Principal ∞-bundles – Presentations

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Abstract

We discuss two aspects of the presentation of the theory of principal ∞-bundles in an ∞-topos, introduced in [NSSa], in terms of categories of simplicial (pre)sheaves.

First we show that over a *cohesive site* C and for G a presheaf of simplicial groups which is C*-acyclic*, G-principal ∞-bundles over any object in the ∞-topos over C are classified by hyper-Čech-cohomology with coefficients in G. Then we show that over a site C with enough points, principal ∞-bundles in the ∞-topos are presented by ordinary simplicial bundles in the sheaf topos that satisfy principality by stalkwise weak equivalences. Finally we discuss explicit details of these presentations for the discrete site (in discrete ∞-groupoids) and the smooth site (in smooth ∞-groupoids, generalizing Lie groupoids and differentiable stacks).

In the companion article [NSSc] we use these presentations for constructing classes of examples of (twisted) principal ∞-bundles and for the discussion of various applications.

Contents

1 Overview

In [NSSa] we have described a general theory of geometric principal ∞-bundles (possibly twisted by local coefficients) and their classification by (twisted) nonabelian cohomology in ∞-toposes. A certain charm of this theory is that, formulated the way it is in the abstract language of ∞-topos theory, it is not only more general but also more elegant than the traditional theory. For instance every ∞-group action is principal over its homotopy quotient, the quotient map is automatically locally trivial, the principal ∞-bundle corresponding to a classifying map is simply its homotopy fiber (hence the universal principal ∞-bundle is the point), and the fact that all principal ∞-bundles arise this way is a fairly direct consequence of the axioms that characterize ∞-toposes in the first place: the Giraud-Rezk-Lurie axioms.

While this abstract formulation provides a useful means to reason about general properties of principal ∞-bundles, it is desireable to complement this with explicit *presentations* of the structures involved (notably of ∞-groups, of ∞-actions and of principal ∞-bundles) by *generators and relations*. This is typically the way that explicit examples are constructed and in terms of which properties of these specific examples are computed in applications.

In recent years it has been well understood that the method of choice for presenting ∞-categories by generators and relations is the homotopical category theory of categories of simplicial presheaves, i.e. presheaves of simplicial sets. The techniques themselves have a long history, dating back to work of Illusie [Il72a], continued in the foundational work of [Bro73] and developed further in [Jo83, Ja87], which will play a prominent role below. Their interpretation as a *generators and relations presentation* for homotopy theoretic structures has been amplified in the exposition of [Dug99], and was formalized in terms of model category theory by the main theorem in [Dug01]. Finally [L09] has provided the general abstract essence of this theorem in terms of the notion of *presentable ∞-categories*. This is the notion of presentation that we are concerned with here.

We formalize and prove the following statements.

1. Over a site C with a terminal object, every ∞-group is presented by a presheaf of simplicial groups G. (Proposition 3.35)

2. If the ambient ∞-topos is locally ∞-connected and local over an *∞-cohesive site* C, and if G is *C-acyclic* (Definition 3.43) then G-principal ∞-bundles over any object X are classified by simplicial hyper-Čech-cohomology of X with coefficients in G. In fact, the ∞-groupoid of geometric G-principal ∞-bundles, morphisms and higher homotopies between these is equivalent to the ∞-groupoid of Čech cocycles, Čech coboundaries and higher order coboundaries (Theorem 3.46).

3. If C is a site with enough points, then principal ∞-bundles over C are presented by ordinary simplicial bundles in sheaves over C which satisfy a weakened notion of principality (Theorem 3.95).

The first and the third statement may be thought of as strictification results, showing that every principal ∞-bundle is equivalent to one that is presented by an ordinary group object with strict group law (not up to homotopy) acting strictly on a simplicial object. This makes available classical principal bundle theory as a tool for constructing and analyzing ∞-bundles. The second statement provides good control over the cocycles underlying principal ∞-bundles.

In Section 4 we discuss details of the presentations for examples of sites that satisfy the assumptions 1, 2 and 3 above:

- the trivial site, modelling *discrete geometry*;
- the site of smooth manifolds, modelling *smooth/differential geometry.*

The presentations of higher principal bundles and their interpretation as cocycles in non-abelian cohomology has a long history. We close this introduction with a short historical overview and indicate how our results both relate to and extend previous works of other authors.

Following the foundational work of Giraud [G71], it seems that the first paper to consider the problem of giving a geometric description of non-abelian cohomology was the paper [Dus82] of Duskin (this paper was intended as a pre-cursor to a more substantial discussion, which unfortunately never materialized). This was followed by the more comprehensive treatment of Breen in [Bre90]. This paper of Breen's is noteworthy in that it treats non-abelian cohomology within the natural context of the homotopy theory of simplicial sheaves and it also introduces the notion of a *pseudo-torseur* for a group stack; a notion which is closely related to our notion of weakly principal simplicial bundle. In [Ulb90, Ulb93] Ulbrich gave a different interpretation of Duskin's work, in particular introducing the notion of *cocycle bitorsor* which is closely related to Murray's later notion of *bundle gerbe* [Mur96]. Joyal and Tierney in [JT93] introduced a notion of *pseudo-torsor* which is again closely related to our notion of weakly principal simplicial bundle; their notion of pseudo-torsor was more general than the corresponding notion of Breen's, since Breen restricted his attention to the case of 1-truncated group objects while Joyal and Tierney worked with simplicial groupoids.

The mid 1990s saw a flurry of interest in interpreting geometrically the standard characteristic classes as higher principal bundles with structure ∞-group of the form $K(\pi, n)$ for some abelian group π; the works [Bry93, BMcL94, BMcL96, Mur96] of Brylinski, Brylinski and McLaughlin and Murray are landmarks from this period. The overarching theme of these papers is to develop and apply a Chern-Weil theory for 'higher line bundles', in particular they focus attention on the ∞-groups $\mathbf{B}U(1)$ and $\mathbf{B}^2U(1)$. Our aim here is less restrictive; we want to develop the theory of principal ∞-bundles as a whole.

To continue the historical discussion, the 2004 thesis [Bar] gave a treatment of 1-truncated principal ∞-bundles — *principal 2-bundles* — while [Jur11] gave such a treatment from the point of view of bundle gerbes (we note that [Jur11] appeared in pre-print form in 2005). In [Bak] these constructions were generalized from structure 2-groups to structure 2-groupoids. The gauge 2-groups of principal 2-bundles were studied in [Woc11]. A comprehensive account is given in [NW11].

Continuing in this vein, a discussion of 2-truncated principal ∞-bundles, *principal 3-bundles*, was given in [Jur09] in the guise of *bundle 2-gerbes*, generalizing the abelian bundle 2-gerbes ($\mathbf{B}^2U(1)$-principal 3-bundles) of [Ste1].

The work that is closest to our discussion in Section 3.7.2 is the paper [JaL04] of Jardine and Luo. This paper goes beyond the previous work of Breen [Bre90] and Joyal and Tierney [JT93]; it introduces a notion of G-torsor for G a group in $\mathrm{sPSh}(C)$ for some site C and shows that isomorphism classes of G-torsors in $\mathrm{sPSh}(C)$ over $*$ are in a bijective correspondence with the set of connected components of $\mathrm{Maps}(*, \overline{W}G)$. The presentation that we discuss in Section 3.7.2 is similar, differing in that it allows the base space to be an *arbitrary* simplicial presheaf and in that it reproduces the *full homotopy type* of the space of cocycles, not just their connected components.

Closely related also is the discussion in [RS12, Ste3], which is concerned with principal ∞-bundles over topological spaces and in particular discusses their classification by traditional classifying spaces.

In summary, our work goes beyond that of all the works cited above in two directions; firstly we show that our notion of weakly principal bundle suffices to interpret the full homotopy type of the cocycle ∞-groupoid, and secondly, we work over arbitrary bases: our base need not be just a space, it could be a 1-stack or even an ∞-stack, or differentiable versions of all of these (we remark that the study of bundles on differentiable stacks plays an important role in recent work on twisted K-theory [FHT, LTX]).

2 Presentations of ∞-toposes

The presentations of principal ∞-bundles and related structures in an ∞-topos, discussed below in section 3, builds on the presentation of the ∞-topos itself by categories of simplicial (pre)sheaves. We assume the reader to be familiar with the basics of this theory (a good starting point is the appendix of [L09], a classical reference is [DK80]), but in order to set up our notation and in order to record some statements, needed below, which are not easily found in the literature in the explicit form in which we will need them, we briefly recall some basics in section 2.1. In 2.2 we discuss a general result about the representability of general objects in an ∞-topos by simplicial objects in the site.

2.1 By simplicial presheaves

The monoidal functor $\pi_0\colon \mathrm{sSet} \to \mathrm{Set}$ that sends a simplicial set to its set of connected components induces a functor $\mathrm{Ho}\colon \mathrm{sSetCat} \to \mathrm{Cat}$, where sSetCat denotes the category of sSet-enriched categories [K82]. Thus if $\mathcal{C}$ is an sSet-enriched category then $\mathrm{Ho}(\mathcal{C})$ is the category with the same underlying objects as $\mathcal{C}$ and with $\mathrm{Ho}(\mathcal{C})(X,Y) := \pi_0\mathcal{C}(X,Y)$ for all objects $X, Y \in \mathcal{C}$. An sSet-enriched functor $f : \mathcal{C} \to \mathcal{D}$ is called a *DK-equivalence* if $\mathrm{Ho}(f)$ is essentially surjective and if for all $X, Y \in \mathcal{C}$ the morphism $f_{X,Y} : \mathcal{C}(X,Y) \to \mathcal{D}(f(X), f(Y))$ is a weak homotopy equivalence. Write $W_{\mathrm{DK}} \subset \mathrm{sSetCat}$ for the inclusion of the full subcategory whose morphisms are DK-equivalences. This is a *wide subcategory*: an inclusion of categories that is bijective on objects.

For D a category and $W \subset D$ a wide subcategory, to be called the subcategory of *weak equivalences*, the *simplicial localization* $L_W D$ is the universal sSet-enriched category with the property that morphisms in $W \subset D$ become homotopy equivalences in $L_W D$ [DK80]. For $X, Y \in C$ two objects, the Kan complex $L_W D(X,Y)$ is called the *derived hom-space* or *derived function complex* or *hom-∞-groupoid* between these objects, in $L_W D$.

We write

$$\mathrm{Grpd}_\infty := L_{W_{\mathrm{wh}}}\mathrm{sSet}$$

for the simplicial localization of the category of simplicial sets at the simplicial weak homotopy equivalences, and we write

$$\mathrm{Cat}_\infty := L_{W_{\mathrm{DK}}}\mathrm{sSetCat}$$

for the simplicial localization of the category of simplicial categories at the Dwyer-Kan equivalences (both Grpd_∞ and Cat_∞ are large sSet-categories).

If a wide subcategory $W \subset D$ on a category D extends to the structure of an $\mathrm{sSet}_{\mathrm{Quillen}}$-enriched model category on D in the sense that W coincides with the class of weak equivalences for the model structure, then the full subcategory D° on the fibrant and cofibrant objects is enriched in Kan complexes and DK-equivalent to its simplicial localization:

$$D^\circ \simeq L_W D \quad \in \mathrm{Cat}_\infty \,.$$

We write $\mathrm{sSet}_{\mathrm{Quillen}}$ for the standard model category structure on simplicial sets, whose weak equivalences are the simplicial weak homotopy equivalences W_{wh} and whose fibrations are the Kan fibrations. Then $\mathrm{Grpd}_\infty \simeq \mathrm{KanCplx} = (\mathrm{sSet}_{\mathrm{Quillen}})^\circ$.

For C any category, there is a model structure $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ on the category of simplicial presheaves over C (the *projective* model structure), whose weak equivalences and fibrations are those transformations that are objectwise so in $\mathrm{sSet}_{\mathrm{Quillen}}$. If C is equipped with the structure of a *site* given by a (pre)topology, then there are corresponding localizations of the simplicial presheaves. We are interested here in the case that C has enough points.

Definition 2.1. A site C has *enough points* if a morphism $(A \xrightarrow{f} B) \in \mathrm{Sh}(C)$ in its sheaf topos is an isomorphism precisely if for every *topos point*, hence for every geometric morphism

$$(x^* \dashv x_*) : \ \mathrm{Set} \underset{x_*}{\overset{x^*}{\rightleftarrows}} \mathrm{Sh}(C)$$

from the topos Set of sets we have that $x^*(f) : x^* A \to x^* B$ is an isomorphism.

Notice here that, by definition of geometric morphism, the functor i^* is left adjoint to i_* – hence preserves all colimits – and in addition preserves all *finite* limits.

Example 2.2. The following sites have enough points.

- The categories Mfd (SmoothMfd) of (smooth) finite-dimensional, paracompact manifolds and smooth functions between them;
- the category CartSp of Cartesian spaces $\mathbb{R}^n$ for $n \in \mathbb{N}$ and continuous (smooth) functions between them.

These examples are discussed in more depth in [NSSc] — we refer the reader there for further details. We restrict from now on attention to the case that C has enough points.

A *C-local weak equivalence* in the category $[C^{\mathrm{op}}, \mathrm{sSet}]$ of simplicial presheaves is a natural transformation which is *stalkwise* a weak homotopy equivalence of simplicial sets. Let $W_C \subset [C^{\mathrm{op}}, \mathrm{sSet}]$ denote the wide sub-category of C-local weak equivalences. The simplicial localization

$$\mathrm{Sh}_\infty(C) := L_{W_C}[C^{\mathrm{op}}, \mathrm{sSet}] \quad \in \mathrm{Cat}_\infty$$

is the *hypercompletion* of the *∞-topos* of *∞-sheaves* or of *∞-stacks* over C.

This is the statement of Proposition 6.5.2.14 of [L09] together with Theorem 17 in [Ja96], which gives a refinement of the above weak equivalences to the *local injective model structure* $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{inj,loc}}$ whose cofibrations are the objectwise simplicial weak equivalences. We will be interested here instead in the *local projective model structure* $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ obtained as the left Bousfield localization of $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ at the covering sieve inclusions. For the *cohesive sites* C considered in Definition 3.37 below this localization will already be hypercomplete and hence we obtain the above ∞-topos equivalently as

$$\mathrm{Sh}_\infty(C) \simeq ([C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}})^\circ .$$

2.2 By simplicial objects in the site

Sometimes it is considered desireable to present an ∞-stack by a simplicial presheaf which in turn is presented by a simplicial object in the underlying site. We observe here that this is always possible provided the site has arbitrary coproducts.

Definition 2.3. Let C be a small site with enough points. Write $\bar{C} \subset [C^{\mathrm{op}}, \mathrm{sSet}]$ for the free coproduct completion. Let $(\bar{C}^{\Delta^{\mathrm{op}}}, W)$ be the category of simplicial objects in $\bar{C}$ equipped with the stalkwise weak equivalences inherited from the canonical embedding

$$i : \bar{C}^{\Delta^{\mathrm{op}}} \hookrightarrow [C^{\mathrm{op}}, \mathrm{sSet}] .$$

Example 2.4. Let C be a category of *connected* topological spaces with given extra structure and properties (for instance smooth manifolds). Then $\bar{C}$ is the category of all such spaces (with arbitrary many connected components).

Proposition 2.5. *The induced ∞-functor*

$$L_{W_C} \bar{C}^{\Delta^{\mathrm{op}}} \to L_{W_C}[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$$

is an equivalence of ∞-categories.

We will prove this shortly, after we have made the following observation.

Proposition 2.6. *Let C be a category and $\bar{C}$ its free coproduct completion. Then the following statements are true:*

1. *Every simplicial presheaf over C is equivalent in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ to a simplicial object in $\bar{C}$ under the image of the degreewise Yoneda embedding $j : \bar{C}^{\Delta^{\mathrm{op}}} \to [C^{\mathrm{op}}, \mathrm{sSet}]$.*

2. *If moreover C has pullbacks and sequential colimits, then the simplicial object in $\bar{C}$ can be taken to be globally Kan, hence fibrant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$.*

This proposition can be interpreted as follows: every ∞-stack over C has a presentation by a simplicial object in $\bar{C}$. Moreover this is true with respect to *any* Grothendieck topology on C, since the weak equivalences in the global projective model structure remain weak equivalences in any left Bousfield localization. If moreover C has all pullbacks (for instance for topological spaces, but not for smooth manifolds) then every ∞-stack over C even has a presentation by a globally Kan simplicial object in $\bar{C}$.

Proof. The first statement is Proposition 2.8 in [Dug01], which says that for every $X \in [C^{\mathrm{op}}, \mathrm{sSet}]$ the canonical morphism $QX \to X$ is a global weak equivalence. Here QX is the simplicial presheaf defined by the formula

$$(QX) : [k] \mapsto \coprod_{U_0 \to \cdots \to U_k \to X_k} j(U_0) ,$$

where the coproduct runs over all sequences of morphisms between representables U_i as indicated and with the evident face and degeneracy maps. The second statement follows by postcomposing with Kan's fibrant replacement functor (see for instance section 3 in [Ja87])

$$\mathrm{Ex}^\infty : \mathrm{sSet} \to \mathrm{KanCplx} \hookrightarrow \mathrm{sSet}.$$

This functor forms new simplices by subdivision, which only involves forming iterated pullbacks over the spaces of the original simplices. □

Proof of Proposition 2.5. Let $Q : [C^{\mathrm{op}}, \mathrm{sSet}] \to \bar{C}^{\Delta^{\mathrm{op}}}$ be Dugger's functor from the proof of Proposition 2.6. In [Dug01] it is shown that for all X the simplicial presheaf QX is cofibrant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ and that the natural morphism $QX \to X$ is a weak equivalence (as we have observed previously). Since left Bousfield localization does not affect the cofibrations and only enlarges the weak equivalences, the same is still true in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$.

Therefore we have a natural transformation

$$i \circ Q \to \mathrm{Id} : [C^{\mathrm{op}}, \mathrm{sSet}] \to [C^{\mathrm{op}}, \mathrm{sSet}]$$

whose components are weak equivalences. From this the claim of Proposition 2.5 follows by Proposition 3.5 in [DK80]. □

Remark 2.7. If the site C is moreover equipped with the structure of a *geometry* as in [L10] then there is a canonical notion of a C*-manifold*: a sheaf on C that is *locally* isomorphic to a representable in C. Write $C\mathrm{Mfd}$ for the full subcategory of the category of presheaves on the C-manifolds.

