Magnetic levels in quasiperiodic superlattices

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With numerical calculations it is shown that in a Fibonacci superlattice, in a magnetic field parallel to the layers, the self-similarity in the length scale is reproduced in the energy-level structure. In particular, the cyclotron-orbit-center dispersion of the energy levels shows a field-dependent structure which is the same at values of the magnetic field, which are related by integer powers of τ, the golden mean $[(\sqrt{5}+1)/2]$. The conditions for which this effect can be observed are discussed.

The one-dimensional Schrödinger equation with a quasiperiodic (incommensurate) potential has been studied theoretically quite intensively, because it is a mathematically accessible model to describe systems with properties that are between periodic and amorphous ones. A particular type of quasiperiodic potential, the Fibonacci potential, has been studied with various theoretical techniques,[1-12] because this particular sequence shows self-similarity, i.e., the properties of the system at different length scales are similar. Further stimulus for such studies arises because with modern crystal-growth techniques, layered materials with sequences of layer thicknesses arranged according to a Fibonacci series can indeed be realized. X-ray studies and optical measurements[13-15] of such quasicrystals have been reported and have revealed peculiar properties.

In this paper we will study theoretically the energy-level structure of a Fibonacci superlattice in a magnetic field, applied parallel to the layers. In such a superlattice the potential consists of barriers in one type of layers and valleys in the other type of layers, whereas the layer thicknesses are arranged according to a Fibonacci sequence. In a parallel magnetic field B the electrons orbit in a plane perpendicular to the layers of the superlattice, with a field-dependent orbit radius $l=\sqrt{\hbar/eB}$. If the barriers are sufficiently thin or shallow, the carriers may tunnel through them and the number of barriers within an orbit depends on the field. Therefore the length scale on which the nonperiodicity is "seen" by the electrons can be varied with an external parameter, i.e., the magnetic field. The magnetic levels under these conditions are broadened because their energy depends on the position of cyclotron orbit with respect to the potential. We will show that this orbit center dispersion, and thereby the density of states, can be self-similar at all values of the magnetic field which are related by a certain factor. It is important to note that the system studied here is completely different from the two-dimensional quasiperiodic networks in which the quantization of the magnetic flux is studied experimentally.[16] In that case the size of the meshes determines particular sets of fields where similarity may be observed. In the one-dimensional potential studied here, there is no closed loop in which the magnetic flux can be quantized, and therefore there are no special values of the field related to

the geometry. Furthermore the properties studied here are basically due to the tunneling through the barriers and are the result of the cooperative effect of the whole structure, as opposed to the confinement in individual elements of the structure. Therefore our work introduces a new type of system with fractal properties.

The Schrödinger equation which describes the motion of the electrons in a potential and a magnetic field B in the x direction can be written as

$$\left[- \left[\frac{d^2}{d(u-u_0)^2} \right] + (u-u_0)^2 + v(u) \right] \Psi(u-u_0)$$

$$= E\Psi(u-u_0) . \quad (1)$$

In this equation the gauge is chosen as $\mathbf{A}=(0,-zB,0)$, the wave functions in the x and y direction are plane waves with wave vector k_x and k_y, respectively. The energy dispersion in the direction of the field (k_x) is unaffected by the potential, and in the following we will discuss the properties only at zero k_x wave vector. Furthermore the potential v and the eigenvalues E are normalized to the cyclotron energy $\hbar eB/m$ and the length $u=z/l$ to the magnetic length; $lu_0=\hbar k_y/eB$, the center coordinate of the cyclotron orbit. Note that this Schrödinger equation has become dimensionless in these units and that it therefore possesses a "trivial" scaling property, namely, that different samples in different fields will have the same properties, provided that the normalized units are the same. In the following we wish to discuss a "nontrivial" scaling in which self-similarity is found in the same sample but at different fields.

In the absence of the potential Eq. (1) is the well-known harmonic-oscillator equation with degenerate eigenvalues $E_N=N+\frac{1}{2}$ for all orbit centers u_0, with N the Landau-level quantum number. This degeneracy will be broken when $v\neq0$ and eigenvalues will in general depend on u_0 leading to broadened Landau levels. In the following we will concentrate on the shape of this cyclotron-orbit-center dispersion.

