sim R$. Then $N(P - R) \subseteq N(P - Q) \cup N(Q - R)$ so $N(P - R) + (1, 0) \subseteq N(P) \cup N(Q) = N(P)$ so $P \sim R$. $\square$

Example 3.7.8 Let P be the differential operator given in example 3.7.2 and let Q be a differential operator with $P \sim Q$. The coefficients of $x^j \tau^1$ in P and Q

must agree for all i and j with (i, j) outside $N(P) + (-1, 0)$, i.e., for all i and j with (i, j) below the upper thick line in figure 3.2. So Q must be of the form

$$\tau^4 + \left(\frac{2}{x^2} - \frac{3}{x} + a_3\right)\tau^3 + \frac{a_2}{x^2}\tau^2 + \left(\frac{1}{x^3} + \frac{a_1}{x^2}\right)\tau + \frac{a_0}{x^3}$$

for some a_0, a_1, a_2 and a_3 in $\mathcal{O}$.

Figure 3.2: picture of the relation $\sim$ on $\mathcal{D}(K)$.

Lemma 3.7.9 *Let $P, Q, R \in \mathcal{D}(K)$ with $R \neq 0$ then*

$$PQ \sim QP \quad \text{and} \quad P \sim Q \iff PR \sim QR.$$

PROOF. Write

$$P = a_m \tau^m + \cdots + a_1 \tau + a_0 \quad \text{and} \quad Q = b_n \tau^n + \cdots + b_1 \tau + b_0$$

with $m, n \in \mathbb{N}$ and $a_0, a_1, \dots, a_m, b_0, b_1, \dots, b_n$ in K . To show that $PQ \sim QP$ it is enough to show that

$$PQ \sim \sum_{i=0}^m \sum_{j=0}^n a_i b_j \tau^{i+j}. \quad (3.20)$$

Notice that

$$a_i \tau^i b_j \tau^j = a_i b_j \tau^{i+j} + a_i \sum_{k=0}^{i-1} \binom{i}{k} \tau^{i-k} (b_j) \tau^{k+j}$$

for all $i \in \{0, \dots, m\}$ and $j \in \{0, \dots, n\}$. It is therefore sufficient to show that

$$N(a_i \tau^{i-k} (b_j) \tau^{k+j}) + (1, 0) \subseteq N(PQ) \quad (3.21)$$

for all $i, j, k \in \mathbb{N}$ with $k < i \leq m$ and $j \leq n$. Lemma 3.7.3 implies that

$$N(a_i b_j \tau^{i+j}) = N(a_i \tau^i) + N(b_j \tau^j) \subseteq N(P) + N(Q) = N(PQ).$$

To prove (3.21) it is therefore sufficient to show

$$N(a_i \tau^{i-k} (b_j) \tau^{k+j}) + (1, 0) \subseteq N(a_i b_j \tau^{i+j}).$$

Using lemma 3.7.3 again this reduces to

$$N(\tau^s(b)\tau) = N(\tau^s(b)) + (1, 0) \subseteq N(b\tau^s) \quad \text{for all } b \in K \text{ and } s \in \mathbb{N}^*.$$

This is clear since $\text{val}(\tau^s(b)\tau) \geq \text{val}(b)$ for all $b \in K$ and $s \in \mathbb{N}^*$.

Remains to prove that $P \sim Q$ if and only if $PR \sim QR$:

$$\begin{aligned} P \sim Q &\iff N(P - Q) + (1, 0) \subseteq N(P) \\ &\iff N(P - Q) + (1, 0) + N(R) \subseteq N(P) + N(R) \\ &\iff N(PR - QR) + (1, 0) \subseteq N(PR) \\ &\iff PR \sim QR. \end{aligned}$$

□

One can define a relation $\sim$ on $\mathcal{L}(K)^*$ by

$$P^{-1}Q \sim R^{-1}S \quad \text{iff} \quad QR \sim PS.$$

Using lemma 3.7.7 and lemma 3.7.9 one can easily show that $\sim$ is a well-defined equivalence relation on $\mathcal{L}(K)^*$ and that the multiplication in $\mathcal{L}(K)^*$ induces the structure of an Abelian group on $\mathcal{L}(K)^*/\sim$. This implies that $\sim$ induces a surjective group homomorphism

$$\mathcal{L}^0(K)^{\text{ab}} \longrightarrow \mathcal{L}^0(K)/\sim.$$

Corollary 3.7.10 *Let $P, Q \in \mathcal{D}^0(K)$ be two elements both representing the Dieudonné determinant of the same differential operator. Then $P \sim Q$.*

Proposition 3.7.11 *Let $P, Q \in \mathcal{D}^0(K)$ and suppose that the $\mathcal{D}(K)$ -modules $\mathcal{D}(K)/\mathcal{D}(K)P$ and $\mathcal{D}(K)/\mathcal{D}(K)Q$ are isomorphic. Then $P \sim Q$.*

PROOF. It follows from proposition 2.6 and the proof of proposition 2.4 that both P^* and Q^* are elements representing the Dieudonné determinant of D , hence by corollary 3.7.10 one has $P^* \sim Q^*$, i.e., $N(P^* - Q^*) + (1, 0) \subseteq N(P^*)$. Notice that $P^* - Q^* = \pm(P - Q)^*$. Now use lemma 3.7.4. □

Lemma 3.7.12 *Let $P, Q \in \mathcal{D}(K)$ and let L be a finite field extension of K . Then $P \sim Q$ in $\mathcal{D}(K)$ if and only if $1 \otimes P \sim 1 \otimes Q$ in $\mathcal{D}(L)$.*

PROOF. Put $e \stackrel{\text{def}}{=} \epsilon_{L/K}$. Then $\text{val}_L(a) = e \cdot \text{val}_K(a)$ for all $a \in K^*$. This implies that

$$N(1 \otimes P) = \{(x, ey) \mid (x, y) \in N(P)\}.$$

It is now clear that

$$N(P - Q) + (1, 0) \subseteq N(P) \iff N(1 \otimes P - 1 \otimes Q) + (1, 0) \subseteq N(1 \otimes P). \quad \square$$

Proposition 3.7.13 *Let (V, D) be a differential operator, P an element in $\mathcal{D}^0(K)$ representing the Dieudonné determinant of D and $h(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0$ an element in $\mathcal{F} \cap R$ representing the characteristic class of D . Then*

$$\tau^n + a_{n-1}\tau^{n-1} + \cdots + a_0 \sim P^*.$$

PROOF. We will prove the statement with induction on $\dim V$. If $\dim V = 0$ then the statement is trivial, so suppose $\dim V = n > 0$. Let L be a splitting field of h over K and $a \in L$ a root of the polynomial h . Then there exists an L -basis $v = (v_1, \dots, v_n)$ of V_L such that

$$M(D, v) = \begin{pmatrix} a & * \\ 0 & A \end{pmatrix},$$

where $A \in \text{Mat}_{n-1}(L)$.

Define a function $\sigma: L[\lambda] \rightarrow \mathcal{D}(L)$ by

$$\sigma(b_m\lambda^m + b_{m-1}\lambda^{m-1} + \cdots + b_0) \stackrel{\text{def}}{=} b_m\tau^m + b_{m-1}\tau^{m-1} + \cdots + b_0$$

for all $m \in \mathbb{N}$ and $b_0, b_1, \dots, b_m$ in L . Then by equation (3.20), $\sigma(fg) \sim \sigma(f)\sigma(g)$ for all $f, g \in L[\lambda]$. Write $h(\lambda) = (\lambda - a)f(\lambda)$ with $f \in L[\lambda]$. Then, by applying the induction hypothesis to $Lv_2 + \cdots + Lv_n$, $\sigma(f) \sim Q^*$ where $Q \in \mathcal{D}^0(L)$ is an element representing the Dieudonné determinant of $\det(\tau + A)$. Since $(\tau + a)Q$ is an element in $\mathcal{D}(L)$ representing the Dieudonné determinant of D , one has $P \sim (\tau + a)Q$ over L by corollary 3.7.10. The proof of proposition 3.7.11 shows that $(\tau - a)Q^* \sim P^*$ over L , hence

$$\sigma(h) = \sigma((\lambda - a)f(\lambda)) \sim \sigma(\lambda - a)\sigma(f) \sim (\tau - a)Q^* \sim P \quad \text{over } L.$$

The statement now follows from lemma 3.7.12. $\square$

Proposition 3.7.14 *Let $P \in \mathcal{D}^0(K)$ and let $\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0$ be an element in $\mathcal{F} \cap R$ representing the characteristic class of the $\mathcal{D}(K)$ -module $\mathcal{D}(K)/\mathcal{D}(K)P$. Then*

$$P \sim \tau^n + a_{n-1}\tau^{n-1} + \cdots + a_0.$$

PROOF. As noted before P^* is an element in $\mathcal{D}^0(K)$ representing the Dieudonné determinant of the $\mathcal{D}(K)$ -module $\mathcal{D}(K)/\mathcal{D}(K)P$. Now use proposition 3.7.13 and the fact that $P^{**} = P$. $\square$

Corollary 3.7.15 *Let (V, D) be a differential operator and let $h \in \mathcal{F} \cap R$ be an element representing the characteristic class of D . Then $N(D) = N(h)$.*

PROOF. This follows immediately from lemma 3.7.6 and proposition 3.7.14. $\square$

In [Die87] Newton-Puiseux polynomials of a differential operator are defined which are used to give some more information about the formal solutions of the differential equation corresponding to the differential operator. We will first give their definition and then show how they can be obtained from the characteristic class of a differential operator.

Definition 3.7.16 (Dietrich) *Let (V, D) be a differential operator and let $v \in V$ be a cyclic vector of D satisfying*

$$D^n v + a_{n-1} D^{n-1} v + \cdots + a_0 v = 0$$

with $n = \dim V$ and $a_0, a_1, \dots, a_{n-1}$ in K . Put $a_n \stackrel{\text{def}}{=} 1$.

Write the Newton-Puiseux polygon $N(D)$ as sum of an invertible element and elements of the form

$$N(\{(0, 0), (x_i, y_i)\}) \quad (i = 1, \dots, r)$$

with $x_i > 0$ for all $i \in \{1, \dots, r\}$ and $y_1/x_1 > \cdots > y_r/x_r > 0$. Let $k \in \{1, \dots, r\}$ and $i \in \{0, \dots, x_k\}$ then the point

$$(t_{k,i}, u_{k,i}) \stackrel{\text{def}}{=} (n - x_1 - \cdots - x_k + i, -y_1 - \cdots - y_k + i y_k / x_k)$$

lies on the border of the Newton-Puiseux polygon $N(D)$. Now define

$$P_k(\lambda) \stackrel{\text{def}}{=} \sum_{i=0}^{x_k} b_{k,i} \lambda^i \quad \text{for all } k \in \{1, \dots, r\},$$

where $b_{k,i}$ is the coefficient of $x^{u_{k,i}}$ in $a_{t_{k,i}}$ (if $u_{k,i} \notin \mathbb{Z}$ then $b_{k,i} = 0$). The polynomials $P_1, P_2, \dots, P_r$ are the *Newton-Puiseux polynomials* of the differential operator D .

Remark 3.7.17 The definition of the Newton-Puiseux polynomials in [Die87] is not the same as our definition, mainly because Dietrich works with differential operators over the field $k((1/x))$. However the polynomials defined in definition 3.7.16 have the same properties w.r.t. formal solutions of the differential equation corresponding to the differential operator.

Proposition 3.7.11 implies that the Newton-Puiseux polynomials of a differential operator are well-defined, i.e., one finds the same Newton-Puiseux polynomials if one takes another cyclic vector.

Example 3.7.18 Consider the $\mathcal{D}(K)$ -module $\mathcal{D}(K)/\mathcal{D}(K)P$, where $P \in \mathcal{D}^0(K)$ is given in example 3.7.2 and whose Newton-Puiseux polygon is drawn in figure 3.1. Then the Newton-Puiseux polynomials are given by

$$\begin{aligned} P_1(\lambda) &= \lambda + 2, \\ P_2(\lambda) &= 2\lambda^2 + 1. \end{aligned}$$

Using proposition 3.7.14 it is immediately clear how to obtain the Newton-Puiseux polynomials of a differential operator from its characteristic class.

3.8 Various invariants

In this section we describe some well-known invariants of a differential operator which are all derivable from its Newton-Puiseux polygon and hence by corollary 3.7.15 from its characteristic class.

3.8.1 The ρ -invariants of Gérard and Levelt

In this subsection we will look at the ρ -invariants defined by Gérard and Levelt in [GL73]. We will start by giving their definition and then show how they can be obtained from the Newton-Puiseux polygon and the characteristic class of a differential operator.

Let (V, D) be a differential operator and let Λ be a lattice in V , i.e., an $\mathcal{O}$ -module of finite type, generating V as a K -vector space. Define

$$F_j^i(\Lambda) \stackrel{\text{def}}{=} \Lambda + (x^j D)\Lambda + \cdots + (x^j D)^i \Lambda \quad \text{for all } i, j \in \mathbb{N}.$$

$F_j^i(\Lambda)$ is a lattice in V for all $i, j \in \mathbb{N}$ and $x^j D$ induces a surjective $\mathcal{O}$ -homomorphism

$$F_j^i(\Lambda)/F_j^{i-1}(\Lambda) \longrightarrow F_j^{i+1}(\Lambda)/F_j^i(\Lambda)$$

for all $i, j \in \mathbb{N}$. Since $F_j^i(\Lambda)/F_j^{i-1}(\Lambda)$ is a finite dimensional k -vector space, this homomorphism is an isomorphism for large i .

Definition 3.8.1 (Gérard and Levelt) Let $j \in \mathbb{N}$ and define

$$\rho_{j+1}(D) \stackrel{\text{def}}{=} \lim_{i \rightarrow \infty} \dim_k F_j^{i+1}(\Lambda)/F_j^i(\Lambda).$$

The right hand side is independent of the lattice Λ . One often writes $\rho(D)$ for $\rho_1(D)$.

Proposition 3.8.2 Let (V, D) be a differential operator and let $v \in V$ be a cyclic vector of D satisfying

$$D^n v + a_{n-1} D^{n-1} v + \cdots + a_0 v = 0$$

with $n = \dim V$ and $a_0, a_1, \dots, a_{n-1}$ in K . Put $a_n \stackrel{\text{def}}{=} 1$. Then

$$\rho_{j+1}(D) = \max_{0 \leq i \leq n} j(i - n) - \text{val}(a_i) \quad \text{for all } j \in \mathbb{N}.$$

PROOF. Let $i, j \in \mathbb{N}$. It is not difficult to show that

$$(x^j D)^i v = x^{ij} (D^i v + b_{i-1,i} D^{i-1} v + \dots + b_{1,i} D v + b_{0,i} v)$$

with all $b_{k,i}$ in $\mathbb{Z}$. This shows that v is also a cyclic vector for $x^j D$. If one writes

$$(x^j D)^n v + c_{n-1} (x^j D)^{n-1} v + \dots + c_0 v = 0$$

with $c_0, c_1, \dots, c_{n-1}$ in K and $c_n \stackrel{\text{def}}{=} 1$ then

$$x^{j(n-i)} a_i = c_i + \sum_{k=i+1}^n b_{ik} c_k x^{j(k-i)}.$$

This shows that

$$\max_{0 \leq i \leq n} j(i - n) - \text{val}(a_i) = \max_{0 \leq i \leq n} -\text{val}(c_i).$$

The statement now follows from [GL73, Thm. 3.1]. □

Using corollary 3.7.15 one obtains the following result.

Corollary 3.8.3 *Let (V, D) be a differential operator and $\lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_0$ with $a_0, a_1, \dots, a_{n-1}$ in K an element representing the characteristic class of D . Put $a_n \stackrel{\text{def}}{=} 1$. Then*

$$\rho_{j+1}(D) = \max_{0 \leq i \leq n} j(i - n) - \text{val}(a_i) = \max_{(x,y) \in N(D)} j(x - n) - y \quad \text{for all } j \in \mathbb{N}.$$

Example 3.8.4 Let (V, D) be a differential operator whose Newton-Puiseux polygon is drawn in figure 3.1 on page 48. Then, by corollary 3.8.3, $\rho_1(D) = 3$, $\rho_2(D) = 1$ and $\rho_i(D) = 0$ for all $i \geq 3$.

3.8.2 The irregularity index of Malgrange

Consider

$$P \stackrel{\text{def}}{=} \sum_{i=0}^n a_i \frac{d^i}{dx^i},$$

where $n \in \mathbb{N}$ and $a_0, a_1, \dots, a_n$ in $\mathbb{C}\{x\}$ (the ring of the power series in one variable over the field of complex numbers $\mathbb{C}$ which are convergent in the

neighborhood of the origin). In [Mal74, prop. 1.1 and prop. 1.3] it is shown that the two functions

$$\begin{aligned} P: \mathbb{C}[[x]] &\longrightarrow \mathbb{C}[[x]] \\ P: \mathbb{C}\{x\} &\longrightarrow \mathbb{C}\{x\} \end{aligned}$$

are of finite index (their kernel and cokernel are both finite dimensional vector spaces over $\mathbb{C}$) and that their indices are given by

$$\begin{aligned} \chi(P, \mathbb{C}[[x]]) &= \max_{0 \leq i \leq n} i - \text{val}(a_i), \\ \chi(P, \mathbb{C}\{x\}) &= n - \text{val}(a_n). \end{aligned}$$

The *irregularity index of Malgrange* (cf. [Mal74, déf. 1.5]), notation $i(P)$, is defined by

$$i(P) \stackrel{\text{def}}{=} \chi(P, \mathbb{C}[[x]]) - \chi(P, \mathbb{C}\{x\}) = \max_{0 \leq i \leq n} \text{val}(a_n) - \text{val}(a_i) + i - n. \quad (3.22)$$

The right hand side of (3.22) still makes sense if all a_i are elements of K . For a differential operator the irregularity index of Malgrange is defined in terms of a cyclic vector. This leads to the following definition (cf. [Mal74, equation (3.2)]).

Definition 3.8.5 Let (V, D) be a differential operator and let $v \in V$ be a cyclic vector of D satisfying

$$D^n v + a_{n-1} D^{n-1} v + \cdots + a_0 v = 0$$

with $n = \dim V$ and $a_0, a_1, \dots, a_{n-1}$ in K . Put $a_n \stackrel{\text{def}}{=} 1$. Then the irregularity index of Malgrange, notation $i(D)$, is defined by

$$i(D) \stackrel{\text{def}}{=} \max_{0 \leq i \leq n} -\text{val}(a_i).$$

Corollary 3.8.6 Let (V, D) be a differential operator and $\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0$ with $a_0, a_1, \dots, a_{n-1}$ in K an element representing the characteristic class of D . Put $a_n \stackrel{\text{def}}{=} 1$. Then

$$i(D) = \rho(D) = \max_{0 \leq i \leq n} -\text{val}(a_i) = \max_{(x,y) \in N(D)} -y.$$

PROOF. The first equality was proved in [GL73, § 5.B] and also follows immediately from proposition 3.8.2. The other equalities follow immediately from corollary 3.7.15. $\square$

3.8.3 The order of the singularity of Katz

The definition of the invariant of Katz is based on the following proposition which is stated in [Del70, lemme II.1.9.1].

Proposition 3.8.7 (Katz) *Let (V, D) be a differential operator. Then there exists a rational integer $r \geq 0$ such that for every lattice Λ in V and every K -basis $(e_1, \dots, e_n)$ of V , there exists an integer C such that for every $i \in \mathbb{N}$*

$$\left| ri + \min_{\substack{1 \leq k \leq n \\ 0 \leq j \leq i}} \text{val}_\Lambda(D^j e_k) \right| \leq C,$$

where val_Λ is defined by

$$\text{val}_\Lambda(v) \stackrel{\text{def}}{=} \sup \{ i \in \mathbb{Z} \mid v \in x^i \Lambda \} \quad \text{for all } v \in V.$$

The rational integer r in proposition 3.8.7 is called the *invariant of Katz* or the *order of the singularity*. In [Del70, prop. II.1.10] the following proposition about the invariant of Katz is proved.

Proposition 3.8.8 (Deligne) *Let (V, D) be a differential operator and let $v \in V$ be a cyclic vector of D satisfying*

$$D^n v + a_{n-1} D^{n-1} v + \dots + a_0 v = 0$$

with $n = \dim V$ and $a_0, a_1, \dots, a_{n-1}$ in K . Then the invariant of Katz is equal to

$$\max \left\{ 0, \max_{0 \leq i < n} -\frac{\text{val}(a_i)}{n-i} \right\}.$$

Using corollary 3.7.15 one obtains the following result.

Corollary 3.8.9 *Let (V, D) be a differential operator and $\lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_0$ with $a_0, a_1, \dots, a_{n-1}$ in K an element representing the characteristic class of D . Then the invariant of Katz is equal to the largest slope of the Newton-Puiseux polygon $N(D)$ and also equal to*

$$\max \left\{ 0, \max_{0 \leq i < n} -\frac{\text{val}(a_i)}{n-i} \right\}.$$

Example 3.8.10 Let (V, D) be a differential operator whose Newton-Puiseux polygon is drawn in figure 3.1 on page 48. Then, by corollary 3.8.9, the invariant of Katz is 2.

The invariant of Katz is directly related to the principal level of Varadarajan, which is defined as follows.

Definition 3.8.11 Let L be a finite field extension of K . Write $L = k'K(t)$ where k' is the field of constants in K and t is a non-zero element satisfying $t^e/x \in k'$ for some $e \in \mathbb{N}^*$. Let $m, n \in \mathbb{N}^*$ and $A \in \text{Mat}_{m,n}(K)$. The *order* of A , sometimes called the *Poincaré rank* of A , denoted by $\text{ord}(A)$ is defined by

$$\text{ord}(A) \stackrel{\text{def}}{=} \max \left\{ r \in \frac{1}{e}\mathbb{Z} \mid r \leq 0 \text{ and } A \in \text{Mat}_{m,n}(t^{er}\mathcal{O}_L) \right\}.$$

Definition 3.8.12 (Varadarajan) Let $n \in \mathbb{N}^*$ and A a matrix in $\text{Mat}_n(K)$. Then the *principal level* of A , notation $\text{pl}(A)$, is defined by (cf. [BV83, equation (4.1)])

$$\text{pl}(A) \stackrel{\text{def}}{=} \sup_{g \in \text{Gl}_n(\bar{K})} \text{ord}(g[A]),$$

where $\bar{K}$ denotes an algebraic closure of K .

In [BV83, § 5] it is shown that $\text{pl}(A) \in (1/m)\mathbb{Z}$ for some $m \in \{1, \dots, n\}$ and that there exists a $g \in \text{Gl}_n(K(x^{1/m}))$ such that $\text{pl}(A) = \text{ord}(g[A])$. Furthermore if

$$A = A_r x^r + A_{r+1} x^{r+1} + \dots$$

with $r \in \mathbb{Z}$ and $r < 0$ and all A_i constant matrices, then $\text{pl}(A) = \text{ord}(A)$ if and only if the matrix A_r is not nilpotent. Following the proof of proposition II.1.10 (our proposition 3.8.8) in [Del70] it is clear that the invariant of Katz of the differential operator $\tau + A$ is equal to $-\text{pl}(A)$ as stated in [BV83, § 5].

3.8.4 The invariants of Moser and Hilali

In this subsection we will give the definition of the super-irreducible form of Hilali (cf. [HW87]) and the invariants related to this form.

Let us start by giving the invariant of Moser which is defined in [Mos60].

Definition 3.8.13 (Moser) Let $n \in \mathbb{N}^*$ and $A \in \text{Mat}_n(K)$. Put $p \stackrel{\text{def}}{=} |\text{ord}(A)|$ and define the rational integer $m(A)$ by

$$m(A) \stackrel{\text{def}}{=} p + \frac{\text{rank}(A_{-p})}{n}.$$

Define the rational integer $\mu(A)$ by

$$\mu(A) \stackrel{\text{def}}{=} \min \{ m(g[A]) \mid g \in \text{Gl}_n(K) \}.$$

The rational integer $\mu(A)$ is called the *invariant of Moser*. For a differential operator (V, D) which has A as matrix w.r.t. some K -basis of V , we will also write $\mu(D)$ instead of $\mu(A)$.

Definition 3.8.14 (Hilali) Let $n \in \mathbb{N}^*$ and $A \in \text{Mat}_n(K)$. Put $p \stackrel{\text{def}}{=} |\text{ord}(A)|$ and define the rational integers $m_1(A), m_2(A), \dots$ by

$$m_k(A) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{for all } k \geq 1 \text{ if } p = 0; \\ p + \frac{n_0}{n} + \frac{n_1}{n^2} + \dots + \frac{n_{k-1}}{n^k} & \text{for all } 1 \leq k \leq p; \\ m_p(A) & \text{for all } k > p > 0; \end{cases}$$

where n_i is the number of columns of A of order $r + i$ for all $0 \leq i < p$.

Now define

$$\mu_k(A) \stackrel{\text{def}}{=} \min \{ m_k(g[A]) \mid g \in \text{Gl}_n(K) \} \quad \text{for all } k \in \mathbb{N}^*.$$

Finally define $\mu^*(A) \stackrel{\text{def}}{=} \mu_{p+1}(A)$. We will call the rational integer $\mu^*(A)$ the *invariant of Hilali*. For a differential operator (V, D) which has A as matrix w.r.t. some K -basis of V , we will also write $\mu^*(D)$ instead of $\mu^*(A)$. The matrix A is called *k-irreducible* if $m_k(A) = \mu_k(A)$ and *super-irreducible* if it is *k-irreducible* for every k , or equivalently if $m_{p+1}(A) = \mu^*(A)$.

From the above definitions it is immediately clear that $\mu_1(A) = \max \{1, \mu(A)\}$.

Remark 3.8.15 Our definition of super-irreducible is not the same as the one given by Hilali in [HW87]. A matrix A is super-irreducible in the sense of Hilali if and only if xA is super-irreducible by our definition. The reason for giving another definition is that Hilali considers differential operators w.r.t. the derivation d/dx while we work with differential operators w.r.t. the derivation $\tau = x \frac{d}{dx}$.

Definition 3.8.16 (Hilali) Let $n \in \mathbb{N}^*$ and $A \in \text{Mat}_n(K)$. Put $p = |\text{ord}(A)|$ and define

$$r_{p-i}(A) \stackrel{\text{def}}{=} - \sum_{j=1}^n \text{ord}(x^i M_j) \quad \text{for all } i \in \mathbb{N},$$

where M_j denotes the j -th column of the matrix A for all $j \in \{1, \dots, n\}$. Notice that

$$0 < r_1(A) - r_0(A) \leq r_2(A) - r_1(A) \leq \dots \leq r_p(A) - r_{p-1}(A) \leq n$$

and $r_i(A) = 0$ for all $i \leq 0$.

From this definition follows immediately that if $A \in \text{Mat}_n(K)$ is super-irreducible, then

$$\mu^*(A) = \begin{cases} 1 & \text{if } p = 0; \\ p + \sum_{i=1}^p \frac{r_i(A) - 2r_{i-1}(A) + r_{i-2}(A)}{n^i} & \text{otherwise;} \end{cases} \quad (3.23)$$

where $p \stackrel{\text{def}}{=} |\text{ord}(A)|$.

In [Hil87a] the following results are proved.

Proposition 3.8.17 (Hilali) Let (V, D) be a differential operator and let $v \in V$ be a cyclic vector of D satisfying

$$D^n v + a_{n-1} D^{n-1} v + \cdots + a_0 v = 0$$

with $n = \dim V$ and $a_0, a_1, \dots, a_{n-1}$ in K . Put $a_n \stackrel{\text{def}}{=} 1$. Define

$$p \stackrel{\text{def}}{=} \min \{ m \in \mathbb{N} \mid \text{val}(a_{n-i}) + im \geq 0 \text{ for all } 1 \leq i \leq n \}$$

and

$$r_{p-j} \stackrel{\text{def}}{=} \max_{0 \leq i \leq n} j(i-n) - \text{val}(a_i) \quad \text{for all } j \in \mathbb{N}.$$

Then

$$\mu^*(D) = \begin{cases} 1 & \text{if } p = 0; \\ p + \sum_{i=1}^p \frac{r_i - 2r_{i-1} + r_{i-2}}{n^i} & \text{otherwise.} \end{cases} \quad (3.24)$$

Proposition 3.8.18 (Hilali) Let $A \in \text{Mat}(K)$ be a super-irreducible matrix. Put $p \stackrel{\text{def}}{=} |\text{ord}(A)|$ and $D \stackrel{\text{def}}{=} \tau + A$. Then

$$\rho_{j+1}(D) = r_{p-j}(A) \quad \text{for all } j \in \mathbb{N}.$$

Corollary 3.8.19 Let (V, D) be a differential operator and

$$h(\lambda) = \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_0 \quad (a_0, a_1, \dots, a_{n-1} \in K)$$

an element representing the characteristic class of D . Put $a_n \stackrel{\text{def}}{=} 1$. Define

$$\begin{aligned} p &\stackrel{\text{def}}{=} \min \{ m \in \mathbb{N} \mid \text{val}(a_{n-i}) + im \geq 0 \text{ for all } 1 \leq i \leq n \} \\ &= \min \{ m \in \mathbb{N} \mid N(D) \subseteq \{ (x, y) \in \mathbb{R}^2 \mid y + (n-x)m \geq 0 \} \} \end{aligned}$$

and

$$r_{p-j} \stackrel{\text{def}}{=} \max_{0 \leq i \leq n} j(i-n) - \text{val}(a_i) = \max_{(x,y) \in N(D)} j(x-n) - y \quad \text{for all } j \in \mathbb{N}.$$

Then the invariant of Hilali is given by equation (3.24).