Then Proposition 2.6 applies to the category $C\mathrm{Mfd}^{\Delta^{\mathrm{op}}}$ of simplicial C-manifolds. Therefore we find that the ∞-topos over C is presented by the simplicial localization of simplicial C-manifolds at the stalkwise weak equivalences:

$$\mathrm{Sh}_\infty(C) \simeq L_{W_C} C\mathrm{Mfd}^{\Delta^{\mathrm{op}}} .$$

Example 2.8. Let $C = \mathrm{CartSp}_{\mathrm{smooth}}$ be the full subcategory of the category SmthMfd of smooth manifolds on the Cartesian spaces, $\mathbb{R}^n$, for $n \in \mathbb{R}$. Then $\bar{C} \subset \mathrm{SmthMfd}$ is the full subcategory on manifolds that are disjoint unions of Cartesian spaces and $C\mathrm{Mfd} \simeq \mathrm{SmthMfd}$. Therefore we have an equivalence of ∞-categories

$$\mathrm{Sh}_\infty(\mathrm{SmthMfd}) \simeq \mathrm{Sh}_\infty(\mathrm{CartSp}) \simeq L_{W_C}\, \mathrm{SmthMfd}^{\Delta^{\mathrm{op}}} .$$

Remark 2.9. While the above gives fairly general conditions on a site C under which every ∞-stack is presented by a simplicial object in the site, and in fact by a simplicial object which is cofibrant in the projective model structure on the simplicial presheaves over the site, this simplicial object is in general not fibrant in that model structure, nor will it be stalkwise fibrant in general.

In parts of the literature special attention is paid to ∞-stacks (or just stacks) that admit a presentation by a simplicial presheaf which is both: 1. represented by a simplicial object in the site and 2. stalkwise Kan fibrant in a suitable sense. (For instance Schommer-Pries discusses this for 1-stacks on manifolds and [Wo] (see there for further references) for ∞-stacks on manifolds.) It is an interesting question – which is open at the time of this writing – what these conditions on the presentation of an ∞-stack mean intrinsically, for instance if they can be interpreted as ensuring an abstract geometricity condition on an ∞-stack, such as considered for instance in [L10].

3 Presentation of structures in an ∞-topos

In the companion article [NSSa] we considered a list of structures present in any ∞-topos, which form the fabric for our discussion of principal (and associated/twisted) ∞-bundles. Here we go through the same list of notions and discuss aspects of their *presentation* in categories of simplicial (pre)sheaves.

3.1 Cones

Proposition 3.1. *Let $A \to C \leftarrow B$ be a cospan diagram in a model category. Sufficient conditions for the ordinary pullback $A \times_C B$ to be a homotopy pullback are*

- *one of the two morphisms is a fibration and all three objects are fibrant;*
- *one of the two morphisms is a fibration and the model structure is right proper.*

This appears for instance as Proposition A.2.4.4 in [L09].

Proposition 3.2. *A finite homotopy limit computed in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ presents also the homotopy limit in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$.*

Proposition 3.3. *For C a model category and $X \in C$ any object, the slice category $C_{/X}$ inherits a model category structure transferred along the forgetful functor $C_{/X} \to C$. If X is fibrant in C, then $C_{/X}$ presents the slice of the ∞-category presented by C:*

$$\left(C_{/X}\right)^{\circ} \simeq (C^{\circ})_{/X}\,.$$

3.2 Effective epimorphisms

We discuss aspects of the presentation of effective epimorphisms in an ∞-topos. We begin with the following observation.

Observation 3.4. *If the ∞-topos $\mathbf{H}$ is presented by a category of simplicial presheaves, Section 2.1, then for X a simplicial presheaf, the canonical morphism $\mathrm{const}X_0 \to X$ in $[C^{\mathrm{op}}, \mathrm{sSet}]$ that includes the presheaf of 0-cells as a simplicially constant simplicial presheaf presents an effective epimorphism in $\mathbf{H}$.*

This follows with Proposition 7.2.1.14 in [L09].

Remark 3.5. In practice the presentation of an ∞-stack by a simplicial presheaf is often taken to be understood, and then Observation 3.4 induces also a canonical *atlas*, i.e. $\mathrm{const}X_0 \to X$.

We now discuss a fibration resolution of the canonical atlas. Write Δ_a for the *augmented simplex category*, which is the simplex category Δ with an initial object $[-1]$ (the empty set) adjoined. The operation of *ordinal sum*

$$[k], [l] \mapsto [k + l + 1]$$

equips Δ_a with the structure of a symmetric monoidal category with unit $[-1]$ (see for instance [McL98]). Write

$$\sigma : \Delta \times \Delta \to \Delta$$

for the restriction of this tensor product along the canonical inclusion $\Delta \subset \Delta_a$.

Definition 3.6. Write

$$\mathrm{Dec}_0 : \mathrm{sSet} \to \mathrm{sSet}$$

for the functor given by precomposition with $\sigma(-,[0]) : \Delta \to \Delta$. This is called the plain *décalage functor* or *shifting functor*.

This functor was introduced in [I72]. A discussion in the present context can be found in section 2.2 of [Ste2], amongst other references.

Proposition 3.7. *Let X be a simplicial set. Then $\mathrm{Dec}_0 X$ is isomorphic to the simplicial set*

$$[n] \mapsto \mathrm{Hom}(\Delta[n] \star \Delta[0], X),$$

where $(-)\star(-) : \mathrm{sSet} \times \mathrm{sSet} \to \mathrm{sSet}$ is the join *of simplicial sets. The canonical inclusions $\Delta[n], \Delta[0] \subset \Delta[n] \star \Delta[0]$ induce morphisms*

$$\begin{tikzcd} \mathrm{Dec}_0 X \arrow[r, two heads] \arrow[d, "\simeq"] & X \\ \mathrm{const} X_0 & \end{tikzcd},$$

where

- *the horizontal morphism is given in degree n by $d_{n+1} : X_{n+1} \to X_n$;*
- *the horizontal morphism is a Kan fibration if X is a Kan complex;*
- *the vertical morphism is a simplicial deformation retraction, in particular a weak homotopy equivalence.*

Proof. The relation to the join of simplicial sets is clear (the point being that the nerve functor sends joins of categories to joins of simplicial sets). The deformation retraction is classical and can be found in many sources. To see that $\mathrm{Dec}_0 X \to X$ is a Kan fibration, using the fact that $(\mathrm{Dec}_0 X)_n = \mathrm{Hom}(\Delta[n] \star \Delta[0], X)$ for any $n \in \mathbb{N}$, we see that the lifting problem for the diagram

$$\begin{tikzcd} \Lambda^i[n] \arrow[r] \arrow[d] & \mathrm{Dec}_0 X \arrow[d] \\ \Delta[n] \arrow[r] & X \end{tikzcd}$$

has a solution if and only if it is the lifting problem for the diagram

$$\begin{tikzcd} (\Lambda^i[n] \star \Delta[n]) \coprod_{\Lambda^i[n]} \Delta[n] \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta[n] \star \Delta[0] \arrow[r] & * \end{tikzcd}$$

has a solution. Here the left hand vertical morphism is an anodyne morphism — in fact an inclusion of an $(n+1)$-horn. Hence a lift exists if X is a Kan complex. (Alternatively, one may argue by observing that $\mathrm{Dec}_0 X$ is the disjoint union of slices $X_{/x}$ for $x \in X_0$, and it is known that $X_{/x} \to X$ is a Kan fibration if X is a Kan complex — see for instance [L09]). □

Corollary 3.8. *For X in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ fibrant, a fibration resolution of the canonical effective epimorphism* $\mathrm{const}X_0 \to X$ *from Observation 3.4 is given by the décalage morphism* $\mathrm{Dec}_0 X \to X$*, Proposition 3.7.*

Proof. It only remains to observe that we have a commuting diagram

$$\begin{tikzcd} \mathrm{const}X_0 \arrow[r, "s"] \arrow[d] & \mathrm{Dec}_0 X \arrow[d] \\ X \arrow[r, "="] & X \end{tikzcd}$$

where the top morphism, given degreewise by the degeneracy maps in X, is a weak homotopy equivalence by classical results. □

3.3 Connected objects

In every ∞-topos $\mathbf{H}$ there is a notion of *connected objects*, which form the objects of the full sub-∞-category $\mathbf{H}_{\geq 1}$. We discuss here presentations of connected and of *pointed* connected objects in $\mathbf{H}$ by means of presheaves of pointed or reduced simplicial sets.

Observation 3.9. *Under the presentation* $\mathrm{Grpd}_\infty \simeq (\mathrm{sSet}_{\mathrm{Quillen}})^\circ$*, a Kan complex* $X \in \mathrm{sSet}$ *presents an n-connected ∞-groupoid precisely if*

1. *X is inhabited (not empty);*
2. *all simplicial homotopy groups $\pi_k(X)$ of X in degree $k \leq n$ are trivial.*

Definition 3.10. For $n \in \mathbb{N}$ a simplicial set $X \in \mathrm{sSet}$ is *n-reduced* if its n-skeleton is the point

$$\mathrm{sk}_n X = * \,,$$

in other words, if it has a single k-simplex for all $k \leq n$. For *0-reduced* we also just say *reduced.* Write

$$\mathrm{sSet}_n \hookrightarrow \mathrm{sSet}$$

for the full subcategory of n-reduced simplicial sets.

Proposition 3.11. *The n-reduced simplicial sets form a reflective subcategory*

$$\begin{tikzcd} \mathrm{sSet}_n \arrow[r, hook, shift right] & \mathrm{sSet} \arrow[l, "\mathrm{red}_n"', shift right] \end{tikzcd}$$

of the category of simplicial sets, with the reflector red_n *given on a simplicial set X by* $\mathrm{red}_n(X) = X/\mathrm{sk}_n X$*, in other words it identifies all the $(k \leq n)$-vertices of X.*

The inclusion $\mathrm{sSet}_n \hookrightarrow \mathrm{sSet}$ *uniquely factors through the forgetful functor* $\mathrm{sSet}^{*/} \to \mathrm{sSet}$ *from pointed simplicial sets, and that factorization is co-reflective*

$$\begin{tikzcd} \mathrm{sSet}_n \arrow[r, hook, shift left] & \mathrm{sSet}^{*/} \arrow[l, shift left, "E_{n+1}"] \end{tikzcd} .$$

Here the co-reflector E_{n+1} *sends a pointed simplicial set* $* \xrightarrow{x} X$ *to the sub-object* $E_{n+1}(X, x)$*, the* $(n+1)$*st Eilenberg subcomplex of the pointed simplicial set* X*.*

Remark 3.12. Recall, see for instance Definition 8.3 in [May67], that for a pointed simplicial set $* \xrightarrow{x} X$, the simplicial set $E_{n+1}(X, x)$ is the subcomplex of X consisting of cells whose n-faces coincide with the base point, hence is the fiber

$$\begin{tikzcd} E_{n+1}(X,x) \arrow[r] \arrow[d] & X \arrow[d] \\ \{*\} \arrow[r] & \mathrm{cosk}_n X \end{tikzcd}$$

of the projection to the n-coskeleton $\mathrm{cosk}_n X$.

For $(* \to X) \in \mathrm{sSet}^{*/}$ such that $X \in \mathrm{sSet}$ is Kan fibrant and n-connected, the counit $E_{n+1}(X, *) \to X$ is a homotopy equivalence. This statement appears for instance as part of Theorem 8.4 in [May67].

Proposition 3.13. *Let* C *be a site with a terminal object and let* $\mathbf{H} := \mathrm{Sh}_\infty(C)$*. Then under the presentation* $\mathbf{H} \simeq ([C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}})^\circ$ *every pointed* n*-connected object in* $\mathbf{H}$ *is presented by a presheaf of* n*-reduced simplicial sets, under the canonical inclusion* $[C^{\mathrm{op}}, \mathrm{sSet}_n] \hookrightarrow [C^{\mathrm{op}}, \mathrm{sSet}]$*.*

Proof. Let $X \in [C^{\mathrm{op}}, \mathrm{sSet}]$ be a simplicial presheaf presenting the given pointed, connected object. Then its objectwise Kan fibrant replacement $\mathrm{Ex}^\infty X$ is still a presentation, fibrant in the global projective model structure. Since the terminal object in $\mathbf{H}$ is presented by the terminal simplicial presheaf and since by assumption on C this is representable and hence cofibrant in the projective model structure, the point inclusion is presented by a morphism of simplicial presheaves $* \to \mathrm{Ex}^\infty X$, hence by a presheaf of pointed simplicial sets $(* \to \mathrm{Ex}^\infty X) \in [C^{\mathrm{op}}, \mathrm{sSet}^{*/}]$. So with Proposition 3.11 we obtain the presheaf of n-reduced simplicial sets

$$E_{n+1}(\mathrm{Ex}^\infty X, *) \in [C^{\mathrm{op}}, \mathrm{sSet}_n] \hookrightarrow [C^{\mathrm{op}}, \mathrm{sSet}]$$

and the inclusion $E_{n+1}(\mathrm{Ex}^\infty X, *) \to \mathrm{Ex}^\infty X$ is a global weak equivalence, hence a local weak equivalence, hence exhibits $E_{n+1}(\mathrm{Ex}^\infty X, *)$ as another presentation of the object in question. □

We next describe a slightly enhanced version of the model structure on reduced simplicial sets introduced by Quillen in [Q69].

Proposition 3.14. *The category* sSet_0 *of reduced simplicial sets carries a left proper combinatorial model category structure whose weak equivalences and cofibrations are those in* $\mathrm{sSet}_{\mathrm{Quillen}}$ *under the inclusion* $\mathrm{sSet}_0 \hookrightarrow \mathrm{sSet}$*.*

Proof. This enhanced version of the classical theorem from [Q69] follows from Proposition A.2.6.13 in [L09], taking the set C_0 there to be

$$C_0 := \{\mathrm{red}(\Lambda^k[n] \to \Delta[n])\}_{n\in\mathbb{N}, 0\leq k\leq n}\,,$$

the image of the generating cofibrations in $\mathrm{sSet}_{\mathrm{Quillen}}$ under the left adjoint red to the inclusion functor (Proposition 3.11). □

Lemma 3.15. *A fibration $f\colon X \to Y$ in sSet_0 (for the model structure of Proposition 3.14) is a Kan fibration precisely if it has the right lifting property against the morphism $(* \to S^1) := \mathrm{red}(\Delta[0] \to \Delta[1])$. In particular every fibrant object in sSet_0 is a Kan complex.*

Proof. The first statement appears as V Lemma 6.6. in [GJ99]. The second (an immediate consequence) as V Corollary 6.8. □

Proposition 3.16. *The adjunction*

$$\mathrm{sSet}_0 \underset{E_1}{\overset{i}{\rightleftarrows}} \mathrm{sSet}^{*/}_{\mathrm{Quillen}}$$

from Proposition 3.11 is a Quillen adjunction between the model structure from Proposition 3.14 and the co-slice model structure, Proposition 3.3, of $\mathrm{sSet}_{\mathrm{Quillen}}$ under the point. This presents the full inclusion

$$(\mathrm{Grpd}_\infty)^{*/}_{\geq 1} \hookrightarrow {\mathrm{Grpd}_\infty}^{*/}$$

of connected pointed ∞-groupoids into all pointed ∞-groupoids.

Proof. It is clear that the inclusion $i\colon \mathrm{sSet}_0 \hookrightarrow \mathrm{sSet}^{*/}_{\mathrm{Quillen}}$ preserves cofibrations and acyclic cofibrations, in fact all weak equivalences. Since the point is cofibrant in $\mathrm{sSet}_{\mathrm{Quillen}}$, the model structure on the right is by Proposition 3.3 indeed a presentation of $\mathrm{Grpd}^{*/}_\infty$.

We claim now that the derived ∞-adjunction of this Quillen adjunction presents a homotopy full and faithful inclusion whose essential image consists of the connected pointed objects. To show this it is sufficient to show that for the derived functors there is a natural weak equivalence

$$\mathrm{id} \simeq \mathbb{R}E_1 \circ \mathbb{L}i\,.$$

This is the case, because by Proposition 3.15 the composite derived functors are computed by the composite ordinary functors precomposed with a fibrant replacement functor P, so that we have a natural morphism

$$X \overset{\simeq}{\to} PX = E_1 \circ i(PX) \simeq (\mathbb{R}E_1) \circ (\mathbb{L}i)(X)\,.$$

Hence $\mathbb{L}i$ is homotopy full-and faithful and by Proposition 3.13 its essential image consists of the connected pointed objects. □

3.4 Groupoids

We discuss aspects of the presentation of *groupoid objects* in an ∞-sheaf topos $\mathbf{H} = \mathrm{Sh}_\infty(C)$, notably of the realization ∞-functor

$$\varinjlim : \mathrm{Grpd}(\mathbf{H}) \to \mathbf{H}$$

given by the ∞-colimit over the underlying simplicial diagram of the groupoid object.

In [Ber08b] a presentation of groupoid objects in $\infty\mathrm{Grpd}$ is discussed in terms of simplicial objects in $\mathrm{sSet}_{\mathrm{Quillen}}$, called 'invertible Segal spaces' in [Ber08b]. This has a straightforward generalization to a presentation of groupoid objects in a sheaf ∞-topos $\mathrm{Sh}_\infty(C)$ by simplicial objects in a category of simplicial presheaves. We discuss here a presentation of homotopy colimits over such simplical diagrams given by the *diagonal simplicial set* or the *total simplicial set* associated with a bisimplicial set. This serves as the basis for the discussion of universal weakly principal simplicial bundles below in Section 3.7.1. For some general background on homotopy colimits the way we need them here, a good survey is [Gam10].

Proposition 3.17. *Write* $[\Delta, \mathrm{sSet}]$ *for the category of cosimplicial simplicial sets. For* sSet *equipped with its cartesian monoidal structure, the tensor unit is the terminal object* $*$.

- *The* simplex functor
$$\Delta : [n] \mapsto \Delta[n] := \Delta(-, [n])$$
is a cofibrant resolution of $*$ *in* $[\Delta, \mathrm{sSet}_{\mathrm{Quillen}}]_{\mathrm{Reedy}}$*;*
- *the* fat simplex functor
$$\mathbf{\Delta} : [n] \mapsto N(\Delta/[n])$$
is a cofibrant resolution of $*$ *in* $[\Delta, \mathrm{sSet}_{\mathrm{Quillen}}]_{\mathrm{proj}}$.