We consider a quasiperiodic (quasi-one-dimensional) superlattice which is constructed in the following way. We generate a "Fibonacci (string) sequence" $\{w_n\}$, with a and b as elementary building blocks, following the prescription

$$w_1 = a , \tag{2a}$$

$$w_2 = b , \tag{2b}$$

and

$$w_n = \begin{cases} w_{n-2} | w_{n-1} & \text{if } n \text{ is odd and } n \geq 3 \tag{2c} \\ w_{n-1} | w_{n-2} & \text{if } n \text{ is even and } n \geq 3 , \tag{2d} \end{cases}$$

thus leading to $\cdots ababbabb \cdots$, where the nonassociative binary operation "|" represents (string) concatenation. It is easy to see that the length, $L(w_n)$, of the pattern w_n (i.e., the total number of elementary building blocks a and b) is given by the nth Fibonacci number F_n, where $F_n = F_{n-1} + F_{n-2}$, for $n \geq 3$, with $F_1 = 1$ and $F_2 = 1$. Therefore the length ratio $L(w_n)/L(w_{n-1})$ tends to $\tau = (\sqrt{5}+1)/2$, the golden mean, for increasing n, as this is a property of the Fibonacci numbers. Under this generation procedure, one can also see the self-similarity of the sequence, namely by the concurrent

$$\begin{cases} abb \mapsto b \tag{3a} \\ ab \mapsto a , \tag{3b} \end{cases}$$

one obtains from w_n the reduced pattern w_{n-2}. Similarly, the concurrent (first-order) reduction-reversal transformation

$$\begin{cases} ab \mapsto b \tag{4a} \\ b \mapsto a \tag{4b} \end{cases}$$

transfers w_n into the reverse pattern of w_{n-1} (i.e., the pattern w_{n-1} read from right to left). To translate the pattern w_n into a one-dimensional potential, we take a as a barrier with width d_a and height V_0 and b as a well

with width d_b. Two successive b's in the pattern combine to give a well $2d_b$ in width. This "comblike" potential, with irregularly spaced "teeth," is shown in Fig. 1 in the bottom half. In order to obtain self-similarity in the length scale, we must require that at a given magnetic field corresponding to a magnetic length l the potential seen by the carriers be the same as the potential transformed according to Eqs. (3) or Eqs. (4), at another field corresponding to another magnetic length l'. This must be true down to the smallest building blocks of the sequence, and using Eqs. (3) one finds

$$\frac{l}{l'} = \frac{d_a + d_b}{d_a} = \frac{d_a + 2d_b}{d_b} . \tag{5}$$

Using the identity $1 + \tau = \tau^2$, one then finds that $d_b = \tau d_a$ and $l' = \tau^2 l$. Since $l \equiv B^2$, scaling of τ in the length implies a scaling of $1/\tau^2$ in the field, and thereby a scaling of $1/\tau^2$ in the energy.