The invariant of Moser is given by

$$\mu(D) = \begin{cases} \tau/n & \text{if } p = 0; \\ p + \tau_1/n & \text{if } p > 1; \end{cases}$$

where

$$\tau \stackrel{\text{def}}{=} \min \{ \deg(g) \mid g \in k[\lambda] \text{ with } g \mid h \text{ and } g(m) \neq 0 \text{ for all } m \in \mathbb{Z} \}.$$

Example 3.8.20 Let (V, D) be a differential operator whose Newton-Puiseux polygon is drawn in figure 3.1 on page 48. Then the invariant of Moser is given by $\mu(D) = 9/4$ and the invariant of Hilali by $\mu^*(D) = 37/16$.

3.9 Connection with the characteristic polynomial

In this section we will give a result which shows to which order of x the coefficients of the characteristic class of a differential operator and the characteristic polynomial of the matrix of the differential operator w.r.t. a K -basis agree. This generalizes a result of Hilali for super-irreducible matrices stated in [Hil87a, theorem 1].

Proposition 3.9.1 (Hilali) *Let $n \in \mathbb{N}^*$ and let A be a super-irreducible matrix in $\text{Mat}_n(K)$. Put $p \stackrel{\text{def}}{=} |\text{ord}(A)|$. Let $v \in K^n$ be a cyclic vector of $D \stackrel{\text{def}}{=} \tau + A$ satisfying*

$$D^n v + a_{n-1} D^{n-1} v + \cdots + a_0 v = 0$$

with $a_0, a_1, \dots, a_{n-1}$ in K . Write

$$\det(\lambda I - A) = \lambda^n + b_{n-1} \lambda^{n-1} + \cdots + b_0$$

with $b_0, b_1, \dots, b_{n-1}$ in K . Then

$$\text{val}(a_{n-i-1} - b_{n-i-1}) \geq -ji - r_{p-j}(A) \quad \text{for all } i, j \in \mathbb{N} \text{ with } i < n.$$

Notice that for given i the exponent $-ji - r_{p-j}(A)$ is maximal for a $j \in \{0, \dots, p\}$ satisfying

$$r_{p-j}(A) - r_{p-j-1}(A) \leq i \leq r_{p-j+1}(A) - r_{p-j}(A),$$

where $r_{p+1}(A) \stackrel{\text{def}}{=} r_p(A) + n$.

Example 3.9.2 Consider the $\mathcal{D}(K)$ -module $\mathcal{D}(K)/\mathcal{D}(K)P$, where $P \in \mathcal{D}^0(K)$ is given in example 3.7.2 and whose Newton-Puiseux polygon is drawn in figure 3.1. If A is a super-irreducible matrix corresponding with this $\mathcal{D}(K)$ -module, then by proposition 3.9.1 one must have

$$\det(\lambda I - A) = \lambda^4 + \left(\frac{2}{x^2} - \frac{3}{x} + a_3\right)\lambda^3 + \frac{a_2}{x^2}\lambda^2 + \frac{a_1}{x^3}\lambda + \frac{a_0}{x^3}$$

for some a_0, a_1, a_2 and a_3 in $\mathcal{O}$. In other words, the coefficients of $x^j \tau^i$ in P and of $x^j \lambda^i$ in $\det(\lambda I - A)$ must agree for all i and j with (i, j) below the upper trick line in figure 3.3. This also shows that in general the characteristic polynomial of a super-irreducible matrix doesn't contain enough information to construct the Newton-Puiseux polygon as was already stated in [Hil87a, § 4].

Let the hypothesis be as in proposition 3.9.1 and suppose that w is another cyclic vector of D satisfying

$$D^n w + a'_{n-1} D^{n-1} w + \cdots + a'_0 w = 0$$

Figure 3.3: picture of the relation with determinant.

with $a'_0, a'_1, \dots, a'_{n-1}$ in K . Then proposition 3.9.1 implies

$$\text{val}(a_{n-i-1} - a'_{n-i-1}) \geq -ji - r_{p-j}(A) \quad \text{for all } i, j \in \mathbb{N} \text{ with } i < n. \quad (3.25)$$

By equation (3.23) and proposition 3.8.17 one has

$$r_{p-j}(A) = \max_{0 \leq i \leq n} j(i - n) - \text{val}(a_i) \quad \text{for all } j \in \mathbb{N} \quad (3.26)$$

with $a_n \stackrel{\text{def}}{=} 1$. Since by proposition 3.7.11 one also has

$$\tau^n + a_{n-1}\tau^{n-1} + \dots + a_0 \sim \tau^n + a'_{n-1}\tau^{n-1} + \dots + a'_0, \quad (3.27)$$

one might expect that (3.27) always implies (3.25) with $r_{p-j}(A)$ replaced by the right hand side of equation (3.26). This is indeed the case as in shown by the following lemma.

Lemma 3.9.3 *Let $a_0, a_1, \dots, a_n, a'_0, a'_1, \dots, a'_n$ in K with $a_n = a'_n = 1$. Suppose that*

$$\tau^n + a_{n-1}\tau^{n-1} + \dots + a_0 \sim \tau^n + a'_{n-1}\tau^{n-1} + \dots + a'_0.$$

Then

$$\text{val}(a_{n-i-1} - a'_{n-i-1}) \geq -ji - s_j \quad \text{for all } i, j \in \mathbb{N} \text{ with } i < n,$$

where the s_j are defined by

$$s_j \stackrel{\text{def}}{=} \max_{0 \leq i \leq n} j(i - n) - \text{val}(a_i) \quad \text{for all } j \in \mathbb{N}.$$

PROOF. Let N_1 be the Newton-Puiseux polygon of $\tau^n + a_{n-1}\tau^{n-1} + \dots + a_0$. We have to show that $(n-i, -ji - s_j)$ lies outside or on the border of the Newton-Puiseux polygon N_1 for all $i, j \in \mathbb{N}$ with $i < n$, or equivalently that

$$(i, \min_{0 \leq k \leq n} (i - k)j + \text{val}(a_k)) \quad (3.28)$$

lies outside or on the border of the Newton-Puiseux polygon N_1 for all $0 < i \leq n$ and $j \in \mathbb{N}$. Let $i \in \mathbb{N}^*$ with $i \leq n$. We will distinguish two cases.

First assume that there exists a $k \in \mathbb{N}$ with $i \leq k \leq n$ such that $\text{val}(a_i) \geq \text{val}(a_k)$ for all $0 \leq l \leq n$. In other words the line segment between $(0, \text{val}(a_k))$ and $(k, \text{val}(a_k))$ lies on the border of the Newton-Puiseux polygon N_1 . Then for every $j \in \mathbb{N}$ we have $(i - k)j + \text{val}(a_k) \leq \text{val}(a_k)$, so the point of (3.28) lies outside or on the border of the Newton-Puiseux polygon N_1 for every $j \in \mathbb{N}$.

Now assume that there exist $s, t \in \mathbb{N}$ with $s < i \leq t \leq n$ such that the line segment between $(s, \text{val}(a_s))$ and $(t, \text{val}(a_t))$ lies on the border of the Newton-Puiseux polygon N_1 . In this case we have to show that

$$\min_{0 \leq k \leq n} (i - k)j + \text{val}(a_k) \leq \frac{(i - s) \text{val}(a_t) + (t - i) \text{val}(a_s)}{t - s}$$

for all $j \in \mathbb{N}$. This is clear since

$$\frac{(i - s) \text{val}(a_t) + (t - i) \text{val}(a_s)}{t - s} \geq \begin{cases} (i - s)j + \text{val}(a_s) & \text{if } j \leq q; \\ (i - t)j + \text{val}(a_t) & \text{if } j \geq q; \end{cases}$$

where

$$q \stackrel{\text{def}}{=} \frac{\text{val}(a_t) - \text{val}(a_s)}{t - s}.$$

□

Proposition 3.9.4 Let $n \in \mathbb{N}^*$ and let A be a super-irreducible matrix in $\text{Mat}_n(K)$. Put $p \stackrel{\text{def}}{=} |\text{ord}(A)|$. Let

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0$$

with $a_0, a_1, \dots, a_{n-1}$ in K be an element representing the characteristic class of the differential operator $\tau + A$ on K^n . Write

$$\det(\lambda I - A) = \lambda^n + b_{n-1}\lambda^{n-1} + \cdots + b_0$$

with $b_0, b_1, \dots, b_{n-1}$ in K . Then

$$\text{val}(a_{n-i-1} - b_{n-i-1}) \geq -ji - r_{p-j}(A) \quad \text{for all } i, j \in \mathbb{N} \text{ with } i < n.$$

PROOF. This follows immediately from proposition 3.7.14, proposition 3.9.1 and lemma 3.9.3. □

We will now generalize proposition 3.9.4. Before doing so, we need to introduce some new notations.

Definition 3.9.5 Let $n \in \mathbb{N}^*$ and $A \in \text{Mat}_n(K)$. Write A in Smith normal form, i.e., $A = PEQ$ with P and Q in $\text{Gl}_n(\mathcal{O})$ and E a diagonal matrix of the form

$$\begin{pmatrix} x^{\varepsilon_1} & 0 & \cdots & 0 \\ 0 & x^{\varepsilon_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & & x^{\varepsilon_n} \end{pmatrix} \quad (3.29)$$

with $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ in $\mathbb{Z}$ and $\varepsilon_1 \leq \varepsilon_2 \leq \dots \leq \varepsilon_n$. Define $\varepsilon_i(A)$ by

$$\varepsilon_i(A) \stackrel{\text{def}}{=} \min \{0, \varepsilon_i\} \quad \text{for all } i \in \{1, \dots, n\}.$$

In particular $\varepsilon_1(A) = \text{ord}(A)$. Notice that

$$r_{p-j}(A) \geq r_{p-j}(PE) = r_{p-j}(E) = - \sum_{k=1}^n \min \{0, j + \varepsilon_k(A)\},$$

where $p \stackrel{\text{def}}{=} |\text{ord}(A)|$. Using this one easily verifies that

$$\sum_{k=1}^i \varepsilon_k(A) \geq -ji - r_{p-j}(A) \quad \text{for all } i, j \in \mathbb{N} \text{ with } i \leq n. \quad (3.30)$$

Define a partial ordering $\succeq$ on $\text{Mat}_n(K)$ by

$$A \succeq B \quad \text{iff} \quad \sum_{k=1}^i \varepsilon_k(A) \geq \sum_{k=1}^i \varepsilon_k(B) \quad \text{for all } i \in \{0, \dots, n\}.$$

Let

$$\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_0$$

with $a_0, a_1, \dots, a_{n-1}$ in K be an element representing the characteristic class of the differential operator $D \stackrel{\text{def}}{=} \tau + A$ on K^n . Write

$$\det(\lambda I - A) = \lambda^n + b_{n-1}\lambda^{n-1} + \dots + b_0$$

with $b_0, b_1, \dots, b_{n-1}$ in K . We will call A a *proper matrix* if

$$\text{val}(a_{n-i-1} - b_{n-i-1}) \geq \sum_{k=1}^i \varepsilon_k(A) \quad \text{for all } i \in \{0, \dots, n-1\}.$$

Remark 3.9.6 For $n \in \mathbb{N}^*$, $A \in \text{Mat}_n(K)$ and $g \in \text{Gl}_n(\mathcal{O})$, one readily verifies that

$$\varepsilon_i(A) = \varepsilon_i(g[A]) \quad \text{for all } i \in \{1, \dots, n\}.$$

This implies that $g[A] \succeq A$ and $A \succeq g[A]$.

We will now show that every matrix is proper, thereby making the notion “proper” superfluous.

Lemma 3.9.7 Let $n \in \mathbb{N}^*$, A a matrix in $\text{Mat}_n(K)$ and C a matrix in $\text{Mat}_n(\mathcal{O})$. Put $p \stackrel{\text{def}}{=} \text{ord}(A)$. Write

$$\det(\lambda I - A) - \det(\lambda I - A + C) = a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \dots + a_0$$

with $a_0, a_1, \dots, a_{n-1}$ in K . Then

$$\text{val}(a_{n-i-1}) \geq \sum_{k=1}^i \varepsilon_k(A) \quad \text{for all } i \in \{0, \dots, n-1\}.$$

PROOF. Write A in Smith normal form, i.e., $A = PEQ$ with P and Q in $\text{Gl}_n(\mathcal{O})$ and E a diagonal matrix of the form given in (3.29). One has

$$\begin{aligned}\det(\lambda I - A) &= \det(\lambda I - QAQ^{-1}), \\ \det(\lambda I - A + C) &= \det(\lambda I - QAQ^{-1} + QCQ^{-1}).\end{aligned}$$

Therefore we may assume that $A = PE$ with $P \in \text{Gl}_n(\mathcal{O})$ and E as in (3.29).

Let $i \in \{0, \dots, n-1\}$ then by Cramer's rule a_{n-i-1} is a sum of elements of the form $c \cdot w_1 \dots w_i$ where c is an element in the matrix C and $w_1, w_2, \dots, w_i$ are elements in different columns of the matrix A or C , so

$$\text{val}(c \cdot w_1 \dots w_i) \geq \sum_{k=1}^i \varepsilon_k(A).$$

□

Lemma 3.9.8 *Let $n \in \mathbb{N}^*$, $A \in \text{Mat}_n(K)$ and $g \in \text{Gl}_n(K)$. Suppose the following conditions are satisfied:*

1. $\tau(g)g^{-1} \in \text{Mat}_n(\mathcal{O})$,
2. $A \succeq g[A]$,
3. A is proper.

Then $g[A]$ is proper.

PROOF. This follows immediately from lemma 3.9.7. □

Lemma 3.9.9 *Lemma 3.9.8 is still valid if condition 1 is not satisfied.*

PROOF. Write g in Smith normal form, i.e., $g = PEQ$ with P and Q in $\text{Gl}_n(\mathcal{O})$ and E a diagonal matrix of the form given in (3.29). The idea is to apply lemma 3.9.8 with Q , E and finally P instead of g . First notice that $\tau(P)P^{-1}$, $\tau(E)E^{-1}$ and $\tau(Q)Q^{-1}$ are all matrices with coefficients in $\mathcal{O}$. By remark 3.9.6 one has $A \succeq Q[A]$, $(EQ)[A] \succeq g[A]$ and

$$Q[A] \succeq A \succeq g[A] \succeq (EQ)[A].$$

This finishes the proof of this lemma. □

Proposition 3.9.10 *The statement in proposition 3.9.4 is still valid if the matrix A is not super-irreducible.*

PROOF. First notice that for any matrix $B \in \text{Mat}_n(K)$ with $p \stackrel{\text{def}}{=} |\text{ord}(B)|$ one has

$$\begin{aligned}r_{p-j}(B) &\geq - \sum_{i=1}^n \min \{0, j + \varepsilon_i(B)\} \\ &= \dim_k (\mathcal{O}^n + (x^j B) \mathcal{O}^n) / \mathcal{O}^n \\ &\geq \rho_{j+1}(\tau + B) \quad \text{for all } j \in \mathbb{N}\end{aligned}\tag{3.31}$$

for all $j \in \mathbb{N}$ with all equalities if (and only if) the matrix B is super-irreducible by proposition 3.8.18.

Now let $g \in \text{Gl}_n(K)$ be a transformation matrix such that the matrix $g[A]$ is super-irreducible. To use lemma 3.9.9 we need to show that $g[A]$ is proper and $g[A] \succeq A$. To prove the first statement, notice that

$$r_{p-j}(g[A]) = - \sum_{i=1}^n \min \{0, j + \varepsilon_i(g[A])\} \quad \text{for all } j \in \mathbb{N}$$

by the remark above and hence for all $i \in \{0, \dots, n\}$

$$\sum_{k=1}^i \varepsilon_k(g[A]) = -ji - r_{p-j}(g[A]) \quad \text{for some } j \in \mathbb{N}.$$

This equality and proposition 3.9.4 show that $g[A]$ is proper.

To show that $g[A] \succeq A$ we need to prove

$$\sum_{k=1}^i \varepsilon_k(g[A]) \geq \sum_{k=1}^i \varepsilon_k(A) \quad \text{for all } i \in \{1, \dots, n\}. \quad (3.32)$$

By equation (3.31) we know that

$$\sum_{i=1}^n \min \{0, j + \varepsilon_i(g[A])\} \geq \sum_{i=1}^n \min \{0, j + \varepsilon_i(A)\}. \quad (3.33)$$

Lemma 3.9.12 below shows that (3.33) implies (3.32).

Now we have shown that the matrix A is proper. Use (3.30) to finish the proof. $\square$

Definition 3.9.11 (cf. [Mac79])

A *Young tableau* λ is a finite sequence $(\lambda_1, \dots, \lambda_n)$ of positive integers in decreasing order:

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n > 0.$$

For convenience one defines $\lambda_i \stackrel{\text{def}}{=} 0$ for all $i \in \mathbb{N}$ with $i > n$. It is graphically represented by a picture of the form

where the number of squares in row i is equal to λ_i for all i .

The conjugate Young tableau of λ , notation λ^t , is the Young tableau whose picture is obtained by transposing the picture of λ , i.e., by reflection in the main diagonal. More algebraically, $\lambda^t = (\lambda_1^t, \dots, \lambda_s^t)$ with $s \stackrel{\text{def}}{=} \lambda_1$ and

$$\lambda_i^t \stackrel{\text{def}}{=} \max \{j \in \mathbb{N}^* \mid \lambda_j \geq i\} \quad \text{for all } i \in \{1, \dots, s\}.$$

Lemma 3.9.12 *Let λ and μ be two Young tableaux with $\sum_{r>i} \lambda_r \leq \sum_{r>i} \mu_r$ for all i . Then $\sum_{c<i} \lambda_c^t \leq \sum_{c<i} \mu_c^t$ for all i .*

PROOF. We will prove this with induction to the numbers of squares in λ . If $\lambda_r \leq \mu_r$ for all r then $\lambda_c^t \leq \mu_c^t$ for all c , hence the statement is obvious in this case.

If there is a j with $\lambda_j = \mu_j > 0$ then remove row j , i.e., consider the Young tableaux

$$\lambda' = (\lambda_1, \dots, \lambda_{j-1}, \lambda_{j+1}, \dots) \quad \text{and} \quad \mu' = (\mu_1, \dots, \mu_{j-1}, \mu_{j+1}, \dots).$$

Otherwise then there exist a j with $\lambda_j > \mu_j$ and $\lambda_r < \mu_r$ for all $r > j$. Now consider the Young tableaux λ' and μ where

$$\lambda' = (\lambda_1, \dots, \lambda_j - 1, \lambda_{j+1} + 1, \dots).$$

It is clear that $\sum_{r>i} \lambda'_r \leq \sum_{r>i} \lambda_r$ and $\sum_{c<i} \lambda_c^t \leq \sum_{c<i} \lambda_c^t$ for all i . Repeating this a finite number of times we get in a situation where $\lambda'_r = \mu_r$ for some r . $\square$

Chapter 4

Rational normal forms

Every linear transformation has a so called rational Jordan normal form, which is briefly described in section 5.1.5. The nilpotent part, the semisimple part and the simple components of the latter follow immediately from this normal form. The uniqueness of this form can be used to determine whether (the modules corresponding to) two linear transformations are isomorphic.

In this chapter a rational normal form of differential operators is defined, which resembles the rational Jordan normal form of linear transformations. This form will have some nice properties:

- One can immediately compute the nilpotent and semisimple part of the differential operator D from a rational normal form. The semisimple part is given by a block diagonal matrix where each block represents a simple component of the semisimple part.
- The ramification indices of all simple components of the semisimple part follow immediately from a rational normal form.
- An element representing the characteristic class of the differential operator D can be computed from a rational normal form.
- A rational normal form is super-irreducible. This is explained in section 4.3.
- A canonical form of Babbitt and Varadarajan over an algebraic closure of K can be obtained by applying a simple “shearing transformation.” This is explained in section 4.4.

In section 4.5 the rational normal form is used to prove that every holonomic $\mathcal{D}_1$ -module without ∂ -torsion and without x -torsion is a holonomic $\mathcal{D}(K)$ -module.

4.1 A rational normal form of a simple differential operator

In this section we will describe a rational normal form of a simple differential operator. In the event that the field of constants k is algebraically closed, one obtains the normal form of [vdEL92] as is shown in proposition 4.1.7.

Lemma 4.1.1 *Let K' be a finite field extension of K and $(\alpha_1, \dots, \alpha_n)$ a K -basis of the vector space K' . Then the following matrix is invertible:*

$$A \stackrel{\text{def}}{=} \begin{pmatrix} \sigma_1(\alpha_1) & \dots & \sigma_n(\alpha_1) \\ \vdots & \ddots & \vdots \\ \sigma_1(\alpha_n) & \dots & \sigma_n(\alpha_n) \end{pmatrix}.$$

Here $\{\sigma_1, \dots, \sigma_n\}$ is the set of K -homomorphisms from K' to L , where L is a normal closure of the field extension K'/K .

PROOF. Let $\alpha \in K'$ be a primitive element, i.e., $K' = K(\alpha)$, and write

$$\alpha_i = \sum_{j=0}^{n-1} \lambda_{ij} \alpha^j \quad \text{for all } i \in \{1, \dots, n\}$$

with all λ_{ij} in K . Then

$$A = \begin{pmatrix} \lambda_{10} & & \lambda_{1,n-1} \\ \vdots & \ddots & \vdots \\ \lambda_{n0} & & \lambda_{n,n-1} \end{pmatrix} \begin{pmatrix} \sigma_1(\alpha)^0 & \dots & \sigma_n(\alpha)^0 \\ \vdots & \ddots & \vdots \\ \sigma_1(\alpha)^{n-1} & \dots & \sigma_n(\alpha)^{n-1} \end{pmatrix}. \quad (4.1)$$

Both matrices on the right hand side of (4.1) are invertible. The first matrix since both $(\alpha_1, \dots, \alpha_n)$ and $(1, \alpha, \dots, \alpha^{n-1})$ are K -bases of the vector space K' and the second since it is a Vandermonde matrix and $\sigma_i(\alpha) \neq \sigma_j(\alpha)$ for all i and j with $i \neq j$. This proves the lemma. $\square$

Theorem 4.1.2 *Let (V, D) be a differential operator. Then the following statements hold:*

1. *If (V, D) is a simple differential operator, there exist $e, f \in \mathbb{N}^*$ with $\dim V = ef$, a K -basis $\mathbf{v}$ of V , $C \in \text{Gl}_f(k)$ and matrices $A_0, A_1, \dots, A_{e-1}$ in $\text{Mat}_f(k[1/x])$ such that*

- (a) *C and all $A_i^{(j)}$ are semisimple and all commute pairwise, where $A_i^{(j)}$ are the constant $f \times f$ matrices satisfying*

$$A_i = \sum_{j=0}^{f-1} A_i^{(j)} x^j \quad \text{for all } i \in \{0, \dots, e-1\}.$$

(b)

$$M(D, v) = \begin{pmatrix} A_0 & CA_{e-1}/x & & CA_1/x \\ A_1 & A_0 - \frac{1}{e}I & & \vdots \\ \vdots & A_1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & CA_{e-1}/x \\ A_{e-1} & A_{e-2} & & A_1 & A_0 - \frac{e-1}{e}I \end{pmatrix}.$$

 (c) The characteristic polynomial h of the matrix

$$\begin{pmatrix} A_0 & CA_{e-1}/x & & CA_1/x \\ A_1 & A_0 & & \vdots \\ \vdots & A_1 & & \vdots \\ \vdots & & \ddots & CA_{e-1}/x \\ A_{e-1} & A_{e-2} & & A_1 & A_0 \end{pmatrix}$$

is irreducible.

2. Conversely, let $e, f \in \mathbb{N}^*$ satisfy $\dim V = ef$ and let v be a K -basis of V , $C \in \text{Gl}_f(k)$ and matrices $A_0, A_1, \dots, A_{e-1}$ in $\text{Mat}_f(k[1/x])$ such that (a), (b) and (c) hold, then (V, D) is a simple differential operator with characteristic class $c(D) = [h]$ and ramification index $\epsilon(D) = e$.
3. If (V, D) is a simple differential operator and $g \in \text{Gl}_{ef}(K)$ a transformation matrix such that

$$g[M(D, v)] = \begin{pmatrix} A'_0 & C'A'_{e-1}/x & & C'A'_1/x \\ A'_1 & A'_0 - \frac{1}{e}I & & \vdots \\ \vdots & A'_1 & & \vdots \\ \vdots & & \ddots & C'A'_{e-1}/x \\ A'_{e-1} & A'_{e-2} & \dots & A'_1 & A'_0 - \frac{e-1}{e}I \end{pmatrix} \quad (4.2)$$

with $C' \in \text{Gl}_f(k)$ and $A'_0, A'_1, \dots, A'_{e-1}$ in $\text{Mat}_f(k[1/x])$. Then the following conditions are equivalent:

- $C', A'_0, A'_1, \dots, A'_{e-1}$ satisfy condition (a).
- There exists an integer $q \in \mathbb{Z}$ and matrices $B, U \in \text{Gl}_f(k)$ such that B is semisimple and commutes with C and with all A_i and

$$g = \begin{pmatrix} U & 0 & & 0 \\ 0 & UB & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & & 0 & UB^{e-1} \end{pmatrix} x^{q/e}.$$

If $C' = C$ and $A'_i = A_i$ for all $i \in \{0, \dots, e-1\}$ then

$$g = \begin{pmatrix} T & 0 & \dots & 0 \\ 0 & T & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & & 0 & T \end{pmatrix}$$

for some semisimple $T \in \text{Gl}_f(k)$ which commutes with C and all A_i .

PROOF. Suppose (V, D) is a simple differential operator. Put $n \stackrel{\text{def}}{=} \dim V$, $e \stackrel{\text{def}}{=} e(D)$, and $f \stackrel{\text{def}}{=} f(D)$. Let L be a Galois extension of K and $a \in L$ a normalized eigenvalue of D_L with corresponding non-zero eigenvector $v \in V_{K(a)}$. Let t be a non-zero element in $K(a)$ with $\tau(t) = t/e$ and let k' be the field of constants of $K(a)$. One has $[k' : k] = f$ by proposition 3.1.5. Put $\gamma \stackrel{\text{def}}{=} x/t^e$, then $\gamma \in k'$. Write

$$v = \sum_{i=0}^{e-1} t^i v_i \quad (v_0, \dots, v_{e-1} \in V_{k'K})$$

and

$$a = \sum_{i=0}^{e-1} t^{-i} a_i \quad (a_0, \dots, a_{e-1} \in k'[1/x]).$$

Define $V' \stackrel{\text{def}}{=} k'Kv_0 + \dots + k'Kv_{e-1} \subseteq V_{k'K}$. The relation $D_L v = av$ gives

$$\sum_{i=0}^{e-1} t^i \left(D_L v_i + \frac{i}{e} v_i \right) = \sum_{i=0}^{e-1} t^i \left(\gamma \sum_{j=0}^{i-1} a_{e+j-i} v_j + \sum_{j=i}^{e-1} a_{j-i} v_j \right).$$

This implies that $D_L(V') \subseteq V'$. The proof of proposition 3.3.1 now shows that the vectors $v_0, v_1, \dots, v_{e-1}$ form a $k'K$ -basis of the vector space V' and

$$M(D_{k'K}|_{V'}, (v_0, \dots, v_{e-1})) = \begin{pmatrix} a_0 & \gamma a_{e-1}/x & \dots & \gamma a_1/x \\ a_1 & a_0 - 1/e & \ddots & \vdots \\ \vdots & a_1 & \ddots & \vdots \\ \vdots & & \ddots & \gamma a_{e-1}/x \\ a_{e-1} & a_{e-2} & & a_0 - (e-1)/e \end{pmatrix}.$$

Let $w = (w_0, \dots, w_{f-1})$ a basis of k' over k . Then w is also a basis of $k'K$ over K . For every $i \in \{0, \dots, e-1\}$ let

$$v_i = \sum_{j=0}^{f-1} w_j v_{i,j} \quad (v_{i,0}, \dots, v_{i,f-1} \in V).$$

Then $v \stackrel{\text{def}}{=} (v_{00}, \dots, v_{0,f-1}, v_{10}, \dots, v_{e-1,f-1})$ is a K -basis of V . To prove this first notice that

$$\{w_j t^i \mid i, j \in \mathbb{N} \text{ with } i < e \text{ and } j < f\}$$

is a K -basis of the vector space $K(a)$. Now let G be the set of K -homomorphisms from $K(a)$ to L . Then $\{\sigma(v) \mid \sigma \in G\}$ is an L -basis of V_L by proposition 3.3.3. We have

$$\sigma(v) = \sum_{i=0}^{e-1} \sum_{j=0}^{f-1} \sigma(w_j t^i) v_{ij} \quad \text{for all } \sigma \in G.$$

Now use lemma 4.1.1.