Proposition 3.18. *Let* C *be a simplicial model category and* $F : \Delta^{\mathrm{op}} \to C$ *a simplicial diagram*

1. *If every monomorphism in* C *is a cofibration, then the homotopy colimit over* F *is given by the realization, i.e.*
$$\mathbb{L}\varinjlim F \simeq \int^{[n]\in\Delta} F([n]) \cdot \Delta[n]\,.$$
2. *If* F *takes values in cofibrant objects, then the homotopy colimit over* F *is given by the fat realization, i.e.*
$$\mathbb{L}\varinjlim F \simeq \int^{[n]\in\Delta} F([n]) \cdot \mathbf{\Delta}[n]\,.$$
3. *If* F *is Reedy cofibrant, then the canonical morphism*
$$\int^{[n]\in\Delta} F([n]) \cdot \mathbf{\Delta}[n] \to \int^{[n]\in\Delta} F([n]) \cdot \Delta[n]$$
(the Bousfield-Kan map*) is a weak equivalence.*

Proposition 3.19. *The homotopy colimit of a simplicial diagram in* $\mathrm{sSet}_{\mathrm{Quillen}}$*, or more generally of a simplicial diagram of simplicial presheaves, is given by the diagonal of the corresponding bisimplicial set / bisimplicial presheaf.*

More precisely, for

$$F : \Delta^{\mathrm{op}} \to [C^{\mathrm{op}}, \mathrm{sSet}_{\mathrm{Quillen}}]_{\mathrm{inj,loc}}$$

a simplicial diagram, its homotopy colimit is given by

$$\mathbb{L} \varinjlim F_\bullet \simeq dF : ([n] \mapsto (F_n)_n)\,.$$

Proof. By Proposition 3.18 the homotopy colimit is given by the coend

$$\mathbb{L} \varinjlim F_\bullet \simeq \int^{[n]\in\Delta} F_n \cdot \Delta[n]\,.$$

By a standard fact (e.g. exercise 1.6 in [GJ99]), this coend is in fact isomorphic to the diagonal. □

Definition 3.20. Let $\sigma\colon \Delta \times \Delta \to \Delta$ denote ordinal sum. Write

$$\sigma^* : \mathrm{sSet} \to [\Delta^{\mathrm{op}}, \mathrm{sSet}]$$

for the operation of precomposition with this functor. By right Kan extension this induces an adjoint pair of functors

$$(\sigma_* \dashv \sigma^*) :\ [\Delta^{\mathrm{op}}, \mathrm{sSet}] \underset{\sigma_*}{\overset{\sigma^*}{\rightleftarrows}} \mathrm{sSet},$$

where

- $\mathrm{Dec} := \sigma^*$ is called the *total décalage* functor;
- σ_* is called the *total simplicial set* functor.

The total simplicial set functor was introduced in [AM66], for further discussion see [CR05, Ste2].

Remark 3.21. By definition, for $X \in \mathrm{sSet}$, its total décalage is the bisimplicial set $\mathrm{Dec}X$ whose set of (k,l) bisimplices is given by

$$(\mathrm{Dec}X)_{k,l} = X_{k+l+1}\,.$$

Remark 3.22. For $X \in [\Delta^{\mathrm{op}}, \mathrm{sSet}]$, the simplicial set $\sigma_* X$ is in each degree given by an equalizer of maps between finite products of components of X (see for instance equation (2) of [Ste2]). Hence forming σ_* is compatible with sheafification and other processes that preserve finite limits.

Proposition 3.23. *The following statements are true:*

- *for every* $X \in [\Delta^{\mathrm{op}}, \mathrm{sSet}]$*, the canonical morphism*

$$dX \to \sigma_* X$$

from the diagonal to the total simplicial set is a weak equivalence in $\mathrm{sSet}_{\mathrm{Quillen}}$*;*

- *for every $X \in \mathrm{sSet}$ the adjunction unit*

$$X \to \sigma_* \sigma^* X$$

is a weak equivalence in $\mathrm{sSet}_{\mathrm{Quillen}}$.

For every $X \in \mathrm{sSet}$

- *there is a natural isomorphism $\sigma_* \mathrm{const} X \simeq X$.*

These statements are due to Cegarra and Remedios in [CR05] and independently Joyal and Tierney (unpublished) — see also [Ste2].

Corollary 3.24. *For*

$$F : \Delta^{\mathrm{op}} \to [C^{\mathrm{op}}, \mathrm{sSet}_{\mathrm{Quillen}}]_{\mathrm{inj,loc}}$$

a simplicial object in simplicial presheaves, its homotopy colimit is given by applying object-wise over each $U \in C$ the total simplicial set functor σ_*, i.e.

$$\mathbb{L} \varinjlim F \simeq (U \mapsto \sigma_* F(U)) \,.$$

Proof. By Proposition 3.23 this follows from Proposition 3.19. □

Remark 3.25. The use of the total simplicial set instead of the diagonal simplicial set in the presentation of simplicial homotopy colimits is useful and reduces to various traditional notions in particular in the context of group objects and action groupoid objects. We discuss this further in Section 3.5 and Section 3.7.1 below.

3.5 Groups

Every ∞-topos $\mathbf{H}$ comes with a notion of ∞-*group object* that generalizes the ordinary notion of group object in a topos as well as that of grouplike A_∞ space in $\mathrm{Top} \simeq \mathrm{Grpd}_\infty$. We discuss presentations of ∞-group objects by presheaves of simplicial groups.

Definition 3.26. One writes $\overline{W}$ for the composite functor from simplicial groups to simplicial sets given by

$$\overline{W} \colon [\Delta^{\mathrm{op}}, \mathrm{Grp}] \xrightarrow{[\Delta^{\mathrm{op}}, \mathbf{B}]} [\Delta^{\mathrm{op}}, \mathrm{Grpd}] \xrightarrow{[\Delta^{\mathrm{op}}, N]} [\Delta^{\mathrm{op}}, \mathrm{sSet}] \xrightarrow{\sigma_*} \mathrm{sSet},$$

where $[\Delta^{\mathrm{op}}, \mathbf{B}] : [\Delta^{\mathrm{op}}, \mathrm{Grp}] \to [\Delta^{\mathrm{op}}, \mathrm{Grpd}]$ is the functor from simplicial groups to simplicial groupoids that degreewise sends a group to the corresponding one-object groupoid.

This simplicial delooping $\overline{W}$ was originally introduced in [McL54]. The above formulation is due to Duskin, see Lemma 15 in [Ste2].

Remark 3.27. The functor $\overline{W}$ takes values in *reduced* simplicial sets, i.e. $\overline{W} \colon [\Delta^{\mathrm{op}}, \mathrm{Grp}] \to \mathrm{sSet}_{\mathrm{red}}$.

Remark 3.28. For G a simplicial group, the simplicial set $\overline{W}G$ is, by Corollary 3.24, the homotopy colimit over a simplicial diagram in simplicial sets. Below in 3.7.2 we see that this simplicial diagram is that presenting the groupoid object $*//G$ which is the action groupoid of G acting trivially on the point.

Proposition 3.29. *The category* sGrp *of simplicial groups carries a cofibrantly generated model structure for which the fibrations and the weak equivalences are those of* $\mathrm{sSet}_{\mathrm{Quillen}}$ *under the forgetful functor* $\mathrm{sGrpd} \to \mathrm{sSet}$.

Proof. This is originally due to [Q67], for a more recent account see V Theorem 2.3 in [GJ99]. Note that since the model structure is therefore transferred along the forgetful functor, it inherits generating (acyclic) cofibrations from those of $\mathrm{sSet}_{\mathrm{Quillen}}$. □

We now consider a presentation of the looping/delooping equivalence $\mathrm{Grp}(\mathbf{H}) \simeq \mathbf{H}^{*/}_{\geq 1}$ due to Lurie, recalled as Theorem 2.14 in [NSSa].

Theorem 3.30 ([Q69])**.** *The functor* $\overline{W}$ *is the right adjoint of a Quillen equivalence*

$$(L \dashv \overline{W}) : \ \mathrm{sGrp} \underset{\overline{W}}{\overset{L}{\rightleftarrows}} \mathrm{sSet}_0$$

with respect to the model structures of Proposition 3.29 and Proposition 3.14. In particular

- *the adjunction unit is a weak equivalence*

$$Y \stackrel{\simeq}{\to} \overline{W} L Y$$

for every reduced simplicial set Y*,*

- $\overline{W}G$ *is a Kan complex for any simplicial group* G*.*

This result is discussed for instance in chapter V of [GJ99]; a new proof that the unit of the adjunction is a weak equivalence is given in [Ste2].

Definition 3.31. For G a simplicial group, write $WG = \mathrm{Dec}_0 \overline{W}G$ (see Definition 3.6) and write

$$WG \to \overline{W}G$$

for the canonical morphism $\mathrm{Dec}_0 \overline{W}G \to \overline{W}G$ of Corollary 3.8.

This morphism is the standard presentation of the *universal* G*-principal simplicial bundle*. We discuss this further in Section 3.7.1 below. The characterization by décalage of the total space WG is made fairly explicit on p. 85 of [Dus75]; a fully explicit statement can be found in [RS12].

Proposition 3.32. *The morphism* $WG \to \overline{W}G$ *is a Kan fibration resolution of the point inclusion* $* \to \overline{W}G$*.*

Proof. This follows directly from the characterization of $WG \to \overline{W}G$ by décalage (Corollary 3.8). □

This statement appears in [May67] as the union of two results there: Lemma 18.2 of [May67] gives the fibration property; Proposition 21.5 of [May67] gives the contractibility of WG.

Corollary 3.33. *For G a simplicial group, the sequence of simplicial sets*

$$G \longrightarrow WG \longrightarrow\!\!\!\!\!\rightarrow \overline{W}G$$

is a presentation in $\mathrm{sSet}_{\mathrm{Quillen}}$ *by a pullback of a Kan fibration of the looping fiber sequence*

$$G \to * \to \mathbf{B}G$$

in Grpd_∞.

Proof. One finds that G is the 1-categorical fiber of $WG \to \overline{W}G$. The statement then follows using Proposition 3.32 together with Proposition 3.1. □

The universality of $WG \to \overline{W}G$ for G-principal simplicial bundles is the topic of section 21 in [May67].

Corollary 3.34. *The Quillen equivalence $(L \dashv \overline{W})$ from Theorem 3.30 is a presentation of the looping/delooping equivalence* $\mathrm{Grp}(\mathbf{H}) \simeq \mathbf{H}^{*/}_{\geq 1}$ *for the ∞-topos $\mathbf{H} = \mathrm{Grpd}_\infty$.*

We now lift all these statements from simplicial sets to simplicial presheaves.

Proposition 3.35. *If the cohesive ∞-topos $\mathbf{H}$ has site of definition C with a terminal object, then*

- *every ∞-group object has a presentation by a presheaf of simplicial groups*

  $$G \in [C^{\mathrm{op}}, \mathrm{sGrp}] \stackrel{U}{\to} [C^{\mathrm{op}}, \mathrm{sSet}]$$

  which is fibrant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$*;*

- *the corresponding delooping object is presented by the presheaf*

  $$\overline{W}G \in [C^{\mathrm{op}}, \mathrm{sSet}_0] \hookrightarrow [C^{\mathrm{op}}, \mathrm{sSet}]$$

  obtained from G by applying the functor $\overline{W}$ objectwise.

Proof. By the fact recalled as Theorem 2.14 in [NSSa], every ∞-group is the loop space object of a pointed connected object. By Proposition 3.13 every such object is presented by a presheaf X of reduced simplicial sets. By the simplicial looping/delooping Quillen equivalence, Theorem 3.30, the presheaf

$$\overline{W}LX \in [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$$

is objectwise weakly equivalent to the simplicial presheaf X. From this the statement follows with Corollary 3.33, combined with Proposition 3.2, which together say that the presheaf LX of simplicial groups presents the given ∞-group. □

Remark 3.36. We may read this as saying that every ∞-group may be *strictified*.

3.6 Cohomology

We discuss presentations of the *hom-∞-groupoids*, hence of *cocycle ∞-groupoids*, hence of the cohomology in an ∞-topos.

We consider two roughly complementary aspects

- In section 3.6.1 we study sufficient conditions on a simplicial presheaf A such that the ordinary simplicial hom $[C^{\mathrm{op}}, \mathrm{sSet}](Y, A)$ out of a split hypercover $Y \longrightarrow X$ is already the correct derived hom out of X. Since this simplicial hom is the Kan complex of simplicial hyper-Čech cocycles relative to Y with coefficients in A, this may be taken to be a sufficient condition for A-Čech cohomology to produce the correct intrinsic cohomology.

- In section 3.6.2 we consider not a full model category structure but just the structure of a *category of fibrant objects*. In this case there is no notion of split hypercover and instead one has to consider all possible covers and refinements between them. A central result of [Bro73] shows that this produces the correct cohomology classes. Here we discuss the refinement of this classical statement to the full cocycle ∞-groupoids.

3.6.1 By hyper-Čech-cohomology in C-acyclic simplicial groups

The condition on an object $X \in [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ to be fibrant models the fact that X is an ∞-*presheaf of ∞-groupoids*. The condition that X is also fibrant as an object in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ models the higher analog of the sheaf condition: it makes X an ∞-*sheaf*/∞-stack. For generic sites, C-fibrancy in the local model structure is a property rather hard to check or establish concretely. But often a given site can be replaced by another site on which the condition is easier to control, without changing the corresponding ∞-topos, up to equivalence. Here we discuss a particularly nice class of sites called ∞-*cohesive sites* [Sch], and describe explicit conditions for a simplicial presheaf over them to be fibrant.

Definition 3.37. A site C is ∞-*cohesive* if

1. it has a terminal object;
2. there is a generating coverage such that for every generating cover $\{U_i \to U\}$ we have
   (a) the Čech nerve $\check{C}(\{U_i\}) \in [C^{\mathrm{op}}, \mathrm{sSet}]$ is degreewise a coproduct of representables;
   (b) the limit and colimit functors, $\varprojlim \colon [C^{\mathrm{op}}, \mathrm{sSet}] \to \mathrm{sSet}$ and $\varinjlim \colon [C^{\mathrm{op}}, \mathrm{sSet}] \to \mathrm{sSet}$ respectively, send the Čech nerve projection $\check{C}(\{U_i\}) \to U$ to a weak homotopy equivalence:
   $$\varinjlim \check{C}(\{U_i\}) \xrightarrow{\simeq} \varinjlim U = *$$
   and
   $$\varprojlim \check{C}(\{U_i\}) \xrightarrow{\simeq} \varprojlim U \,.$$

We call the generating covers satisfying the conditions of 2 (b) the *good covers* in C.

Remark 3.38. Since C is assumed to have a terminal object, the limit over a functor $C^{\mathrm{op}} \to \mathrm{Set}$ is the evaluation on that object:

$$\varprojlim U = C(*, U) \,.$$

On the other hand, the colimit of a representable Set-valued functor is the singleton set: $\varinjlim U \simeq *$. Therefore together with the assumption that the Čech nerve is degreewise representable the condition $\varinjlim \check{C}(\{U_i\}) \xrightarrow{\simeq} \varinjlim U$ says that the simplicial set obtained from the Čech nerve by replacing each k-fold intersection with an abstract k-simplex is contractible.

This last condition is familiar from the *nerve theorem* [Bor48]:

Theorem 3.39. *Let X be a paracompact topological space. Let $\{U_i \to X\}$ be a good open cover (all non-empty k-fold intersections $U_{i_1} \cap \cdots U_{i_k}$ for $k \in \mathbb{N}$ are homeomorphic to an open ball). Then the simplicial set*

$$\Pi(X) := \mathrm{contr}\left(\int^{[k]\in\Delta} \coprod_{i_0,\cdots,i_k} U_{i_0} \cap \cdots \cap U_{i_k}\right) = \int^{[k]\in\Delta} \coprod_{i_0,\cdots,i_k} * \in \mathrm{sSet}\,,$$

where contr *is the functor that degreewise sends contractible spaces to points, is weakly homotopy equivalent to the singular simplicial set of X:*

$$\Pi(X) \xrightarrow{\simeq} \mathrm{Sing}X,$$

and hence presents the homotopy type of X.

Remark 3.40. The conditions on an ∞-cohesive site ensure that the Čech nerve of a good cover is cofibrant in the projective model structure $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ and hence also in its localization $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$.

In order to discuss descent over C it is convenient to introduce the following notation for 'cohomology over the site C'. For the moment this is just an auxiliary technical notion. Later we will see how it relates to an intrinsically defined notion of cohomology.

Definition 3.41. For C an ∞-cohesive site, $A \in [C^{\mathrm{op}}, \mathrm{Set}]_{\mathrm{proj}}$ fibrant, and $\{U_i \to U\}$ a good cover in U, we write

$$H^0_C(\{U_i\}, A) := \pi_0 \mathrm{Maps}(\check{C}(\{U_i\}), A)\,.$$

Moreover, if A is equipped with the structure of a group object (respectively an abelian group object) we write

$$H^n_C(\{U_i\}, A) := \pi_0 \mathrm{Maps}(\check{C}(\{U_i\}), \overline{W}^n A),$$

if $n = 1$ (respectively $n \geq 1$). Here $\mathrm{Maps}(-,-)$ denotes the usual simplicial mapping space in $[C^{\mathrm{op}}, \mathrm{sSet}]$.