With the potential constructed in this way we have calculated the energy levels of Eq. (1) numerically, using the finite-element method. The results are shown in Figs. 1(a) and 1(b). For the sake of illustration we have chosen parameters corresponding to GaAs/Ga$_{1-x}$Al$_x$As superlattices, and reasonable layer thicknesses of 1.12 and 1.68 nm to have $d_a/d_b \simeq \tau$ and at the same time an integer number of lattice planes per layer. The barrier height in the calculation was 0.2 eV and the corresponding range of the fields was up to 20 T. It can be seen from Fig. 1 that the energy levels all show a dispersion as a function of the cyclotron-orbit-center position. However, the striking result is, that this dispersion for a given field B and at $\tau^4 B$ is the same when the reduced units, introduced in Eq. (1), are used for the energies and lengths.
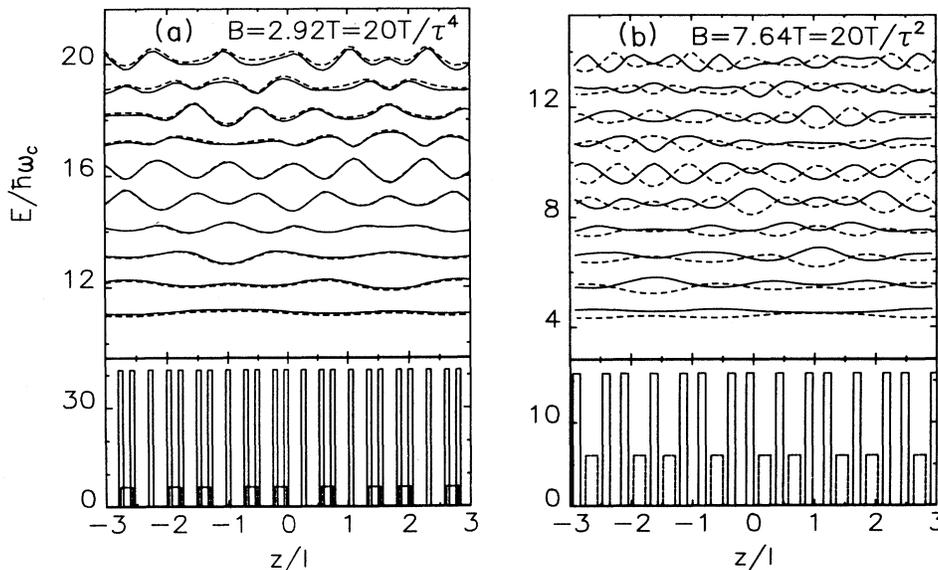

FIG. 1. (a) Magnetic energy-level structure of a quasiperiodic superlattice at $B = 2.92$ T $= (20$ T$)/\tau^4$ as function the cyclotron-orbit-center coordinate (top) at zero wave vector for motion in the direction of the field. In the lower part, a section of the Fibonacci potential at the same fields. Energies are all in units of the cyclotron energy and lengths in units of the magnetic length. (b) As in (a), but for $B = 7.64$ T $= (20$ T$)/\tau^2$.

For $B\tau^2$, the levels are seen to be "anti-self-similar," with respect to those of B, which means self-similar if the coordinate system is inverted. These observations are illustrated more clearly in Fig. 2 which shows the calculated density of states for several values of the magnetic field. It can be noted that only scaling with τ^{2n}, with n an integer, leads to self-similarity. It can now also be seen clearly that the τ^2 scaling leads to the same density of states as that at B, because for the density of states the inversion of the coordinate system has no consequences. For values of the field with different powers of τ no similarity is observed. It should be emphasized that, although in this example the levels are calculated for a base field of 2.92 T and integer powers of τ times this field, the choice of this base field is of course arbitrary, because any set of fields related by even powers of τ would show the same behavior.

To understand the results discussed above it is most easy to consider the Fibonacci potential as a perturbation of the energy levels in a magnetic field. The criterion for the applicability of perturbation theory is that the unperturbed wave function (the solutions for the magnetic field only) are only slightly altered by the Fibonacci potential. This criterion implies that $l \gg d$, with $d = d_a + \tau d_b$, the average periodicity and $d_a^2 < \hbar^2/2mV_0$, which means that the width and the height of the barriers must be sufficiently small to allow coupling between neighboring wells (easy tunneling). In first order the energy levels are
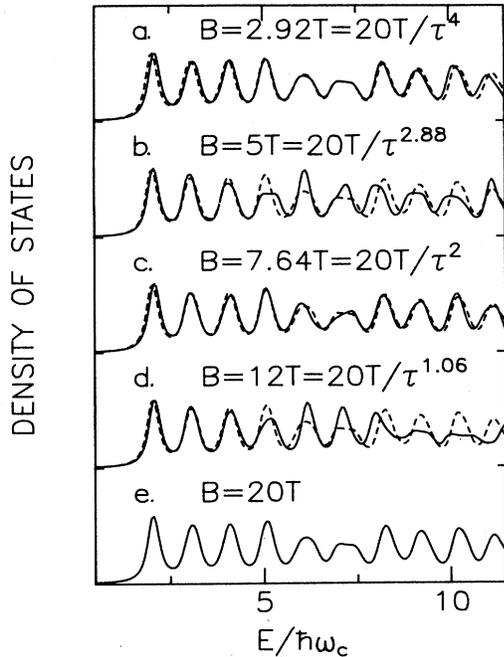