Let C be the $f \times f$ matrix of the multiplication by γ w.r.t. the k -basis w of k' and for every $i \in \{0, \dots, e-1\}$ let A_i be the $f \times f$ matrix of the multiplication by a_i w.r.t. the K -basis w of $k'K$. Then one easily verifies that statements (a) and (b) of the theorem are satisfied. Define $A \stackrel{\text{def}}{=} M(D, v)$.

At this point we forget all the notations and definitions made so far in this proof. The notations and definitions introduced from this point on will remain valid throughout the rest of the proof.

To show statement (c) we will first show that statements (a) and (b) of the theorem imply that $c(D_L) = [h]$ for some finite field extension L of K with $e_{L/K} = e$. So assume that we have (a) and (b).

Since C and all $A_i^{(j)}$ are semisimple and all commute pairwise, there exists a Galois extension ℓ of k and a matrix $Y \in \text{GL}_f(\ell)$ such that $Y^{-1}CY$ and all $Y^{-1}A_i Y$ are diagonal matrices. Let $w \in \ell^f$ be the first column of the matrix Y and write $Cw = \gamma w$ with $\gamma \in \ell$ and $A_i w = a_i w$ with $a_i \in \ell[1/x]$ for all $i \in \{0, \dots, e-1\}$. Without loss of generality one may assume that the field ℓ contains a primitive e -th root of unity ζ and that there is a diagonal matrix $\Gamma \in \text{Mat}_f(\ell)$ with $\Gamma_{1,1} = 1$ such that $Y^{-1}CY = \gamma \Gamma^e$. Define $L \stackrel{\text{def}}{=} \ell K(t)$ where $t \in L$ satisfies $x = \gamma t^e$. Define

$$g_1 \stackrel{\text{def}}{=} \begin{pmatrix} Y & 0 & 0 \\ 0 & Y\Gamma^{-1}t & \vdots \\ \vdots & \ddots & \ddots \\ 0 & 0 & Y\Gamma^{1-e}t^{e-1} \end{pmatrix}$$

and

$$g_2 \stackrel{\text{def}}{=} \begin{pmatrix} I & I & I \\ I & \zeta I & \zeta^{e-1} I \\ \vdots & \vdots & \vdots \\ I & \zeta^{e-1} I & \zeta^{(e-1)^2} I \end{pmatrix}.$$

Then $(g_1 g_2)^{-1}[A]$ is a diagonal matrix with entries in L . This implies that

$$c(D_L)(\lambda) = [\det(\lambda I - (g_1 g_2)^{-1}[A])].$$

Now notice that

$$(g_1 g_2)((g_1 g_2)^{-1}[A])(g_1 g_2)^{-1} = \begin{pmatrix} A_0 & CA_{e-1}/x & & CA_1/x \\ A_1 & A_0 & & \vdots \\ \vdots & A_1 & \ddots & \vdots \\ \vdots & & \ddots & CA_{e-1}/x \\ A_{e-1} & A_{e-2} & A_1 & A_0 \end{pmatrix}.$$

So statements (a) and (b) of the theorem imply that $c(D_L) = [h]$ for some finite field extension L of K with $\epsilon_{L/K} = e$.

Let $g \in \mathcal{F} \cap R$ with $c(D) = [g]$. Since we assumed that D is simple, g is irreducible by corollary 3.5.1. Using proposition 3.5.3 this implies that $g \equiv_e h$, so h is also irreducible.

Now assume that statements (a), (b) and (c) hold for the differential operator D . We want to show that this implies that D is a simple differential operator. We have already seen that statements (a) and (b) imply that $c(D_L) = [h]$ for some finite field extension L of K with $\epsilon_{L/K} = e$. Let $g \in \mathcal{F} \cap R$ with $c(D) = [g]$. Using proposition 3.5.3 again we have $g \equiv_e h$. Since h is irreducible by statement (c), g is also irreducible, hence D is simple by corollary 3.5.1.

Now we will show that $c(D) = [h]$ and $\epsilon(D) = e$. The first element of the diagonal matrix $(g_1 g_2)^{-1}[A]$ is

$$a \stackrel{\text{def}}{=} a_0 + a_1 t^{-1} + \cdots + a_{e-1} t^{1-e}.$$

So $a \in \mathcal{P}_L$ is an eigenvalue of D_L , hence $\epsilon(D) = \epsilon_{K(a)/K} \leq \epsilon_{L/K} = e$. Write $a_i = \sum_{j \leq 0} a_{ij} x^j$ for all $i \in \{0, \dots, e-1\}$ and all a_{ij} in ℓ . Put

$$k' \stackrel{\text{def}}{=} k(\gamma, a_{00}, a_{0,-1}, \dots, a_{e-1,0}, a_{e-1,-1}, \dots).$$

If $p \in k[y, x_{00}, x_{0,-1}, \dots, x_{e-1,0}, x_{e-1,-1}, \dots]$ is a polynomial, then

$$p(C, A_0^{(0)}, A_0^{(-1)}, \dots, A_{e-1}^{(0)}, A_{e-1}^{(-1)}, \dots)w = p(\gamma, a_{00}, a_{0,-1}, \dots, a_{e-1,0}, a_{e-1,-1}, \dots)w$$

since C and all $A_i^{(j)}$ commute pairwise. This shows that every element of k' is an eigenvalue of some matrix in $\text{Mat}_f(k)$. Hence $f(D) = f_{K(a)/K} \leq [k' : k] \leq f$. We have $\epsilon(D)f(D) = ef$ and therefore $\epsilon(D) = e$. Applying proposition 3.5.4 shows that $c(D) = [h]$. This also shows that a is a normalized eigenvalue of D_L .

To show the last part of the theorem first assume that

$$g = \begin{pmatrix} U & 0 & \cdots & 0 \\ 0 & UB & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & UB^{e-1} \end{pmatrix} x^{q/e},$$

where q is an integer, U and B are matrices in $\text{Gl}_f(k)$ and B is semisimple and commutes with C and all A_i . Equation (4.2) implies that

$$\begin{aligned} C' &= UB^{-e}CU^{-1}, \\ A'_0 &= UA_0U^{-1} - \frac{q}{e}I, \\ A'_i &= UB^iA_iU^{-1} \quad \text{for all } i \in \{1, \dots, e-1\}. \end{aligned}$$

One now easily verifies the "primed" statement (a), using the fact that B is semisimple and commutes with C and all A_i .

Conversely, assume that equation (4.2) holds for some $g \in \text{Gl}_n(K)$, $C' \in \text{Gl}_f(k)$ and $A'_0, A'_1, \dots, A'_{e-1}$ in $\text{Mat}_f(k[1/x])$ satisfying condition (a) of the theorem. Let $\{\sigma_1, \dots, \sigma_f\}$ be the set of k -homomorphisms from k' to ℓ . We may assume that $\sigma_1 = 1$. By the same sort of argument as in the proof of proposition 3.2.5, we may assume that $w \in (k')^f$. Suppose $\sigma_1(w), \dots, \sigma_f(w)$ are linearly dependent over ℓ . Then there is an element $h \in \{1, \dots, f-1\}$ such that $\{\sigma_1(w), \dots, \sigma_h(w)\}$ are linearly independent over ℓ and

$$\sigma_{h+1}(w) = \sum_{j=1}^h \lambda_j \sigma_j(w) \quad \text{for some } \lambda_1, \dots, \lambda_h \in \ell. \quad (4.3)$$

At least one of the λ_j is non-zero. Suppose $\lambda_{j'} \neq 0$. Applying C and A_i to (4.3) gives $\sigma_{j'}(\gamma) = \sigma_{h+1}(\gamma)$ and $\sigma_{j'}(a_{ij}) = \sigma_{h+1}(a_{ij})$ for all i and j . This implies that $\sigma_{j'} = \sigma_{h+1}$, a contradiction. So we may replace Y by the matrix whose columns are $\sigma_1(w), \dots, \sigma_f(w)$.

Everything that was done for the matrix A can be repeated for the matrix $g[A]$. Denote the corresponding elements with an extra prime, so $k'', \ell', L', Y', w', \gamma', a', a'_i, t'$ and σ'_i . We may assume that $\ell' = \ell$ and $L' = L$.

Now both a and a' are normalized eigenvalues of D_L , so by proposition 3.3.6 there exists an element $j \in \{1, \dots, f\}$, an element $t'' \in L$ with $x = \sigma_j(\gamma)(t'')^e$ and an integer $q \in \mathbb{Z}$ such that

$$a' + \frac{q}{e} = \sigma_j(a_0) + \sigma_j(a_1)(t'')^{-1} + \dots + \sigma_j(a_{e-1})(t'')^{1-e}.$$

Notice that $\sigma_j(k') = k''$. Define $\delta \stackrel{\text{def}}{=} t'/t''$, then $\tau(\delta) = 0$, so $\delta \in k''$. Replacing Y by $\sigma_j Y$, t by t'' , γ by $\sigma_j(\gamma)$ and a_i by $\sigma_j(a_i)$ for all $i \in \{0, \dots, e-1\}$, one may assume that $k' = k''$, $\sigma_i = \sigma'_i$ for all $i \in \{1, \dots, f\}$ and

$$\begin{aligned} t' &= \delta t, \\ \gamma' &= \delta^{-e} \gamma, \\ a'_0 &= a_0 - \frac{q}{e}, \\ a'_i &= \delta^i a_i \quad \text{for all } i \in \{1, \dots, e-1\}. \end{aligned}$$

Define $Z \in \text{Gl}_f(k')$ by

$$Z \stackrel{\text{def}}{=} \begin{pmatrix} \sigma_1(\delta) & 0 & 0 \\ 0 & \sigma_2(\delta) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & & 0 & \sigma_f(\delta) \end{pmatrix}.$$

Then

$$\begin{aligned} (Y')^{-1}C'Y' &= Z^{-e}Y^{-1}CY, \\ (Y')^{-1}A'_0Y' &= Y^{-1}A_0Y - \frac{q}{e}I, \\ (Y')^{-1}A'_iY' &= Z^iY^{-1}A_iY \quad \text{for all } i \in \{1, \dots, e-1\}. \end{aligned}$$

Now define $B \stackrel{\text{def}}{=} YZY^{-1}$ and $U \stackrel{\text{def}}{=} Y'Y^{-1}$. It is clear that B is semisimple and that B commutes with C and all A_i and

$$\begin{aligned} C' &= U'B^{-e}C(U')^{-1}, \\ A'_0 &= U'A_0(U')^{-1} - \frac{q}{e}I, \\ A'_i &= U'B^iA_i(U')^{-1} \quad \text{for all } i \in \{1, \dots, e-1\}. \end{aligned}$$

Now define

$$g' \stackrel{\text{def}}{=} \begin{pmatrix} U' & 0 & 0 \\ 0 & U'B & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & & 0 & U'B^{e-1} \end{pmatrix} x^{q/e}.$$

Then $g'[A] = g[A]$.

We will now show that $B \in \text{Gl}_f(k)$. Let $\sigma \in \text{Gal}(\ell/k)$. It is enough to show that $\sigma(B) = B$. Notice that $\sigma(Y) = YP$ and $\sigma(Z) = P^{-1}ZP$ for some $f \times f$ permutation matrix P . Hence $\sigma(B) = \sigma(Y)\sigma(Z)\sigma(Y)^{-1} = B$.

Notice that in general $U' \notin \text{Gl}_f(k)$. Let $\alpha \in \ell$ with $\ell = k(\alpha)$ and write

$$U' = U_0 + U_1\alpha + \dots + U_{m-1}\alpha^{m-1}$$

where $m \stackrel{\text{def}}{=} [\ell : k]$ and $U_0, U_1, \dots, U_{m-1}$ are $f \times f$ matrices with entries in k . Let $b(X) \stackrel{\text{def}}{=} U_0 + U_1X + \dots + U_{m-1}X^{m-1} \in \text{Mat}_f(k[X])$. Since $b(\alpha) \in \text{Gl}_f(\ell)$, the determinant of $b(X)$ is a non-zero polynomial in $k[X]$. The field k has infinitely many elements, so there is also an element $\beta \in k$ such that $\det b(\beta) \neq 0$. Define $U'' \stackrel{\text{def}}{=} b(\beta)$ and

$$g'' \stackrel{\text{def}}{=} \begin{pmatrix} U'' & 0 & 0 \\ 0 & U''B & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & & 0 & U''B^{e-1} \end{pmatrix} x^{q/e}.$$

Then $U'' \in \text{Gl}_f(k)$ and $g''[A] = g[A]$. To finish the proof of the theorem, it is enough to show the last statement of the theorem, i.e.,

$$g(g'')^{-1} = \begin{pmatrix} T & 0 & 0 \\ 0 & T & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & & 0 & T \end{pmatrix} \quad (4.4)$$

for some semisimple $T \in \text{Gl}_f(k)$ which commutes with C and all A_i . Define $R \stackrel{\text{def}}{=} \{ \tilde{g} \in \text{Mat}_n(K) \mid \tilde{g}(\tau + A) = (\tau + A)\tilde{g} \}$. Define the ring homomorphism $\phi: k' \rightarrow \text{Mat}_f(\ell)$ by

$$\phi: \alpha \longmapsto Y \begin{pmatrix} \sigma_1(\alpha) & 0 & 0 \\ 0 & \sigma_2(\alpha) & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & & 0 & \sigma_f(\alpha) \end{pmatrix} Y^{-1},$$

where $Y \in \text{Gl}_f(\ell)$ and $\sigma_1, \sigma_2, \dots, \sigma_f$ are defined as before. It is clear that for every $\alpha \in k'$, $\phi(\alpha)$ is semisimple and commutes with C and all A_i . Using the same argument as before, one proves that $\phi(k') \subseteq \text{Mat}_f(k)$. Therefore

$$\left\{ \begin{pmatrix} \phi(\alpha) & 0 & 0 \\ 0 & \phi(\alpha) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \phi(\alpha) \end{pmatrix} \mid \alpha \in k' \right\} \subseteq R. \quad (4.5)$$

Since k' and R are isomorphic by proposition 3.3.9 and ϕ is a ring homomorphism (and therefore injective), one must have equality in equation (4.5). We have $g(g'')^{-1} \in R$, so equation (4.4) holds with $T = \phi(\alpha)$ for some $\alpha \in (k')^*$. $\square$

Definition 4.1.3 If (V, D) is a simple differential operator and v is a K -basis of V such that the conditions in theorem 4.1.2 are fulfilled then we will say that $M(D, v)$ is a *rational normal form* of the differential operator D .

Proposition 4.1.4 Let $e, A_i, A_i^{(j)}$ and C as in theorem 4.1.2. Then:

- If $d > 1$ is a divisor of e , then $A_i \neq 0$ for some i not divisible by d .
- The characteristic polynomials of C and all $A_i^{(j)}$ are powers of irreducible polynomials. In particular, if $A_i^{(j)} \neq 0$ then $A_i^{(j)}$ is invertible.

PROOF. Let L be a finite field extension of K and $a \in \mathcal{P}_L$. Let k' be the field of constants of $K(a)$ and let $t \in K(a)$ be a non-zero element satisfying $\gamma \stackrel{\text{def}}{=}} x/t^e \in k'$. Write

$$a = \sum_{i=0}^{e-1} \sum_{j \leq 0} a_{ij} x^j t^{-i}$$

with all a_{ij} in k' . Let $w = (w_1, \dots, w_n)$ be a k -basis of the vector space k' and let C be the $f \times f$ matrix of the multiplication by γ w.r.t. the k -basis w and let $A_i^{(j)}$ be the $f \times f$ matrix of the multiplication by a_{ij} w.r.t. the k -basis w for all i and j . It follows from the proof of theorem 4.1.2 that every rational normal form can be obtained in this way.

Suppose there is a natural number d with $d > 1$ and $A_i = 0$ for all $i \in \{0, \dots, e-1\}$ with $d \nmid i$. Then $a \in k'K(t^d)$, so $e_{K(a)/K} \leq e/d$, a contradiction.

The fact that C and all $A_i^{(j)}$ are matrices of the multiplication by some element of k' w.r.t. some k -basis, immediately implies that their characteristic polynomials are powers of irreducible polynomials. $\square$

Proposition 4.1.5 *Every simple differential operator has a rational normal form such that the characteristic polynomial of the matrix C in the rational normal form of theorem 4.1.2 is irreducible.*

PROOF. Let (V, D) be a simple differential operator, L a finite field extension of K , $a \in L$ a normalized eigenvalue of D_L and k' the field of constants in $K(a)$. The matrix C in theorem 4.1.2 is constructed as the matrix of multiplication by x/t^e w.r.t. some k -basis of the vector space k' where $t \in L^*$ satisfies $\tau(t) = t/e$. By proposition 3.1.12 one can choose this $t \in K(a)$ such that $k' = k(x/t^e)$. This proves the proposition. $\square$

Proposition 4.1.6 *Let $A \in \text{Mat}(K)$ be a rational normal form of a simple differential operator. Then A is the 1×1 zero matrix or A is invertible and $A^{-1} \in \text{Mat}(\mathcal{O})$.*

PROOF. Write

$$A = \begin{pmatrix} A_0 & CA_{e-1}/x & & CA_1/x \\ A_1 & A_0 - \frac{1}{e}I & & \vdots \\ \vdots & A_1 & & \vdots \\ \vdots & & \ddots & CA_{e-1}/x \\ A_{e-1} & A_{e-2} & A_1 & A_0 - \frac{e-1}{e}I \end{pmatrix}$$

with $e, f \in \mathbb{N}^*$, $C \in \text{Gl}_f(k)$ and $A_0, A_1, \dots, A_{e-1}$ in $\text{Mat}_f(k[1/x])$ as in theorem 4.1.2. Put $r \stackrel{\text{def}}{=} \text{ord}(A)$ and write

$$A = \sum_{j=r}^0 A^{(j)} x^j;$$

$$A_i = \sum_{j=r}^0 A_i^{(j)} x^j \quad \text{for all } i \in \{0, \dots, e-1\};$$

with all $A_i^{(j)}$ and $A_i^{(j)}$ constant matrices.

If $r = 0$ then A_0 is a constant matrix and $A_i = 0$ for all $i \in \{1, \dots, e-1\}$. By the second statement of proposition 4.1.4 one has $e = 1$, so $A = A_0$. Hence the characteristic polynomial of A is an irreducible polynomial with coefficients in k . The statement now follows.

Now assume that $r < 0$ and $A_0^{(r)} \neq 0$. Then

$$A^{(r)} = \begin{pmatrix} A_0^{(r)} & * & * \\ 0 & A_0^{(r)} & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & & 0 & A_0^{(r)} \end{pmatrix}.$$

By proposition 4.1.4 the matrix $A_0^{(r)}$ is invertible, so the matrix $A^{(r)}$ is also invertible and therefore the matrix A is invertible and $A^{-1} \in \text{Mat}(x^{-r}\mathcal{O})$.

Finally assume that $r < 0$ and $A_0^{(r)} = 0$. Let i' be the largest $i \in \{1, \dots, e-1\}$ such that $A_i^{(r+1)} \neq 0$. Then

$$A^{(r)} = \begin{pmatrix} 0 & U \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad A^{(r+1)} = \begin{pmatrix} V & W \\ X & Y \end{pmatrix},$$

where U is a constant $fi' \times fi'$ matrix of the form

$$U = \begin{pmatrix} CA_{i'}^{(r+1)} & * & * \\ 0 & CA_{i'}^{(r+1)} & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & & 0 & CA_{i'}^{(r+1)} \end{pmatrix}$$

and X is a constant $(n - fi') \times (n - fi')$ matrix of the form

$$X = \begin{pmatrix} A_{i'}^{(r+1)} & * & * \\ 0 & A_{i'}^{(r+1)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & & 0 & A_{i'}^{(r+1)} \end{pmatrix}.$$

The matrix C is invertible and by proposition 4.1.4 the matrix $A_{i'}^{(r+1)}$ is also invertible, so the matrices U and X are invertible. We will now show that this implies that A is invertible and that $A^{-1} \in \text{Mat}(\mathcal{O})$ by constructing its inverse B . Write

$$B = \begin{pmatrix} 0 & E_r \\ 0 & F_r \end{pmatrix} x^{-r-1} + \begin{pmatrix} G_r & E_{r+1} \\ H_r & F_{r+1} \end{pmatrix} x^{-r} + \dots$$

where all E_i , F_i , G_i , and H_i are constant matrices of the same size as resp. X , V , Y and U . For $i = r, r+1, \dots$ one can compute E_i , F_i , G_i , and H_i by putting $BA = I$ and comparing the coefficients of x^{i-r} on both sides. $\square$

Proposition 4.1.7 Assume that k is an algebraically closed field. Let (V, D) be a differential operator.

Then: (V, D) is simple if and only if there exists a K -basis $v = (v_0, \dots, v_{n-1})$ of V and elements $a_0, a_1, \dots, a_{n-1}$ in $k[1/x]$ such that

1.

$$M(D, v) = \begin{pmatrix} a_0 & a_{n-1}/x & \dots & a_1/x \\ a_1 & a_0 - \frac{1}{n} & \ddots & \vdots \\ \vdots & a_1 & \ddots & \vdots \\ \vdots & & \ddots & a_{n-1}/x \\ a_{n-1} & a_{n-2} & a_1 & a_0 - \frac{n-1}{n} \end{pmatrix}.$$

2. If $d > 1$ is a divisor of n , then $a_i \neq 0$ for some i not divisible by d .

Furthermore:

$$g[M(D, v)] = \begin{pmatrix} a'_0 & a'_{n-1}/x & & a'_1/x \\ a'_1 & a'_0 - \frac{1}{n} & & \vdots \\ \vdots & a'_1 & & \vdots \\ \vdots & & \ddots & a'_{n-1}/x \\ a'_{n-1} & a'_{n-2} & a'_1 & a'_0 - \frac{n-1}{n} \end{pmatrix}$$

for some $g \in \text{Gl}_n(K)$ and elements $a'_0, a'_1, \dots, a'_{n-1}$ in $k[1/x]$ satisfying condition 2

if and only if

there exists an integer $q \in \mathbb{Z}$ and an element $s \in \{0, \dots, n-1\}$ such that

$$g = \begin{pmatrix} 1 & 0 & & 0 \\ 0 & \zeta^s & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & & 0 & \zeta^{(n-1)s} \end{pmatrix} x^{q/n},$$

where ζ is a primitive n -th root of unity.

PROOF. Suppose (V, D) is a simple differential operator. Since k is an algebraically closed field, one has $\epsilon(D) = \dim V$ and $\mathfrak{f}(D) = 1$. Applying theorem 4.1.2 shows that there exists a K -basis $v = (v_0, \dots, v_{n-1})$ of V , $c \in k^*$ and

elements $a_0, a_1, \dots, a_{n-1}$ in $k[1/x]$ such that

$$M(D, v) = \begin{pmatrix} a_0 & ca_{n-1}/x & & ca_1/x \\ a_1 & a_0 - \frac{1}{n} & & \vdots \\ \vdots & a_1 & \ddots & \vdots \\ \vdots & & \ddots & ca_{n-1}/x \\ a_{n-1} & a_{n-2} & a_1 & a_0 - \frac{n-1}{n} \end{pmatrix}.$$

Since k is an algebraically closed field, one has $c = 1/r^n$ for some $r \in k^*$. W.r.t. the K -basis $(v_0, rv_1, \dots, r^{n-1}v_{n-1})$ of V the differential operator D has the required form. The condition on the a_i follows immediately from proposition 4.1.4.

Now suppose that there is an element $n \in \mathbb{N}^*$ and a K -basis $v = (v_0, \dots, v_{n-1})$ of V and elements $a_0, a_1, \dots, a_{n-1}$ in $k[1/x]$ such that the two statements in the proposition hold. Let $L \stackrel{\text{def}}{=} K(t)$ where t satisfies $t^n = x$ and define

$$a \stackrel{\text{def}}{=} \sum_{i=0}^{n-1} t^{-i} a_i \quad \text{and} \quad v \stackrel{\text{def}}{=} \sum_{i=0}^{n-1} t^i v_i.$$

Then the form of $M(D, v)$ implies that $Dv = av$. The condition on the a_i implies that $L = K(a)$, so $[K(a) : K] = n = \dim V$. Hence, by corollary 3.3.2, (V, D) is a simple differential operator.

The last statement of the proposition follows immediately from statement (c) of theorem 4.1.2. $\square$

Remark 4.1.8 Proposition 4.1.7 also appears in [vdEL92], but the last statement about the uniqueness is not stated there. In [vdEL92] the result is generalized to simple holonomic $\mathcal{D}_1$ -modules. In section 4.5 we will state a result on holonomic $\mathcal{D}_1$ -modules which generalizes the statement about simple holonomic $\mathcal{D}_1$ -modules given in [vdEL92].

4.2 A rational normal form of a general differential operator

Proposition 4.2.1 *Let (V, D) be a semisimple differential operator. Then*

$$\text{Hom}_K(V, V) = \ker \text{ad}(D) \oplus \text{im ad}(D)$$

as vector spaces over k , where $\text{ad}(D): \text{Hom}_K(V, V) \rightarrow \text{Hom}_K(V, V)$ is defined by $f \mapsto Df - fD$ for all $f \in \text{Hom}_K(V, V)$.

PROOF. By corollary 3.6.3 the differential operator $\text{ad}(D)$ on $\text{Hom}(V, V)$ is semi-simple. To prove the proposition it is therefore enough to show that

$$V = \ker D \oplus \text{im } D$$

as vector spaces over k for all semisimple differential operators (V, D) . But this follows from proposition 3.3.5. $\square$

Proposition 4.2.2 *Let (V, D_1) and (W, D_2) be two non-isomorphic simple differential operators. Define the function $\varphi: \text{Hom}_K(V, W) \rightarrow \text{Hom}_K(V, W)$ by $g \mapsto D_2g - gD_1$ for all $g \in \text{Hom}_K(V, W)$. Then φ is an isomorphism of vector spaces over k .*

PROOF. By corollary 3.6.3 the differential operator φ on $\text{Hom}(V, W)$ is semisimple. By proposition 3.3.5 one has

$$\text{Hom}(V, W) = \ker \varphi \oplus \text{im } \varphi$$

as vector spaces over k . Therefore, to prove the proposition, it is enough to show that $\ker \varphi = 0$. Let $g \in \ker \varphi$, then $gD_1 = D_2g$, so g is a $\mathcal{D}(K)$ -homomorphism from V to W . We have to show that $g = 0$. Notice that $\ker g$ is a $\mathcal{D}(K)$ -submodule of V and $\text{im } g$ is a $\mathcal{D}(K)$ -submodule of W . Since (V, D_1) and (W, D_2) are simple differential operators, one has $\ker g = 0$ or $\ker g = V$ and also $\text{im } g = 0$ or $\text{im } g = W$. Suppose that $g \neq 0$ then $\ker g = 0$ and $\text{im } g = W$, so g is a $\mathcal{D}(K)$ -isomorphism from V to W , a contradiction. Conclusion: $g = 0$. $\square$

Theorem 4.2.3 *Let (V, D) be a differential operator. Then there exists a K -basis $v = (v_1, \dots, v_n)$ of V such that*

$$M(D, v) = \begin{pmatrix} B_1 & 0 & & 0 \\ 0 & B_2 & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & & 0 & B_s \end{pmatrix},$$

where $s \in \mathbb{N}^*$ and every B_i is of the form

$$\begin{pmatrix} S_i & I & 0 & 0 \\ 0 & S_i & I & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \vdots & & \ddots & I \\ 0 & & 0 & S_i \end{pmatrix}.$$

Here I stands for the identity matrix and S_i is a square matrix such that $\tau + S_i$ is a simple differential operator. This form is unique in the sense that any other such form can be obtained by permuting the blocks B_i and replacing the S_i by $g_i[S_i]$ with $g_i \in \text{Gl}(K)$.