As is described in [Ja07] the homotopy groups of a simplicial set X have a base-point free interpretation as group objects over X_0: one defines $\pi_0(X)$ as a colimit in the usual way as

$$\pi_0(X) = \varinjlim (X_1 \rightrightarrows X_0),$$

and, for any integer $n \geq 1$, one defines

$$\pi_n(X) = \coprod_{x \in X_0} \pi_n(X, x),$$

so that $\pi_n(X) \to X_0$ has a natural structue as a group object over X_0. If now $X \in [C^{\mathrm{op}}, \mathrm{sSet}]$ we can perform these constructions object-wise to form presheaves $\pi_0^{\mathrm{PSh}}(X)$ and $\pi_n^{\mathrm{PSh}}(X)$, so that

$$\pi_n^{\mathrm{PSh}}(X)(U) = \coprod_{x_U \in X_0(U)} \pi_n(X(U), x_U)$$

for instance. Note that both constructions are functorial in X, and that $\pi_n^{\mathrm{PSh}}(X) \to X_0$ is a group object over X_0 in $[C^{\mathrm{op}}, \mathrm{sSet}]$. If $x_U \in X_0(U)$ then we define the presheaf $\pi_n(X, x_U)$ by the pullback diagram

$$\begin{tikzcd} \pi_n(X, x_U) \arrow[r] \arrow[d] & \pi_n(X) \arrow[d] \\ U \arrow[r, "x_U"] & X_0 \end{tikzcd}$$

so that $\pi_n(X, x_U)$ is naturally a presheaf of groups on the slice $C_{/U}$. Following [Ja07] we make the following definition.

Definition 3.42. Let C be a site, and let

$$\pi_0^{\mathrm{PSh}} \colon [C^{\mathrm{op}}, \mathrm{sSet}] \to \mathrm{PSh}(C)$$

and

$$\pi_n^{\mathrm{PSh}} \colon [C^{\mathrm{op}}, \mathrm{sSet}] \to \mathrm{PSh}(C),$$

for $n \geq 1$, denote the functors described above. We write

$$\pi_0 \colon [C^{\mathrm{op}}, \mathrm{sSet}] \to \mathrm{Sh}(C)$$

and, for $n \geq 1$,

$$\pi_n \colon [C^{\mathrm{op}}, \mathrm{sSet}] \to \mathrm{Sh}(C)$$

for their sheafified versions.

Note that if X is a simplicial presheaf on C, then $\pi_n(X)$ is naturally a group object over the sheaf associated to X_0. Using this we can state the main definition of this section.

Definition 3.43. An object $A \in [C^{\mathrm{op}}, \mathrm{sSet}]$ is called *C-acyclic* if

1. it is fibrant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$;
2. for all $n \in \mathbb{N}$ we have $\pi_n^{\mathrm{PSh}}(A) = \pi_n(A)$, in other words the homotopy group presheaves from Definition 3.42 are already sheaves;
3. the sheaves $\pi_n^{PSh}(A)$ are acyclic with respect to good covers; i.e. for every object U, for every point $a_U \in A_0(U)$, and for all good covers $\{U_i \to U\}$ of U, we have

   $$H^1_C(\{U_i\}, \pi_1(A, a_U)) = 1$$

   and

   $$H^k_C(\{U_i\}, \pi_n(A, a_U)) = 1$$

   for all $k \geq 1$ if $n \geq 2$.

Remark 3.44. This definition can be formulated and the following statements about it are true over any site whatsoever. However, on generic sites C the C-acyclic objects are not very interesting. They become interesting on sites such as the ∞-cohesive sites considered here, whose topology sees all their objects as being contractible.

Observation 3.45. *If A is C-acyclic then $\Omega_x A$ is C-acyclic for every point $x : * \to A$ (for any model of the loop space object in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$).*

Proof. The standard statement in $\mathrm{sSet}_{\mathrm{Quillen}}$

$$\pi_n \Omega X \simeq \pi_{n+1} X$$

directly prolongs to $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$. □

Theorem 3.46. *Let C be an ∞-cohesive site. Sufficient conditions for an object $A \in [C^{\mathrm{op}}, \mathrm{sSet}]$ to be fibrant in the local model structure $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ are*

- *A is 0-C-truncated and C-acyclic;*
- *A is C-connected and C-acyclic;*
- *A is a group object and C-acyclic.*

Here and in the following "C-truncated" and "C-connected" means: as simplicial presheaves (not after sheafification of homotopy presheaves). So for example, here and in the following a simplicial presheaf X is C-connected if it takes values in connected simplicial sets.

Remark 3.47. This means that with A satisfying the conditions of Theorem 3.46 above, with X any simplicial presheaf and $Y \xrightarrow{\simeq} X$ a *split* hypercover (see Definition 4.8 of [DHI04]), the cocycle ∞-groupoid $\mathbf{H}(X, A)$ is presented by simplicial function complex $[C^{\mathrm{op}}, \mathrm{sSet}](Y, A)$. The vertices of this simplicial set are simplicial hyper-Čech cocycles with coefficients in A, the edges are Čech coboundaries and so on. Specifically, if $\{U_i \to X\}$ is a *good cover* in that all finite non-empty intersections of patches are representable, then the Čech nerve $\check{C}(\{U_i\}) \to X$ is a split hypercover, and a morphism of simplicial presheaves $\check{C}(\{U_i\}) \to A$ is a hyper-Čech cocycle with respect to the given cover.

We demonstrate Theorem 3.46 in several stages in the following list of propositions.

Lemma 3.48. *A 0-C-truncated object is fibrant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ precisely if it is fibrant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ and weakly equivalent to a sheaf: to an object in the image of the canonical inclusion*

$$\mathrm{Sh}(C) \hookrightarrow [C^{\mathrm{op}}, \mathrm{Set}] \hookrightarrow [C^{\mathrm{op}}, \mathrm{sSet}]\,.$$

Proof. From general facts of left Bousfield localization we have that the fibrant objects in the local model structure are necessarily fibrant also in the global structure. Since moreover $A \to \pi_0(A)$ is a weak equivalence in the global model structure by assumption, we have for every covering $\{U_i \to U\}$ in C a sequence of weak equivalences

$$\begin{aligned} \mathrm{Maps}(\check{C}(\{U_i\}), A) &\simeq \mathrm{Maps}(\check{C}(\{U_i\}), \pi_0(A)) \\ &\simeq \mathrm{Maps}(\pi_0 \check{C}(\{U_i\}), \pi_0(A)) \\ &\simeq \mathrm{Sh}_C(S(\{U_i\}), \pi_0(A)), \end{aligned}$$

where $S(\{U_i\}) \hookrightarrow U$ is the sieve corresponding to the cover. Therefore the descent condition

$$\mathrm{Maps}(U, A) \xrightarrow{\simeq} \mathrm{Maps}(\check{C}(\{U_i\}), A)$$

is precisely the sheaf condition for $\pi_0(A)$. □

Lemma 3.49. *A pointed and C-connected fibrant object $A \in [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ is fibrant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ if for all objects $U \in C$*

1. *$H^0_C(U, A) \simeq *$;*
2. *$\Omega_* A$ is fibrant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$,*

where $\Omega_* A$ is any fibrant object in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ representing the simplicial looping of A.

Proof. For $\{U_i \to U\}$ a good covering of an object U we need to show that the canonical morphism

$$\mathrm{Maps}(U, A) \to \mathrm{Maps}(\check{C}(\{U_i\}), A)$$

is a weak homotopy equivalence. This is equivalent to the two morphisms

1. $\pi_0 \mathrm{Maps}(U, A) \to \pi_0 \mathrm{Maps}(\check{C}(\{U_i\}), A)$
2. $\Omega_* \mathrm{Maps}(U, A) \to \Omega_* \mathrm{Maps}(\check{C}(\{U_i\}), A)$

being weak equivalences. Since A is C-connected the first of these says that there is a weak equivalence $* \xrightarrow{\simeq} H^0_C(U, A)$. The second condition is equivalent to $\mathrm{Maps}(U, \Omega_* A) \to \mathrm{Maps}(\check{C}(\{U_i\}), \Omega_* A)$, being a weak equivalence, hence to the descent of $\Omega_* A$. □

Lemma 3.50. *An object A which is C-connected, 1-C-truncated and C-acyclic is fibrant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$.*

Proof. The first condition of Lemma 3.49 holds by the third condition of C-acyclicity. The second condition in Lemma 3.49 is that $\pi_1(A)$ satisfies descent. By C-acyclicity this is a sheaf and it is 0-truncated by assumption, therefore it satisfies descent by Lemma 3.48. □

Proposition 3.51. *Every pointed C-connected and C-acyclic object $A \in [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ is fibrant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$.*

Proof. We first show the statement for truncated A and afterwards for the general case. The k-truncated case in turn we consider by induction over k. If A is 1-truncated the proposition holds by Lemma 3.50. Assuming then that the statement has been shown for k-truncated A, we need to show it for $(k+1)$-truncated A.

We achieve this by decomposing A into its Moore-Postnikov tower

$$A \to \cdots \to A(n+1) \to A(n) \to \cdots \to * .$$

It is a standard fact (shown in [GJ99], VI Theorem 3.5 for simplicial sets, which generalizes immediately to the global model structure $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$) that for all $n > 1$ we have sequences

$$K(n) \to A(n) \to A(n-1) ,$$

where $A(n-1)$ is $(n-1)$-truncated with homotopy groups in degree $\leq n-1$ those of A, and where the right morphism is a Kan fibration and the left morphism is its kernel, such that

$$A = \varprojlim A(n)\,.$$

Moreover, there are canonical weak homotopy equivalences

$$K(n) \to \Xi((\pi_{n-1}A)[n])$$

to the Eilenberg-MacLane object on the $(n-1)$-st homotopy group in degree n. Since $A(n-1)$ is $(n-1)$-truncated and connected, the induction assumption implies that it is fibrant in the local model structure.

Moreover we see that $K(n)$ is fibrant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$: the first condition of 3.49 holds by the assumption that A is C-connected. The second condition is implied again by the induction hypothesis, since $\Omega K(n)$ is $(n-1)$-truncated, connected and still C-acyclic, by Observation 3.45.

Therefore in the diagram (where $\mathrm{Maps}(-,-)$ denotes the simplicial hom complex)

$$\begin{tikzcd}
\mathrm{Maps}(U, K(n)) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Maps}(U, A(n)) \arrow[r] \arrow[d] & \mathrm{Maps}(U, A(n-1)) \arrow[d, "\simeq"] \\
\mathrm{Maps}(\check{C}(\{U_i\}), K(n)) \arrow[r] & \mathrm{Maps}(\check{C}(\{U_i\}), A(n)) \arrow[r] & \mathrm{Maps}(\check{C}(\{U_i\}), A(n-1))
\end{tikzcd}$$

for $\{U_i \to U\}$ any good cover in C the top and bottom rows are fiber sequences (notice that all simplicial sets in the top row are connected because A is connected) and the left and right vertical morphisms are weak equivalences in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ (the right one since $A(n-1)$ is fibrant in the local model structure by induction hypothesis, as remarked before, and the left one by C-acyclicity of A). It follows that also the middle morphism is a weak equivalence. This shows that $A(n)$ is fibrant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$. By completing the induction the same then follows for the object A itself.

This establishes the claim for truncated A. To demonstrate the claim for general A notice that the limit over a sequence of fibrations between fibrant objects is a homotopy limit. Therefore we have

$$\begin{tikzcd}
\mathrm{Maps}(U, A) \arrow[r, phantom, "\simeq"] \arrow[d] & \varprojlim_n \mathrm{Maps}(U, A(n)) \arrow[d, "\simeq"] \\
\mathrm{Maps}(\check{C}(\{U_i\}), A) \arrow[r, phantom, "\simeq"] & \varprojlim_n \mathrm{Maps}(\check{C}(\{U_i\}), A(n))
\end{tikzcd}$$

where the right vertical morphism is a morphism between homotopy limits in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ induced by a weak equivalence of diagrams, hence is itself a weak equivalence. Therefore A is fibrant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$. □

Lemma 3.52. *For $G \in [C^{\mathrm{op}}, \mathrm{sSet}]$ a group object, the canonical sequence*

$$G_0 \to G \to G/G_0$$

is a homotopy fiber sequence in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$.

Proof. Since homotopy pullbacks of presheaves are computed objectwise, it is sufficient to show this for $C = *$, hence in $\mathrm{sSet}_{\mathrm{Quillen}}$. One checks that generally, for X a Kan complex and G a simplicial group acting on X, the quotient morphism $X \to X/G$ is a Kan fibration. Therefore the homotopy fiber of $G \to G/G_0$ is presented by the ordinary fiber in sSet. Since the action of G_0 on G is free, this is indeed $G_0 \to G$. $\square$

Proposition 3.53. *Every C-acyclic group object $G \in [C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ for which G_0 is a sheaf is fibrant in $[C^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$.*

Proof. By lemma 3.52 we have a fibration sequence

$$G_0 \to G \to G/G_0 \,.$$

Since G_0 is assumed to be a sheaf it is fibrant in the local model structure by Lemma 3.48. Since G/G_0 is evidently connected and C-acyclic it is fibrant in the local model structure by Proposition 3.51. As before in the proof there this implies that also G is fibrant in the local model structure. $\square$

This completes the proof of Theorem 3.46.

3.6.2 By cocycles in a category of fibrant objects

We discuss here a presentation of the hom-∞-groupoids of an ∞-category which itself is presented by the homotopical structure known as a *category of fibrant objects* [Bro73]. The resulting presentation is much 'smaller' than the general Dwyer-Kan simplicial localization [DK80]: where the latter encodes a morphism in the localization by a zig-zag of arbitrary length (of morphisms in the presenting category), the following Theorem 3.61 asserts that with the structure of a category of fibrant objects, we may restrict to zig-zags of length 1. A slight variant of this statement has been proven by Cisinski in [Cis]. The following subsumes this variant and provides a maybe more direct proof.

Before describing the hom-∞-groupoids, we briefly recall some basic notions and facts from [Bro73].

Definition 3.54 (Brown)**.** A *category of fibrant objects* is a category $\mathcal{C}$ with finite products, which comes equipped with two distinguished full subcategories $\mathcal{C}_F$ and $\mathcal{C}_W$, whose morphisms are called *fibrations* and *weak equivalences* respectively, such that the following properties hold:

1. $\mathcal{C}_F$ and $\mathcal{C}_W$ contain all of the isomorphisms of $\mathcal{C}$,
2. weak equivalences satisfy the 2-out-of-3 property,
3. the subcategories $\mathcal{C}_F$ and $\mathcal{C}_F \cap \mathcal{C}_W$ are stable under pullback,
4. there exist functorial path objects in $\mathcal{C}$.

Morphisms in $\mathcal{C}_F \cap \mathcal{C}_W$ are called *acyclic fibrations.*

The axioms for a category of fibrant objects give roughly half of the structure of a model category, however these axioms still suffice to give a calculus-of-fractions description of the associated homotopy category.

Examples 3.55. We have the following well known examples of categories of fibrant objects.

- For any model category (with functorial factorization) the full subcategory of fibrant objects is a category of fibrant objects.
- The category of stalkwise Kan simplicial presheaves on any site with enough points. In this case the fibrations are the stalkwise fibrations and the weak equivalences are the stalkwise weak equivalences.

Remark 3.56. Notice that (over a non-trivial site) the second example above is *not* a special case of the first: while there are model structures on categories of simplicial presheaves whose weak equivalences are the stalkwise weak equivalences, their fibrations (even between fibrant objects) are much more restricted than just being stalkwise fibrations.

We will use repeatedly the following consequence of the axioms of a category of fibrant objects (this is called the *cogluing lemma* in [GJ99] where it appears as Lemma 8.10, Chapter II).

Lemma 3.57. *Let $\mathcal{C}$ be a category of fibrant objects. Suppose given a diagram*

$$\begin{tikzcd} A_1 \arrow[r, "p_1"] \arrow[d, "f_A"'] & B_1 \arrow[d, "f_B"'] & C_1 \arrow[l] \arrow[d, "f_C"'] \\ A_2 \arrow[r, "p_2"'] & B_2 & C_2 \arrow[l] \end{tikzcd}$$

in which p_1 and p_2 are fibrations, and f_A, f_B and f_C are weak equivalences. Then the induced map

$$A_1 \times_{B_1} C_1 \to A_2 \times_{B_2} C_2$$

is also a weak equivalence.

We now come to the discussion of the hom-∞-groupoids presented by $\mathcal{C}$

Definition 3.58. Let $\mathcal{C}$ be a category of fibrant objects and let X and A be objects of $\mathcal{C}$. Write $\mathrm{Cocycle}(X, A)$ for the category whose

- objects are spans, hence diagrams in $\mathcal{C}$ of the form

$$X \underset{\simeq}{\overset{p}{\twoheadleftarrow}} Y \xrightarrow{g} A \ ,$$

  such that the left morphism is an acyclic fibration;

- morphisms $f : (p_1, g_1) \to (p_2, g_2)$ are given by morphisms $f : X \to Y$ in $\mathcal{C}$, making the diagram

$$\begin{tikzcd} & Y_1 \arrow[dl, "\simeq"'] \arrow[dr, "g_1"] \arrow[dd] & \\ X & & A \\ & Y_2 \arrow[ul, "\simeq"] \arrow[ur, "g_2"'] & \end{tikzcd}$$

  commute.

Similarly write $\mathrm{wCocycle}(X, A)$ for the category defined analogously, where however the left legs are only required to be weak equivalences, not necessarily fibrations.

Remark 3.59. In Section 3.3 of [Cis] the category Cocycle(X, A) is denoted $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, A)$. In Section 1 of [Ja06] the category wCocycle(X, A) (under different assumptions on $\mathcal{C}$) is denoted $H(X, A)$ (and there only the connected components are analyzed).

Remark 3.60. The morphisms f in Cocycle(X, A) and wCocycle(X, A) are necessarily weak equivalences by the 2-out-of-3 property. The evident composition of spans under fiber product in $\mathcal{C}$ induces a functor

$$\mathrm{Cocycle}(X, A) \times \mathrm{Cocycle}(A, B) \to \mathrm{Cocycle}(X, B)\,,$$

which defines the structure of a bicategory whose objects are the objects of $\mathcal{C}$.

Theorem 3.61. *Let $\mathcal{C}$ be a category of fibrant objects. Then for all objects $X, A \in \mathcal{C}$ the canonical inclusions*

$$N\mathrm{Cocycle}(X, A) \to N\mathrm{wCocycle}(X, A) \to L^H\mathcal{C}(X, A)$$

of the simplicial nerves of the categories of cocycles into the hom-space $L^H(X, A)$ of the hammock localization *[DK80] of $\mathcal{C}$ are weak homotopy equivalences.*

As remarked above, a variant of this statement has been proven by Cisinski [Cis] — more precisely he has shown that the inclusion $N\mathrm{Cocycle}(X, A) \to L^H\mathcal{C}(X, A)$ is a weak homotopy equivalence (see Proposition 3.23 of [Cis]]). We give here a direct proof of this result, which also establishes that $N\mathrm{Cocycle}(X, A) \to N\mathrm{wCocycle}(X, A)$ is also a weak equivalence.