FIG. 2. Calculated density of states at zero wave vector for motion in the direction of the field. The energy scale is in units of the cyclotron energy for all curves. The zero of the energy scale for all but the lower curve is linearly shifted and aligned with the first peak. For comparison the dashed curves, all representing the results at 20 T, are drawn on the other results. Note that only for even powers of τ is self-similarity observed.

given by

$$(N + \tfrac{1}{2}) + \langle \Psi_N(u - u_0) | v(u) | \Psi_N(u - u_0) \rangle$$

and if v is a rapidly varying function with respect to Ψ the correction, $\langle v(u_0) \rangle$ is a strong function of u_0 in a nonperiodic system[17] and leads to the dispersion of the Landau levels. Conversely if v is constant everywhere, this leads only to a rigid shift in all Landau levels and does not contribute to the dispersion. To prove the self-similarity in the dispersion we have to show that spatially varying part of $\langle v(u_0) \rangle$ remains the same at B and $\tau^{2n}B$. We therefore calculate this matrix element for the transformation (3) which corresponds to scaling with $n = 2$ [an analogous calculation can also be performed for the transformation (4)].

Assuming that at a given field corresponding to a given length scale the barrier and well have widths d_a and d_b with an actual ($=$not in reduced units) barrier height V_0 giving rise to some $\langle v(u_0) \rangle$ (expressed in the cyclotron energy corresponding to that field). At τ^4 times lower field (τ^2 longer magnetic length or τ^2 larger extension of the wave function) it may be assumed that the wave function does not change substantially during the variation of the potential between a barrier and a well. Therefore we may replace a barrier of width d_a followed by a single well of width d_b ($ab \mapsto a$) by a new single barrier with width d_a^{new} given by

$$d_a^{\text{new}} = d_a + d_b = \tau^2 d_a$$

with a new average barrier height

$$v_{\text{barrier}}^{\text{new}} = \frac{V_0 d_a}{d_a + d_b} = \frac{V_0}{\tau^2} . \tag{6}$$

Similarly, every barrier of width d_b followed by a well of width $2d_b$ ($abb \mapsto b$) may be replaced by a new well with width d_b^{new} given by

$$d_b^{\text{new}} = d_a + 2d_b ,$$

with a new average barrier height

$$V_{\text{well}}^{\text{new}} = \frac{V_0 d_a}{d_a + 2d_b} = \frac{V_0}{\tau^3} . \tag{7}$$

In the length scale this new potential is self-similar to the previous one, because this is a property of the Fibonacci sequence. The amplitude of this new potential has a part which does not vary throughout the sample (V_0/τ^3) and a spatially varying part which is just the difference between (6) and (7) and, using the identity $\tau - 1 = 1/\tau$, equals V_0/τ^4. This spatially varying part gives rises to the dispersion $\langle v(u_0) \rangle$. Therefore the new effective potential for a scaling in the length of τ^2 is just smaller by a factor of τ^4. Thus, the spatially varying part of the matrix element $\langle v(u_0) \rangle$ has not changed, and thus the dispersion is self-similar, both in amplitude as in the length scale.

In Figs. 1(a) and 1(b) we have drawn, in the lower part, the potential in reduced units of energy and lengths. The previous reasoning shows that the averaged value of the drawn potential in Fig. 1(a) is the same as the dashed po-

tential. The remarkable property of this Fibonacci potential is therefore that a scaling in the length by τ^n leads to a scaling of $1/\tau^{2n}$ in the spatially varying part of the average potential. This is exactly the same scaling property that is required by the magnetic Hamiltonian (1), and it is this identical scaling behavior that is responsible for the self-similarity in the energy-level structure as shown in Figs. 1 and 2.