PROOF. Let

$$0 = V_0 \subset V_1 \subset \cdots \subset V_m = V \quad (4.6)$$

be a composition series of the $\mathcal{D}(K)$ -module V . Define $i(j) \stackrel{\text{def}}{=} \dim V_j$ for all $j \in \{0, \dots, m\}$. Let $\mathbf{v} = (v_1, \dots, v_n)$ be a K -basis of V such that $(v_1, \dots, v_{i(j)})$ is a K -basis of V_j for all $j \in \{1, \dots, m\}$. Let D_j be the simple differential operator on V_j/V_{j-1} induced by D for all $j \in \{1, \dots, m\}$. Then

$$A \stackrel{\text{def}}{=} M(D, \mathbf{v}) = \begin{pmatrix} C_{11} & C_{12} & & C_{1m} \\ 0 & C_{22} & & \vdots \\ \vdots & \ddots & \ddots & C_{m-1,m} \\ 0 & & 0 & C_{mm} \end{pmatrix},$$

where $C_{jj} = M(D_j, (v_{i(j-1)+1} + V_{j-1}, \dots, v_{i(j)} + V_{j-1}))$ for all $j \in \{1, \dots, m\}$. One can choose $\mathbf{v}$ in such a way that $C_{ii} = C_{jj}$ if the differential operators D_i and D_j are isomorphic.

Let $g \in \text{Gl}_n(K)$ be a "block elementary" matrix of the form

$$\begin{pmatrix} I & 0 & \cdots & 0 \\ 0 & I & & T \\ \vdots & \ddots & \ddots & \vdots \\ 0 & & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$$

with T in position (i, j) . Then

$$g[A] = \begin{pmatrix} C_{11} & C'_{12} & & C'_{1m} \\ 0 & C_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & C'_{m-1,m} \\ 0 & & 0 & C_{mm} \end{pmatrix}.$$

Then $C'_{ij} = C_{ij} + T(\tau + C_{jj}) - (\tau + C_{ii})T$ and $C'_{kl} = C_{kl}$ for all k and l with $k > i$ or $l < j$.

If $C_{ii} = C_{jj}$ then by proposition 4.2.1 one can choose a matrix T such that C'_{ij} commutes with the differential operator $\tau + C_{ii}$, otherwise by proposition 4.2.2 one can choose a matrix T such that $C'_{ij} = 0$.

Use this construction for all positions (i, j) with $j = i + 1$ and then for all positions (i, j) with $j = i + 2$, etc. Continuing in this way one can construct a matrix $g_1 \in \text{Gl}_n(K)$ of the form

$$g_1 = \begin{pmatrix} I & T_{12} & \cdots & T_{1m} \\ 0 & I & \ddots & \vdots \\ \vdots & \ddots & \ddots & T_{m-1,m} \\ 0 & & 0 & I \end{pmatrix}$$

and a permutation matrix $g_2 \in \text{Gl}_n(K)$ such that

$$(g_2 g_1)[A] = \begin{pmatrix} \tilde{B}_1 & 0 & 0 \\ 0 & \tilde{B}_2 & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & & 0 & \tilde{B}_r \end{pmatrix},$$

where $r \in \mathbb{N}^*$ and every $\tilde{B}_i$ is of the form

$$\begin{pmatrix} S_i & S_{12}^{(i)} & & S_{1s(i)}^{(i)} \\ 0 & S_i & & \vdots \\ \vdots & \ddots & \ddots & S_{s(i)-1, s(i)}^{(i)} \\ 0 & & 0 & S_i \end{pmatrix} \quad (m = \sum_{i=1}^r s(i)).$$

Here S_i and all $S_{jk}^{(i)}$ are square matrices and

- $\tau + S_i$ is a simple differential operator for all $i \in \{1, \dots, r\}$;
- $\tau + S_i$ and $S_{jk}^{(i)}$ commute for all $i \in \{1, \dots, r\}$ and $j, k \in \mathbb{N}^*$ with $j < k \leq s(i)$.

Let $i \in \{1, \dots, r\}$ and define

$$R_i \stackrel{\text{def}}{=} \{ C \in \text{Mat}_{q_i}(K) \mid C \text{ and } \tau + S_i \text{ commute} \},$$

where q_i is the size of the matrix S_i ($n = \sum_{i=1}^r s(i)q_i$). By proposition 3.3.9 R_i is a (commutative) field. By the theorem of Jordan there exists a matrix $g'_i \in \text{Gl}_{s(i)}(R_i)$ such that

$$g'_i \begin{pmatrix} 0 & S_{12}^{(i)} & & S_{1s(i)}^{(i)} \\ 0 & 0 & & \vdots \\ \vdots & \ddots & \ddots & S_{s(i)-1, s(i)}^{(i)} \\ 0 & & 0 & 0 \end{pmatrix} (g'_i)^{-1} = \begin{pmatrix} 0 & * & 0 & 0 \\ 0 & 0 & * & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 0 & * \\ 0 & & & 0 & 0 \end{pmatrix},$$

where every asterisk stands for either zero or the identity matrix. Since, by definition, every element of R_i commutes with $\tau + S_i$, one has

$$g'_i[\tilde{B}_i] = \begin{pmatrix} S_i & * & 0 & 0 \\ 0 & S_i & * & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \vdots & & \ddots & * \\ 0 & & 0 & S_i \end{pmatrix}.$$

This proves the first part of the theorem. Remains to prove the uniqueness of the form.

After permuting the blocks B_i one may assume that there exists natural numbers $j(0), j(1), \dots, j(r)$ with $r \in \mathbb{N}^*$ and $0 = j(0) < j(1) < \dots < j(r) = s$ such that the differential operators $\tau + S_i$ and $\tau + S_j$ are isomorphic if and only if $i, j \in \{j(k-1) + 1, \dots, j(k)\}$ for some $k \in \{1, \dots, s\}$. Let $V_1, V_2, \dots, V_r$ be the corresponding subspaces of V , i.e.,

$$M(D|_{V_i}, v_i) = \begin{pmatrix} B_{j(i-1)+1} & 0 & 0 \\ 0 & B_{j(i-1)+2} & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & B_{j(i)} \end{pmatrix} \quad \text{for all } i \in \{1, \dots, r\}$$

where v_i is the subset of v spanning V_i . It is clear that

$$M(s(D)|_{V_i}, v_i) = \begin{pmatrix} S_{j(i-1)+1} & 0 & 0 \\ 0 & S_{j(i-1)+1} & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & S_{j(i)} \end{pmatrix} \quad \text{for all } i \in \{1, \dots, r\}.$$

Hence the V_i are the isotypical components of the semisimple part of the differential operator (V, D) and therefore independent of the K -basis v , cf. [Bou58, § 3, n° 4].

Remains to show that for every $i \in \{1, \dots, r\}$ the number and sizes of the blocks $B_{j(i-1)+1}, \dots, B_{j(i)}$ corresponding with the isotypical component V_i are invariant. This follows directly from the fact that these are completely determined by the nilpotent function $n(D)|_{V_i}$, e.g.,

$$\dim V_i = \dim \ker n(D)|_{V_i} + (j(i) - j(i-1))q_i \quad \text{for all } i \in \{1, \dots, r\}$$

where q_i denotes the size of the matrix $S_{j(i)}$. □

Definition 4.2.4 If (V, D) is a differential operator and v is a K -basis of V such that the matrix $M(D, v)$ has the same form as in theorem 4.2.3 and all B_i are rational normal forms of simple differential operators (see definition 4.1.3), then we will say that the matrix $M(D, v)$ is a *rational normal form* of the differential operator D .

Proposition 4.2.5 Let $A \in \text{Mat}(K)$ be a rational normal form of a differential operator (V, D) . Then A has a zero column or A is invertible and $A^{-1} \in \text{Mat}(\mathcal{O})$.

PROOF. The matrix A is a block diagonal matrix with blocks of the form

$$B = \begin{pmatrix} S & I & 0 & & 0 \\ 0 & S & I & & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & I \\ 0 & & & 0 & S \end{pmatrix},$$

where S is a rational normal form of a simple differential operator and I denotes the identity matrix. It is enough to show the proposition for the rational normal form B . If S is the 1×1 zero matrix then B has a zero column, otherwise S is invertible and $S^{-1} \in \text{Mat}(\mathcal{O})$ by proposition 4.1.6. In the last case the matrix B is also invertible and $B^{-1} \in \text{Mat}(\mathcal{O})$, since

$$B^{-1} = \begin{pmatrix} S^{-1} & -S^{-2} & S^{-3} & \dots & -(-S)^{-r} \\ 0 & S^{-1} & -S^{-2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & S^{-3} \\ \vdots & & \ddots & \ddots & -S^{-2} \\ 0 & & & 0 & S^{-1} \end{pmatrix}.$$

□

4.3 Comparison with the super-irreducible form

In this section we will prove that a rational normal form of a differential operator as defined in section 4.2 is super-irreducible. The definition of super-irreducible is given in definition 3.8.14.

There is a simple criterion to decide whether a matrix $A \in \text{Mat}(K)$ is super-irreducible, cf [HW87]. We will repeat this criterion here since it will be needed to prove that every rational normal form is super-irreducible.

Let $A \in \text{Mat}_n(K)$ and $p \stackrel{\text{def}}{=} |\text{ord}(A)|$. Define

$$\theta_{p-i}(A, \lambda) \stackrel{\text{def}}{=} (x^{r_{p-i}}(A) \det(\lambda I + x^i A)) \Big|_{x=0} \quad \text{for all } i \in \mathbb{N},$$

where $r_{p-i}(A)$ is given in definition 3.8.16. Then $\theta_{p-i}(A, \lambda) \in k[\lambda]$ for all $i \in \mathbb{N}$. Notice that $\theta_0(A, \lambda) = \det(\lambda I + A_{-p})$ and $\theta_i(A, \lambda) = \lambda^n$ for all $i < 0$.

Proposition 4.3.1 (Hilali) *Let $A \in \text{Mat}(K)$ and $p \stackrel{\text{def}}{=} |\text{ord}(A)|$. Then:*

$$A \text{ is super-irreducible} \quad \text{iff} \quad \theta_i(A, \lambda) \neq 0 \quad \text{for all } i \in \{1, \dots, p\}.$$

We will now show that a rational normal form of a differential operator is super-irreducible.

Lemma 4.3.2 *Let $n \in \mathbb{N}^*$ and $A \in \text{Mat}_n(K)$. Put $p \stackrel{\text{def}}{=} |\text{ord}(A)|$ and write*

$$A = A_{-p}x^{-p} + A_{1-p}x^{1-p} + \cdots \quad (A_{-p}, A_{1-p}, \dots \in \text{Mat}_n(k)).$$

If $p = 0$ or A_p is invertible, then A is super-irreducible.

PROOF. If $p = 0$ then it is immediately clear by proposition 4.3.1 that A is super-irreducible. Now assume that $p > 0$ and A_{-p} is invertible. Then one easily checks that $r_i(A) = in$ and $\theta_i(A, \lambda) = \det(A_{-p}) \neq 0$ for all $i \in \{1, \dots, p\}$, so A is super-irreducible by proposition 4.3.1. $\square$

Lemma 4.3.3 *Let $n \in \mathbb{N}^*$ and $A \in \text{Mat}_n(K)$ with $p \stackrel{\text{def}}{=} |\text{ord}(A)| > 0$. Suppose*

$$A = \begin{pmatrix} 0 & U \\ 0 & 0 \end{pmatrix} x^{-p} + \begin{pmatrix} V & W \\ X & Y \end{pmatrix} x^{1-p} + \cdots,$$

where $U \in \text{Gl}_m(k)$, $V \in \text{Mat}_{m, n-m}(k)$, $W \in \text{Mat}_m(k)$, $X \in \text{Gl}_{n-m}(k)$ and $Y \in \text{Mat}_{n-m, m}(k)$ for some $m \in \mathbb{N}^$ with $m < n$. Then A is super-irreducible.*

PROOF. One easily checks that

$$r_i(A) = m + (i-1)n \quad \text{and} \quad \theta_i(A, 0) = \det(U) \det(X) \neq 0$$

for all $i \in \{1, \dots, p\}$, so A is super-irreducible by proposition 4.3.1. $\square$

Lemma 4.3.4 *Let*

$$A \stackrel{\text{def}}{=} \begin{pmatrix} A_1 & C \\ 0 & A_2 \end{pmatrix}$$

with $A_1 \in \text{Mat}_m(K)$, $A_2 \in \text{Mat}_n(K)$, $C \in \text{Mat}_{m, n}(\mathcal{O})$ and $m, n \in \mathbb{N}^$. Then A is super-irreducible if and only if A_1 and A_2 are both super-irreducible.*

PROOF. Put $p \stackrel{\text{def}}{=} |\text{ord}(A)|$, $p_1 \stackrel{\text{def}}{=} |\text{ord}(A_1)|$ and $p_2 \stackrel{\text{def}}{=} |\text{ord}(A_2)|$. It is not difficult to check that

$$r_{p-i}(A) = r_{p_1-i}(A_1) + r_{p_2-i}(A_2) \quad \text{and} \quad \theta_{p-i}(A, \lambda) = \theta_{p_1-i}(A_1, \lambda) \theta_{p_2-i}(A_2, \lambda)$$

for all $i \in \mathbb{N}$. The proposition now follows from proposition 4.3.1. $\square$

Proposition 4.3.5 *Let $A \in \text{Mat}(K)$ be a rational normal form of a differential operator. Then the matrix A is super-irreducible.*

PROOF. By lemma 4.3.4 it is enough to prove that a rational normal form of a simple differential operator is super-irreducible. So let A be a rational normal form of a simple differential operator. Put $r \stackrel{\text{def}}{=} \text{ord}(A)$ and write

$$A = \sum_{i=r}^0 A_i \quad (A_r, A_{r+1}, \dots, A_0 \in \text{Mat}(k)).$$

The proof of proposition 4.1.6 shows that there are three possibilities:

1. $r = 0$.
2. $r < 0$ and A_r is invertible.
3. $r < 0$ and

$$A_r = \begin{pmatrix} 0 & U \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad A_{r+1} = \begin{pmatrix} V & W \\ X & Y \end{pmatrix},$$

where U and X are constant invertible matrices.

In the first two cases A is super-irreducible by lemma 4.3.2. In the last case it is super-irreducible by lemma 4.3.3. $\square$

We will conclude this section by mentioning the connection of a super-irreducible form with special lattices as defined in [Lev91].

Definition 4.3.6 (Levelt) Let (V, D) be a differential operator and let Λ be a lattice in V , i.e., an $\mathcal{O}$ -module of finite type, generating V as a K -vector space. Let $j \in \mathbb{N}$. We say that the lattice Λ *stabilizes* $x^j D$ if

$$\rho_{j+1}(D) = \dim_k (\Lambda + (x^j D)\Lambda) / \Lambda.$$

Λ is called a *special lattice* for the differential operator D if the lattice Λ stabilizes $x^j D$ for all $j \in \mathbb{N}$.

Proposition 4.3.7 (Levelt) Let (V, D) be a differential operator, Λ a lattice in V and $\mathbf{v} = (v_1, \dots, v_n)$ an $\mathcal{O}$ -basis of Λ . Put $A \stackrel{\text{def}}{=} \text{Mat}(D, \mathbf{v})$. Then the following two statements are equivalent:

1. The matrix $g[A]$ is super-irreducible for some $g \in \text{Gl}_n(\mathcal{O})$;
2. Λ is a special lattice for D .

PROOF. Cf. [Lev91, section 6]. It also follows from proposition 3.8.18 since the last statement is equivalent with

$$\rho_{j+1}(D) = - \sum_{i=1}^n \min \{0, j + \varepsilon_i(A)\} \quad \text{for all } j \in \mathbb{N}.$$

$\square$

Corollary 4.3.8 Let (V, D) be a differential operator and let $\mathbf{v} = (v_1, \dots, v_n)$ be a K -basis of V such that $A \stackrel{\text{def}}{=} \text{Mat}(D, \mathbf{v})$ is a rational normal form of D . Then $\mathcal{O}v_1 + \dots + \mathcal{O}v_n$ is a special lattice for D .

4.4 Comparison with the canonical form of Varadarajan

In this section we will show the connection between a rational normal form of a differential operator and the canonical form of a differential operator over an algebraic closure of K as defined by Babbitt and Varadarajan in [BV83, section 6.3].

Definition 4.4.1 (Babbitt and Varadarajan) Let $\bar{K}$ be an algebraic closure of K and let $\bar{k}$ be the field of constants of $\bar{K}$. ($\bar{k}$ is an algebraic closure of the field k .) Then one calls a matrix A in $\text{Mat}(\bar{K})$ of the form

$$A = D_1 x^{r_1} + \cdots + D_m x^{r_m} + C$$

canonical if

- $r_1, \dots, r_m$ are rational numbers satisfying $r_1 < \cdots < r_m < 0$;
- $D_1, \dots, D_m, C$ are matrices in $\text{Mat}(\bar{k})$ which commute pairwise;
- $D_1, \dots, D_m$ are semisimple and non-zero.

One calls

$$A_{\text{irr}} \stackrel{\text{def}}{=} D_1 x^{r_1} + \cdots + D_m x^{r_m}$$

the *irregular part* of A , while the r_i are called the *canonical levels* of A .

In [Bab] an algorithm is given to compute a canonical form of a given differential operator over an algebraic closure of K . A slightly improved version of this algorithm can be found in section 5.2.

Proposition 4.4.2 Let $n \in \mathbb{N}^*$ and $A \in \text{Mat}_n(K)$ be a rational normal form of a differential operator and let $\bar{K}$ be an algebraic closure of K . Then one can construct a matrix $g \in \text{Gl}_n(\bar{K})$ of the form

$$g = \begin{pmatrix} x^{u_1} & 0 & \cdots & 0 \\ 0 & x^{u_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x^{u_n} \end{pmatrix}$$

with $u_i \in \cup_{1 \leq q \leq n} q^{-1}\mathbb{Z}$ and $-1 < u_i \leq 0$ for all $i \in \{1, \dots, n\}$ such that $g[A]$ is canonical in the sense of definition 4.4.1.

PROOF. Let

$$A = \begin{pmatrix} A_0 & CA_{e-1}/x & \cdots & CA_1/x \\ A_1 & A_0 - \frac{1}{e}I & \ddots & \vdots \\ \vdots & A_1 & & \vdots \\ \vdots & & \ddots & CA_{e-1}/x \\ A_{e-1} & A_{e-2} & A_1 & A_0 - \frac{e-1}{e}I \end{pmatrix}$$

be a rational normal form of a simple differential operator with $e, f \in \mathbb{N}^*$, $C \in \text{Gl}_f(k)$ and $A_0, A_1, \dots, A_{e-1}$ in $\text{Mat}_f(k[1/x])$. By looking at the part of the proof of theorem 4.1.2 on page 73 where the transformation matrices g_1 and g_2 are defined, one easily verifies that $g[A]$ is canonical in the sense of definition 4.4.1, where

$$g = \begin{pmatrix} I & 0 & \dots & 0 \\ 0 & x^{-1/e}I & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & & 0 & x^{(1-e)/e}I \end{pmatrix}.$$

For the rational normal form

$$\begin{pmatrix} A & I & 0 & \dots & 0 \\ 0 & A & I & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & I \\ 0 & & & 0 & A \end{pmatrix}, \quad (4.7)$$

where I denotes the identity matrix, one should take

$$\begin{pmatrix} g & 0 & \dots & 0 \\ 0 & g & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & & 0 & g \end{pmatrix}.$$

Since every rational normal form is a block diagonal matrix with blocks of the same form as in (4.7) the statement follows. $\square$

4.5 Holonomic $\mathcal{D}_1$ -modules

In this section we will give some results on holonomic $\mathcal{D}_1$ -modules.

Definition 4.5.1 $\mathcal{D}_1$ or $\mathcal{O}[\partial]$ will denote the subring of $\mathcal{D}(K)$ generated by ∂ and the elements of $\mathcal{O}$.

Definition 4.5.2 A $\mathcal{D}_1$ -module M of finite type is called *holonomic* if the following equivalent statements are verified (cf. [vdE89, prop. 2.7 and 2.9]):

1. M is torsion, i.e., every $m \in M$ is annihilated by some non-zero element of $\mathcal{D}_1$.
2. M is a $\mathcal{D}_1$ -module of finite length.

3. M_x is a finite dimensional vector space over K .

Proposition 4.5.3 *Let M be a simple $\mathcal{D}_1$ -module.*

- *If M has ∂ -torsion, i.e., $\partial m = 0$ for some non-zero $m \in M$, then M is isomorphic to $\mathcal{O}$.*
- *If M has x -torsion, i.e., $xm = 0$ for some non-zero $m \in M$, then M is isomorphic to $K/\mathcal{O}$.*

PROOF. For completeness we will give a proof of this well-known result. Suppose M has ∂ -torsion and let m be a non-zero element in M with $\partial m = 0$. Define the function $\phi: \mathcal{O} \rightarrow M$ by $a \mapsto am$ for all $a \in \mathcal{O}$. It is not difficult to verify that ϕ is an isomorphism of $\mathcal{D}_1$ -modules. The injectivity follows from the fact that $\mathcal{D}_1\partial$ is a maximal left ideal of $\mathcal{D}_1$.

Now suppose M has x -torsion and let m be a non-zero element in M with $xm = 0$. Define the function $\phi: K \rightarrow M$ by

$$\sum_{n=r}^{\infty} a_n x^n \mapsto \sum_{n=0}^{-r-1} a_{-n-1} \frac{(-\partial)^n m}{n!} \quad (a_r, a_{r+1}, \dots \in k).$$

One easily verifies that ϕ is a surjective homomorphism of $\mathcal{D}_1$ -modules and that $\ker \phi = \mathcal{O}$. So M and $K/\mathcal{O}$ are isomorphic $\mathcal{D}_1$ -modules. $\square$

Proposition 4.5.4 *Let M be a holonomic $\mathcal{D}_1$ -module without ∂ -torsion and without x -torsion. Then M is a holonomic $\mathcal{D}(K)$ -module.*

PROOF. Define $V \stackrel{\text{def}}{=} K \otimes_{\mathcal{O}} M$, then by definition 4.5.2, V is a finite dimensional vector space over K . Since M has no x -torsion, V is a non-zero holonomic $\mathcal{D}(K)$ -module. Define the differential operator D on V by $Dv \stackrel{\text{def}}{=} \tau \cdot v$ for all $v \in V$.

By theorem 4.2.3 there exists a K -basis $v = (v_1, \dots, v_n)$ of V such that $M(D, v)$ is a rational normal form. We will now show that $x^r v \in M^n$ for all $r \in \mathbb{Z}$.

It is clear that this is true for large r . Now suppose that $x^r v \in M^n$ for some given $r \in \mathbb{Z}$. The equation $Dv = vA$ implies $D(x^r v) = x^r v(A + rI)$. From the fact that A is a rational normal form one easily deduces that $A + rI$ is also a rational normal form. Suppose that column i of $A + rI$ is the zero column for some $i \in \{1, \dots, n\}$. Then $x\partial(x^r v_i) = 0$, but this is impossible since we assumed that M has no ∂ -torsion (and no x -torsion). Therefore $A + rI$ has no zero columns and hence by proposition 4.2.5 $A + rI$ is invertible and $(A + rI)^{-1} \in \text{Mat}_n(\mathcal{O})$. So

$$x^{r-1} v = \partial(x^r v)(A + rI)^{-1} \in M^n.$$

This shows that $V = \sum_{i=1}^n K v_i \subseteq M \subseteq V$. Hence $M = V$, so M is a holonomic $\mathcal{D}(K)$ -module. $\square$

Corollary 4.5.5 *Let M be a $\mathcal{D}_1$ -module without ∂ -torsion. Then the following two statements are equivalent:*

- M is a simple $\mathcal{D}_1$ -module without x -torsion.
- M is a simple $\mathcal{D}(K)$ -module.

PROOF. Suppose M is a simple $\mathcal{D}_1$ -module without ∂ -torsion and without x -torsion. Since M is a simple $\mathcal{D}_1$ -module, it has finite length and hence is holonomic. Therefore M is a holonomic $\mathcal{D}(K)$ -module by proposition 4.5.4. It is clear that M is a simple $\mathcal{D}(K)$ -module.

Now suppose that M is a simple $\mathcal{D}(K)$ -module without ∂ -torsion. Let S be a simple $\mathcal{D}_1$ -submodule of M . Then S is a simple $\mathcal{D}(K)$ -module by the first part of the proof. Hence $M = S$ and therefore M is a simple $\mathcal{D}_1$ -module without x -torsion. $\square$

4.6 The trace of a differential operator

In this section we will define a group homomorphism from the Grothendieck group $K_0(\mathcal{V})$ to $k[1/x]/\mathbb{Z}$ and show the connection with the trace of the matrix of the differential operator w.r.t. a K -basis.

Let (V, D) be a differential operator and let $h \in \mathcal{F} \cap R$ be an element representing the characteristic class of D . Write

$$h(\lambda) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0 \quad (c_0, \dots, c_{n-1} \in k[1/x])$$

and define the *trace* of D , notation $\text{Tr}(D)$, by

$$\text{Tr}(D) \stackrel{\text{def}}{=} -c_{n-1} + \mathbb{Z} \in k[1/x]/\mathbb{Z}.$$

To show that the right hand side doesn't depend on the element representing the characteristic class of D , let $f_1 \dots f_m$ be the prime factor decomposition of h in $K[\lambda]$. If $g \in \mathcal{F} \cap R$ is another element with $c(D) = [g]$ then

$$g(\lambda) = f_1(\lambda + \frac{r_1}{\epsilon(f_1)}) \dots f_m(\lambda + \frac{r_m}{\epsilon(f_m)}) \quad \text{for some } r_1, \dots, r_m \in \mathbb{Z}.$$

Since $\epsilon(f_i) \mid \deg(f_i)$ for all $i \in \{1, \dots, m\}$, one has

$$g(\lambda) = f_1(\lambda + \frac{s_1}{\deg(f_1)}) \dots f_m(\lambda + \frac{s_m}{\deg(f_m)}) \quad \text{for some } s_1, \dots, s_m \in \mathbb{Z}.$$

Now it is clear that the coefficient of λ^{n-1} in $h - g$ is an integer.

Since there is a group homomorphism (even a ring isomorphism) from $K_0(\mathcal{V})$ to $\mathcal{C}$, one can define the trace on $K_0(\mathcal{V})$. This gives a (surjective) group homomorphism $\text{Tr}: K_0(\mathcal{V}) \rightarrow k[1/x]/\mathbb{Z}$.

Proposition 4.6.1 *Let (V, D) be a differential operator and let $e = (e_1, \dots, e_n)$ be a K -basis of V . Let s be the trace of the matrix $M(D, e)$. Then $\text{Tr}(D) = p(s)$ or $\text{Tr}(D) = p(s) + \frac{1}{2}$ where $p: K \rightarrow k[1/x]/\mathbb{Z}$ is the function defined by*

$$p: a_r x^r + \dots + a_{-1} x^{-1} + a_0 + a_1 x + \dots \mapsto a_r x^r + \dots + a_{-1} x^{-1} + a_0 + \mathbb{Z}$$

for all $a_r, a_{r+1}, \dots \in k$.

PROOF. We will first show that $p(s)$ doesn't depend on the K -basis e of V . Let $f = (f_1, \dots, f_n)$ be another K -basis of V and let $g \in \text{Gl}_n(K)$ be the transformation matrix from e to f . Then

$$gM(D, f)g^{-1} - \tau(g)g^{-1} = M(D, e).$$

Since every nonsingular matrix with entries in K can be written as product of elementary matrices and diagonal matrices, it is enough to show that the trace of the matrix $\tau(g_0)g_0^{-1}$ is an element of $\mathcal{O} = \mathbb{Z} + x\mathcal{O}$ for all elementary and diagonal matrices g_0 . This is easily verified.

To prove the proposition we may therefore assume that $M(D, e)$ is a rational normal form of the differential operator D . So $M(D, e)$ is of the form

$$\begin{pmatrix} S_1 & * & 0 & \dots & 0 \\ 0 & S_2 & * & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & * \\ 0 & & & 0 & S_m \end{pmatrix}.$$

Here every S_i is a square matrix such that $\tau + S_i$ is a simple differential operator. The trace of the matrix $M(D, e)$ is the sum of the traces of the matrices S_i for $i \in \{1, \dots, m\}$. Since Tr is additive on $K_0(\mathcal{V})$ one also has

$$\text{Tr}(D) = \text{Tr}(\tau + S_1) + \dots + \text{Tr}(\tau + S_m).$$

Therefore it is sufficient to prove the proposition for a simple differential operator.

So assume that (V, D) is a simple differential operator and let $e = (e_1, \dots, e_n)$ be a K -basis of V such that $M(D, e)$ is a rational normal form as given in theorem 4.1.2. Let L be a finite field extension of K and let $a \in L$ be a normalized eigenvalue of D_L . Define $e \stackrel{\text{def}}{=} e(D)$ and $f \stackrel{\text{def}}{=} f(D)$. Let t be a non-zero element in $K(a)$ with $\tau(t) = t/e$ and let k' be the field of constants of $K(a)$. Write

$$a = \sum_{i=0}^{e-1} t^{-i} a_i \quad (a_0, \dots, a_{e-1} \in k'[1/x]).$$

The proof of the first part of theorem 4.1.2 implies that for A_0 in a rational normal form one can take the $f \times f$ matrix of the multiplication by a_0 w.r.t. a K -basis of the vector space $k'K$. In particular $\text{Tr}_{k'K/K}(a_0) = \text{Tr}(A_0)$. By looking at the rational normal form in theorem 4.1.2 one finds

$$\text{Tr}(M(D, e)) = e \text{Tr}(A_0) + \frac{1}{2}(e-1)f.$$

On the other hand, let h be the minimal polynomial of a over K . Write

$$h(\lambda) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0 \quad (c_0, \dots, c_{n-1} \in k[1/x]).$$

Then $c(D) = [h]$ and

$$-c_{n-1} = \text{Tr}_{K(a)/K}(a) = e \text{Tr}_{k'K/K}(a_0) = e \text{Tr}(A_0).$$

This proves the proposition. □

Chapter 5

Some algorithms

In this chapter we will give some remarks on algorithms for computing the characteristic class and a rational normal form of a differential operator given by its matrix w.r.t. some basis.