In order to write out the proof, we first need a little bit of notation. By $W^{-1}\mathcal{C}^i(A, B)$ we shall mean the category which has as objects zig-zags of the form

$$A \xleftarrow{\sim} X_1 \longrightarrow X_2 \longrightarrow \ldots \longrightarrow X_i \longrightarrow B\ ,$$

where the morphism to the left is a weak equivalence, and as morphisms ladders of weak equivalences. By $W^{-1}W^i(A, B)$ we denote the full subcategory where also the arrows going to the right are weak equivalences. Analogously we have similar categories $W^{-1}\mathcal{C}^iW^{-1}\mathcal{C}^j(A, B)$ and $W^{-1}W^iW^{-1}W^j(A, B)$ for pair of integers $i, j > 0$. There are obvious functors

$$\begin{aligned} &W^{-1}\mathcal{C}^{i+j}(A, B) \to W^{-1}\mathcal{C}^iW^{-1}\mathcal{C}^j(A, B) \qquad \text{and} \\ &W^{-1}W^{i+j}(A, B) \to W^{-1}W^iW^{-1}W^j(A, B). \end{aligned}$$

given by filling in identity morphisms. If these inclusions induce weak equivalences on nerves, then $\mathcal{C}$ is said to admit a *homotopy calculus of left fractions*, see [DK80, Section 6]. In this case they show that the canonical morphism

$$N\mathrm{wCocycle}(A, B) = N\big(W^{-1}\mathcal{C}(A, B)\big) \to L^H(A, B)$$

is a weak homotopy equivalence [DK80, Proposition 6.2]. Therefore we want to show that each category of fibrant objects $\mathcal{C}$ admits a homotopy calculus of left fractions.

Let $F^{-1}\mathcal{C}^i(A, B)$ be the full subcategory of $W^{-1}\mathcal{C}^i(A, B)$ consisting of the zig-zags where the left going morphim is as acyclic fibration rather than a weak equivalence. Analogously we write $F^{-1}\mathcal{C}^iF^{-1}\mathcal{C}^j(A, B)$. Note that in either case the morphisms of these spans still consist of weak equivalences and not necessarily of acyclic fibrations.

Lemma 3.62. *Let* $A \xleftarrow{\sim} X \longrightarrow B$ *be a span in* $\mathcal{C}$ *where the left leg is a weak equivalence. Then we can find* $Y \in \mathcal{C}$ *and a commuting diagram*

$$\begin{tikzcd} & X \arrow[dl, "\sim"'] \arrow[d, "\sim"] \arrow[dr] & \\ A & & B \\ & Y \arrow[ul, two heads, "\sim"] \arrow[ur] & \end{tikzcd} .$$

In other words we find another span $A \overset{\sim}{\twoheadleftarrow} Y \longrightarrow B$ *where the left leg is an acyclic fibration and a morphism of spans between them. Moreover this assignment is functorial in the original span.*

Proof. We first note that in a category of fibrant objects we can always factor a morphism $X \to Z$ as

$$X \xrightarrow{s} Y \overset{p}{\twoheadrightarrow} Z .$$

where p is a fibration and s is a weak equivalence which is a section of an acyclic fibration $\hat{Z} \overset{\sim}{\twoheadrightarrow} X$. To see this set $Y := X \times_Z \times Z^I$.

Now a span between A and B is the same as a morphism $X \to A \times B$. Applying the factorization to this morphism yields a diagram

$$X \to Y \to A \times B$$

which tranlates exactly into the diagram from above. It only remains to check that the left leg is indeed a fibration since it is clearly a weak equivalence by the 2-out-of-3 property. This follows by the fact that it can be expressed as the composition $Y \to A \times B \xrightarrow{p} A$ where p is the projection to the first factor which is a fibration since B is fibrant. □

Lemma 3.63. *The following inclusion functors induce weak equivalences on nerves for all* $i, j > 0$.

$$\begin{aligned} F^{-1}\mathcal{C}^i(A,B) &\longrightarrow W^{-1}\mathcal{C}^i(A,B) \\ F^{-1}W^i(A,B) &\longrightarrow W^{-1}W^i(A,B) \\ F^{-1}\mathcal{C}^iF^{-1}\mathcal{C}^j(A,B) &\longrightarrow W^{-1}\mathcal{C}^iW^{-1}\mathcal{C}^j(A,B) \\ F^{-1}W^iF^{-1}W^j(A,B) &\longrightarrow W^{-1}W^iW^{-1}W^j(A,B) \\ F^{-1}\mathcal{C}^{i+j}(A,B) &\longrightarrow F^{-1}\mathcal{C}^iF^{-1}\mathcal{C}^j(A,B) \\ F^{-1}W^{i+j}(A,B) &\longrightarrow F^{-1}W^iF^{-1}W^j(A,B). \end{aligned}$$

Proof. We explicitly construct functors which are homotopy inverses on nerves. For the first functor $K_1 : F^{-1}\mathcal{C}^i(A,B) \longrightarrow W^{-1}\mathcal{C}^i(A,B)$ we define an inverse $L_1 : W^{-1}\mathcal{C}^i(A,B) \longrightarrow F^{-1}\mathcal{C}^i(A,B)$ using the factorization from the last lemma as

$$\begin{aligned} &\Big(A \xleftarrow{\sim} X_1 \longrightarrow X_2 \longrightarrow \ldots \longrightarrow X_i \longrightarrow B \Big) \\ &\quad \mapsto \qquad \Big(A \overset{\sim}{\twoheadleftarrow} Y_1 \longrightarrow X_2 \longrightarrow \ldots \longrightarrow X_i \longrightarrow B \Big). \end{aligned}$$

One checks that this indeed forms a functor and that the morphism $X_1 \xrightarrow{\sim} Y_1$ from the last lemma form natural transformations $id \Rightarrow K_1 \circ G_1$ and $id \Rightarrow G_1 \circ K_1$. This shows that on nerves NK_1 and NL_1 are homotopy inverses.

Now the functor L_1 restricts to a functor $W^{-1}W^i(A,B) \to F^{-1}W^i(A,B)$ which is homotopy inverse to the second functor of the lemma.

For the third funtor $K_2 : F^{-1}\mathcal{C}^iF^{-1}\mathcal{C}^j(A,B) \to W^{-1}\mathcal{C}^iW^{-1}\mathcal{C}^j(A,B)$ in the lemma an inverse $L_2 : W^{-1}\mathcal{C}^iW^{-1}\mathcal{C}^j(A,B) \longrightarrow F^{-1}\mathcal{C}^iF^{-1}\mathcal{C}^j(A,B)$ is similarily constructed as

$$\begin{aligned}
&\Big(A \xleftarrow{\sim} X_1 \to X_2 \to \quad \to X_i \xleftarrow{\sim} X_{i+1} \to X_{i+2} \to \quad \to B \Big) \\
\mapsto \quad &\Big(A \overset{\sim}{\twoheadleftarrow} Y_1 \to X_2 \to \quad \to X_i \overset{\sim}{\twoheadleftarrow} Y_{i+1} \to X_{i+2} \to \quad \to B \Big).
\end{aligned}$$

using again the factorization of Lemma 3.62. The morphisms $X_1 \xrightarrow{\sim} Y_1$ and $X_{i+1} \xrightarrow{\sim} Y_{i+1}$ provide natural transformations $id \Rightarrow K_2 \circ L_2$ and $id \Rightarrow L_2 \circ K_2$. As before the functor L_2 restrict to an inverse for the fourth functor in the lemma.

Now we come to the functor $K_3 : F^{-1}\mathcal{C}^{i+j}(A,B) \longrightarrow F^{-1}\mathcal{C}^iF^{-1}\mathcal{C}^j(A,B)$. Its homotopy inverse L_3 is constructed by iterated pullbacks as indicated in the following diagram

$$\begin{tikzcd}
 & & X'_1 \arrow[r, dashed] \arrow[dl, dashed, two heads, "\sim"'] & X'_2 \arrow[r, dashed] & \quad \arrow[r, dashed] & X'_{i-1} \arrow[r, dashed] & X_{i+1} \arrow[r] \arrow[dl, two heads, "\sim"'] & X_{i+2} \arrow[r] & \quad \arrow[r] & B \\
 & X_1 \arrow[r] \arrow[dl, two heads, "\sim"'] & X_2 \arrow[r] & \quad \arrow[r] & X_{i-1} \arrow[r] & X_i & & & & \\
A & & & & & & & & &
\end{tikzcd}$$

So, using this notiation, the functor $L_3 : F^{-1}\mathcal{C}^iF^{-1}\mathcal{C}^j(A,B) \to F^{-1}\mathcal{C}^{i+j}(A,B)$ can be defined as

$$\begin{aligned}
&\Big(A \overset{\sim}{\twoheadleftarrow} X_1 \to X_2 \to \quad \to X_{i-1} \to X_i \overset{\sim}{\twoheadleftarrow} X_{i+1} \to \quad \to B \Big) \\
\mapsto \quad &\Big(A \overset{\sim}{\twoheadleftarrow} X'_1 \dashrightarrow X'_2 \dashrightarrow \quad \dashrightarrow X'_{i-1} \dashrightarrow X_{i+1} \to \quad \to B \Big).
\end{aligned}$$

The structure maps of the pullbacks $X'_i \to X_i$ and the map $X_{i+1} \to X_i$ provide a natural transformation $K_3 \circ L_3 \Rightarrow id$. The other composition $L_3 \circ K_3$ consists essentially of pullbacks along the identity and is therefor also naturally isomorphic to the identity. Finally we note that the functor L_3 restricts to a functor $F^{-1}W^iF^{-1}W^j(A,B) \to F^{-1}W^{i+j}(A,B)$ by iterated use of the 2-out-of-3 property. Hence it also provides an inverse for the last functor of the lemma. $\square$

Lemma 3.64. *Each category of fibrant objects $\mathcal{C}$ admits a homotopy calculus of fractions and the chain of inclusions $N\mathrm{Cocycle}(A,B) \to N\mathrm{wCocycle}(A,B) \to L^H\mathcal{C}(A,B)$ are all homotopy equivalences.*

Proof. In order to show that $\mathcal{C}$ admits a homotopy calculus of fractions we consider the following commuting diagrams of inclusions

$$\begin{tikzcd}
F^{-1}\mathcal{C}^{i+j}(A,B) \arrow[r] \arrow[d] & W^{-1}\mathcal{C}^{i+j}(A,B) \arrow[d] \\
F^{-1}\mathcal{C}^iF^{-1}\mathcal{C}^j(A,B) \arrow[r] & W^{-1}\mathcal{C}^iW^{-1}\mathcal{C}^j(A,B)
\end{tikzcd}
\qquad
\begin{tikzcd}
F^{-1}W^{i+j}(A,B) \arrow[r] \arrow[d] & W^{-1}W^{i+j}(A,B) \arrow[d] \\
F^{-1}W^iF^{-1}W^j(A,B) \arrow[r] & W^{-1}W^iW^{-1}W^j(A,B)
\end{tikzcd}$$

By definition of a homotopy calculus of fraction we have to show that the two right vertical maps induce weak equivalences on nerves. But this follows since we know from the last

Lemma 3.63 that all the other maps in the diagrams are weak equivalences. From the fact that $\mathcal{C}$ admits a homotopy calculus of fractions and [DK80, Proposition 6.2] we know that the canonical map $N\mathrm{wCocycle}(A, B) \to L^H\mathcal{C}(A, B)$ is a weak equivalence of simplicial sets. The map $N\mathrm{Cocycle}(A, B) \to N\mathrm{wCocycle}(A, B)$ is just the nerve of the functor $F^{-1}\mathcal{C}(A, B) \to W^{-1}\mathcal{C}(A, B)$ which is a weak equivalence by the last lemma. $\square$

This completes the proof of Theorem 3.61.

3.7 Principal bundles

We discuss a presentation of the theory of *principal* ∞*-bundles* from section 3 in [NSSa].

3.7.1 Universal simplicial principal bundles and the Borel construction

By Proposition 3.35 every ∞-group in an ∞-topos over an ∞-cohesive site is presented by a (pre)sheaf of simplicial groups, hence by a strict group object G in a 1-category of simplicial (pre)sheaves. We have seen in Section 3.5 that, for such a presentation, the abstract delooping $\mathbf{B}G$ is presented by $\overline{W}G$. By Theorem 3.19 in [NSSa], the theory of G-principal ∞-bundles is essentially that of homotopy fibers of morphisms into $\mathbf{B}G$, and hence, for such a presentation, that of homotopy fibers of morphisms into $\overline{W}G$. By Proposition 3.1 such homotopy fibers are computed as ordinary pullbacks of fibration resolutions of the point inclusion into $\overline{W}G$. Here we discuss these fibration resolutions. They turn out to be the classical *universal simplicial principal bundles* $WG \to \overline{W}G$ of Definition 3.31.

Let C be a site; we consider group objects in $[C^{\mathrm{op}}, \mathrm{sSet}]$. In the following let $P \in [C^{\mathrm{op}}, \mathrm{sSet}]$ be an object equipped with an action $\rho : P \times G \to P$ by a group object G. Since sheafification preserves finite limits, all of the following statements hold verbatim also in the category $\mathrm{sSh}(C)$ of simplicial sheaves over C.

Definition 3.65. The *action groupoid object*

$$P//G \in [\Delta^{\mathrm{op}}, [C^{\mathrm{op}}, \mathrm{sSet}]]$$

is the simplicial object in $[C^{\mathrm{op}}, \mathrm{sSet}]$ whose n-simplices are

$$(P//G)_n := P \times G^{\times^n} \quad \in [C^{\mathrm{op}}, \mathrm{sSet}]\,,$$

whose face maps are given on elements by

$$d_i(p, g_1, \ldots, g_n) = \begin{cases} (pg_1, g_2, \ldots, g_n) & \text{if } i = 0, \\ (p, g_1, \ldots, g_i g_{i+1}, \ldots, g_n) & \text{if } 1 \leq i \leq n-1, \\ (p, g_1, \ldots, g_{n-1}) & \text{if } i = n, \end{cases}$$

and whose degeneracy maps are given on elements by

$$s_i(p, g_1, \ldots, g_n) = (p, g_1, \ldots, g_{i-1}, 1, g_i, \ldots, g_n)\,.$$

Definition 3.66. Write

$$P/_hG := \sigma_*(P//G) \in [C^{\mathrm{op}}, \mathrm{sSet}]$$

for the total simplicial object, Definition 3.20.

Remark 3.67. According to Corollary 3.24 the object $P/_hG$ presents the homotopy colimit over the simplicial object $P//G$. We say that $P/_hG$ is the *homotopy quotient* of P by the action of G.

Example 3.68. The unique trivial action of a group object G on the terminal object $*$ gives rise to a canonical action groupoid $*//G$. According to Definition 3.26 we have

$$*/_hG = \overline{W}G\,.$$

The multiplication morphism $G \times G \to G$ regarded as an action of G on itself gives rise to a canonical action groupoid $G//G$. The terminal morphism $G \to *$ induces a morphism of simplicial objects

$$G//G \to *//G\,.$$

Defined this way $g \in G$ acts naturally from the *left* on $G//G$. To adhere to our convention that actions on bundles are right actions, we consider instead the right action of $g \in G$ on G given by left multiplication by g^{-1}. With respect to this action, the action groupoid object $G//G$ is canonically equipped with the right G-action by multiplication from the right. Whenever in the following we write

$$G//G \to *//G$$

we are referring to this latter definition.

Definition 3.69. Given a group object in $[C^{\mathrm{op}}, \mathrm{sSet}]$, write $WG \to \overline{W}G$ for the morphism of simplicial presheaves

$$G/_hG \to */_hG$$

induced on homotopy quotients, Definition 3.66, by the morphism of canonical action groupoid objects of example 3.68.

We will call this the *universal weakly G-principal bundle.*

Remark 3.70. Traditionally, at least over the trivial site, this is known as a presentation of the *universal G-principal simplicial bundle*; we review this traditional theory below in Section 4.1. However, when prolonged to presheaves of simplicial sets as considered here, it is not quite accurate to speak of a genuine *universal principal bundle*: because the pullbacks of this bundle to hypercovers will in general only be "weakly principal" in a sense that we discuss in a moment in Section 3.7.2. Therefore it is more accurate to speak of the *universal weakly G-principal bundle.*

The following proposition (which appears as Lemma 10 in [RS12]) justifies this terminology and the notation WG (which, recall, has already been used in Definition 3.31).

Proposition 3.71. *For G a group object in $[C^{\mathrm{op}}, \mathrm{sSet}]$, the morphism $WG \to \overline{W}G$ from Definition 3.69 has the following properties:*

1. *it is isomorphic to the décalage morphism $\mathrm{Dec}_0\overline{W}G \to \overline{W}G$, Definition 3.31,*
2. *WG is canonically equipped with a right G-action over $\overline{W}G$ that makes $WG \to \overline{W}G$ a G-principal bundle.*

In particular it follows from 2 that $WG \to \overline{W}G$ is an objectwise Kan fibration replacement of the point inclusion $* \to \overline{W}G$.

We now discuss some basic properties of the morphism $WG \to \overline{W}G$.

Definition 3.72. For $\rho : P \times G \to P$ a G-action in $[C^{\mathrm{op}}, \mathrm{sSet}]$, we write

$$P \times_G WG := (P \times WG)/G \in [C^{\mathrm{op}}, \mathrm{sSet}]$$

for the quotient by the diagonal G-action with respect to the given right G action on P and the canonical right G-action on WG from Proposition 3.71. We call this quotient the *Borel construction* of the G-action on P.

Proposition 3.73. *For $P \times G \to P$ an action in $[C^{\mathrm{op}}, \mathrm{sSet}]$, there is an isomorphism*

$$P/_h G = P \times_G WG \,,$$

between the homotopy quotient, Definition 3.66, and the Borel construction. In particular, for all $n \in \mathbb{N}$ there are isomorphisms

$$(P/_h G)_n = P_n \times G_{n-1} \times \cdots \times G_0 \,.$$

Proof. This follows by a straightforward computation. □

Lemma 3.74. *Let P be a Kan complex, G a simplicial group and $\rho : P \times G \to P$ a free action. The following holds.*

1. *The quotient map $P \to P/G$ is a Kan fibration.*
2. *The quotient P/G is a Kan complex.*

The second statement is for instance Lemma 3.7 in Chapter V of [GJ99].