The scaling property that we have derived is quite general, because the argument in the previous paragraph makes clear that the self-similar properties in a field will be observed as long as it is possible to obtain the eigenvalues of (1) with perturbation theory and to replace the potential by its average value. The observed behavior will therefore break down at very high fields, for very high or very thick barriers. It should be realized that the self-similarity of the energy-level structure in a magnetic field is a direct consequence of the fractal character of the sequence of Fibonacci patterns; therefore the properties that we have derived are not general for all quasiperiodic systems. However, some aspects of our results have a more general significance; as an example it is worthwhile mentioning that, as can be seen from Fig. 2, the width of the peaks in the reduced units of the energy are the same at different magnetic fields. This implies that in absolute units of the energy in a nonperiodic superlattice (SL) the width will increase proportional with the field, contrary to that in a periodic superlattice,[18,19] where flat Landau levels were seen with a field-independent width.

In summary, we have shown that quasiperiodic Fibonacci superlattices can exhibit a self-similar behavior in their energy-level structure as a function of the parallel magnetic field. We have also derived criteria on the parameters for which this behavior occurs, and have shown that the fractal property is a consequence of the identical scaling behavior of the Hamiltonian and the Fibonacci potential. Our calculations constitute a new example of a system showing self-similar behavior. Of special significance is the fact that the scaling property in this example varies with an external parameter (the magnetic field) in contrast to many other cases where basically the scattering of light with a variable frequency or scattering angle is observed. We believe that our results may also contribute to the understanding of the magnetic levels in a perturbed potential.

ACKNOWLEDGMENTS

[1]M. Kohmoto, L. P. Kadanoff, and C. Tang, Phys. Rev. Lett. **50**, 1870 (1983).

[2]M. Kohmoto and J. Banavar, Phys. Rev. B **34**, 563 (1986).

[3]J. M. Luck and D. Petritis, J. Stat. Phys. **42**, 289 (1986); J. M. Luck, J. Phys. (Paris) Colloq. **47**, C3-205 (1986).

[4]M. Kohmoto, B. Sutherland, and C. Tang, Phys. Rev. B **35**, 1020 (1987).

[5]S. Ostlund, R. Pandit, R. Rand, H. J. Schellnhuber, and E. D. Siggia, Phys. Rev. Lett. **50**, 1873 (1983).

[6]M. Kohmoto and Y. Oono, Phys. Lett. **102A**, 145 (1984).

[7]Q. Niu and F. Nori, Phys. Rev. Lett. **57**, 2057 (1986).

[8]F. Nori and J. P. Rodriquez, Phys. Rev. B **34**, 2207 (1986).

[9]M. Fujita and K. Machida, Solid State Commun. **59**, 61 (1986).

[10]H. Xu, J. Phys. C **20**, 5999 (1987).

[11]J. P. Lu, T. Odagaki, and J. L. Birman, Phys. Rev. B **33**, 4809 (1986).

[12]J. M. Luck and Th. M. Nieuwenhuizen, Europhys. Lett. **2**, 257 (1986); Th. M. Nieuwenhuizen, J. Phys. (Paris) Colloq. **47**, C3-211 (1986).

[13]R. Merlin, K. Bajema, and R. Clarke, Phys. Rev. Lett. **55**, 1768 (1985).

[14]J. Todd, R. Merlin, and R. Clarke, Phys. Rev. Lett. **57**, 1157 (1986).

[15]R. Merlin, K. Bajema, J. Nagle, and K. Ploog, J. Phys. (Paris) Colloq. **48**, C5-503 (1987).

[16]A. Behrooz, M. J. Burn, H. Deckman, D. Levine, B. Whitehead, and P. M. Chaikin, Phys. Rev. Lett. **57**, 369 (1986).

[17]Note that in a periodic superlattice this situation gives rise to flat Landau levels, because (v) does not depend on u_0 (Refs. 18 and 19).

[18]G. Belle, J. C. Maan, and G. Weimann, Solid State Commun. **56**, 65 (1985).

[19]J. C. Maan, in *Festkörperprobleme, Advances in Solid State Physics,* edited by P. Grosse (Vieweg, Braunschweig, 1987), p. 137.