We start with a brief overview of some known algorithms to compute various invariants of differential operators and of an algorithm to compute a rational normal form of a linear transformation. In section 5.2 we will give a slightly improved version of the algorithm by Babbitt and Varadarajan which leads to a canonical form. In some cases this will also yield a rational normal form of the differential operator. In section 5.3 we will give a description of a complete algorithm to compute the characteristic class and a rational normal form of a differential operator over a two and three dimensional vector space. In section 5.4 we will describe some difficulties which arise when trying to make a completely rational algorithm for differential operators over vector spaces of higher dimension. We conclude this chapter with some items for future research.

5.1 Some existing algorithms

5.1.1 Super-irreducible form of Hilali

In his Ph.D. thesis [Hil87c] Hilali describes an efficient algorithm to compute a super-irreducible form of a given differential operator which is based on proposition 4.3.1. This algorithm doesn't require computations in field extensions of the constant field. A description of the algorithm can be found in for instance [HW87].

From a super-irreducible form of a differential operator one can immediately obtain some invariants like the minimal pole order and the ρ -invariants of Gérard and Levelt (see proposition 3.8.18). With some more work one can obtain the Newton-Puiseux polygon of the differential operator (cf. [Hil87b])

and hence the invariant of Katz (see corollary 3.8.9). However not much more information can be obtained from the super-irreducible form. In particular, if A is an arbitrary matrix of order r , then the matrix $x^{r-1}I + A$ is always super-irreducible, while one would like to have some connection between a normal form of A and a normal form of $x^{r-1}I + A$ as is the case for the rational normal form of section 4.2 and the canonical form of Babbitt and Varadarajan described in section 4.4.

An algorithm to compute a super-irreducible form of a differential operator has been implemented by the author of this thesis in the computer algebra system Maple.

5.1.2 Canonical form of Babbitt and Varadarajan

In [Bab] one can find a detailed algorithm to compute the canonical form of Babbitt and Varadarajan of a differential operator as given in section 4.4. However this algorithm is not very practical since it requires computations in finite field extensions of the constant field. It also requires often unnecessary ramifications which means that information about the isomorphism class of the differential operator gets lost. Still this algorithm comes the closest to actually computing the characteristic class and a rational normal form of a differential operator and therefore we will study it in some more detail in section 5.2.

5.1.3 Stabilizing differential operators of Levelt

In [Lev91] the following theorem is proved.

Theorem 5.1.1 (Levelt) *Let (V, D) be a differential operator and let Λ be a lattice in V . Put $n \stackrel{\text{def}}{=} \dim V$ and let $j \in \mathbb{N}$. Then the lattice $F_j^{n-1}(\Lambda)$ stabilizes $x^j D$ (see definition 4.3.6), where*

$$F_j^i(\Lambda) \stackrel{\text{def}}{=} \Lambda + (x^j D)\Lambda + \cdots + (x^j D)^i \Lambda \quad \text{for all } i \in \mathbb{N}.$$

This theorem is the basis of an algorithm to compute the minimal pole order and the ρ -invariants of Gérard and Levelt of a differential operator. The following lemma from [Lev91] allows one to compute a special lattice for a differential operator.

Lemma 5.1.2 *Let (V, D) be a differential operator and let Λ be a lattice in V . Let $i, j \in \mathbb{N}$ with $i \geq j$. If the lattice Λ stabilizes $x^i D$, then $\Lambda + (x^j D)\Lambda$ has the same property.*

Together with proposition 4.3.7 this gives another, in general less efficient, way to compute a super-irreducible form of a differential operator.

The algorithms to compute the minimal pole order, the ρ -invariants of Gérard and Levelt and a special lattice have been implemented by Levelt in the computer algebra system Maple.

5.1.4 Dieudonné determinant algorithm of Adjamagbo

As shown in chapter 2 the Dieudonné determinant determines the semisimple part of a differential operator and thus, by the results in section 3.7, invariants like the Newton-Puiseux polygon and polynomials. An algorithm to compute an element of $\mathcal{D}^0(K)$ representing the Dieudonné determinant of a differential operator is described in [Adj85] and has been implemented by Adjamagbo in the computer algebra system Macsyma and by the author of this thesis in the computer algebra system Maple. This algorithm is based on the fact that $\mathcal{D}(K)$ is a constructive (left) Euclidean ring, i.e., given elements a and b in $\mathcal{D}(K)$ with $b \neq 0$, one can compute elements q and r in $\mathcal{D}(K)$ such that $a = qb + r$ with either $r = 0$ or $\deg(r) < \deg(b)$, where $\deg$ denotes the degree in τ . This enables one to perform Gauss elimination on the matrix.

5.1.5 Rational normal form algorithm for linear transformations

Let $f: V \rightarrow V$ be a linear transformation of a finite dimensional vector space over some field k of characteristic zero. A square matrix A with entries in k is called a *rational (Jordan) normal form* of f if it is the matrix of f w.r.t. some k -basis of V and

$$A = \begin{pmatrix} B_1 & 0 & & 0 \\ 0 & B_2 & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & & 0 & B_s \end{pmatrix}$$

with

$$B_i = \begin{pmatrix} P_i & * & 0 & \dots & 0 \\ 0 & P_i & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & P_i & * \\ 0 & & & 0 & P_i \end{pmatrix} \quad (i = 1, \dots, s).$$

All P_i are square matrices such that $\det(\lambda I - P_i)$ is an irreducible polynomial in $k[\lambda]$ and every asterisk stands for either the identity matrix or zero.

There exists an efficient algorithm to compute the rational Jordan normal form of a square matrix over a field of characteristic zero. This algorithm is based on the algorithm described in [Oze87]. There exists a good implementation of the rational Jordan normal form algorithm by Mulders and Levelt in the computer algebra language Maple.

5.2 Algorithms for some special cases

In this section we will mainly describe a slightly improved version of the algorithm by Babbitt and Varadarajan for computing a canonical form of a differential operator as given in section 4.4. Where possible we will mention the connection with the characteristic class and the rational normal form.

The algorithm we will describe leads to a matrix of the form

$$D_1 x^{r_1} + \cdots + D_m x^{r_m} + C,$$

where $r_1, \dots, r_m \in \mathbb{Q}$ with $r_1 < \cdots < r_m < 0$ and $D_1, D_2, \dots, D_m, C$ are square matrices with entries in k satisfying:

- $D_1, D_2, \dots, D_m, C$ commute pairwise;
- $D_1, D_2, \dots, D_m$ are semisimple and non-zero.

Notice that the original algorithm of Babbitt and Varadarajan as given in [Bab] leads to a form where the entries of the matrices $D_1, D_2, \dots, D_m$ and C are in general not in the constant field k but in some finite field extension of k .

From the form given above one can determine the characteristic class $c(D)$ up to similarity over some field extension of K .

However if the so-called nilpotent case described in section 5.2.7 doesn't occur during execution of the algorithm then one can obtain the characteristic class and a rational normal form of the differential operator. Some of the difficulties which arise when trying to make a complete rational algorithm to compute the characteristic class and a rational normal form in all cases are described in section 5.4.

5.2.1 An overview

Throughout section 5.2 we will assume that A is an $n \times n$ matrix with entries in K and

$$A = A_r x^r + A_{r+1} x^{r+1} + A_{r+2} x^{r+2} + \cdots$$

with $r \stackrel{\text{def}}{=} \text{ord}(A)$ and all A_i in $\text{Mat}_n(k)$. Let D be the differential operator $\tau + A$ on K^n .

By first applying a constant transformation matrix one may assume that the matrix A_r is in the rational Jordan normal form described in section 5.1.5.

We will now distinguish six different cases:

1. If $r = 0$ then the differential operator D has a *regular singularity* in the origin. This type of differential operator has been studied at length in the literature, cf. [Was65, Ch. II and Ch. V, sec. 17].

In this case one can give an algorithm to compute a rational normal form of the differential operator D and also the transformation matrix up to a given order.

The first step consists of constructing a transformation matrix $g \in \text{Gl}_n(\mathcal{O})$ such that

$$g[A] = A'_0 + A'_1x + A'_2x^2 + \cdots$$

with all A'_i in $\text{Mat}_n(k)$ and

$$\gcd(f(\lambda), f(\lambda - 1)) = 1$$

where $f(\lambda) \stackrel{\text{def}}{=} \det(\lambda I - A'_0)$.

The second step consists of constructing (up to a certain order) a transformation matrix $\tilde{g} \in \text{Gl}_n(\mathcal{O})$ such that

$$(\tilde{g}g)[A] = A'_0.$$

It will follow from the construction of this rational normal form that

$$c(D)(\lambda) = [\det(\lambda I - A'_0)] = [\det(\lambda I - A_0)].$$

A detailed algorithm can be found in section 5.2.3.

2. If $r < 0$ and the matrix A_r is of the form

$$\begin{pmatrix} P_r & 0 \\ 0 & Q_r \end{pmatrix},$$

where P_r and Q_r are square matrices whose characteristic polynomials are relatively prime, then one can use the so-called *splitting lemma* to construct (up to a certain order) a transformation matrix $g \in \text{Gl}_n(\mathcal{O})$ such that

$$g[A] = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix},$$

where P and Q are square matrices with entries in K such that $P - P_r \in \text{Mat}(x^{r+1}\mathcal{O})$ and $Q - Q_r \in \text{Mat}(x^{r+1}\mathcal{O})$. One can now recursively apply the algorithm to the differential operators $\tau + P$ and $\tau + Q$.

The characteristic class of the differential operator D is given by

$$c(D) = c(\tau + P)c(\tau + Q).$$

A detailed algorithm can be found in section 5.2.4.

3. If $r < 0$ and $\det(\lambda I - A_r)$ is squarefree then the first step consists of constructing (up to a certain order) a transformation matrix $g \in \text{Gl}_n(\mathcal{O})$ such that

$$g[A] = A'_rx^r + A'_{r+1}x^{r+1} + A'_{r+2}x^{r+2} + \cdots$$

with $A'_i \in \text{Mat}_n(k)$, $A'_r = A_r$ and $A'_r A'_i = A'_i A'_r$ for all $i \geq r$. The construction of this g is based on the following fact. There exists an algorithm that

for a given $B \in \text{Mat}_n(k)$ constructs $b_0, b_1, \dots, b_{n-1} \in k$ and $Y \in \text{Mat}_n(k)$ such that

$$B = b_0 I + b_1 A_r + \dots + b_{n-1} A_r^{n-1} + A_r Y - Y A_r.$$

One can show that the characteristic class $c(D)$ satisfies

$$c(D)(\lambda) = \left[\det \left(\lambda I - \sum_{i \leq 0} A'_i x^i \right) \right].$$

The second step consists of constructing (up to a certain order) a transformation matrix $\tilde{g} \in \text{Gl}_n(\mathcal{O})$ such that

$$(\tilde{g}g)[A] = A'_r x^r + A'_{r+1} x^{r+1} + \dots + A'_1 x^{-1} + A'_0.$$

If $\det(\lambda I - A_r)$ is irreducible then the differential operator D is simple and a rational Jordan normal form of D is given by

$$A'_r x^r + A'_{r+1} x^{r+1} + \dots + A'_0.$$

Detailed algorithms can be found in section 5.2.5.

4. If $r < 0$ and $\det(\lambda I - A_r) = (\lambda - a)^n$ for some $a \in k^*$ then, by recursively applying the algorithm to the differential operator $D - ax^r I$, a transformation matrix $g \in \text{Gl}_n(K)$ is constructed such that $g[A - ax^r I]$ has the required form. In this case $g[A]$ is a canonical form (resp. rational normal form) of the differential operator D .

The characteristic class of the differential operator D is given by

$$c(D)(\lambda) = c(D - ax^r I)(\lambda - a).$$

5. If $r < 0$ and $\det(\lambda I - A_r) = f(\lambda)^s$ where $f(\lambda)$ is an irreducible polynomial of degree $m > 1$ in $k[\lambda]$ and $s > 1$ then (the rational normal form of) the matrix A_r is of the form

$$\begin{pmatrix} P & * & 0 & \dots & 0 \\ 0 & P & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & P & * \\ 0 & & & 0 & P \end{pmatrix},$$

where P is a square matrix such that $\det(\lambda I - P) = f(\lambda)$ and every asterisk stands for either the identity matrix or zero. The first step consists of constructing (up to a certain order) a transformation matrix $g \in \text{Gl}_n(\mathcal{O})$ such that

$$g[A] = A_r x^r + A'_{r+1} x^{r+1} + A'_{r+2} x^{r+2} + \dots$$

and every A'_{r+i} ($i \in \mathbb{N}^*$) commutes with the matrix

$$\begin{pmatrix} P & 0 & \dots & 0 \\ 0 & P & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & & 0 & P \end{pmatrix}$$

or equivalently

$$g[A] = \begin{pmatrix} f_{11}(P) & f_{1s}(P) \\ \vdots & \vdots \\ f_{s1}(P) & f_{ss}(P) \end{pmatrix},$$

where all f_{ij} are univariate polynomials of degree less than m with coefficients in K . This can be done by a generalization of the algorithm needed in case 3 of the algorithm.

Let $k' = k(\alpha)$ be a finite field extension of k where α is an eigenvalue of the matrix P . Define the matrix $A' \in \text{Mat}_s(k'K)$ by

$$A' \stackrel{\text{def}}{=} \begin{pmatrix} f_{11}(\alpha) & f_{1s}(\alpha) \\ \vdots & \vdots \\ f_{s1}(\alpha) & f_{ss}(\alpha) \end{pmatrix}$$

and recursively apply the algorithm on the differential operator $D' \stackrel{\text{def}}{=} \tau + A'$. Replacing every α by P in the transformation matrix and in the resulting matrix gives a canonical form (resp. rational normal form) of the differential operator D .

The characteristic class of the differential operator D is given by

$$c(D)(\lambda) = N_{k'K(\lambda)/K(\lambda)}(c(D')(\lambda)) = \text{Res}_y(f(y), c(D')(\lambda)|_{\alpha=y}),$$

where $N_{k'K(\lambda)/K(\lambda)}$ is the norm function of the field extension $k'K(\lambda)/K(\lambda)$ and Res_y denotes the resultant w.r.t. the variable y .

Detailed algorithms can be found in section 5.2.6.

6. If $r < 0$ and A_r is nilpotent, then there are (at least) two ways to proceed. For both methods ramification might be necessary, i.e., replacing x by t^m for some $m \in \mathbb{N}$ with $m > 1$.

The first method consists of computing a super-irreducible form, which seems by the way always a good thing to do in practice since the algorithm to compute such a form is quite efficient. If for the resulting matrix A one still has $r \stackrel{\text{def}}{=} \text{ord}(A) < 0$ and A_r is nilpotent, then compute the principal level (or equivalently the invariant of Katz) by one of the methods described in [HW86] or [Hil87b]. Now replace x by t^m where $m \in \mathbb{N}^*$ is

the smallest integer such that the principal level is an element of $(1/m)\mathbb{Z}$. Using the algorithm of Hilali over the field $K(t)$ to make the matrix 1-irreducible (see definition 3.8.14) yields a matrix A with $r \stackrel{\text{def}}{=} \text{ord}(A)$ equal to the principal level and A_r not nilpotent.

Another method described in [BV83, § 4] uses the theory of orbits of the adjoint representation of the algebraic group $\text{Gl}_n(k)$. We will describe this method in some more detail.

Let $g\ell_n(k)$ denote the general linear algebra $\text{Mat}_n(k)$ with bracket operation $[x, y] = xy - yx$ for all $x, y \in g\ell_n(k)$ and let $s\ell_n(k)$ denote the special linear algebra: the subalgebra of $g\ell_n(k)$ consisting of the matrices having trace zero.

The first step consists of computing matrices H and X in $s\ell_n(k)$ such that

$$[H, X] = 2X, \quad [H, A_r] = -2A_r, \quad [X, A_r] = H.$$

In other words $\{X, A_r, H\}$ span a Lie subalgebra of $g\ell_n(k)$ which is isomorphic to $s\ell_2(k)$. The triplet (A_r, H, X) is called a *standard triplet*. It follows from the representation theory of $s\ell_2(k)$ that

$$g\ell_n(k) = \text{im ad}(A_r) \oplus \ker \text{ad}(X).$$

(See definition 5.2.1 for the definition of the ad operator.)

The second step consists of constructing a transformation matrix $g \in \text{Gl}_n(\mathcal{O})$ such that

$$g[A] = A_r x^r + A'_{r+1} x^{r+1} + \cdots + A'_{r-rn-1} x^{r-rn-1} + \cdots$$

with $A'_i \in \ker \text{ad}(X)$ for all $i \in \{r+1, \dots, r-rn-1\}$. The next step consists of applying a transformation matrix $\tilde{g}$ of the form $x^{-(\delta/2)H}$ for some $\delta \in \mathbb{Q}$ with $0 < \delta \leq |r|$ such that

$$(\tilde{g}g)[A] = \tilde{A}_{r+\delta} x^{r+\delta} + \cdots$$

with $0 \neq \tilde{A}_{r+\delta} \neq A_r$ or $r + \delta = 0$. Now recursively apply the algorithm to $(\tilde{g}g)[A]$. One can prove that if $r + \delta < 0$ and $\tilde{A}_{r+\delta}$ is nilpotent, then $d(A'_{r+\delta}) > d(A_r)$, where $d(\cdot)$ denotes the dimension of the orbit, i.e., for any M in $g\ell_n(k)$, $d(M)$ is the dimension of

$$\{gMg^{-1} \mid g \in \text{Gl}(k)\}.$$

This guarantees termination in this case.

Detailed algorithms can be found in section 5.2.7

We will now study each of the steps in more detail and prove the claims about the characteristic class which were made in the outline of the steps above.

5.2.2 A few basic algorithms

In this subsection we will give some basic algorithms which will be used further on to describe the steps of the algorithm of section 5.2.1 in more detail.

Definition 5.2.1 Let V be a finite dimensional vector space over a field k and let $f: V \rightarrow V$ be a linear transformation. Define $\text{ad}: \text{End}(V) \rightarrow \text{End}(V)$ by

$$\text{ad}(f)(g) \stackrel{\text{def}}{=} [f, g] = fg - gf \quad \text{for all } g \in \text{End}(V).$$

Lemma 5.2.2 (Babbitt and Varadarajan) Let $n \in \mathbb{N}^*$ and $A \in \text{Mat}_n(K)$. Suppose that $r = \text{ord}(A) < 0$. Write

$$A = A_r x^r + A_{r+1} x^{r+1} + \cdots \quad (A_r, A_{r+1}, \dots \in g\ell_n(k)).$$

Let W be a linear subspace of $g\ell_n(k)$ satisfying $W + \text{im ad}(A_r) = g\ell_n(k)$. Then there exists a matrix $g = I + g_1 x + g_2 x^2 + \cdots$ in $\text{Gl}_n(\mathcal{O})$ such that

$$g[A] = A_r x^r + A'_{r+1} x^{r+1} + \cdots \quad (\text{with } A'_{r+i} \in W \text{ for all } i \in \mathbb{N}).$$

Furthermore for all $i \in \mathbb{N}^*$, the matrices A'_{r+i} and g_i only depend on $A_r, A_{r+1}, \dots, A_{r+i}$.

PROOF. A constructive proof of this result is given in [BV83, section 4.3 and 4.4]. For completeness we will repeat the proof here. Let $i \in \mathbb{N}^*$ and suppose that $g_1, g_2, \dots, g_{i-1}$ and $A'_{r+1}, A'_{r+2}, \dots, A'_{r+i-1}$ have been chosen already. Then the relation determining A'_{r+i} is

$$A'_{r+i} + A_r g_i - g_i A_r = A_{r+i} - (r+i)g_{r+i} + \sum_{j=1}^{i-1} (g_j A_{r+i-j} - A'_{r+i-j} g_j),$$

where $g_{r+i} \stackrel{\text{def}}{=} 0$ if $r+i < 0$. Since $W + \text{im ad}(A_r) = g\ell_n(k)$ one can find $A'_{r+i} \in W$ and $g_i \in g\ell_n(k)$ satisfying this equation. It is obvious that A'_{r+i} and g_i only depend on $A_r, A_{r+1}, \dots, A_{r+i}$. $\square$

An algorithm corresponding to this lemma is:

Algorithm lift.

Input: $m \in \mathbb{N}$, $A \in \text{Mat}_n(K)$ with $r = \text{ord}(A) < 0$ and an algorithm *split* such that if $M \in \text{Mat}_n(k)$ and $(T, U) := \text{split}(A_r, M)$ then $T, U \in \text{Mat}_n(k)$ and $M = A_r T - T A_r + U$.

Output: a matrix $g = I + g_1 x + \cdots + g_m x^m$ and a matrix

$$A' = A_r x^r + A'_{r+1} x^{r+1} + \cdots + A'_{r+m} x^{r+m} \text{ such that}$$

$$g[A] - A' \in \text{Mat}_n(x^{r+m+1}\mathcal{O}) \text{ and for all } i \in \{1, \dots, m\} \text{ one has}$$

$$(g_i, A'_{r+i}) = \text{split}(A_r, B_i) \text{ for some } B_i \in \text{Mat}_n(k).$$

```

lift( $m, A, \text{split}$ ) :=
   $r := \text{ord}(A)$ ;
   $g := I$ ;  $A' := A_r x^r$ ;
  for  $i := 1, 2, \dots, m$  do
     $B := A_{r+i} + \sum_{j=1}^{i-1} (g_j A_{r+i-j} - A'_{r+i-j} g_j)$ ;
    if  $r+i > 0$  then  $B := B - (r+i)g_{r+i}$  fi;
     $(T, U) := \text{split}(A_r, B)$ ;
     $g := g + T x^i$ ;  $A' := A' + U x^{r+i}$ 
  od;
  return( $g, A'$ ).

```

Proposition 5.2.3 *Let V and W be finite dimensional vector spaces over an arbitrary field k and let $f: W \rightarrow W$ and $g: V \rightarrow V$ linear transformations with relatively prime characteristic polynomials. Then the linear transformation from $\text{Hom}(V, W)$ to $\text{Hom}(V, W)$ defined by $h \mapsto fh - hg$ is an isomorphism.*

PROOF. Below is a short proof of this well-known result.

Let p (resp. q) be the characteristic polynomial of f (resp. g). Since p and q are relatively prime there exist polynomials a and b such that $1 = ap + bq$. Hence $b(f)q(f) = 1_W$ by the Cayley-Hamilton theorem. It suffices to prove the injectivity of the function $h \mapsto fh - hg$. So assume $fh - hg = 0$. Then we have $q(f)h = hq(g) = 0$. Since $q(f)$ is invertible, it follows that $h = 0$. $\square$

Let $m, n \in \mathbb{N}^*$, $P \in \text{Mat}_m(k)$, $Q \in \text{Mat}_n(k)$ and $M \in \text{Mat}_{m,n}(k)$ where k is an arbitrary field. If the characteristic polynomials of P and Q are relatively prime then by proposition 5.2.3 there is one and only one matrix $X \in \text{Mat}_{m,n}(k)$ satisfying $M = PX - XQ$. This matrix X can be determined by solving a sparse system of mn linear equations in mn unknowns. The corresponding algorithm will be called *solvemateqn*. So after the assignment

$$X := \text{solvemateqn}(P, Q, M)$$

the matrix X satisfies $M = PX - XQ$.

Let $n \in \mathbb{N}^*$ and $P, M \in \text{Mat}_n(k)$ where k is a perfect field. If the characteristic polynomial of P is squarefree then there exists a matrix $X \in \text{Mat}_n(k)$ and a univariate polynomial f satisfying

$$\deg(f) < n \quad \text{and} \quad M = PX - XP + f(P). \quad (5.1)$$

The polynomial f is completely determined by (5.1). This follows from the following three lemmas which we will state without proof.

Lemma 5.2.4 *Let V be a finite dimensional vector space over a perfect field k and let $f: V \rightarrow V$ be a linear transformation. Then $\text{ad}(f)$ is semisimple if and only if f is semisimple.*

Lemma 5.2.5 *Let V be a finite dimensional vector space over an arbitrary field k and let $f: V \rightarrow V$ be a semisimple linear transformation. Then*

$$V = \ker f \oplus \operatorname{im} f.$$

Lemma 5.2.6 *Let V be a finite dimensional vector space over an arbitrary field k and let $f: V \rightarrow V$ be a linear transformation with a squarefree characteristic polynomial. Then $(1, f, \dots, f^{n-1})$ is a k -basis of $\ker \operatorname{ad}(f)$, where $n \stackrel{\text{def}}{=} \dim V$.*

A polynomial f and a matrix X satisfying equation (5.1) can be determined by solving a sparse system of n^2 linear equations in $n(n+1)$ unknowns. The corresponding algorithm will be called *solvemateqn2*. So after the assignment

$$(X, f) := \text{solvemateqn2}(P, M)$$

the matrix X and the univariate polynomial f satisfy equation (5.1). If the characteristic polynomial of P is *irreducible*, then one may assume that the matrix X is of the form

$$\begin{pmatrix} x_{11} & x_{1n} \\ \vdots & \vdots \\ x_{n-1,1} & x_{n-1,n} \\ 0 & 0 \end{pmatrix}. \quad (5.2)$$

The matrix X is now completely determined by (5.1) and (5.2). To prove this, it is enough to show that the last rows of the matrices $I, P, \dots, P^{n-1}$ are independent over k . Suppose that there exists a univariate polynomial h with $\deg(h) < n$ such that $h(P)$ is not invertible. By the Cayley-Hamilton theorem, this implies that $\det(\lambda I - P)$ and h are not relatively prime. Since $\det(\lambda I - P)$ is irreducible and $\deg(h) < n$, this is only possible when $h = 0$.

5.2.3 Regular singularity

In this section we will assume that

$$A = A_0 + A_1x + A_2x^2 + \dots$$

with $A_0, A_1, \dots$ in $\operatorname{Mat}_n(k)$.

In lemma 5.2.7 we will give a constructive proof of the claim that $c(D)(\lambda) = [\det(\lambda I - A_0)]$ and that there exists a transformation matrix $g \in \operatorname{GL}_n(K)$ such that $g[A] \in \operatorname{Mat}_n(k)$.

Lemma 5.2.7 *Let (V, D) be a differential operator and let $v = (v_1, \dots, v_n)$ be a K -basis of V such that $A \stackrel{\text{def}}{=} M(D, v) \in \operatorname{Mat}_n(\mathcal{O})$. Write*

$$A = A_0 + A_1x + A_2x^2 + \dots$$

with $A_0, A_1, \dots$ in $\operatorname{Mat}_n(k)$. Then the characteristic class of D satisfies $c(D)(\lambda) = [\det(\lambda I - A_0)]$.

PROOF. Put $m \stackrel{\text{def}}{=} \text{maxdiff}(\det(\lambda I - A_0))$, where

$$\text{maxdiff}(f) \stackrel{\text{def}}{=} \max \{ p \in \mathbb{N} \mid \gcd(f(\lambda), f(\lambda - p)) \neq 1 \} \quad \text{for all } f \in k[\lambda] \setminus \{0\}.$$

Write $\det(\lambda I - A_0) = f(\lambda)h(\lambda)$ with f and g monic polynomials satisfying

$$q \mid f \Rightarrow q(\lambda + m) \mid \det(\lambda I - A_0) \quad \text{and} \quad q \mid h \Rightarrow q(\lambda + m) \nmid \det(\lambda I - A_0)$$

for all irreducible polynomials $q \in k[\lambda]$. It is clear that f and h are relatively prime and that $f \neq 1$ and $h \neq 1$. So there exists a matrix $g_1 \in \text{Gl}_n(k)$ such that

$$g_1 A_0 g_1^{-1} = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix},$$

with $\det(\lambda I - P) = f(\lambda)$ and $\det(\lambda I - Q) = h(\lambda)$. Now let

$$g_2 \stackrel{\text{def}}{=} \begin{pmatrix} xI & 0 \\ 0 & I \end{pmatrix}.$$

Then $(g_2 g_1)[A] = A'_0 + A'_1 x + \dots$ with

$$A'_0 = \begin{pmatrix} P - I & 0 \\ * & Q \end{pmatrix}.$$

Now notice that $\det(\lambda I - A'_0) \equiv \det(\lambda I - A_0)$ and

$$\text{maxdiff}(\det(\lambda I - A'_0)) = m - 1.$$

Repeating this process a finite number of times, we find a matrix $g \in \text{Gl}_n(K)$ such that

1. $g[A] = A''_0 + A''_1 x + \dots$;
2. the eigenvalues of A''_0 don't differ by positive integers;
3. $\det(\lambda I - A''_0) \equiv \det(\lambda I - A_0)$.