Lemma 3.75. *For P a Kan complex and $P \times G \to P$ an action by a group object, the homotopy quotient $P/_h G$, Definition 3.66, is itself a Kan complex.*

Proof. By Proposition 3.73 the homotopy quotient is isomorphic to the Borel construction. Since G acts freely on WG it acts freely on $P \times WG$. The statement then follows with Lemma 3.74. □

Remark 3.76. Let $\hat{X} \to \overline{W}G$ be a morphism in $[C^{\mathrm{op}}, \mathrm{sSet}]$, presenting, by Proposition 3.35, a morphism $X \to \mathbf{B}G$ in the ∞-topos $\mathbf{H} = \mathrm{Sh}_\infty(C)$. By theorem 3.19 of [NSSa] every G-principal ∞-bundle over X arises as the homotopy fiber of such a morphism. By using Proposition 3.71 together with Proposition 3.1 it follows that the principal ∞-bundle classified by $\hat{X} \to \overline{W}G$ is presented by the ordinary pullback of $WG \to \overline{W}G$. This is the defining property of the universal principal bundle.

In section 3.7.2 below we show how this observation leads to a complete presentation of the theory of principal ∞-bundles by *weakly principal simplicial bundles*.

3.7.2 Presentation in locally fibrant simplicial sheaves

We discuss a presentation of the general notion of principal ∞-bundles, by *weakly principal bundles* in a 1-category of simplicial sheaves.

Let $\mathbf{H}$ be a hypercomplete ∞-topos (for instance a cohesive ∞-topos), which admits a 1-site C with enough points.

Observation 3.77. *From Section 2.1 a category with weak equivalences that presents* $\mathbf{H}$ *under simplicial localization is the category* $\mathrm{sSh}(C)$ *of simplicial 1-sheaves on* C *with the weak equivalences* $W \subset \mathrm{sSh}(C)$ *being the stalkwise weak equivalences:*

$$\mathbf{H} \simeq L_W \mathrm{sSh}(C)\,.$$

Also the full subcategory

$$\mathrm{sSh}(C)_{\mathrm{lfib}} \hookrightarrow \mathrm{sSh}(C)$$

on the locally fibrant objects is a presentation.

Corollary 3.78. *Regard* $\mathrm{sSh}(C)_{\mathrm{lfib}}$ *as a category of fibrant objects, Definition 3.54, with weak equivalences and fibrations the stalkwise weak equivalences and fibrations in* $\mathrm{sSet}_{\mathrm{Quillen}}$*, respectively, as in Example 3.55. Then for any two objects* $X, A \in \mathbf{H}$ *there are simplicial sheaves, to be denoted by the same symbols, such that the hom* ∞*-groupoid in* $\mathbf{H}$ *from* X *to* A *is presented in* $\mathrm{sSet}_{\mathrm{Quillen}}$ *by the Kan complex of cocycles from Section 3.6.2.*

Proof. By theorem 3.61. □

We now discuss, for the general theory of principal ∞-bundles in $\mathbf{H}$ from [NSSa] a corresponding realization in the presentation for $\mathbf{H}$ given by $(\mathrm{sSh}(C), W)$.

By Proposition 3.35 every ∞-group in $\mathbf{H}$ is presented by an ordinary group in $\mathrm{sSh}(C)$. It is too much to ask that also every G-principal ∞-bundle is presented by a principal bundle in $\mathrm{sSh}(C)$. But something close is true: every principal ∞-bundle is presented by a *weakly principal* bundle in $\mathrm{sSh}(C)$.

Definition 3.79. Let $X \in \mathrm{sSh}(C)$ be any object, and let $G \in \mathrm{sSh}(C)$ be equipped with the structure of a group object. A *weakly G-principal bundle* is

- an object $P \in \mathrm{sSh}(C)$ (the *total space*);
- a local fibration $\pi\colon P \to X$ (the *bundle projection*);
- a right action

  $$\begin{tikzcd} P \times G \arrow[rr, "\rho"] \arrow[dr] & & P \arrow[dl] \\ & X & \end{tikzcd}$$

  of G on P over X

such that

- the action of G is *weakly principal* in the sense that the *shear map*

  $$(p_1, \rho) : P \times G \to P \times_X P \qquad (p,g) \mapsto (p, pg)$$

  is a local weak equivalence.

Remark 3.80. We do not ask the G-action to be degreewise free as in [JaL04], where a similar notion is considered. However we show in Corollary 3.97 below that each weakly G-principal bundle is equivalent to one with free G-action.

Definition 3.81. A morphism of weakly G-principal bundles $(\pi, \rho) \to (\pi', \rho')$ over X is a morphism $f : P \to P'$ in $\mathrm{sSh}(C)$ that is G-equivariant and commutes with the bundle projections, hence such that it makes this diagram commute:

$$\begin{tikzcd}
P \times G \arrow[rr, "(f,\mathrm{id})"] \arrow[d, "\rho"] & & P' \times G \arrow[d, "\rho'"] \\
P \arrow[rr, "f"] \arrow[dr, "\pi"'] & & P' \arrow[dl, "\pi'"] \\
& X &
\end{tikzcd}$$

Write

$$\mathrm{w}G\mathrm{Bund}(X) \in \mathrm{sSet}_{\mathrm{Quillen}}$$

for the nerve of the category of weakly G-principal bundles and morphisms as above. The ∞-groupoid that this presents under $\mathrm{Grpd}_\infty \simeq (\mathrm{sSet}_{\mathrm{Quillen}})^\circ$ (i.e. its Kan fibrant replacement), we call the *∞-groupoid of weakly G-principal bundles over X*.

Lemma 3.82. *Let $\pi : P \to X$ be a weakly G-principal bundle. Then the following statements are true:*

1. *for any point $p : * \to P$ the action of G induces a weak equivalence*

$$G \longrightarrow P_x$$

   where $x = \pi(p)$ and where P_x is the fiber of $P \to X$ over x,

2. *for all $n \in \mathbb{N}$, the multi-shear maps*

$$P \times G^n \to P^{\times_X^{n+1}} \qquad (p, g_1, ..., g_n) \mapsto (p, pg_1, ..., pg_n)$$

   are weak equivalences.

Proof. We consider the first statement. Regard the weak equivalence $P \times G \xrightarrow{\sim} P \times_X P$ as a morphism over P where in both cases the map to P is given by projection onto the first factor. By basic properties of categories of fibrant objects, both of these projections are fibrations. Therefore, by the cogluing lemma (Lemma 3.57) the pullback of the shear map along p is still a weak equivalence. But this pullback is just the map $G \to P_x$, which proves the claim.

For the second statement, we use induction on n. Suppose that $P \times G^n \to P^{\times_X^{n+1}}$ is a weak equivalence. By Lemma 3.57 again, the pullback $P^{\times_X^n} \times_X (P \times G) \to P^{\times_X^{n+2}}$ of the shear map $P \times G \to P \times_X P$ along the fibration $P^{\times_X^n} \to X$ is again a weak equivalence. Similarly the product $P \times G^n \times G \to P^{\times_X^{n+1}} \times G$ of the n-fold shear map with G is also a weak equivalence. The composite of these two weak equivalences is the multi-shear map $P \times G^{n+1} \to P^{\times_X^{n+2}}$, which is hence also a weak equivalence. $\square$

Proposition 3.83. *Let $P \to X$ be a weakly G-principal bundle and let $f : Y \to X$ be an arbitrary morphism. Then the pullback $f^*P \to Y$ exists and is also canonically a weakly G-principal bundle. This operation extends to define a pullback morphism*

$$f^* : \mathrm{w}G\mathrm{Bund}(X) \to \mathrm{w}G\mathrm{Bund}(Y)\,.$$

Proof. Again this follows by basic properties of a category of fibrant objects: the pullback f^*P exists and the morphism $f^*P \to Y$ is again a local fibration; thus it only remains to show that f^*P is weakly principal, i.e. that the morphism $f^*P \times G \to f^*P \times_Y f^*P$ is a weak equivalence. This follows from Lemma 3.57 again. □

Remark 3.84. The functor f^* associated to the map $f : Y \to X$ above is the restriction of a functor $f^* : \mathrm{sSh}(C)/X \to \mathrm{sSh}(C)/Y$ mapping from simplicial sheaves over X to simplicial sheaves over Y. This functor f^* has a left adjoint $f_! : \mathrm{sSh}(C)_{/Y} \to \mathrm{sSh}_{/X}$ given by composition along f, in other words

$$f_!(E \to Y) = E \to Y \xrightarrow{f} X.$$

Note that the functor $f_!$ does not usually restrict to a functor $f_! : \mathrm{w}G\mathrm{Bund}(Y) \to \mathrm{w}G\mathrm{Bund}(X)$. But when it does, we say that principal ∞-bundles *satisfy descent along* f. In this situation, if P is a weakly G-principal bundle on Y, then P is weakly equivalent to the pulled-back principal ∞-bundle $f^*f_!P$ on Y, in other words P 'descends' to $f_!P$.

The next result says that weakly G-principal bundles satisfy descent along local acyclic fibrations (hypercovers).

Proposition 3.85. *Let $p : Y \to X$ be a local acyclic fibration in* $\mathrm{sSh}(C)$. *Then the functor $p_!$ defined above restricts to a functor $p_! : \mathrm{w}G\mathrm{Bund}(Y) \to \mathrm{w}G\mathrm{Bund}(X)$, left adjoint to $p^* : \mathrm{w}G\mathrm{Bund}(X) \to \mathrm{w}G\mathrm{Bund}(Y)$, hence to a homotopy equivalence in* $\mathrm{sSet}_{\mathrm{Quillen}}$.

Proof. Given a weakly G-principal bundle $P \to Y$, the first thing we have to check is that the map $P \times G \to P \times_X P$ is a weak equivalence. This map can be factored as $P \times G \to P \times_Y P \to P \times_X P$. Hence it suffices to show that the map $P \times_Y P \to P \times_X P$ is a weak equivalence. But this follows from Lemma 3.57, since both pullbacks are along local fibrations and $Y \to X$ is a local weak equivalence by assumption. This establishes the existence of the functor $p_!$. It is easy to see that it is left adjoint to p^*. This implies that it induces a homotopy equivalence in $\mathrm{sSet}_{\mathrm{Quillen}}$. □

Corollary 3.86. *For $f : Y \to X$ a local weak equivalence, the induced functor $f^* : \mathrm{w}G\mathrm{Bund}(X) \to \mathrm{w}G\mathrm{Bund}(Y)$ is a homotopy equivalence.*

Proof. Using the Factorization Lemma of [Bro73] we can factor the weak equivalence f into a composite of a local acyclic fibration and a right inverse to a local acyclic fibration. Therefore, by Proposition 3.85, f^* may be factored as the composite of two homotopy equivalences, hence is itself a homotopy equivalence. □

We discuss now how weakly G-principal bundles arise from the universal G-principal bundle (Definition 3.69) by pullback, and how this establishes their equivalence with G-cocycles.

Proposition 3.87. *For G a group object in* $\mathrm{sSh}(C)$*, the map $WG \to \overline{W}G$ from Definition 3.69 equipped with the G-action of Proposition 3.71 is a weakly G-principal bundle.*

Indeed, it is a genuine (strictly) G-principal bundle, in that the shear map is an isomorphism. This is a classical fact, for instance around Lemma 4.1 in chapter V of [GJ99]. In terms of the total simplicial set functor it is observed in Section 4 of [RS12].

Proof. By inspection one finds that

$$\begin{array}{ccc} (G//G)\times G & \longrightarrow & G//G \\ \downarrow & & \downarrow \\ G//G & \longrightarrow & *//G \end{array}$$

is a pullback diagram in $[\Delta^{\mathrm{op}}, \mathrm{sSh}(C)]$. Since the total simplicial object functor σ_* of Definition 3.20 is right adjoint it preserves this pullback. This shows the principality of the shear map. □

Definition 3.88. For $Y \to X$ a morphism in $\mathrm{sSh}(C)$, write

$$\check{C}(Y) \in [\Delta^{\mathrm{op}}, \mathrm{sSh}(C)]$$

for its *Čech nerve*, given in degree n by the n-fold fiber product of Y over X

$$\check{C}(Y)_n := Y^{\times_X^{n+1}} .$$

Observation 3.89. *Under σ_* the canonical morphism of simplicial objects $\check{C}(Y) \to X$, with X regarded as a constant simplicial object induces (by Proposition 3.23) canonical morphism*

$$\sigma_* \check{C}(Y) \to X \quad \in \mathrm{sSh}(C) .$$

Lemma 3.90. *For $p : Y \to X$ a local acyclic fibration, the morphism $\sigma_* \check{C}(Y) \to X$ from Observation 3.89 is a local weak equivalence.*

Proof. By pullback stability of local acyclic fibrations, for each $n \in \mathbb{N}$ the morphism $Y^{\times_X^n} \to X$ is a local weak equivalence. By Remark 3.22 and Proposition 3.23 this degreewise local weak equivalence is preserved by the functor σ_*. □

The main statement now is the following.

Theorem 3.91. *For $P \to X$ a weakly G-principal bundle in* $\mathrm{sSh}(C)$*, the canonical morphism*

$$P/_h G \longrightarrow X$$

is a local acyclic fibration.

Proof. To see that the morphism is a local weak equivalence, factor $P//G \to X$ in $[\Delta^{\mathrm{op}}, \mathrm{sSh}(C)]$ via the multi-shear maps from Lemma 3.82 through the Čech nerve, Definition 3.88, as

$$P//G \to \check{C}(P) \to X .$$

Applying the total simplicial object functor σ_* (Definition 3.20) yields a factorization

$$P/_hG \to \sigma_*\check{C}(P) \to X\,.$$

The left morphism is a weak equivalence because, by Lemma 3.82, the multi-shear maps are weak equivalences and by Corollary 3.24 σ_* preserves sends degreewise weak equivalences to weak equivalences. The right map is a weak equivalence by Lemma 3.90.

We now prove that $P/_hG \to X$ is a local fibration. We need to show that for each topos point p of $\mathrm{Sh}(C)$ the morphism of stalks $p(P/_hG) \to p(X)$ is a Kan fibration of simplicial sets. By Proposition 3.73 this means equivalently that the morphism

$$p(P \times_G WG) \to p(X)$$

is a Kan fibration. By definition of *topos point*, p commutes with all the finite products and colimits involved here. Therefore equivalently we need to show that

$$p(P) \times_{p(G)} Wp(G) \to p(X)$$

is a Kan fibration for all topos points p. Observe that this morphism factors the projection $p(P) \times W(p(G)) \to p(X)$ as

$$p(P) \times W(p(G)) \to p(P) \times_{p(G)} W(p(G)) \to p(X)$$

in sSet. Here the first morphism is a Kan fibration by Lemma 3.74, which in particular is also surjective on vertices. Also the total composite morphism is a Kan fibration, since $W(p(G))$ is Kan fibrant. From this the desired result follows with the next Lemma 3.92. $\square$

Lemma 3.92. *Suppose that $X \xrightarrow{p} Y \xrightarrow{q} Z$ is a diagram of simplicial sets such that p is a Kan fibration surjective on vertices and qp is a Kan fibration. Then q is also a Kan fibration.*

This is Exercise V3.8 in [GJ99].

We now discuss the equivalence between weakly G-principal bundles and G-cocycles. For $X, A \in \mathrm{sSh}(C)$, write $\mathrm{Cocycle}(X, A)$ for the category of cocycles from X to A, according to 3.6.

Definition 3.93. Let $X \in \mathrm{sSh}(C)$ be locally fibrant, and let $G \in \mathrm{sSh}(C)$ be a group object. Define a functor

$$\mathrm{Extr} : \mathrm{w}G\mathrm{Bund}(X) \to \mathrm{Cocycle}(X, \overline{W}G)$$

("extracting" a cocycle) on objects by sending a weakly G-principal bundle $P \to X$ to the cocycle

$$X \overset{\sim}{\twoheadleftarrow} P/_hG \longrightarrow \overline{W}G\ ,$$

where the left morphism is the local acyclic fibration from Theorem 3.91, and where the right morphism is the image under σ_* (Definition 3.20) of the canonical morphism $P//G \to *//G$ of simplicial objects.

Define also a functor

$$\mathrm{Rec} : \mathrm{Cocycle}(X, \overline{W}G) \to \mathrm{w}G\mathrm{Bund}(X)$$

("reconstruction" of the bundle) which on objects takes a cocycle $X \xleftarrow{\pi} Y \xrightarrow{g} \overline{W}G$ to the weakly G-principal bundle

$$g^*WG \to Y \xrightarrow{\pi} X\,,$$

which is the pullback of the universal G-principal bundle (Definition 3.69) along g, and which on morphisms takes a coboundary to the morphism between pullbacks induced from the corresponding morphism of pullback diagrams.

Observation 3.94. *The functor* Extr *sends the universal G-principal bundle $WG \to \overline{W}G$ to the cocycle*

$$\overline{W}G \simeq * \times_G WG \xleftarrow{\simeq} WG \times_G WG \xrightarrow{\simeq} WG \times_G * \simeq \overline{W}G\,.$$

Write

$$q : \mathrm{Cocycle}(X, \overline{W}G) \to \mathrm{Cocycle}(X, \overline{W}G)$$

for the functor given by postcomposition with this universal cocycle. This has an evident left and right adjoint $\bar{q}$. Therefore under the simplicial nerve these functors induce homotopy equivalences in $\mathrm{sSet}_{\mathrm{Quillen}}$.

Theorem 3.95. *The functors* Extr *and* Rec *from Definition 3.93 induce weak equivalences*

$$N\mathrm{w}G\mathrm{Bund}(X) \simeq N\mathrm{Cocycle}(X, \overline{W}G) \quad \in \mathrm{sSet}_{\mathrm{Quillen}}$$

between the simplicial nerves of the category of weakly G-principal bundles and of cocycles, respectively.

Proof. We construct natural transformations

$$\mathrm{Extr} \circ \mathrm{Rec} \Rightarrow q$$

and

$$\mathrm{Rec} \circ \mathrm{Extr} \Rightarrow \mathrm{id}\,,$$

where q is the homotopy equivalence from Observation 3.94.