It is well-known (cf. [Was65, Ch. II and Ch. V, sec. 17]) that there exists a matrix $\tilde{g} \in \text{Gl}_n(\mathcal{O})$ such that $\tilde{g}[g[A]] = (\tilde{g}g)[A] = A''_0$. This shows that

$$c(D)(\lambda) = [\det(\lambda I - A''_0)] = [\det(\lambda I - A_0)]. \quad \square$$

Below are the complete algorithms to compute everything.

Algorithm maxdiff.

Input: a non-zero univariate polynomial f with coefficients in k .

Output: $\max \{ p \in \mathbb{N} \mid \gcd(f(\lambda), f(\lambda - p)) \neq 1 \}$.

```

maxdiff( $f$ ) :=
  { $f_1, \dots, f_r$ } := the set of monic prime factors of  $f$ ;
   $m := 0$ ;
  for each  $i \in \{1, \dots, r\}$  do
     $d := \deg(f_i)$ ;  $c :=$  coefficient of  $\lambda^{d-1}$  in  $f_i(\lambda)$ ;
    for each  $j \in \{i+1, \dots, r\}$  do
      if  $d = \deg(f_j)$  then
         $c' :=$  coefficient of  $\lambda^{d-1}$  in  $f_j(\lambda)$ ;  $\delta := (c' - c)/d$ ;
        if  $\delta \in \mathbb{Z}$  and  $|\delta| > m$  and  $f_i(\lambda + \delta) = f_j(\lambda)$  then
           $m := |\delta|$ 
        fi
      fi
    od
  od;
  return( $m$ ).

```

Algorithm maxdiff.

Input: a non-zero univariate polynomial f with coefficients in k .

Output: $\max \{ p \in \mathbb{N} \mid \gcd(f(\lambda), f(\lambda - p)) \neq 1 \}$.

```

maxdiff( $f$ ) :=
   $R := \text{resultant}_y(f(y - \lambda), f(y))$ ;
   $F :=$  the set of monic prime factors of  $R$ ;
   $m := 0$ ;
  for each  $g \in F$  do
    if  $\deg(g) = 1$  and  $g(0) \in \mathbb{Z}$  and  $g(0) > m$  then  $m := g(0)$  fi
  od;
  return( $m$ ).

```

Algorithm regsingl.

Input: $A \in \text{Mat}_n(\mathcal{O})$.

Output: $g \in \text{Gl}_n(k(x))$ and $A' \in \text{Mat}_n(K)$ with $g[A] = A'$ and $\text{maxdiff}(\det(\lambda I - A'_0)) = 0$.

```

regsingl( $A$ ) :=
   $f := 1$ ;  $h := \det(\lambda I - A_0)$ ;
   $m := \text{maxdiff}(h)$ ;
   $g := I$ ;  $A' := A$ ;

```

```

for  $i := m, m-1, \dots, 1$  do
   $f' := \gcd(h(\lambda), h(\lambda - i)); f' := \gcd(h, f^{\deg(h)/\deg(f')});$ 
   $f := ff'; h := h/f';$ 
   $(v_1, \dots, v_r) :=$  a  $k$ -basis of  $\text{im } h(A'_0)$ ;
   $(v_{r+1}, \dots, v_n) :=$  a  $k$ -basis of  $\text{im } f(A'_0)$ ;
   $g' :=$  the matrix with columns  $v_1, \dots, v_r, xv_{r+1}, \dots, xv_n$ ;
   $g' := (g')^{-1}; g := g'g; A' := g'[A'];$ 
   $h := h(\lambda - 1)$ 
od;
return( $g, A'$ ).

```

Algorithm regsing2.

Input: $m \in \mathbb{N}$ and $A \in \text{Mat}_n(\mathcal{O})$ satisfying $\maxdiff(\det(\lambda I - A_0)) = 0$.

Output: a matrix $g = I + g_1x + \dots + g_mx^m$ such that $g[A] - A_0 \in \text{Mat}_n(x^{m+1}\mathcal{O})$.

$\text{regsing2}(m, A) :=$

```

 $g := I;$ 
for  $i := 1, 2, \dots, m$  do
   $Y := \text{solvemateqn}(A_0 + iI, A_0, A_i + g_1A_{i-1} + \dots + g_{i-1}A_1);$ 
   $g := g + Yx^i$ 
od;
return( $g$ ).

```

5.2.4 Splitting lemma

In this section we will assume that

$$A = A_r x^r + A_{r+1} x^{r+1} + A_{r+2} x^{r+2} + \dots$$

with $A_r, A_{r+1}, \dots$ in $\text{Mat}_n(k)$, $r \stackrel{\text{def}}{=} \text{ord}(A) < 0$ and

$$A_r = \begin{pmatrix} P_r & 0 \\ 0 & Q_r \end{pmatrix}$$

where P_r and Q_r are square matrices with entries in k whose characteristic polynomials are relatively prime. We will construct (up to a certain order) a transformation matrix $g \in \text{GL}_n(\mathcal{O})$ such that

$$g[A] = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix},$$

where P and Q are square matrices with entries in K such that

$$P - P_r \in \text{Mat}(x^{r+1}\mathcal{O}) \quad \text{and} \quad Q - Q_r \in \text{Mat}(x^{r+1}\mathcal{O}).$$

In the literature this construction is known as the *splitting lemma*, cf. [Was65, Ch. II and Ch. V, sec. 17] and [Lev75, § 2].

By algorithm *lift* in section 5.2.2 it is enough to give an algorithm *splitting* such that if $M \in \text{Mat}_n(k)$ and $(T, U) := \text{splitting}(A_r, M)$ then $T, U \in \text{Mat}_n(k)$ with $M = A_r T - T A_r + U$ and

$$U = \begin{pmatrix} W & 0 \\ 0 & Y \end{pmatrix}$$

where W is a square matrix of the same size as P_r , and Y is a square matrix of the same size as Q_r . This is done by the following algorithm:

Algorithm splitting.

Input: $A_r, M \in \text{Mat}_n(k)$ with $A_r = \begin{pmatrix} P_r & 0 \\ 0 & Q_r \end{pmatrix}$ and the characteristic polynomials of P_r and Q_r are relatively prime.

Output: $T, U \in \text{Mat}_n(k)$ satisfying $M = A_r T - T A_r + U$ and $U = \begin{pmatrix} W & 0 \\ 0 & Y \end{pmatrix}$.

splitting(A_r, M) :=

Write $M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$ where M_{11} has the same size as P and M_{22} has the same size as Q ;

$T_1 := \text{solvemateqn}(P_r, Q_r, M_{12})$; $T_2 := \text{solvemateqn}(Q_r, P_r, M_{21})$;

$T := \begin{pmatrix} 0 & T_1 \\ T_2 & 0 \end{pmatrix}$; $U := \begin{pmatrix} M_{11} & 0 \\ 0 & M_{22} \end{pmatrix}$;

return(T, U).

It is clear that the characteristic polynomial $c(D) = c(\tau + P)c(\tau + Q)$.

5.2.5 Squarefree case

In this section we will assume that

$$A = A_r x^r + A_{r+1} x^{r+1} + A_{r+2} x^{r+2} + \dots$$

with $A_r, A_{r+1}, \dots$ in $\text{Mat}_n(k)$, $r \stackrel{\text{def}}{=} \text{ord}(A) < 0$ and $\det(\lambda I - A_r)$ squarefree.

We will first construct a transformation matrix $g \in \text{Gl}_n(\mathcal{O})$ such that for given $m \in \mathbb{N}$

$$g[A] = A'_r x^r + A'_{r+1} x^{r+1} + A'_{r+2} x^{r+2} + \dots$$

with $A'_i \in \text{Mat}_n(k)$, $A'_r = A_r$ and $A_r A'_i = A'_i A_r$ for all $i \in \{r, \dots, m\}$.

By algorithm *lift* in section 5.2.2 it is enough to give an algorithm *solvemateqn3* such that if $M \in \text{Mat}_n(k)$ and $(T, U) := \text{solvemateqn3}(A_r, M)$ then $T, U \in \text{Mat}_n(k)$ with $M = A_r T - T A_r + U$ and $A_r U = U A_r$. This is done by the following algorithm:

Algorithm solvemateqn3.

Input: $A_r, M \in \text{Mat}_n(k)$ with $\det(\lambda I - A_r)$ squarefree.

Output: $T, U \in \text{Mat}_n(k)$ satisfying $M = A_r T - T A_r + U$ and $A_r U = U A_r$.

solvemateqn3(A_r, M) :=

$(T, f) := \text{solvemateqn2}(A_r, M);$

$U := f(A_r);$

return(T, U).

In proposition 5.2.8 we will prove the claim that the characteristic class $c(D)$ satisfies

$$c(D)(\lambda) = \left[\det \left(\lambda I - \sum_{i \leq 0} A'_i x^i \right) \right].$$

First notice that $A'_i A'_j = A'_j A'_i$ for all $i, j \in \{r, \dots, m\}$, since the characteristic polynomial of A_r is squarefree and hence by lemma 5.2.6 every matrix which commutes with A_r is a polynomial in A_r .

Proposition 5.2.8 *Let (V, D) be a differential operator and let $v = (v_1, \dots, v_n)$ be a K -basis of V . Write $A \stackrel{\text{def}}{=} M(D, v)$. Suppose*

1. $A = A_r x^r + A_{r+1} x^{r+1} + \dots$ with $r \stackrel{\text{def}}{=} \text{ord}(A)$ and $A_r, A_{r+1}, \dots$ in $\text{Mat}_n(k)$;
2. $A_r, A_{r+1}, \dots, A_{-1}$ are semisimple matrices;
3. $A_r, A_{r+1}, \dots, A_0$ commute pairwise.

Then

$$c(D)(\lambda) = \left[\det \left(\lambda I - \sum_{i \leq 0} A_i x^i \right) \right].$$

PROOF. We will prove the proposition by induction on $n + |r|$. If $r = 0$ then the proposition follows from lemma 5.2.7, so assume that $r < 0$. Let f be the characteristic polynomial of the matrix A_r .

First suppose that f has only one eigenvalue $\alpha \in k$. Since A_r is semisimple there exists a matrix $g \in \text{Gl}_n(k)$ such that $g A_r g^{-1} = \alpha I$. In this case it is not hard to see that

$$c(D)(\lambda) = c(\tau + A)(\lambda) = c(\tau + g[A])(\lambda) = (c(\tau + g[A] - \alpha x^r I))(\lambda - \alpha x^r).$$

By induction

$$c(\tau + g[A] - \alpha x^r I)(\lambda) = \left[\det \left(\lambda I - \sum_{i=r+1}^0 (gA_i g^{-1}) x^i \right) \right] = \left[\det \left(\lambda I - \sum_{i=r+1}^0 A_i x^i \right) \right],$$

$$\text{so } c(D)(\lambda) = \left[\det \left(\lambda I - \sum_{i=r}^0 A_i x^i \right) \right].$$

Now suppose that f has at least two different eigenvalues. Let k' be a finite field extension of k and $g \in \text{Gl}_n(k')$ such that

$$gA_r g^{-1} = \begin{pmatrix} P_r & 0 \\ 0 & Q_r \end{pmatrix},$$

where P_r and Q_r are square matrices with entries in k' whose characteristic polynomials are relatively prime. Since $A_{r+1}, \dots, A_0$ commute with A_r , we must have in virtue of proposition 5.2.3

$$gA_i g^{-1} = \begin{pmatrix} P_i & 0 \\ 0 & Q_i \end{pmatrix} \quad \text{for all } i \in \{r+1, \dots, 0\},$$

where P_i has the same dimensions as P_r and Q_i has the same dimensions as Q_r for all $i \in \{r+1, \dots, 0\}$. By the splitting lemma described in section 5.2.4 there exists a matrix $g' \in \text{Gl}_n(k'K)$ of the form

$$g' = I + g'_{1-r} x^{1-r} + g'_{2-r} x^{2-r} + \dots \quad \text{with } g'_{1-r}, g'_{2-r}, \dots \in \text{Mat}_n(k')$$

such that

$$(g'g)[A] = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}$$

with $P = \sum_{i=r}^{\infty} P_i x^i$, $Q = \sum_{i=r}^{\infty} Q_i x^i$ and all P_i and Q_i are square matrices with entries in k' ($P_r, \dots, P_0, Q_r, \dots, Q_0$ are the same as before). Now use proposition 3.5.4 and the multiplicativity property of the characteristic class to see that $c(D) = c(\tau + P)c(\tau + Q)$. By induction

$$c(\tau + P)(\lambda) = \left[\det \left(\lambda I - \sum_{i \leq 0} P_i x^i \right) \right] \quad \text{and} \quad c(\tau + Q)(\lambda) = \left[\det \left(\lambda I - \sum_{i \leq 0} Q_i x^i \right) \right],$$

so

$$\begin{aligned} c(D)(\lambda) &= \left[\det \left(\lambda I - \sum_{i \leq 0} \begin{pmatrix} P_i & 0 \\ 0 & Q_i \end{pmatrix} x^i \right) \right] \\ &= \left[\det \left(\lambda I - \sum_{i \leq 0} (gA_i g^{-1}) x^i \right) \right] \\ &= \left[\det \left(\lambda I - \sum_{i \leq 0} A_i x^i \right) \right]. \end{aligned}$$

□

We will now show how to construct a transformation matrix $\tilde{g} \in \text{Gl}_n(\mathcal{O})$ such that

$$(\tilde{g}g)[A] = A'_r x^r + A'_{r+1} x^{r+1} + \cdots + A'_0 + A''_{m+1} x^{m+1} + \cdots$$

This is done by the following algorithm:

Algorithm chopoff.

Input: $m \in \mathbb{N}$ and $A \in \text{Mat}_n(K)$ with $A = A_r x^r + A_{r+1} x^{r+1} + \cdots$ and $A_i A_j = A_j A_i$ for all $i, j \in \{r, \dots, m\}$.

Output: $g = I + g_1 x + \cdots + g_m x^m \in \text{Gl}_n(\mathcal{O})$ such that $g[A] - p(A) \in \text{Mat}_n(x^{m+1}\mathcal{O})$, where $p(A) \stackrel{\text{def}}{=} A_r x^r + \cdots + A_0$.

chopoff(m, A) :=

$g := I;$

for $i := 1, 2, \dots, m$ **do**

$$g := g + \left(\frac{A_1 g_{i-1} + \cdots + A_{i-1} g_1 + A_i}{i} \right) x^i$$

od;

return(g).

Remains to prove the claim that if $\det(\lambda I - A_r)$ is irreducible then the differential operator D is simple and a rational normal form of D is given by

$$A' \stackrel{\text{def}}{=} A'_r x^r + A'_{r+1} x^{r+1} + \cdots + A'_0.$$

By the above algorithms it is clear that $A' = M(D, v)$ for some basis v of K^n . Assume that $\det(\lambda I - A_r)$ is irreducible. To prove that D is a simple differential operator and that A' is a rational normal form of D , one has to verify the three statements in theorem 4.1.2 for the matrix A' . Take $e = 1$ in theorem 4.1.2. We already noticed that all A'_i are polynomials in A_r and hence all A'_i are semisimple and commute pairwise. Remains to show that $\det(\lambda I - A')$ is irreducible. Notice that

$$\det(\lambda I - x^{-r} A')|_{x=0} = \det(\lambda I - A_r).$$

This shows that $\det(\lambda I - A')$ is irreducible.

If $\det(\lambda I - A_r)$ is reducible, then the matrix A' is a rational normal form of D if and only if the matrix A_r is in the rational Jordan normal form described in section 5.1.5.

On the other hand, if $\det(\lambda I - A_r)$ is reducible (and squarefree), then one could also skip the squarefree case algorithm altogether and start with the splitting lemma algorithm.

5.2.6 Field extension case

In this subsection we will assume that

$$A = A_r x^r + A_{r+1} x^{r+1} + A_{r+2} x^{r+2} + \dots$$

with $A_r, A_{r+1}, \dots$ in $\text{Mat}_n(k)$, $r \stackrel{\text{def}}{=} \text{ord}(A) < 0$ and

$$A_r = \begin{pmatrix} P & B_1 & 0 & & 0 \\ 0 & P & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & P & B_{s-1} \\ 0 & & & 0 & P \end{pmatrix}$$

with P and all B_i in $\text{Mat}_m(k)$, the characteristic polynomial of P is irreducible and every B_i is either the identity matrix or zero. We will construct (up to a certain order) a transformation matrix $g \in \text{Gl}_n(\mathcal{O})$ such that

$$g[A] = A_r x^r + A'_{r+1} x^{r+1} + A'_{r+2} x^{r+2} + \dots$$

and for every $i \in \mathbb{N}^*$ the matrix A'_{r+i} commutes with the matrix

$$\begin{pmatrix} P & 0 & \dots & 0 \\ 0 & P & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & P \end{pmatrix}. \quad (5.3)$$

By algorithm *lift* in section 5.2.2 it is enough to give an algorithm *sizereduction* such that if $M \in \text{Mat}_n(k)$ and $(T, U) := \text{sizereduction}(A_r, M)$ then $T, U \in \text{Mat}_n(k)$ with $M = A_r T - T A_r + U$ and U commutes with (5.3). A matrix which commutes with (5.3) is of the form

$$\begin{pmatrix} f_{11}(P) & f_{1s}(P) \\ \vdots & \vdots \\ f_{s1}(P) & f_{ss}(P) \end{pmatrix},$$

where all f_{ij} are univariate polynomials of degree less than m with coefficients in k . These polynomials are needed in the rest of the algorithm so I will give a slight adaptation of the algorithm *sizereduction*:

Algorithm sizereduction.

Input: $A_r, M \in \text{Mat}_n(k)$ with A_r of the form described above.

Output: $T \in \text{Mat}_n(k)$ and $U \in \text{Mat}_s(k[\lambda])$ such that $M = A_r T - T A_r + U|_{\lambda=P}$.

sizereduction(A_r, M) :=

Write $M = \begin{pmatrix} M_{11} & \dots & M_{1s} \\ \vdots & \ddots & \vdots \\ M_{s1} & \dots & M_{ss} \end{pmatrix}$ with all $M_{ij} \in \text{Mat}_m(k)$;

for $i := s, s-1, \dots, 1$ **do**

for $j := 1, 2, \dots, s$ **do**

$N := M_{ij}$;

if $i < s$ **then** $N := N - B_i T_{i+1,j}$ **fi**;

if $j > 1$ **then** $N := N + T_{i,j-1} B_{j-1}$ **fi**;

$(T_{ij}, f_{ij}) := \text{solvemateqn2}(P, N)$

od

od;

$T := \begin{pmatrix} T_{11} & \dots & T_{1s} \\ \vdots & \ddots & \vdots \\ T_{s1} & \dots & T_{ss} \end{pmatrix}; U := \begin{pmatrix} f_{11} & f_{1s} \\ \vdots & \vdots \\ f_{s1} & f_{ss} \end{pmatrix};$

return(T, U).

Let $k' = k(\alpha)$ be a finite field extension of k where α is an eigenvalue of the matrix P . Let $w = (w_1, \dots, w_m)$ a basis of k' over k such that P is the matrix of multiplication by α w.r.t. the basis w . Let $\psi: (k'K)^s \rightarrow K^n$ be the group isomorphism defined by

$$\left(\sum_{i=1}^m v_{1i} w_i, \dots, \sum_{i=1}^m v_{si} w_i \right) \mapsto (v_{11}, \dots, v_{1m}, \dots, v_{s1}, \dots, v_{sm}),$$

where all v_{ji} are elements in K . Notice that

$$\psi(\alpha v) = \begin{pmatrix} P & 0 & \dots & 0 \\ 0 & P & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & P & \end{pmatrix} \psi(v) \quad \text{for all } v \in (k'K)^s. \quad (5.4)$$

Define the differential operator D' from $(k'K)^s$ to $(k'K)^s$ by $D' \stackrel{\text{def}}{=} \psi^{-1} g D g^{-1} \psi = \tau + \psi^{-1} g[A] \psi$.

Replace every polynomial in P in $g[A]$ by the same polynomial in α and call the resulting $s \times s$ matrix A' . Using equation (5.4) one easily verifies that A' is the matrix of D' w.r.t. the natural basis of $(k'K)^s$.

Let A'' be a rational normal form of the differential operator D' and let B be the $n \times n$ matrix derived from A'' by replacing every α by P . We have to show that B is a rational normal form of D . By looking at a general rational normal

form as given in theorem 4.2.3 we can restrict ourselves to the case where A'' is a rational normal form of a simple differential operator. So assume that

$$A'' = \begin{pmatrix} A_0 & CA_{e-1}/x & & CA_1/x \\ A_1 & A_0 - \frac{1}{e}I & & \vdots \\ \vdots & A_1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & CA_{e-1}/x \\ A_{e-1} & A_{e-2} & & A_1 & A_0 - \frac{e-1}{e}I \end{pmatrix}$$

with $e, f \in \mathbb{N}^*$, $C \in \text{Gl}_f(k')$ and $A_0, A_1, \dots, A_{e-1}$ in $\text{Mat}_f(k'[1/x])$ as in theorem 4.1.2. Write

$$A_i = \sum_{j \leq 0} A_i^{(j)} x^j \quad \text{for all } i \in \{0, \dots, e-1\}.$$

where all $A_i^{(j)}$ are constant $f \times f$ matrices. Let $B_i^{(j)}$ be the $(fm) \times (fm)$ matrix derived from $A_i^{(j)}$ by replacing every α by P and let C' be the $(fm) \times (fm)$ matrix derived from C in the same way. Let $v \in (k')^m$ be a non-zero eigenvector of P with eigenvalue α and let $\{\sigma_1, \dots, \sigma_m\}$ be the set of K -homomorphisms from $k'K$ to L , where L is a normal closure of the field extension $k'K/K$. Let T_0 be the $m \times m$ matrix with columns $\sigma_1(v), \sigma_2(v), \dots, \sigma_m(v)$ and let T_1 be the $(fm) \times (fm)$ block diagonal matrix with f blocks T_0 on the diagonal. Then

$$Q_1^{-1} T_1^{-1} B_i^{(j)} T_1 Q_1 = \begin{pmatrix} \sigma_1(A_i^{(j)}) & 0 & & 0 \\ 0 & \sigma_2(A_i^{(j)}) & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & & 0 & \sigma_m(A_i^{(j)}) \end{pmatrix},$$

where Q_1 is the $(fm) \times (fm)$ permutation matrix corresponding with the permutation on $\{0, \dots, fm-1\}$ satisfying

$$im + j \mapsto jf + i \quad \text{for all } i \in \{0, \dots, f-1\} \text{ and } j \in \{0, \dots, m-1\}.$$

This shows that C' and all $B_i^{(j)}$ are semisimple and all commute pairwise.

Remains to show that the characteristic polynomial of the $n \times n$ matrix

$$\tilde{B} \stackrel{\text{def}}{=} B + \begin{pmatrix} 0 & 0 & & 0 \\ 0 & \frac{1}{e}I & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & & 0 & \frac{e-1}{e}I \end{pmatrix}$$

is irreducible. This follows from the identity

$$Q_2^{-1}T_2^{-1}\tilde{B}T_2Q_2 = \begin{pmatrix} \sigma_1(\tilde{A}) & 0 & 0 \\ 0 & \sigma_2(\tilde{A}) & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \sigma_m(\tilde{A}) \end{pmatrix}, \quad (5.5)$$

where T_2 is the $n \times n$ block diagonal matrix with e blocks T_1Q_1 , Q_2 is an $n \times n$ permutation matrix and

$$\tilde{A} \stackrel{\text{def}}{=} A'' + \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & \frac{1}{e}I & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \frac{e-1}{e}I \end{pmatrix}.$$

Equation (5.5) also shows that

$$c(D)(\lambda) = \prod_{i=1}^m \sigma_i(c(D')(\lambda)).$$

Here

$$\sigma_i(a_p\lambda^p + \dots + a_1\lambda + a_0) \stackrel{\text{def}}{=} \sigma_i(a_p)\lambda^p + \dots + \sigma_i(a_1)\lambda + \sigma_i(a_0) \quad (a_0, \dots, a_p \in k'K).$$

Using

$$f(y) = \prod_{i=1}^m (y - \sigma_i(\alpha))$$

one derives

$$c(D)(\lambda) = N_{k'K(\lambda)/K(\lambda)}(c(D')(\lambda)) = \text{Res}_y(f(y), c(D')(\lambda)|_{\alpha=y}),$$

where $N_{k'K(\lambda)/K(\lambda)}$ is the norm function of the field extension $k'K(\lambda)/K(\lambda)$ and Res_y denotes the resultant w.r.t. the variable y .

5.2.7 Nilpotent case

In this section we will assume that

$$A = A_r x^r + A_{r+1} x^{r+1} + A_{r+2} x^{r+2} + \dots$$

with $A_r, A_{r+1}, \dots$ in $\text{Mat}_n(k)$, $r \stackrel{\text{def}}{=} \text{ord}(A) < 0$ and A_r nilpotent.

After applying a linear transformation from $\text{Gl}_n(k)$, one may assume that

$$A_r = \begin{pmatrix} Y_1 & 0 & 0 \\ 0 & Y_2 & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & & 0 & Y_s \end{pmatrix}$$

with $s \in \mathbb{N}^*$,

$$Y_i = \begin{pmatrix} 0 & 0 & & 0 \\ b_1^{(i)} & 0 & & \vdots \\ 0 & b_2^{(i)} & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & & 0 & b_{n_i-1}^{(i)} \end{pmatrix} \quad \text{for all } i \in \{1, \dots, s\},$$

and $b_j^{(i)} = j(n_i - j)$ for all $i \in \{1, \dots, s\}$ and $j \in \{1, \dots, n_i\}$. One easily verifies that (A_r, H, X) is a standard triplet, where

$$X \stackrel{\text{def}}{=} \begin{pmatrix} X_1 & 0 & \dots & 0 \\ 0 & X_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & & 0 & X_s \end{pmatrix}, \quad H \stackrel{\text{def}}{=} \begin{pmatrix} H_1 & 0 & \dots & 0 \\ 0 & H_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & & 0 & H_s \end{pmatrix}$$

and for all $i \in \{1, \dots, s\}$:

$$X_i \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & 0 & 1 \\ 0 & & & 0 & 0 \end{pmatrix}, \quad H_i = \begin{pmatrix} n_i - 1 & 0 & \dots & 0 \\ 0 & n_i - 3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & & 0 & 1 - n_i \end{pmatrix}.$$

It follows from the representation theory of $\mathfrak{sl}_2(k)$ that

$$\mathfrak{gl}_n(k) = \text{im ad}(A_r) \oplus \ker \text{ad}(X).$$

We will now construct a transformation matrix $g \in \text{Gl}_n(\mathcal{O})$ such that

$$g[A] = A_r x^r + A'_{r+1} x^{r+1} + \dots + A'_{r-rN-1} x^{r-rN-1} + \dots$$

with $N \stackrel{\text{def}}{=} \max\{n_1, \dots, n_s\}$ and $A'_i \in \ker \text{ad}(X)$ for all $i \in \{r+1, \dots, r-rN-1\}$. By algorithm *lift* in section 5.2.2 it is enough to give an algorithm that determines for a given matrix $M \in \text{Mat}_n(k)$, matrices T and U in $\text{Mat}_n(k)$ such

that $M = A_r T - T A_r + U$ and $XU = UX$. This can be done by solving a sparse set of $2n^2$ linear equations in $2n^2$ unknowns.

The next step consists of applying a transformation matrix $\tilde{g}$ of the form $x^{-(\delta/2)H}$ for some $\delta \in \mathbb{Q}$ with $0 < \delta \leq |r|$. Notice that $x^{-(\delta/2)H}[A_r] = A_r x^\delta + (\delta/2)H$. Now choose δ such that

$$(\tilde{g}g)[A] = \tilde{A}_{r+\delta} x^{r+\delta} + \dots$$

with $0 \neq \tilde{A}_{r+\delta} \neq A_r$ or $r+\delta = 0$. If a_{ijk} denotes the entry in row j and column k of A'_{r+i} and h_j denotes the entry in row j and column j of H , then it is not difficult to verify that $h_k > h_j$ implies $a_{ijk} = 0$ and

$$\delta = \min \{-r\} \cup \left\{ \frac{2i}{h_j - h_k + 2} \mid i \in \{1, \dots, -rN - 1\}, j, k \in \{1, \dots, n\}, a_{ijk} \neq 0 \right\}.$$

It is possible to combine the construction of the transformation matrix g with the computation of δ . This can save some steps in the *lift* algorithm.

The following lemma guarantees that the nilpotent case occurs only a finite number of times in succession. This implies termination of the algorithm in this case.