For

$$X \xleftarrow{\pi} Y \xrightarrow{f} \overline{W}G.$$

a cocycle, its image under $\mathrm{Extr} \circ \mathrm{Rec}$ is

$$X \leftarrow (f^*WG)/_hG \to \overline{W}G\,.$$

The morphism $(f^*WG)/_hG \to X$ factors through Y by construction, so that the left triangle in the diagram

$$\begin{tikzcd} & (f^*WG)/_hG \arrow[dl, "\sim"'] \arrow[d] \arrow[dr] & \\ X & Y \arrow[l, "\sim"] \arrow[r, dashed, "q(f)"] & \overline{W}G \end{tikzcd}$$

commutes. The top right morphism is by definition the image under σ_* (Definition 3.20) of $(f^*WG)//G \to *//G$. This factors the top horizontal morphism as

$$\begin{tikzcd} (f^*WG)//G \arrow[r] \arrow[d] & (WG)//G \arrow[r] \arrow[d] & *//G \\ Y \arrow[r, "f"] & \overline{W}G\,. & \end{tikzcd}$$

Applying the total simplicial object functor to this diagram gives the above commuting triangle on the right. Clearly this construction is natural and hence provides a natural transformation Extr Rec $\Rightarrow q$.

For the other natural transformation, let now $P \to X$ be a weakly G-principal bundle. This induces the following commutative diagram of simplicial objects (with P and X regarded as constant simplicial objects)

$$\begin{tikzcd} P \arrow[d] & (P \times_X P)//G \arrow[l] \arrow[d] & (P \times G)//G \arrow[l, "\phi", "\sim"'] \arrow[r] \arrow[d] & G//G \arrow[d] \\ X & P//G \arrow[l] \arrow[r, equal] & P//G \arrow[r] & *//G \end{tikzcd},$$

where the left and the right square are pullbacks, and where the top horizontal morphism ϕ is the degreewise local weak equivalence which is degreewise induced by the shear map, composed with exchange of the two factors.

The image of the above diagram under σ_*, which preserves all the pullbacks and weak equivalences involved, is

$$\begin{tikzcd} P \arrow[d] & P \times_X P/_hG \arrow[l, two heads, "\sim"'] \arrow[d] & (P \times G)/_hG \arrow[l, "\sim"'] \arrow[r] \arrow[d] & WG \arrow[d] \\ X & P/_hG \arrow[l, two heads, "\sim"'] \arrow[r, equal] & P/_hG \arrow[r] & \overline{W}G \end{tikzcd}.$$

Here the total bottom span is the cocycle $\mathrm{Extr}(P)$, and so the object $(P \times G)/_hG$ over X is $\mathrm{Rec}(\mathrm{Extr}(P))$. Therefore this exhibits a natural morphism $\mathrm{Rec}\,\mathrm{Extr}P \to P$. $\square$

Remark 3.96. By Theorem 3.61 the simplicial set $N\mathrm{Cocycle}(X, \overline{W}G)$ is a presentation of the intrinsic cocycle ∞-groupoid $\mathbf{H}(X, \mathbf{B}G)$ of the hypercomplete ∞-topos $\mathbf{H} = \mathrm{Sh}_\infty^{\mathrm{hc}}(C)$. Therefore the equivalence of Theorem 3.95 is a presentation of theorem 3.19 in [NSSa],

$$G\mathrm{Bund}_\infty(X) \simeq \mathbf{H}(X, \mathbf{B}G)$$

between the ∞-groupoid of G-principal ∞-bundles in $\mathbf{H}$ and the intrinsic cocycle ∞-groupoid of $\mathbf{H}$.

Corollary 3.97. *For each weakly G-principal bundle $P \to X$ there is a weakly G-principal bundle P^f with a levelwise free G-action and a weak equivalence $P^f \xrightarrow{\sim} P$ of weakly G-principal bundles over X. In fact, the assignment $P \mapsto P^f$ is an homotopy inverse to the full inclusion of weakly G-principal bundles with free action into all weakly G-principal bundles.*

Proof. Note that the universal bundle $WG \to \overline{W}G$ carries a free G-action, in the sense that the levelwise action of G_n on $(WG)_n$ is free. This means that the functor Rec from the proof of Theorem 3.95 indeed takes values in weakly G-principal budles with free action. Hence we can set

$$P^f := \mathrm{Rec}(\mathrm{Extr}(P)) = (P \times G)/_hG\,.$$

By the discussion there we have a natural morphism $P^f \to P$ and one checks that this exhibits the homomotopy inverse. $\square$

3.8 Associated bundles

In Section 4.1 of [NSSa] is discussed a general notion of V-fiber bundles which are *associated* to a G-principal ∞-bundle via an action of G on some V. Here we discuss presentations of these structures in terms of the weakly principal simplicial bundles from Section 3.7.2.

Let C be a site with terminal object. By Proposition 3.35 every ∞-group over C has a presentation by a sheaf of simplicial groups $G \in \mathrm{Grp}(\mathrm{sSh}(C)_{\mathrm{lfib}})$. Moreover, by Theorem 3.95 every ∞-action of G on an object V according to Definition 3.1 of [NSSa], is exhibited by a weakly principal simplicial bundle $V \to V/_h G$ which is classified by a morphism $\mathbf{c} : V/_h \to \overline{W}G$. The resulting fiber sequence of simplicial presheaves

$$\begin{tikzcd} V \arrow[r] & V/_h G \arrow[d, "\mathbf{c}"] \\ & \overline{W}G \end{tikzcd}$$

is therefore a presentation for the *universal ρ-associated V-bundle* from Section 4.1 of [NSSa].

In terms of this presentation, Proposition 4.6 in [NSSa] has the following "strictification".

Proposition 3.98. *Let $P \to X$ in $\mathrm{sSh}(C)_{\mathrm{lfib}}$ be a weakly G-principal bundle with classifying cocycle $X \xleftarrow{\simeq} Y \longrightarrow \overline{W}G$ according to Theorem 3.95. Then the ρ-associated simplicial V-bundle $P \times_G V$ is locally weakly equivalent to the pullback of $\mathbf{c}$ along g.*

Proof. By the same argument as in the proof of Theorem 3.91, the morphism $\mathbf{c} : V/_h G \to \overline{W}G$ is a local fibration. By Proposition 3.73 this in turn is isomorphic to the pullback of $V \times_G WG \to \overline{W}G$. Since $\mathrm{sSh}(C)$ is a 1-topos, pullbacks preserve quotients, and so this pullback finally is

$$g^*(WG \times_G V) \simeq (g^* WG) \times_G V \simeq P \times_G V \,.$$

$\square$

Remark 3.99. According to Theorem 4.11 in [NSSa], every V-fiber bundle in an ∞-topos is associated to an $\mathbf{Aut}(V)$-principal ∞-bundle. We observe that the main result of [We11] is a presentation of this general theorem for 1-localic ∞-toposes (with a 1-site of definition) in terms of simplicial presheaves.

4 Models

So far we have discussed presentations of the theory of principal ∞-bundles over arbitrary sites. Here we consider certain examples of sites and discuss aspects of the resulting presentations.

- The trivial site models higher *discrete geometry*. We show how in this case the general theory reduces to the classical theory of ordinary simplicial principal bundles in Section 4.1.
- The site of smooth manifolds models higher *smooth geometry/differential geometry*. Since this site does not have all pullbacks, item 2 of Proposition 2.6 does not apply,

and so it is of interest to identify conditions under which a given principal ∞-bundle is presentable not just by a simplicial smooth manifold, but by a *locally Kan* simplicial smooth manifolds. This we discuss in Section 4.2.

4.1 Discrete geometry

The terminal ∞-topos is the ∞-category Grpd_∞ of ∞-groupoids, the one presented by the standard model category structures on simplicial sets and on topological spaces. Regarded as a *gros* ∞-topos akin to that of smooth ∞-groupoids discussed below in 4.2, we are to think of Grpd_∞ as describing *discrete* geometry: an object in Grpd_∞ is an ∞-groupoid without extra geometric structure. In order to amplify this geometric perspective, we will sometimes speak of *discrete ∞-groupoids*.

We have $\mathrm{Grpd}_\infty \simeq \mathrm{Sh}_\infty(*)$, for $*$ the trivial site. For this site the category of locally fibrant simplicial sheaves from Observation 3.77 is equivalent simply to the category of Kan complexes

$$\begin{tikzcd} \mathrm{sSh}(*)_{\mathrm{lfib}} \arrow[r, hook] \arrow[d, "\simeq"] & \mathrm{sSh}(*) \arrow[d, "\simeq"] \\ \mathrm{KanCpx} \arrow[r, hook] & \mathrm{sSet} \end{tikzcd}$$

and local fibrations/equivalences are simply Kan fibrations and weak homotopy equivalences of simplicial sets respectively. A group object G in $\mathrm{sSh}(C)_{\mathrm{lfib}}$ is a simplicial group; therefore over the trivial site the presentation of principal ∞-bundles from 3.7.2 is by weakly principal Kan simplicial bundles.

There is a traditional theory of *strictly* principal Kan simplicial bundles, i.e. simplicial bundles with G action for which the shear map is an *isomorphism* instead of, more generally, a weak equivalence, see also Remark 3.70. A classical reference for this is [May67]. A standard modern reference is Chapter V of [GJ99]. We now compare this classical theory of strictly principal simplicial bundles to the theory of weakly principal simplicial bundles according to Section 3.7.2.

Definition 4.1. Let G be a simplicial group and X a Kan simplicial set. A *strictly G-principal bundle* over X is a morphism of simplicial sets $P \to X$ equipped with a G-action on P over X such that

1. the G action is degreewise free;
2. the canonical morphism $P/G \to X$ out of the ordinary (1-categorical) quotient is an isomorphism of simplicial sets.

A morphism of strictly G-principal bundles over X is a map $P \to P'$ respecting both the G-action as well as the projection to X. Write $\mathrm{s}G\mathrm{Bund}(X)$ for the category of strictly G-principal bundles.

In [GJ99] this is Definitions 3.1 and 3.2 of Chapter V.

Lemma 4.2. *Every morphism in* $\mathrm{s}G\mathrm{Bund}(X)$ *is an isomorphism.*

In [GJ99] this is Remark 3.3 of Chapter V.

Observation 4.3. *Evidently every strictly G-principal bundle is also a weakly G-principal bundle, Definition 3.79; in fact the strictly principal G-bundles are precisely those weakly G-principal bundles for which the shear map is an isomorphism. This identification induces a full inclusion of categories*

$$\mathrm{s}G\mathrm{Bund}(X) \hookrightarrow \mathrm{w}G\mathrm{Bund}(X)\,.$$

Lemma 4.4. *Every morphism of weakly principal simplicial bundles in* KanCpx *is a weak homotopy equivalence on the underlying Kan complexes.*

Proposition 4.5. *For G a simplicial group, the category* sSet_G *of G-actions on simplicial sets and G-equivariant morphisms carries the structure of a simplicial model category where the fibrations and weak equivalences are those of the underlying simplicial sets.*

This is Theorem 2.3 of Chapter V in [GJ99].

Corollary 4.6. *For G a simplicial group and X a Kan complex, the slice category* sSet_G/X *carries a simplicial model structure where the fibrations and weak equivalences are those of the underlying simplicial sets, after forgetting the map to X.*

Lemma 4.7. *Let G be a simplicial group and $P \to X$ a weakly G-principal simplicial bundle. Then the loop space* $\Omega_{(P\to X)}\mathrm{Ex}^\infty N(\mathrm{w}G\mathrm{Bund}(X))$ *has the same homotopy type as the derived hom space* $\mathbb{R}\mathrm{Hom}_{\mathrm{sSet}_G/X}(P,P)$.

Proof. By Theorem 2.3, Chapter V of [GJ99] and Lemma 4.4 the free resolution P^f of P from Corollary 3.97 is a cofibrant-fibrant resolution of P in the slice model structure of Corollary 4.6. Therefore the derived hom space is presented by the simplicial set of morphisms $\mathrm{Hom}_{\mathrm{sSet}_G/X}(P^f \cdot \Delta^\bullet, P^f)$ and all these morphisms are equivalences. Therefore by Proposition 2.3 in [DKc] this simplicial set is equivalent to the loop space of the nerve of the subcategory of sSet_G/X on the weak equivalences connected to P^f. By Lemma 4.4 this subcategory is equivalent (isomorphic even) to the connected component of $\mathrm{w}G\mathrm{Bund}(X)$ on P. □

Proposition 4.8. *Under the nerve, the inclusion of observation 4.3 yields a morphism*

$$N\mathrm{s}G\mathrm{Bund}(X) \to N\mathrm{w}G\mathrm{Bund}(X)$$

which is

- *for all G and X an isomorphism on connected components;*
- *not in general a weak equivalence in* $\mathrm{sSet}_{\mathrm{Quillen}}$.

Proof. Let $P \to X$ be a weakly G-principal bundle. To see that it is connected in $\mathrm{w}G\mathrm{Bund}(X)$ to some strictly G-principal bundle, first observe that by Corollary 3.97 it is connected via a morphism $P^f \to P$ to the bundle

$$P^f := \mathrm{Rec}(X \leftarrow P/_h G \xrightarrow{f} \overline{W}G)\,,$$

which has free G-action, but does not necessarily satisfy strict principality. Since, by Theorem 3.91, the morphism $P/_h G \to X$ is an acyclic fibration of simplicial sets it has a section $\sigma : X \to P/_h G$ (every simplicial set is cofibrant in $\mathrm{sSet}_{\mathrm{Quillen}}$). The bundle

$$P^s := \mathrm{Rec}(X \xleftarrow{\mathrm{id}} X \xrightarrow{f\circ\sigma} \overline{W}G)$$

is strictly G-principal, and with the morphism

$$
(P^s \to P^f) := \mathrm{Rec} \left(\begin{tikzcd} & P/_h G \arrow[dl, "\sim"'] \arrow[dr, "f"] & \\ X & & \overline{W}G \\ & X \arrow[ul, "\mathrm{id}"] \arrow[uu, "\sigma"] \arrow[ur, "f\circ\sigma"'] & \end{tikzcd} \right)
$$

we obtain (non-naturally, due to the choice of section) in total a morphism $P^s \to P^f \to P$ of weakly G-principal bundles from a strictly G-principal replacement P^s to P.

To see that the full embedding of strictly G-principal bundles is also injective on connected components, notice that by Lemma 4.7 if a weakly G-principal bundle P with degreewise free G-action is connected by a zig-zag of morphisms to some other weakly G-principal bundle P, then there is already a direct morphism $P \to P'$. Since all strictly G-principal bundles have free actions by definition, this shows that two of them that are connected in $\mathrm{w}G\mathrm{Bund}(X)$ are already connected in $\mathrm{s}G\mathrm{Bund}(X)$.

To see that in general $N\mathrm{s}G\mathrm{Bund}(X)$ nevertheless does not have the correct homotopy type, it is sufficient to notice that the category $\mathrm{s}G\mathrm{Bund}(X)$ is always a groupoid, by Lemma 4.2. Therefore $N\mathrm{s}G\mathrm{Bund}(X)$ is always a homotopy 1-type. But by Theorem 3.95 the object $N\mathrm{w}G\mathrm{Bund}(X)$ is not an n-type if G is not an $(n-1)$-type. □

Corollary 4.9. *For all Kan complexes X and simplicial groups G there is an isomorphism*

$$\pi_0 N\mathrm{s}G\mathrm{Bund} \simeq H^1(X,G) := \pi_0 \mathrm{Grpd}_\infty(X, \mathbf{B}G)$$

between the isomorphism classes of strictly G-principal bundles over X and the first non-abelian cohomology of X with coefficients in G (but this isomorphism on cohomology does not in general lift to an equivalence on cocycle spaces).

Proof. By Proposition 4.8 and Remark 3.96. □

Remark 4.10. The first statement of corollary 4.9 is the classical classification result for strictly principal simplicial bundles, for instance Theorem V3.9 in [GJ99].

4.2 Smooth geometry

We discuss the canonical homotopy theoretic context for higher differential geometry.

Definition 4.11. Let SmthMfd be the category of finite dimensional smooth manifolds. We regard this as a site with the covers being the standard open covers. Write

$$\mathrm{CartSp} \hookrightarrow \mathrm{SmthMfd}$$

for the full subcategory on the Cartesian spaces $\mathbb{R}^n$, for all $n \in \mathbb{N}$, equipped with their canonical structure of smooth manifolds.

Proposition 4.12. *The inclusion* $\mathrm{CartSp} \hookrightarrow \mathrm{SmthMfd}$ *exhibits* CartSp *as a* dense subsite *of* $\mathrm{SmthMfd}$. *Accordingly, there is an equivalence of categories between the sheaf toposes over both sites,* $\mathrm{Sh}(\mathrm{CartSp}) \simeq \mathrm{Sh}(\mathrm{SmthMfd})$.

Lemma 4.13. *The sheaf topos* $\mathrm{Sh}(\mathrm{CartSp}) \simeq \mathrm{Sh}(\mathrm{SmthMfd})$ *has enough points. A complete set of points*

$$\left\{ \mathrm{Set} \underset{}{\overset{p^n}{\rightleftarrows}} \mathrm{Sh}(\mathrm{CartSp}) \;\middle|\; n \in \mathbb{N} \right\}$$

is given by the stalks at the origin of the open n-disk, for all $n \in \mathbb{N}$.

This statement was first highlighted in [Dug99]. In more detail, p_n is given as follows. Let $D^n_k \subset \mathbb{R}^n$ denote the smooth manifold given by the standard open n-disk of radius $1/k$ centered at the origin. For $X \in \mathrm{Sh}(\mathrm{CartSp})$ and $n \in \mathbb{N}$ the *n-stalk* of X is the colimit

$$p_n(X) = \varinjlim_{k \to \infty} \mathrm{Hom}(D^n_k, X).$$

of the values on X on these disks. In particular the set $p_0(X)$ is the set of global sections of X.