Lemma 5.2.9 *Let G be a reductive Lie algebra and (Y, H, X) a standard triplet. Let Z be a non-zero element of the centralizer of X in G . If $Y + Z$ is nilpotent, then $d(Y + Z) > d(Y)$, where $d(\cdot)$ denotes the dimension of the orbit.*

PROOF. Cf. [BV83, section 2.2]. □

Remark 5.2.10 Let $Y \in \text{Mat}_n(k)$ be a nilpotent matrix of the form

$$\begin{pmatrix} Y_1 & 0 & \dots & 0 \\ 0 & Y_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & & 0 & Y_s \end{pmatrix},$$

where Y_i is a nilpotent matrix of dimension n_i and rank $n_i - 1$. Assume $n_1 \geq n_2 \geq \dots \geq n_s$. Then

$$d(Y) = n^2 - \dim \ker \text{ad}(Y) = n^2 - (n_1 + 3n_2 + \dots + (2s - 1)n_s).$$

5.3 Algorithm for dimension two and three

In this section we will give a description of a complete rational algorithm to compute the characteristic class and a rational normal form of differential operators over two and three dimensional vector spaces.

We only have to give a complete rational algorithm for the so-called nilpotent case, as the other cases are already completely covered in section 5.2. So we will assume that the matrix A is of the form

$$A_r x^r + A_{r+1} x^{r+1} + \dots$$

with $r < 0$, A_r nilpotent and non-zero. Furthermore we may assume that A is super-irreducible.

The idea is to first apply a transformation of the form

$$h = \begin{pmatrix} x^{\varepsilon_1} & 0 & \dots & 0 \\ 0 & x^{\varepsilon_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & & 0 & x^{\varepsilon_n} \end{pmatrix}$$

such that $h[A]$ can be brought in a rational normal form over $K(x^{1/m})$ by a transformation matrix g with coefficients in $K(x^{1/m})$, where m will be 2 or 3 in this case. It will turn out that the algorithm given in section 5.2 to bring $h[A]$ in a rational normal form will not use the nilpotent case in this case, so further ramification will not take place. The last step is to apply the transformation h^{-1} to $(gh)[A]$. We have to prove that g can be constructed in such a way that $h^{-1}gh$ is a matrix with coefficients in K and $(h^{-1}gh)[A]$ is a rational normal form.

For a differential operator over a two dimensional vector space we will give a detailed algorithm to construct a rational normal form without first introducing ramification, but the algorithm is still based on the idea given above.

5.3.1 Algorithm for dimension two

In this section we will give a description of an algorithm for a differential operator over a two dimensional vector space. By the remarks above we may assume that $r \stackrel{\text{def}}{=} \text{ord}(A) < 0$, A_r is nilpotent and A is super-irreducible. After applying a constant transformation matrix to A one may assume that

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x^r + \begin{pmatrix} a_{r+1} & b_{r+1} \\ c_{r+1} & d_{r+1} \end{pmatrix} x^{r+1} + \begin{pmatrix} a_{r+2} & b_{r+2} \\ c_{r+2} & d_{r+2} \end{pmatrix} x^{r+2} + \dots$$

The assumption that A is super-irreducible is by proposition 4.3.1 equivalent with $c_{r+1} \neq 0$. Put $c \stackrel{\text{def}}{=} c_{r+1}$.

For given $m \in \mathbb{N}$ we will first construct a transformation matrix

$$g = I + g_1 x + g_2 x^2 + \dots + g_m x^m$$

with

$$g_i = \begin{pmatrix} s_i & 0 \\ t_i & 0 \end{pmatrix} \quad \text{for all } i \in \{1, \dots, m\}$$

such that $g[A] = A'$ and

$$A' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x^r + \begin{pmatrix} a'_{r+1} & b'_{r+1} \\ c'_{r+1} & d'_{r+1} \end{pmatrix} x^{r+1} + \begin{pmatrix} a'_{r+2} & b'_{r+2} \\ c'_{r+2} & d'_{r+2} \end{pmatrix} x^{r+2} + \dots$$

with $c'_{r+1} = c$, $b'_{r+m} = 0$, $a'_i = d'_i + \delta_{0i}/2$ for all $i \in \{r+1, \dots, r+m\}$ and $c'_{i+1} = cb'_i$ for all $i \in \{r+1, \dots, r+m-1\}$. Here $\delta_{..}$ denotes the Kronecker delta.

In step i of the algorithm the coefficients $s_i, t_i, a'_{r+i}, b'_{r+i}, d'_{r+i}$ and c'_{r+i+1} will be computed. Comparing the coefficient of x^{r+i} in the equation $A'g = gA - \tau(g)$ gives

$$\begin{aligned} A'_{r+i} + \begin{pmatrix} t_i & -s_i \\ 0 & -t_i \end{pmatrix} + \begin{pmatrix} * & * \\ cs_{i-1} + (d'_{r+1} - a_{r+1})t_{i-1} & * \end{pmatrix} = \\ = A_{r+i} - (r+i)g_{r+i} + \sum_{j=1}^{i-2} (g_j A_{r+i-j} - A'_{r+i-j} g_j), \end{aligned}$$

where $g_{r+i} \stackrel{\text{def}}{=} 0$ if $r+i < 0$. This leads to the following algorithm which should be compared with the algorithm *lift* in section 5.2.2.

Algorithm nilpotent2.

Input: $m \in \mathbb{N}$, $A \in \text{Mat}_2(K)$ with $r = \text{ord}(A) < 0$, A super-irreducible and

$$A_r = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Output: a matrix $g = I + g_1x + \dots + g_mx^m$ and a matrix

$A' = A_r x^r + A'_{r+1} x^{r+1} + \dots + A'_{r+m} x^{r+m}$ such that

$g[A] - A' \in \text{Mat}_n(x^{r+m+1}\mathcal{O})$. Put $A'_i := \begin{pmatrix} a'_i & b'_i \\ c'_i & d'_i \end{pmatrix}$ for all

$i \in \{r+1, \dots, r+m\}$. Then the following equalities will be satisfied:

$b'_{r+m} = 0$, $a'_i = d'_i + \delta_{0i}/2$ for all $i \in \{r+1, \dots, r+m\}$ and $c'_{i+1} = cb'_i$ for all $i \in \{r+1, \dots, r+m-1\}$. Here $c \stackrel{\text{def}}{=} c'_{r+1}$ and $\delta_{..}$ denotes the Kronecker delta.

nilpotent2(m, A) :=

$r := \text{ord}(A)$;

Write $A_{r+1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$;

$w := (a - d)/2$;

if $r+1 = 0$ **then** $w := w + \frac{1}{4}$ **fi**;

$g := I$; $A' := A_r x^r$;

for $i := 1, 2, \dots, m$ **do**

if $i = 1$ **then**

$U := A_{r+1}$

else

$$U := V + g_{i-1}A_{r+1} - A'_{r+1}g_{i-1}$$

fi;

$$\text{Write } U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix};$$

$$t := \frac{1}{2}(u_{11} - u_{22});$$

if $r + i = 0$ **then** $t := t - \frac{1}{4}$ **fi**;

if $i = m$ **then**

$$s := -u_{12}$$

else

$$V := A_{r+i+1} + \sum_{j=1}^{i-1} (g_j A_{r+i+1-j} - A'_{r+i+1-j} g_j);$$

if $r + i + 1 > 0$ **then**

$$\text{if } r + 1 < 0 \text{ then } h := g_{r+i+1} \text{ else } h := \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix} \text{ fi};$$

$$V := V - (r + i + 1)h$$

fi;

$$\text{Write } V = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix};$$

$$s := (v_{21} + tw - cu_{12})/(2c);$$

$$\text{if } r + 1 = 0 \text{ and } i > 0 \text{ then } V := V - \begin{pmatrix} is & 0 \\ 0 & 0 \end{pmatrix} \text{ fi}$$

fi;

$$g_i := \begin{pmatrix} s & 0 \\ t & 0 \end{pmatrix};$$

$$A'_{r+i} := U + \begin{pmatrix} -t & s \\ 0 & t \end{pmatrix};$$

$$g := g + g_i x^i; \quad A' := A' + A'_{r+i} x^{r+i}$$

od;

return(g, A').

If $(g, A') := \text{nilpotent2}(|r|, A)$ then A' is a rational normal form of the differential operator $\tau + A$. To prove this, we will give another algorithm that computes a transformation matrix $\tilde{g}$ such that $\tilde{g}[A] - A' \in \text{Mat}_n(x^{m+1}\mathcal{O})$ for any given $m \in \mathbb{N}$.

The idea is to compute $(g, A'') := \text{nilpotent2}(|r| + m + 1, A)$, then to apply the transformation matrix

$$h \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ 0 & x^{-1/2} \end{pmatrix}$$

to A'' , then to apply the transformation matrix g' obtained by algorithm *chopoff* in section 5.2.5 and finally applying h^{-1} . The special form of A'' implies that

$$g' = I + \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} x^{1/2} + \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} x + \cdots + \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} x^{m+1/2} + \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} x^{m+1}.$$

This guarantees that $h^{-1}g'h \in \text{Gl}_2(\mathcal{O})$.

5.3.2 Algorithm for dimension three

In this section we will give a description of an algorithm for a differential operator over a two dimensional vector space. By the remarks above we may assume that $r \stackrel{\text{def}}{=} \text{ord}(A) < 0$, A_r is nilpotent and A is super-irreducible. We will now distinguish three cases.

First assume that the rank of the matrix A_r is two. After applying a constant transformation matrix to A one may assume that

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} x^r + \begin{pmatrix} a_{r+1} & b_{r+1} & c_{r+1} \\ d_{r+1} & e_{r+1} & f_{r+1} \\ g_{r+1} & h_{r+1} & i_{r+1} \end{pmatrix} x^{r+1} + \cdots$$

The assumption that A is super-irreducible is by proposition 4.3.1 equivalent with $g_{r+1} \neq 0$. Put $c \stackrel{\text{def}}{=} g_{r+1}$. Let h be the matrix

$$h \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 & 0 \\ 0 & x^{-1/3} & 0 \\ 0 & 0 & x^{-2/3} \end{pmatrix}.$$

Then

$$\begin{aligned} h[A] &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ c & 0 & 0 \end{pmatrix} x^{r+1/3} + \begin{pmatrix} 0 & 0 & 0 \\ d_{r+1} & 0 & 0 \\ 0 & h_{r+1} & 0 \end{pmatrix} x^{r+2/3} + \\ &+ \begin{pmatrix} a_{r+1} & 0 & 0 \\ 0 & e_{r+1} & 0 \\ 0 & 0 & i_{r+1} \end{pmatrix} x^{r+1} + \begin{pmatrix} 0 & b_{r+1} & 0 \\ 0 & 0 & f_{r+1} \\ g_{r+2} & 0 & 0 \end{pmatrix} x^{r+4/3} + \cdots \end{aligned}$$

Since the characteristic polynomial of the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ c & 0 & 0 \end{pmatrix}$$

is squarefree, we can apply the algorithm described in section 5.2.5. To see what form the transformation matrix can have, we will call a matrix of the

form

$$\begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & * \\ * & 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 & * \\ * & 0 & 0 \\ 0 & * & 0 \end{pmatrix}$$

a matrix of type 0, 1 or 2 respectively. One immediately verifies that the product of a matrix of type i with a matrix of type j gives a matrix of type $i + j \bmod 3$. Using this fact together with

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ c & 0 & 0 \end{pmatrix} \begin{pmatrix} t & u & 0 \\ v & w & 0 \\ y & z & 0 \end{pmatrix} - \begin{pmatrix} t & u & 0 \\ v & w & 0 \\ y & z & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ c & 0 & 0 \end{pmatrix} = \begin{pmatrix} v & w - t & -u \\ y & z - v & -w \\ ct & cu - y & -z \end{pmatrix}$$

one can verify that one can take a transformation matrix of the form

$$g = I + \begin{pmatrix} 0 & 0 & 0 \\ v_0 & 0 & 0 \\ 0 & z_0 & 0 \end{pmatrix} x^{1/3} + \begin{pmatrix} 0 & u_1 & 0 \\ 0 & 0 & 0 \\ y_0 & 0 & 0 \end{pmatrix} x^{2/3} + \begin{pmatrix} t_1 & 0 & 0 \\ 0 & w_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} x + \dots$$

such that

$$\begin{aligned} (gh)[A] &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ c & 0 & 0 \end{pmatrix} x^{r+1/3} + \mu_{r+2/3} \begin{pmatrix} 0 & 0 & 1 \\ c & 0 & 0 \\ 0 & c & 0 \end{pmatrix} x^{r+2/3} \\ &+ \mu_{r+1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} x^{r+1} + \mu_{r+4/3} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ c & 0 & 0 \end{pmatrix} x^{r+4/3} + \dots + \mu_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

One now readily verifies that $h^{-1}gh \in \text{Gl}_3(K)$ and that $(h^{-1}gh)[A]$ is a rational normal form of a simple differential operator with ramification index three.

Now assume that the rank of the matrix A_r is one. After applying a constant transformation matrix to A one may assume that

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x^r + \begin{pmatrix} a_{r+1} & b_{r+1} & c_{r+1} \\ d_{r+1} & e_{r+1} & f_{r+1} \\ g_{r+1} & h_{r+1} & i_{r+1} \end{pmatrix} x^{r+1} + \dots$$

The assumption that A is super-irreducible is by proposition 4.3.1 equivalent with

$$\begin{vmatrix} 0 & 0 & 1 \\ d_{r+1} & \lambda + e_{r+1} & 0 \\ g_{r+1} & h_{r+1} & 0 \end{vmatrix} \neq 0.$$

First assume that $g_{r+1} = 0$ and $d_{r+1}h_{r+1} \neq 0$. After applying a constant transformation one may assume that

$$A = \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x^r + \begin{pmatrix} a_{r+1} & b_{r+1} & c_{r+1} \\ 1 & e_{r+1} & f_{r+1} \\ 0 & 1 & i_{r+1} \end{pmatrix} x^{r+1} + \dots$$

with $c \neq 0$. The rest of the algorithm is completely analogous to the first case where $\text{rank}(A_r) = 2$ and also leads to a rational normal form of a simple differential operator with ramification index three.

The last case to consider is when $g_{r+1} \neq 0$. After applying a constant transformation one may assume that

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x^r + \begin{pmatrix} a_{r+1} & b_{r+1} & c_{r+1} \\ d_{r+1} & e_{r+1} & f_{r+1} \\ g_{r+1} & h_{r+1} & i_{r+1} \end{pmatrix} x^{r+1} + \dots$$

with $d_{r+1} \neq 0$ and $f_{r+1} = g_{r+1} = 0$. The idea is now to first apply the transformation

$$h \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 & 0 \\ 0 & x^{-1/2} & 0 \\ 0 & 0 & x^{-1/2} \end{pmatrix},$$

then to use the splitting lemma algorithm of section 5.2.4 to construct a transformation matrix g and finally to apply h^{-1} . The transformation matrix g should be constructed in such a way that $h^{-1}gh$ is a matrix with coefficients in K . In this particular case it seems easier to describe the construction of the transformation matrix $h^{-1}gh$ directly.

For given $m \in \mathbb{N}$ we will first construct a transformation matrix

$$g = I + g_1x + g_2x^2 + \dots + g_mx^m$$

with

$$g_i = \begin{pmatrix} 0 & 0 & t_i \\ 0 & 0 & u_i \\ v_i & w_i & 0 \end{pmatrix} \quad \text{for all } i \in \{1, \dots, m\}$$

such that $g[A] = A'$ and

$$\begin{aligned} A' &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x^r + \begin{pmatrix} a_{r+1} & b_{r+1} & 0 \\ d_{r+1} & e_{r+1} & 0 \\ 0 & 0 & i_{r+1} \end{pmatrix} x^{r+1} \\ &+ \begin{pmatrix} a'_{r+2} & b'_{r+2} & 0 \\ d'_{r+2} & e'_{r+2} & 0 \\ 0 & 0 & i'_{r+2} \end{pmatrix} x^{r+2} + \dots + \begin{pmatrix} a'_{r+m} & b'_{r+m} & 0 \\ d'_{r+m} & e'_{r+m} & 0 \\ 0 & 0 & i'_{r+m} \end{pmatrix} x^{r+m} + \dots \end{aligned}$$

That this can be done follows from the following two commutator equations:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & t_i \\ 0 & 0 & u_i \\ v_i & w_i & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & t_i \\ 0 & 0 & u_i \\ v_i & w_i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & u_i \\ 0 & 0 & 0 \\ 0 & -v_i & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{pmatrix} \begin{pmatrix} 0 & 0 & t_i \\ 0 & 0 & u_i \\ v_i & w_i & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & t_i \\ 0 & 0 & u_i \\ v_i & w_i & 0 \end{pmatrix} \begin{pmatrix} a & b & * \\ c & d & 0 \\ 0 & * & e \end{pmatrix} \\ = \begin{pmatrix} * & * & * \\ * & * & (d-e)u_i + ct_i \\ (e-a)v_i - cw_i & * & * \end{pmatrix}.$$

This shows that by choosing t_i , u_i , v_i and w_i one can make the coefficients c_{r+i} , h_{r+i} , f_{r+i+1} and g_{r+i+1} equal to zero. For completeness we will give the detailed algorithm to do this.

Algorithm nilpotent3.

Input: $m \in \mathbb{N}$, $A \in \text{Mat}_3(K)$ with $r = \text{ord}(A) < 0$ and

$$A_r = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A_{r+1} = \begin{pmatrix} a & b & * \\ c & d & 0 \\ 0 & * & e \end{pmatrix}$$

and $c \neq 0$.

Output: a matrix $g = I + g_1x + \cdots + g_mx^m$ and a matrix

$A' = A_rx^r + A'_{r+1}x^{r+1} + A'_{r+2}x^{r+2} + \cdots + A'_{r+m}x^{r+m}$ such that

$g[A] - A' \in \text{Mat}_n(x^{r+m+1}\mathcal{O})$ and A'_{r+i} is of the form $\begin{pmatrix} P_i & 0 \\ 0 & Q_i \end{pmatrix}$ with

$P_i \in \text{Mat}_2(k)$ and $Q_i \in k$ for all $i \in \{r+1, \dots, r+m\}$.

nilpotent3(m, A) :=

$r := \text{ord}(A);$

$g := I; A' := A_rx^r;$

for $i := 1, 2, \dots, m$ **do**

if $i = 1$ **then**

$U := A_{r+1}$

else

$U := V + g_{i-1}A_{r+1} - A'_{r+1}g_{i-1}$

fi;

Write $U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix};$

$V := A_{r+i+1} + \sum_{j=1}^{i-1} (g_j A_{r+i+1-j} - A'_{r+i+1-j} g_j);$

if $r+i+1 > 0$ **then**

```

if  $r + 1 < 0$  then  $h := g_{r+i+1}$  else  $h := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & u_{13} \\ -u_{32} & 0 & 0 \end{pmatrix}$  fi;

 $V := V - (r + i + 1)h$ 
fi;

Write  $V = \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix}$ ;

 $g_i := \begin{pmatrix} 0 & 0 & ((e - d)u_{13} + v_{23})/c \\ 0 & 0 & u_{13} \\ -u_{32} & ((a - e)u_{32} - v_{31})/c & 0 \end{pmatrix}$ ;

if  $r + 1 = 0$  and  $i > 0$  then  $V := V + i(h - g_i)$  fi;

 $A'_{r+i} := \begin{pmatrix} u_{11} & u_{12} & 0 \\ u_{21} & u_{22} & 0 \\ 0 & 0 & u_{33} \end{pmatrix}$ ;

 $g := g + g_i x^i$ ;  $A' := A' + A'_{r+i} x^{r+i}$ 
od;

return( $g, A'$ ).

```

The last step is to apply to algorithm of section 5.3.1 to the 2×2 matrix in the upper left hand corner of A' . This leads to a rational normal form with two blocks which also shows that in this case the differential operator is a direct sum of a simple differential operator over a vector space of dimension two which has ramification index two and a differential operator over a vector space of dimension one.

5.4 Various remarks on a complete algorithm

The algorithm described in section 5.2 doesn't lead to a rational normal form if the nilpotent case occurs during the execution of the algorithm. Furthermore the canonical form which it produces doesn't contain enough information to compute the characteristic class over the field K .

It is possible to use always the minimal ramification in the nilpotent case by first computing the principal level using either the algorithm of Hilali described in [HW86] or [Hil87b], or by applying the algorithm of Babbitt and Varadarajan described in section 5.2.7 and going back. The denominator of the principal level gives the minimal ramification m necessary to make the matrix of the lowest order non-nilpotent. After replacing x by t^m one can then apply the algorithm of Moser and Hilali to make the matrix 1-irreducible (see definition 3.8.14).

Even if one always uses the minimal ramification in the nilpotent case, information about the isomorphism class might still get lost in this way. For

suppose that the invariant of Katz is $-\frac{3}{2}$, but that the differential operator is semisimple with simple parts having ramification 1 and 2. Then information about the isomorphism class of the differential operator can be lost by introducing $\sqrt{x}$ before splitting the matrix in its simple parts. This information is of course somehow hidden in the transformation matrix but it is not clear how to use that to transform the resulting canonical form of Babbitt and Varadarajan into a rational normal form.

It seems therefore wise to postpone ramification as long as possible so that we can split the matrix into blocks of different ramification before introducing ramification.

Let $A, B \in \text{Mat}(K)$ with $r \stackrel{\text{def}}{=} \text{ord}(A) = \text{ord}(B) < 0$ be two rational normal forms of the same differential operator and write

$$\begin{aligned} A &= A_r x^r + A_{r+1} x^{r+1} + \dots \\ B &= B_r x^r + B_{r+1} x^{r+1} + \dots \end{aligned}$$

with all A_i and B_i constant matrices. It follows from theorem 4.1.2 and theorem 4.2.3 that the matrices A_r and B_r are conjugate. It seems therefore a good idea to first try to transform a given matrix A in $\text{Mat}(K)$ to another matrix A' in $\text{Mat}(K)$ such that either its order r is zero or A'_r is conjugate with the lowest order matrix in a rational normal form.

A super-irreducible form does not fulfill this requirement. A counterexample is given by the following matrix:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} x^{-1} + \frac{1}{4} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 4 & -3 & 0 & 1 \\ 0 & 4 & -6 & -4 \\ 4 & 1 & 0 & -3 \end{pmatrix}.$$

The matrix A is super-irreducible by proposition 4.3.1 since

$$\theta_1(A, \lambda) = \begin{vmatrix} \lambda & 1 & 0 & 0 \\ 1 & 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & \lambda - \frac{3}{4} \end{vmatrix} = 1.$$

Applying the transformation matrix hg to A where

$$g = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad h = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/x & 0 \\ 0 & 0 & 0 & 1/x \end{pmatrix},$$

gives the rational normal form

$$(hg)[A] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} x^{-1} + \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & -\frac{1}{2} & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} \end{pmatrix}.$$

The matrices

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

are not conjugate since only the square of the first matrix is non-zero.

Another more practical shortcoming of the algorithm described in section 5.2 is that the “field extension case” described in section 5.2.6 requires computations in finite field extensions of the constant field. It can become necessary to factorize polynomials over towers of field extensions in later stages of the algorithm. It would be an improvement if we could replace these factorizations by factorizations of polynomials in two variables, namely λ and $1/x$, with coefficients in k .

Another problem is to determine an a priori upper bound for the number of matrices in the Laurent series which are sufficient to compute the characteristic class. The following conjecture is likely to be true.

Conjecture 5.4.1 *Let $n \in \mathbb{N}^*$ and $A \in \text{Mat}_n(K)$. Write*

$$A = A_r x^r + A_{r+1} x^{r+1} + \dots$$

with $r \stackrel{\text{def}}{=} \text{ord}(A)$ and all A_i matrices with coefficients in k . Then the characteristic class of the differential operator $\tau + A$ only depends on the matrices $A_r, A_{r+1}, \dots, A_{r-n}$.

The upper bound given in this conjecture is sharp as can be seen in the following example.

Example 5.4.2 Let $r, n \in \mathbb{Z}$ with $r \leq 0$ and $n > 0$ and define

$$A(c) \stackrel{\text{def}}{=} \begin{pmatrix} 0 & x^r & 0 & 0 \\ 0 & r & x^r & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & x^r \\ cx^{r-\tau n} & 0 & 0 & rn - r \end{pmatrix} \quad \text{for all } c \in k.$$

Let $g \in \text{GL}_n(K)$ be the transformation matrix

$$g \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 & 0 \\ 0 & x^r & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & x^{rn-r} \end{pmatrix}.$$

Then

$$g[A] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \vdots \\ \vdots & & \ddots & 0 \\ 0 & 0 & & \ddots & 1 \\ c & 0 & & 0 & 0 \end{pmatrix}.$$

Hence the characteristic class of the differential operator $\tau + A(c)$ is equal to $[\lambda^n - c]$. Since $\lambda^n \not\equiv \lambda^n - \frac{1}{2}$, this shows that the characteristic class of $\tau + A(c)$ depends on c .

I can't prove conjecture 5.4.1, but it seems reasonable to believe that the conjecture is true. There are several reasons to support this believe. One of this reasons is the following proposition.

Proposition 5.4.3 *Let r , n and A as in conjecture 5.4.1. Then there exists a finite field extension L of K with ramification index dividing $\text{lcm}(1, 2, \dots, n-1)$ such that the characteristic class of the differential operator $\tau + A$ over L only depends on the matrices $A_r, A_{r+1}, \dots, A_{r-rn}$.*

PROOF. The proof of this proposition is based on a result by Varadarajan in [BV83, § 6.6, prop. 1] which shows that the irregular part of the differential operator $\tau + A$ only depends on the matrices $A_r, A_{r+1}, \dots, A_{r-rn-1}$. It is convenient to introduce a new definition, called the *lag* of a transformation matrix, cf. [BV83, § 1].

If L is a finite field extension of K and $g \in \text{Gl}_n(L)$ then the lag of g is defined by

$$\text{lag}(g) \stackrel{\text{def}}{=} \sup \{ \text{ord}(A) - \text{ord}(g[A]) \mid A \in \text{Mat}_n(L) \}.$$

If one writes g in Smith normal form, i.e., $g = PEQ$ with P and Q in $\text{Gl}_n(L)$ and E a diagonal matrix of the form given in (3.29) with $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ in $\mathbb{Q}$ and $\varepsilon_1 \leq \varepsilon_2 \leq \dots \leq \varepsilon_n$, then $\text{lag}(g) = \varepsilon_n - \varepsilon_1$. Furthermore, $\text{lag}(g_1 g_2) \leq \text{lag}(g_1) + \text{lag}(g_2)$ for all g_1 and g_2 in $\text{Gl}_n(L)$.

Studying the algorithm described in section 5.2 shows that there exists a finite field extension L of K and a transformation matrix g in $\text{Gl}_n(L)$ with $\text{lag}(g) \leq r - rn$ such that

$$g[A] - (A_{r_m} x^{r_m} + \dots + A_{r_1} x^{r_1} + A_{r_0}) \in \text{Mat}_n(\mathfrak{m}_L),$$

where $r_0, \dots, r_m \in \mathbb{Q}$ with $r_m < \dots < r_1 < r_0 = 0$ and all A_{r_i} are square matrices with entries in k satisfying

- $A_{r_0}, A_{r_1}, \dots, A_{r_m}$ commute pairwise;
- $A_{r_1}, A_{r_2}, \dots, A_{r_m}$ are semisimple.

Proposition 5.2.8 now implies that

$$c(D_L)(\lambda) = \left[\det \left(\lambda I - \sum_{i=0}^m A_{r_i} x^{r_i} \right) \right].$$

A bound for the lag of the transformation matrix needed in the nilpotent case can either be deduced from the algorithm by Babbitt and Varadarajan described in section 5.2.7 or from the algorithm by Moser and Hilali to make a matrix 1-irreducible, as by studying their algorithm one can show that in each step a transformation matrix of lag one is used either to reduce the order of the matrix or to lower the rank of the matrix of the lowest order. This implies that at most $r - rn$ steps are possible, hence the lag of the final transformation matrix is less or equal to $r - rn$.

The statement about the ramification of the field extension follows by always using the minimal ramification necessary in the nilpotent case. $\square$

Careful examination of the algorithms in section 5.3 shows that conjecture 5.4.1 is true for differential operators over a vector space of dimension less than four.

5.5 Items for future research

To conclude this thesis we would like to give some of the main streams along which lines future research could be developed.

- Development of a completely rational algorithm to compute characteristic classes and rational normal forms and full implementation of these algorithms in some computer algebra system.
- To find more properties and applications of the characteristic ring (cf. the results of Almkvist). For example to investigate whether the characteristic ring has nilpotent elements.
- To prove conjecture 5.4.1 about the number of matrices in the Laurent series which are sufficient to compute the characteristic class.
- Definition of “analytic” characteristic classes (if possible) as opposed to the “formal” characteristic classes in this thesis.
- Extension of the research to families of differential operators depending on parameters.

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Glossary of notations

$\mathbb{N}^*$	The set of natural numbers with zero excluded	7
$\text{Mat}_{m,n}(A)$	The additive group of $m \times n$ matrices with coefficients in A .	7
$\text{Mat}_n(A)$	The ring of $n \times n$ matrices with coefficients in A .	7
$\text{Mat}(A)$	The union of all $\text{Mat}_n(A)$.	7
$\text{Gl}_n(A)$	The multiplicative subgroup of $\text{Mat}_n(A)$ consisting of matrices which have a left and a right inverse.	7
$\text{Gl}(A)$	The union of all $\text{Gl}_n(A)$.	7
G^{ab}	The commutative group G/G' , where G' is the commutator subgroup of G .	7
$K(M)$	The Grothendieck group of the commutative monoid M . The monoid homomorphism $M \rightarrow K(M)$ has the universal property with respect to homomorphisms of M into commutative groups.	7
K	A differential field of characteristic zero with a non-trivial derivation τ . From page 19 onwards it will be the field of formal Laurent series over k , i.e., $K = k((x))$.	7, 19

τ	7, 19
The non-trivial derivation on the differential field K . From page 19 onwards it is defined as the derivation $x \frac{d}{dx}$ on $k((x))$.	
k	7
The field of constants of the differential field K .	
$\mathcal{V}$	7
The Abelian category of differential operators or equivalent the category of holonomic $\mathcal{D}(K)$ -modules.	
$\mathcal{D}(K)$	8
The subring of $\text{End}_k(K)$ generated by K and τ , also denoted by $K[\tau]$.	
$\mathcal{L}(K)$	8
The left (skew) quotient field of the Ore domain $\mathcal{D}(K)$.	
$\mathcal{D}^0(K)$	8
The multiplicative subset of $\mathcal{D}(K)$ containing the elements which are monic in τ .	
$\mathcal{L}^0(K)$	8
The multiplicative subgroup of $\mathcal{L}(K)$ of elements of the form $P^{-1}Q$ with $P, Q \in \mathcal{D}^0(K)$.	
D^*	9
The dual differential operator of D defined on V^* by $D^*(f)(v) \stackrel{\text{def}}{=} \tau(fv) - f(Dv)$ for all $f \in V^*$ and $v \in V$.	
$M(D, v)$	9
The matrix of the differential operator D w.r.t. the K -basis v .	
$g[A]$	9
An abbreviation for $gAg^{-1} - \tau(g)g^{-1}$.	
$K_0(C)$	11
The Grothendieck group of the Abelian category C .	
$E_n(A)$	12
The subgroup of $\text{Gl}_n(A)$ generated by the elementary matrices.	
$\det(D)$	12
The Dieudonné determinant of the differential operator D .	

$P^\otimes$	15
The adjoint operator of $P \in \mathcal{D}(K)$.	
P^*	15
$P^* \stackrel{\text{def}}{=} (-1)^{\deg P} P^\otimes$.	
V_L	17
An abbreviation for $L \otimes_K V$.	
D_L	17
The differential operator on V_L defined by $D_L(a \otimes v) \stackrel{\text{def}}{=} \tau(a) \otimes v + a \otimes Dv$ for all $a \in K$ and $v \in V$.	
$\mathcal{O}_L$	18
The discrete valuation ring $\{a \in L \mid \text{val}_L(a) \geq 0\}$.	
$\mathfrak{m}_L$	18
The maximal ideal of the discrete valuation ring $\mathcal{O}_L$.	
$\text{val}_L(a)$	18
The discrete valuation of the element a in the local field L .	
$\mathcal{O}$	18
An abbreviation for $\mathcal{O}_K$.	
$\mathfrak{m}$	18
An abbreviation for $\mathfrak{m}_K$.	
$\text{val}(a)$	18
An abbreviation for $\text{val}_K(a)$.	
$e_{L/K}$	19
The ramification index of the field extension L/K , i.e., $\mathcal{O}_L \mathfrak{m}_K = \mathfrak{m}_L^e$ where $e \stackrel{\text{def}}{=} e_{L/K}$.	
$f_{L/K}$	19
The inertial degree of the field extension L/K , i.e., $f_{L/K} = [\mathcal{O}_L / \mathfrak{m}_L : \mathcal{O}_K / \mathfrak{m}_K]$.	
∂	19
The derivation $\frac{d}{dx}$ on $k((x))$.	
$\dot{\mathcal{O}}_L$	20
The $\mathbb{Z}$ -module $\{\tau(a)/a \mid a \in L^*\}$. It is equal to $(1/e_{L/K})\mathbb{Z} + \mathfrak{m}_L$.	

$\mathcal{P}_L$	20
If $L = k'((t))$ with $\tau(t)/t \in k'$ and k' is the field of constants of L , then $\mathcal{P}_L \stackrel{\text{def}}{=} k'[1/t]$.	
$\epsilon(D)$	29
The ramification index of the simple differential operator D , which is defined as the ramification index of the field extension $K(a)/K$ where a is a normalized eigenvalue of D .	
$f(D)$	29
The inertial degree of the simple differential operator D , which is defined as the inertial degree of the field extension $K(a)/K$ where a is a normalized eigenvalue of D .	
$\mathcal{F}$	32
The multiplicative monoid of the monic polynomials in one variable over the field K .	
$\mathcal{I}$	32
The subset of $\mathcal{F}$ consisting of the irreducible polynomials.	
$\equiv_m$	32
An equivalence relation on $\mathcal{F}$. For $f, g \in \mathcal{I}$ it is defined by $f \equiv_m g$ if and only if $f(\lambda) = g(\lambda + r/m)$ for some $r \in \mathbb{Z}$.	
$\equiv$	32
An equivalence relation on $\mathcal{F}$. For $f, g \in \mathcal{I}$ it is defined by $f \equiv g$ if and only if $f(\lambda) = g(\lambda + r/\epsilon(f))$ for some $r \in \mathbb{Z}$.	
$[f]$	32, 39
The equivalence class in $\mathcal{F}/\equiv$ or $\mathcal{C}$ of the element f .	
$\epsilon_K(f)$	32
The ramification index of the field extension L/K , where L is a splitting field of the polynomial f over K .	
$\mathcal{R}(f, g)$	33
$\mathcal{R}(f, g)(\lambda) \stackrel{\text{def}}{=} \text{Resultant}_y(f(y), g(\lambda - y))$.	
$\star$	36, 39
The multiplication in the characteristic ring $\mathcal{C}$.	

R	39
The multiplicative subgroup of $k[1/x](\lambda)$ consisting of the elements of the form f/g with $f, g \in \mathcal{F}$ and all roots of f and g are in P_L , where L is a splitting field of fg .	
$\mathcal{C}$	39
The characteristic ring. It is isomorphic to the Grothendieck ring of the category of differential operators.	
$c(D)$	40
The characteristic class of the differential operator D . For a simple differential operator it is defined as the equivalence class in $\mathcal{C}$ of the minimal polynomial of a normalized eigenvalue of the differential operator over a finite field extension.	
$s(D)$	45
The semisimple part of the differential operator D .	
$n(D)$	45
The nilpotent part of the differential operator D .	
$\mathcal{N}$	47
The commutative monoid of all Newton-Puiseux polygons.	
$\mathcal{N}_0$	47
An abbreviation for $\mathcal{N} \setminus \{\emptyset\}$.	
$N(P)$	47
The Newton-Puiseux polygon of the element $P \in \mathcal{D}(K)$ or $P \in K[\lambda]$.	
$\tilde{\mathcal{N}}$	49
The Grothendieck group of the monoid $\mathcal{N}_0$.	
$N(D)$	49
The Newton-Puiseux polygon of the differential operator D . It is defined as $N(P)$ where P is an element in $\mathcal{D}^0(K)$ such that the $\mathcal{D}(K)$ -modules (V, D) and $\mathcal{D}(K)/\mathcal{D}(K)P$ are isomorphic.	
$\sim$	49
An equivalence relation on $\mathcal{D}(K)$ defined by $P \sim Q$ iff $N(P - Q) + (1, 0) \subseteq N(P)$.	

$\rho_i(D)$	54
The i -th Gérard-Levelt invariant of the differential operator D .	
$\mathbb{C}\{x\}$	55
The ring of the power series in one variable over the field of complex numbers $\mathbb{C}$ which are convergent in the neighborhood of the origin.	
$i(D)$	56
The irregularity index of Malgrange. It is equal to $\rho_1(D)$.	
val_Λ	57
$\text{val}_\Lambda(v) \stackrel{\text{def}}{=} \sup \{ i \in \mathbb{Z} \mid v \in x^i \Lambda \}$ for all $v \in V$, where Λ is a lattice in V .	
$\text{ord}(A)$	58
$\text{ord}(A) \stackrel{\text{def}}{=} \max \{ r \in \mathbb{Q} \mid r \leq 0 \text{ and } \text{val}(a) \geq r \text{ for all entries } a \text{ of } A \}$.	
$\text{pl}(A)$	58
The principal level of the matrix A . It is defined as $\text{pl}(A) \stackrel{\text{def}}{=} \sup_{g \in \text{GL}(K)} \text{ord}(g[A])$.	
$\mu(A)$	58
The invariant of Moser.	
$\mu^*(A)$	59
The invariant of Hilali.	
$r_i(A)$	59
An integer defined by Hilali. If A is super-irreducible then $\rho_{i+1}(\tau + A) = r_{p-i}(A)$ for all $i \in \mathbb{N}$, where $p \stackrel{\text{def}}{=} \text{ord}(A) $.	
$\varepsilon_i(A)$	64
An integer related to the Smith normal form of the matrix A .	
$\succeq$	64
A partial relation on $\text{Mat}_n(K)$ defined by $A \succeq B$ if and only if $\varepsilon_1(A) + \cdots + \varepsilon_i(A) \geq \varepsilon_1(B) + \cdots + \varepsilon_i(B)$ for all $i \in \{0, \dots, n\}$.	
λ^t	66
The conjugate of the Young tableau λ .	

$\text{ad}(D)$	81
The differential operator defined on $\text{Hom}(V, V)$ by $f \mapsto Df - fD$ for all $f \in \text{Hom}(V, V)$.	
$\theta_i(A, \lambda)$	86
A polynomial with constant coefficients, which is related to the super-irreducible form.	
$\mathcal{D}_1$	90
The subring of $\mathcal{D}(K)$ generated by ∂ and the elements of $\mathcal{O}$, also denoted by $\mathcal{O}[\partial]$.	
$\text{Tr}(D)$	92
The trace of the differential operator D , which is an element in $k[1/x]/\mathbb{Z}$.	
$gl_n(k)$	102
The general linear algebra $\text{Mat}_n(k)$ with bracket operation $[x, y] = xy - yx$ for all $x, y \in gl_n(k)$.	
$sl_n(k)$	102
The special linear algebra: the subalgebra of $gl_n(k)$ consisting of the matrices having trace zero.	
$d(M)$	102
The dimension of the orbit of the element M in $gl_n(k)$, i.e., the dimension of $\{gMg^{-1} \mid g \in \text{Gl}(k)\}$.	
$\text{ad}(f)$	103
The linear transformation defined on $\text{End}(V)$ by $g \mapsto fg - gf$ for all $g \in \text{End}(V)$.	
$\text{lag}(g)$	129
$\sup \{ \text{ord}(A) - \text{ord}(g[A]) \mid A \in \text{Mat}_n(L) \}$.	

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Samenvatting

Voor een lineaire afbeelding kan men invarianten en normaalvormen definiëren zoals de determinant, eigenwaarden, de karakteristieke veelterm, de Jordan normaalvorm en de rationale Jordan normaalvorm.

In dit proefschrift ‘karakteristieke klassen voor irreguliere singulariteiten’ wordt iets analoogs gedaan voor differentiaaloperatoren. Met een differentiaaloperator wordt in dit verband een additieve afbeelding $D: V \rightarrow V$ bedoeld, die voldoet aan de regel van Leibniz:

$$D(av) = aDv + \tau(a)v \quad \text{voor alle } a \in K \text{ en } v \in V.$$

Hierbij is V een eindig dimensionale vectorruimte over K en K is een differentiaallichaam van karakteristiek nul met een niet triviale derivatie τ . In veel gevallen nemen we voor K het lichaam der formele Laurentreeksen $k((x))$ en voor τ de derivatie $x \frac{d}{dx}$. Een voorbeeld van een differentiaaloperator op K^2 met $K = k((x))$ en $\tau = x \frac{d}{dx}$ is:

$$D: \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mapsto \begin{pmatrix} \tau y_1 \\ \tau y_2 \end{pmatrix} + \begin{pmatrix} 1 & 1/x^2 \\ x & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \quad (1)$$

De matrix van een differentiaaloperator D op V ten opzichte van een K -basis $v = (v_1, \dots, v_n)$ van V wordt genoteerd met $M(D, v)$. Voor de differentiaaloperator in (1) geldt derhalve

$$M(D, e) = \begin{pmatrix} 1 & 1/x^2 \\ x & -\frac{1}{2} \end{pmatrix},$$

waarbij $e = (e_1, e_2)$ de standaardbasis van K^2 is.

Een differentiaaloperator $D: V \rightarrow V$ geeft V de structuur van een $K[\tau]$ -moduul door de werking $\tau \cdot v \stackrel{\text{def}}{=} Dv$ voor alle $v \in V$. Hierbij wordt met $K[\tau]$ de K -algebra aangegeven voortgebracht door τ met als relatie $\tau \cdot a = a\tau + \tau(a)$ voor alle a in K . Bij vaste K en τ zal de Abelse categorie van paren (V, D) waarbij D een differentiaaloperator op V is, met $\mathcal{V}$ worden genoteerd.

Bij een gegeven een differentiaaloperator $D: V \rightarrow V$ en K -basis v van V kan men de Dieudonné determinant van de matrix $\tau I + M(D, v)$ bekijken. Dit is een

element van $\mathcal{L}(K)^{\text{ab}} \stackrel{\text{def}}{=} \mathcal{L}(K)/\mathcal{L}(K)'$ waarbij $\mathcal{L}(K)$ het (scheef) breukenlichaam van $K[\tau]$ is en $\mathcal{L}(K)'$ de commutator ondergroep van $\mathcal{L}(K)$. De Dieudonné determinant hangt niet van de K -basis v af en er bestaat een algoritme om de Dieudonné determinant van een gegeven differentiaaloperator te bepalen (zie [Adj85]). Een element dat de Dieudonné determinant van D in (1) representerendeert is $\tau^2 - \frac{1}{2}\tau - 1/x$.

In hoofdstuk 2 wordt bewezen dat de afbeelding $\det: \mathcal{V} \rightarrow \mathcal{L}(K)^{\text{ab}}$ een *isomorfisme* induceert van de Grothendieck groep $K_0(\mathcal{V})$ naar $\mathcal{L}^0(K)^{\text{ab}}$, waarbij $\mathcal{L}^0(K)$ de multiplicatieve ondergroep van de monische differentiaaloperatoren in $\mathcal{L}(K)$ is. Dit resultaat moet worden vergeleken met [KS68, p. 327] en [Alm74].

Vanaf hoofdstuk 3 nemen we aan dat $K = k((x))$ (de formele Laurentreeksen) en $\tau = x \frac{d}{dx}$. Uit de Dieudonné determinant kunnen dan direct het Newton-Puiseux diagram en enkele andere bekende invarianten van de differentiaaloperator worden bepaald, maar het is bijvoorbeeld niet duidelijk hoe men kan vaststellen of twee gegeven elementen in $\mathcal{L}^0(K)^{\text{ab}}$ hetzelfde zijn, noch hoe een vermenigvuldiging op $\mathcal{L}^0(K)^{\text{ab}}$ moet worden gedefinieerd die correspondeert met de vermenigvuldiging op $K_0(\mathcal{V})$ (van het tensorproduct). In paragraaf 3.5 definiëren we echter een andere invariant, die net als de Dieudonné determinant een groepsisomorfisme op de Grothendieck groep $K_0(\mathcal{V})$ induceert, maar waarvan men makkelijker kan vaststellen of twee gegeven elementen gelijk zijn en waarmee ook de vermenigvuldiging op $K_0(\mathcal{V})$ goed beschreven kan worden.

Deze invariant zullen we de *karakteristieke klasse* van de differentiaaloperator noemen omdat er een sterke analogie bestaat met de karakteristieke veelterm van een lineaire afbeelding en ook met de constructie van karakteristieke klassen in algebraïsche topologie. We zullen nu schetsen hoe de karakteristieke klasse en de vermenigvuldiging van karakteristieke klassen worden gedefinieerd.

In [Lev75] werd bewezen dat iedere differentiaaloperator over $K = k((x))$ een eigenwaarde heeft over een eindige lichaamsuitbreiding van K , m.a.w. er bestaat een eindige lichaamsuitbreiding L van K , een $a \in L$ en een vektor $v \in L \otimes_K V$ met $v \neq 0$ zodat $D_L v = av$. In (1) is bijvoorbeeld $1/\sqrt{x}$ een eigenwaarde van de differentiaaloperator D_L met $L = k((\sqrt{x}))$, want

$$D_L \left(\frac{1}{\sqrt{x}} \right) = \frac{1}{\sqrt{x}} \left(\frac{1}{\sqrt{x}} \right).$$

In tegenstelling tot de situatie bij lineaire afbeeldingen is de verzameling van eigenwaarden over een eindige lichaamsuitbreiding van een differentiaaloperator niet eindig: als a een eigenwaarde is dan is $a + r$ ook een eigenwaarde voor alle $r \in \mathbb{Z} + xk[[x]]$.

Daarom wordt in paragraaf 3.2 het begrip *genormaliseerde eigenwaarde* ingevoerd; een eigenwaarde $a \in L$ van D_L heet *genormaliseerd* als $a \in k'[1/t]$ en $D_L v = av$ voor een zekere $v \in K(a) \otimes_K V$ met $v \neq 0$. Hierbij is k' het constantenlichaam van $K(a)$ en $t \in K(a)$ voldoet aan $x/t^e \in k'$ voor zekere

$e \in \mathbb{N}^*$. De eigenwaarden $1/\sqrt{x}$ en $-1/\sqrt{x}$ van de differentiaaloperator D in (1) zijn genormaliseerd.

In paragraaf 3.3 wordt aangetoond dat een simpele differentiaaloperator op isomorfie na bepaald is door een genormaliseerde eigenwaarde. Dit komt in propositie 3.3.6 tot uitdrukking. De vertakkingsindex van de lichaamsuitbreiding $K(a)/K$ (dit is het natuurlijke getal e in de vorige alinea), waarbij a een genormaliseerde eigenwaarde is, speelt hierbij een belangrijke rol.

Dit leidt tot de definitie van de *karakteristieke klasse* van een differentiaaloperator in paragraaf 3.5. Voor een simpele differentiaaloperator is de karakteristieke klasse de verzameling van minimumveeltermen van al zijn genormaliseerde eigenwaarden. In het algemene geval wordt de karakteristieke klasse gedefinieerd als de verzameling van alle produkten van elementen in de karakteristieke klassen van de simpele componenten in een Jordan-Hölder decompositierij van het $K[\tau]$ -moduul V . De karakteristieke klasse van D in (1) zal

$$[\lambda^2 - 1/x] = \{ (\lambda + a)^2 - 1/x \mid a \in \frac{1}{2}\mathbb{Z} \}$$

blijken te zijn.

De verzameling van alle karakteristieke klassen vormt een Abelse monoïde waarin de schrapwet geldt. De Grothendieck groep van deze monoïde (de Abelse groep van de breuken) zullen we met C aanduiden. De karakteristieke klasse induceert een groepsisomorfisme van de Grothendieck groep $K_0(V)$ naar C .

Het tensorproduct van differentiaaloperatoren geeft $K_0(V)$ de structuur van een commutatieve ring. In paragraaf 3.4 wordt de corresponderende vermenigvuldiging op de Abelse groep C gedefinieerd zonder verwijzing naar deze produktstructuur op $K_0(V)$. In tegenstelling tot de situatie bij lineaire afbeeldingen, of meer algemeen bij endomorfismen van eindig voortgebrachte projektieve modulen over een Abelse ring (zie o.a. [Alm74]), kan de vermenigvuldiging op C , uitgaande van het produkt op elementen van graad één, niet m.b.v. universele veeltermen worden voortgezet.

De zaak is in ons geval een stuk gecompliceerder omdat, in tegenstelling tot de situatie bij lineaire afbeeldingen, twee simpele differentiaaloperatoren die over K niet isomorf zijn, over een eindige lichaamsuitbreiding wel isomorf kunnen zijn. Het eenvoudigste voorbeeld hiervan wordt gegeven door de differentiaaloperatoren $x \frac{d}{dx}$ en $x \frac{d}{dx} + \frac{1}{2}$ die over $K = k((x))$ niet isomorf, maar over $k((\sqrt{x}))$ wel isomorf zijn. Dit zorgt er o.a. voor dat de ring C nuldelers heeft.

De vertakkingsindex van de lichaamsuitbreiding $K(a)/K$, waarbij a een genormaliseerde eigenwaarde is, speelt weer een belangrijke rol bij de definitie van de vermenigvuldiging op C . In stelling 3.5.5 wordt bewezen dat de vermenigvuldiging zoals gedefinieerd in paragraaf 3.4 de vermenigvuldiging is die wordt geïnduceert door het tensorproduct op $K_0(V)$, m.a.w. de karakteristieke klasse induceert een *ringisomorfisme* van $K_0(V)$ naar de ring C .

In [Lev75] werd bewezen dat iedere differentiaaloperator op precies één manier te schrijven is als som van een halfenkelvoudige (= semisimpele) diffe-

rentiaaloperator S en een nilpotente lineaire afbeelding N met de extra eigenschap dat S en N commuteren. Men noemt S het halfenkelvoudige deel en N het nilpotente deel van de differentiaaloperator.

In paragraaf 3.6 wordt bewezen dat de afbeelding die aan een differentiaaloperator zijn halfenkelvoudige deel toevoegt, een exakte funktor is. Dit impliceert dat de Dieudonné determinant en de karakteristieke klasse beiden het halfenkelvoudige deel van een differentiaaloperator karakteriseren.

In paragraaf 3.7 onderzoeken we tot op welke orde van x de coëfficiënten van een element dat de Dieudonné determinant van een differentiaaloperator representeert onafhankelijk zijn van de gekozen representant. Dit wordt gedaan met behulp van een equivalentierelatie $\sim$ op $K[\tau]$ waarvan de definitie gebruik maakt van het Newton-Puiseux diagram. Deze equivalentierelatie wordt ook gebruikt om de coëfficiënten van de karakteristieke klasse en de Dieudonné determinant van een differentiaaloperator met elkaar te vergelijken. Dat verband maakt het mogelijk om het Newton-Puiseux diagram en de Newton-Puiseux veeltermen (cf. [Die87]) van een differentiaaloperator uit de karakteristieke klasse te halen.

In paragraaf 3.8 beschrijven we enkele bekende invarianten van een differentiaaloperator die allemaal uit het Newton-Puiseux diagram volgen en daarom ook uit de karakteristieke klasse.

In paragraaf 3.9 wordt een relatie gegeven tussen de coëfficiënten van de karakteristieke klasse van een differentiaaloperator en de karakteristieke veelterm van de matrix van de differentiaaloperator t.o.v. een K -basis. Dit generaliseert een resultaat van Hilali over super-irreducibele matrices geformuleerd in [Hil87a, theorem 1].

Een lineaire afbeelding heeft een zgn. rationale Jordan normaalvorm. Het nilpotente deel, het halfenkelvoudige deel en de simpele componenten hiervan volgen direct uit zo'n normaalvorm. De uniciteit van die normaalvorm kan worden gebruikt om vast te stellen of (de modulen behorende bij) twee lineaire afbeeldingen isomorf zijn.

In hoofdstuk 4 van dit proefschrift wordt iets analoogs gedaan voor differentiaaloperatoren. In stelling 4.1.2 wordt een rationale normaalvorm voor een simpele differentiaaloperator gegeven. Deze rationale normaalvorm is uniek tot op een zeer speciale blokdiagonaal transformatie, die in het tweede deel van de stelling wordt beschreven. In het geval dat het constantenlichaam k algebraïsch gesloten is, krijgt men de normaalvorm van [vdEL92]. De differentiaaloperator D in (1) blijkt een simpele differentiaaloperator te zijn en

$$\begin{pmatrix} 0 & 1/x \\ 1 & -\frac{1}{2} \end{pmatrix}$$

is een rationale normaalvorm van D .

In stelling 4.2.3 wordt een rationale normaalvorm van een algemene differentiaaloperator gegeven. De blokstructuur is in dit geval geheel analoog aan

de rationale Jordan normaalvorm van een lineaire afbeelding. De karakteristieke klasse is direkt uit een rationale normaalvorm te berekenen.

In paragraaf 4.3 wordt aangetoond dat een rationale normaalvorm superirreducibel is, een term gedefinieerd in [HW87], en in paragraaf 4.4 staat het verband met de kanonieke vorm van Babbitt en Varadarajan zoals gedefinieerd in [BV83].

In paragraaf 4.5 wordt de rationale normaalvorm gebruikt om te bewijzen dat ieder holonoom $\mathcal{D}_1$ -moduul zonder ∂ -torsie en zonder x -torsie een holonoom $\mathcal{D}(K)$ -moduul is.

In paragraaf 4.6 wordt het spoor van een differentiaaloperator gedefinieerd en de rationale normaalvorm wordt gebruikt om een resultaat te bewijzen dat het verband geeft met het spoor van de matrix van de differentiaaloperator t.o.v. een K -basis.

Hoofdstuk 5 begint met een overzicht van verschillende bekende algoritmen om de matrix van een differentiaaloperator in een speciale vorm te brengen of om enige invarianten te kunnen uitrekenen.

De bewijzen voor het bestaan van de karakteristieke klasse en een rationale normaalvorm zijn grotendeels constructief, dus in principe is het mogelijk deze te berekenen. Dat vereist echter wel bewerkingen, waaronder factorisatie van veeltermen, in eindige lichaamsuitbreidingen van het constantenlichaam, dat in de praktijk vaak het lichaam van de rationale getallen $\mathbb{Q}$ is. Deze berekeningen zijn zeer bewerkelijk en moeilijk te implementeren in de meeste bestaande computeralgebrasystemen. Verder is het a priori niet duidelijk hoeveel termen uit de Laurentreeks ontwikkeling van de matrix nodig zijn om de karakteristieke klasse en rationale normaalvorm te kunnen berekenen.

In paragraaf 5.2 wordt een ietwat verbeterde versie van het algoritme van Babbitt en Varadarajan (zie [Bab]) beschreven die leidt tot een kanonieke vorm:

$$x \frac{d}{dx} + A_r x^{r/m} + A_{r+1} x^{(r+1)/m} + \dots + A_0$$

Hierbij is $m \in \mathbb{N}^*$ en alle A_i 's zijn matrices met coëfficiënten in het constantenlichaam k . Bovendien is A_i halfenkelvoudig voor alle $i < 0$ en A_i en A_j commuteren voor alle i and j . De karakteristieke veelterm van deze kanonieke vorm is een element dat de karakteristieke klasse van de differentiaaloperator over één of ander eindig lichaam representeert. Als echter het zogenaamde 'nilpotente geval' nooit voorkomt gedurende de uitvoering van het algoritme dan kan men de karakteristieke klasse en een rationale normaalvorm verkrijgen.

In paragraaf 5.3 wordt een compleet rationaal algoritme beschreven om de karakteristieke klasse en een rationale normaalvorm van differentiaaloperatoren over twee en drie dimensionale vectorruimten te berekenen.

Tenslotte worden enige open vragen en punten voor toekomstig onderzoek gegeven.

Curriculum Vitae

Ik ben geboren op 5 juni 1961 te Oosterbeek. Op het Christelijk Atheneum te Arnhem behaalde ik in 1979 mijn diploma VWO Atheneum B. Daarna studeerde ik wiskunde aan de Katholieke Universiteit te Nijmegen, waar ik in 1985 cum laude voor mijn doctoraal examen wiskunde slaagde. Na een jaar als wetenschappelijk onderzoeker bij de afdeling informatica van de Katholieke Universiteit Nijmegen te hebben gewerkt, heb ik me in het academisch jaar 1986–1987 bij prof. B. F. Caviness aan de universiteit van Delaware verder in de computeralgebra bekwaamd. Het jaar daarna heb ik voor 20 uur per week op het CWI gewerkt. Daar heb ik o.l.v. dr. A. M. Cohen de eerste versie van het computeralgebrapakket LiE ontwikkeld. Vanaf juni 1988 was ik projectmedewerker bij het NWO project ‘karakteristieke veeltermen voor differentiaaloperatoren’ o.l.v. prof. dr. A. H. M. Levelt. Dat resulteerde in dit proefschrift. Vanaf september 1991 heb ik een postdoc. plaats voor twee jaar op het CWI bij het NWO project ‘computeralgebra in de coderingstheorie’.