Definition 4.14. The ∞-topos of *smooth ∞-groupoids* is

$$\mathrm{SmoothGrpd}_\infty := \mathrm{Sh}_\infty(\mathrm{CartSp})\,.$$

Proposition 4.15. *The ∞-topos* $\mathrm{SmoothGrpd}_\infty$ *has the following properties:*

1. *It is hypercomplete.*
2. *It is equivalent to* $\mathrm{Sh}_\infty(\mathrm{SmthMfd})$.
3. *The site C is a* ∞-cohesive site *(Definition 3.37).*

These and the following statements are discussed in detail in Section 4.4. of [Sch]. In particular we have

Observation 4.16. *The ∞-topos of smooth ∞-groupoids is presented by the local projective model structure on simplicial presheaves over* CartSp

$$\mathrm{SmoothGrpd}_\infty \simeq ([\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}})\,.$$

For

$$X \in \mathrm{SmthMfd} \hookrightarrow [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]$$

a smooth manifold and $\{U_i \to U\}$ *a* good open cover *in the sense that every non-empty finite intersection of the U_i is diffeomorphic to an open ball, the Čech nerve $\check{C}(\{U_i\}) \to X$ is a split hypercover. Hence every morphism out of* $X \in \mathrm{SmoothGrpd}_\infty$ *is presented by a hyper-Čech cocycle with respect to this cover.*

Definition 4.17. Write

$$\mathrm{sSh}(\mathrm{CartSp}) := [\Delta^{\mathrm{op}}, \mathrm{Sh}(\mathrm{CartSp})]$$

for the category of simplicial objects in the sheaf topos over CartSp. As in Section 2.1, we say that a morphism in sSh(CartSp) is

- a *local weak equivalence* if it is stalkwise a weak equivalence of simplicial sets;
- a *local fibration* if it is stalkwise a Kan fibration of simplicial sets,

where the stalks $\{p_n\}_{n\in\mathbb{N}}$ are those of Lemma 4.13. Write

$$\mathrm{sSh}(\mathrm{CartSp})_{\mathrm{lfib}} \hookrightarrow \mathrm{sSh}(\mathrm{CartSp})$$

for the full subcategory on the locally fibrant objects.

Proposition 4.18. *The* ∞*-topos* SmoothGrpd$_\infty$ *is presented by the category* sSh(CartSp)$_{\mathrm{lfib}}$ *from Definition 4.17 with weak equivalences the local weak equivalences*

$$\mathrm{SmoothGrpd}_\infty \simeq L_W \mathrm{sSh}(\mathrm{CartSp})_{\mathrm{lfib}}\,.$$

Together with the local fibrations this is a category of fibrant objects, Definition 3.54.

Therefore the hom-∞-groupoids are equivalently given by the cocycle categories of Proposition 3.61.

4.2.1 Locally fibrant simplicial manifolds

By Proposition 4.18 smooth ∞-groupoids are presented by locally fibrant simplicial sheaves on CartSp. Every simplicial manifold represents a simplicial sheaf over this site. We discuss now the full sub-∞-category of SmoothGrpd$_\infty$ on those objects that are presented by locally Kan fibrant simplicial smooth manifolds.

Definition 4.19. Let the category of *locally fibrant simplicial smooth manifolds* be the full subcategory

$$\mathrm{sSmthMfd}_{\mathrm{lfib}} \hookrightarrow \mathrm{sSh}(\mathrm{CartSp})_{\mathrm{lfib}}$$

of the category of locally fibrant simplicial sheaves over smooth manifolds, Definition 4.17, on those simplicial sheaves that are represented by simplicial smooth manifolds.

The structure of a category of fibrant objects on sSh(CartSp)$_{\mathrm{lfib}}$, Proposition 4.18, does not quite transfer along this inclusion, because pullbacks in SmthMfd do not generally exist. Pullbacks in SmthMfd do however exist, notably, along surjective submersions.

Following [He08] we will take advantage of this last fact and give the following enhanced definition of the notion of local fibration between smooth simplicial manifolds. Before we do this however we briefly review the notion of matching object for simplicial objects in SmthMfd. Recall (see for example [GJ99] Section VIII) that if X is a simplicial object in SmthMfd and K is a simplicial set, then the limit

$$\varprojlim_{\Delta^n \to K} X_n$$

in SmthMfd (if it exists) is denoted $M_K X$ and is called the *(generalized) matching object* of X at K. Here the limit is taken over the simplex category Δ/K of K. This notion has a straightforward generalization to simplicial objects in an arbitrary category $\mathcal{C}$. To talk about matching objects we need to confront the afore-mentioned problem that SmthMfd does not have all of the limits that one would like — very often we would like to talk about the limit $M_K X$ without knowing that it actually exists. In this situation, we will (as in [He08]) interpret $M_K X$ as the matching object of the simplicial sheaf on SmthMfd represented by X. If the sheaf $M_K X$ is representable then the matching objects exists in SmthMfd. With these remarks out of the way we can state the following definition.

Definition 4.20. A morphism $f : X \to Y$ in sSmthMfd is

- a *submersive local fibration* if $f_0 \colon X_0 \to Y_0$ is a surjective submersion and for all $0 \leq k \leq n$, $n \geq 1$ the canonical morphism

$$X_n \to Y_n \times_{M_{\Lambda^k[n]}Y} M_{\Lambda^k[n]}X$$

  is a surjective submersion;

- a *submersion* if $f_n : X_n \to Y_n$ is a submersion for each $n \in \mathbb{N}$;

- A simplicial smooth manifold X is said to be a *Lie ∞-groupoid* if $X \to *$ is a submersive local fibration and all of the face maps of X are submersions.

Example 4.21. Let X be a smooth manifold and $\{U_i \to X\}$ an open cover. Then the Čech nerve projection $\check{C}(\{U_i\}) \to X$ is a submersive local acyclic fibration between locally fibrant simplicial smooth manifolds.

Lemma 4.22 ([He08]). *Let $A \hookrightarrow B$ be an acyclic cofibration between finite simplicial sets. Suppose that $f \colon X \to Y$ is a submersive local fibration and that*

$$M_A X \times_{M_A Y} M_B Y$$

is a manifold. Then $M_B X$ is a manifold and

$$M_B X \to M_A X \times_{M_A Y} M_B Y$$

is a surjective submersion.

As a corollary we have the following statement.

Corollary 4.23. *If $f \colon X \to Y$ is a submersive local fibration, then f is a surjective submersion.*

Proof. Consider the acyclic cofibration $\Delta[0] \subset \Delta[n]$ corresponding to the vertex 0 of $\Delta[n]$. Then the diagram

$$\begin{array}{ccc} X_0 \times_{Y_0} Y_n & \longrightarrow & X_0 \\ \downarrow & & \downarrow \\ Y_n & \longrightarrow & Y_0 \end{array}$$

is a pullback in SmthMfd and $X_0 \times_{Y_0} Y_n \to Y_n$ is a surjective submersion. From Lemma 4.22 we see that $X_n \to X_0 \times_{Y_0} Y_n$ is a surjective submersion. Since the map $f_n \colon X_n \to Y_n$ factors through $X_0 \times_{Y_0} Y_n$ we see that f_n is a surjective submersion. □

Proposition 4.24. *The pullback of a (locally acyclic) submersive local fibration in* sSmthMfd *exists and is again a (locally acyclic) submersive local fibration.*

Proof. Suppose that $p \colon X \to Y$ is a submersive local fibration. Then by Lemma 4.23 $p_n \colon X_n \to Y_n$ is a surjective submersion for all n and hence the pullback $X \times_Y Z$ exists in sSmthMfd. We need to show that the projection $X \times_Y Z \to Z$ is a submersive local fibration. Clearly $X_0 \times_{Y_0} Z_0 \to Z_0$ is a submersion. Next, observe that we have isomorphisms of topological spaces

$$\begin{aligned} M_{\Lambda^k[n]}(X \times_Y Z) \times_{M_{\Lambda^k[n]}Z} Z_n &= M_{\Lambda^k[n]}X \times_{M_{\Lambda^k[n]}Y} Z_n \\ &= (M_{\Lambda^k[n]}X \times_{M_{\Lambda^k[n]}Y} Y_n) \times_{Y_n} Z_n. \end{aligned}$$

Since the surjective submersion $X_n \to Y_n$ factors as

$$X_n \to M_{\Lambda^k[n]}X \times_{M_{\Lambda^k[n]}Y} Y_n \to Y_n$$

and $X \to Y$ is a submersive local fibration, it follows that

$$M_{\Lambda^k[n]}X \times_{M_{\Lambda^k[n]}Y} Y_n \to Y_n$$

is also a surjective submersion. Hence

$$M_{\Lambda^k[n]}(X \times_Y Z) \times_{M_{\Lambda^k[n]}Z} Z_n = (M_{\Lambda^k[n]}X \times_{M_{\Lambda^k[n]}Y} Y_n) \times_{Y_n} Z_n$$

is a manifold and

$$X_n \times_{Y_n} Z_n \to M_{\Lambda^k[n]}(X \times_Y Z) \times_{M_{\Lambda^k[n]}Z} Z_n$$

is a surjective submersion.

To check the statement about local weak equivalences, use the facts that stalks commute with pullbacks and that acyclic fibrations in sSet are stable under pullback. ☐

4.2.2 Groups

By Theorem 3.30 every ∞-group in $\mathrm{SmoothGrpd}_\infty$ is presented by some group object in sSh(CartSp). In view of the discussion in Section 4.2.1 it is of interest to determine those which are in the inclusion $\mathrm{sSmthMfd}_{\mathrm{lfib}} \hookrightarrow \mathrm{sSh(CartSp)}$ from Definition 4.19.

Proposition 4.25. *Let G be a simplicial Lie group. Then G is a Lie ∞-groupoid, and so in particular is a locally fibrant simplicial smooth manifold, Definition 4.19.*

Proof. Clearly all of the face maps of G are surjective submersions. Therefore we need to prove that G is locally fibrant. Our proof is based on the observation in Lemma 3.3 of [Ste2] that for any smooth manifold Y, the simplicial set $\mathrm{SmthMfd}(Y, G)$ whose set of n-simplices is the set $\mathrm{SmthMfd}(Y, G_n)$ has the structure of a simplicial group and hence the various maps

$$\mathrm{SmthMfd}(Y, G_n) \to M_{\Lambda^n_k}\mathrm{SmthMfd}(Y, G)$$

are all surjective. Therefore, if the limit $M_{\Lambda^n_k}G$ exists in SmthMfd then we can identify

$$M_{\Lambda^n_k}\mathrm{SmthMfd}(Y, G) = \mathrm{SmthMfd}(Y, M_{\Lambda^n_k}G)$$

and hence conclude that

$$G_n \to M_{\Lambda^n_k}G$$

admits a global section and hence is a surjective submersion. The details are more delicate here than in [Ste2] since we need to show that all of the requisite limits exist in SmthMfd.

When $n = 1$ we need to show that the two face maps $d_0, d_1 \colon G_1 \to G_0$ are surjective submersions which is again clear since $s_0 \colon G_0 \to G_1$ is a global section of both of these maps. When $n = 2$ the matching objects $M_{\Lambda^2_k} G$ for $0 \leq k \leq 2$ can be identified with pullbacks $G_1 \times_{G_0} G_1$ which exist in SmthMfd since $d_0, d_1 \colon G_1 \to G_0$ are submersions. The Yoneda argument above then shows that $G_2 \to M_{\Lambda^2_k} G$ is a surjective submersion in these cases.

The case $n = 3$ makes the general pattern clear: in this case any of the matching objects $M_{\Lambda^3_k} G$ for $0 \leq k \leq 3$ can be identified with pullbacks of the form

$$G_2 \times_{G_1} G_2 \times_{G_1 \times_{G_0} G_1} G_2$$

in which the map $G_2 \to G_1 \times_{G_0} G_1$ is the canonical map $G_2 \to M_{\Lambda^2_k} G$. Likewise the pullback $G_2 \times_{G_1} G_2$ is the matching object $M_{\Lambda^2_k} \mathrm{Dec}_0 G$ where $\mathrm{Dec}_0 G$ is the simplicial Lie group which is the décalage of G (Definition 3.6).

This observation forms the basis for a proof by induction on $n \geq 1$ that for any simplicial Lie group G, and any integer $0 \leq k \leq n$, the limit $M_{\Lambda^n_k} G$ exists in SmthMfd (the Yoneda argument above then shows that G is locally fibrant). The case $n = 1$ is clear.

For the inductive step, first suppose that $k < n$. Observe that we have an identification

$$\Lambda^n_k = C\Lambda^{n-1}_k \cup_{\Lambda^{n-1}_k} \Delta^{n-1}$$

where $C\Lambda^{n-1}_k$ denotes the usual cone construction on Λ^{n-1}_k (see [GJ99] Chapter III). It follows, using Corollary 2.2 of [Ste2] and the fact that the matching objects functor $M_{(-)} G \colon s\mathrm{Set}^{\mathrm{op}} \to \mathrm{Sh}(\mathrm{CartSp})$ on the representable simplicial sheaf G preserves limits, that the diagram

$$\begin{array}{ccc} M_{\Lambda^n_k} G & \longrightarrow & G_{n-1} \\ \downarrow & & \downarrow \\ M_{\Lambda^{n-1}_k} \mathrm{Dec}_0 G & \longrightarrow & M_{\Lambda^{n-1}_k} G \end{array}$$

in Sh(CartSp) is a pullback. Hence $M_{\Lambda^n_k} G$ acquires the unique structure of a smooth manifold so that this diagram is a pullback in the category SmthMfd. It follows that with this unique smooth structure $M_{\Lambda^n_k} G$ is a model for the corresponding limit in SmthMfd.

For the case $k = n$ we apply the statement just proven with G replaced by its *opposite* simplicial Lie group G^o; this has the property that $M_{\Lambda^n_0} G^o = M_{\Lambda^n_n} G$, which shows that the limit $M_{\Lambda^n_n} G$ exists, completing the inductive step. □

4.2.3 Principal bundles

By the discussion in 3.7.2 and using Proposition 4.18 we have a presentation of principal ∞-bundles in the ∞-topos $\mathrm{SmoothGrpd}_\infty$ by weakly principal bundles in the category $\mathrm{sSh}(\mathrm{CartSp})_{\mathrm{lfib}}$ of locally fibrant simplicial sheaves. Here we discuss how parts of this construction may be restricted further along the inclusion $\mathrm{sSmthMfd}_{\mathrm{lfib}} \hookrightarrow \mathrm{sSh}(\mathrm{CartSp})_{\mathrm{locfib}}$ of locally fibrant simplicial smooth manifolds, Definition 4.2.1.

Proposition 4.26. *Let G be a simplicial lie group. Then the following statements are true:*

1. *the object $\overline{W}G \in \mathrm{sSh}(\mathrm{CartSp})$, Definition 3.26, is presented by a submersively locally fibrant simplicial smooth manifold.*

2. *the universal G-principal bundle $WG \to \overline{W}G$, Definition 3.31, formed in* sSh(CartSp) *is presented by a* submersive *local fibration of simplicial smooth manifolds.*

Proof. We first prove 1. Our proof of this essentially follow the proof of the corresponding result (Lemma 4.3) in [Ste2], some extra care is needed however since it is not immediately clear that all of the requisite limits exist in SmthMfd. Therefore we will prove by induction on $n \geq 1$ that for any simplicial Lie group G and any integer $0 \leq k \leq n$, the limit $M_{\Lambda^n_k}\overline{W}G$ exists in SmthMfd and the canonical map $\overline{W}G_n \to M_{\Lambda^n_k}\overline{W}G$ is a surjective submersion.

Suppose we have shown that for any simplicial Lie group G, the canonical map $\overline{W}G_{n-1} \to M_{\Lambda^{n-1}_k}\overline{W}G$ is a surjective submersion for all $0 \leq k \leq n-1$. Let $0 \leq k < n$. We claim that, under this assumption, the following statements are true:

(a) the limit $M_{\Lambda^{n-1}_{k-1}}WG$ exists in SmthMfd,

(b) the map
$$WG_{n-1} \to M_{\Lambda^{n-1}_{k-1}}WG \times_{M_{\Lambda^{n-1}_{k-1}}\overline{W}G} \overline{W}G_{n-1}$$
is a surjective submersion

Granted these statements, we shall show that the map in (b) is the canonical map
$$\overline{W}G_n \to M_{\Lambda^n_k}\overline{W}G.$$

As in [Ste2] and the proof of Proposition 4.25 above observe that we have an identification
$$\Lambda^n_k = C\Lambda^{n-1}_k \cup_{\Lambda^{n-1}_k} \Delta^{n-1}.$$

It follows that we have an identification of sheaves on CartSp
$$M_{\Lambda^n_k}\overline{W}G = M_{\Lambda^{n-1}_k}WG \times_{M_{\Lambda^{n-1}_k}\overline{W}G} \overline{W}G_{n-1}$$

which belongs to the image of SmthMfd $\hookrightarrow$ Sh(CartSp). It follows that the limit $M_{\Lambda^n_k}\overline{W}G$ exists in SmthMfd, as required.

To complete the inductive step we need to deal with the case when $k = n$. Just as in the proof of Proposition 4.25 above, we can settle this case by replacing the group G with its opposite simplicial group G^o.

It remains to prove the statements (a) and (b) above and the second statement of the Proposition. Before we do so, let us note that in analogy with Definition 4.1 we have a notion of a strictly principal bundle in simplicial manifolds, the only difference being that we require the bundle projection to be a submersion.

Definition 4.27. Let G be a simplicial Lie group and let X be a simplicial manifold. A *strictly principal G-bundle* on X is a simplicial manifold P together with a submersion $P \to X$ and an action of G on P such that for every $n \geq 0$, the action of G_n on P_n equips $P_n \to X_n$ with the structure of a (smooth) principal G_n bundle.

To prove the second statement of the Proposition, and the statements (a) and (b) above, it is enough to prove the following lemmas.

Lemma 4.28. *Suppose that P is a strictly principal G-bundle on X in* SmthMfd *such that $P_n \to X_n$ admits a section for all $n \geq 0$. If for some $0 \leq k \leq n$ and some $n \geq 1$, $X_n \to M_{\Lambda^n_k} X$ is a surjective submersion, and the limit $M_{\Lambda^n_k} P$ exists in* SmthMfd, *then the map*

$$P_n \to M_{\Lambda^n_k} P \times_{M_{\Lambda^n_k} X} X_n$$

is a surjective submersion and hence

$$P_n \to M_{\Lambda^n_k} P$$

is also a surjective submersion.

Lemma 4.29. *Suppose that P is a strictly principal G-bundle on X in* SmthMfd. *Suppose that for some $0 \leq k \leq n$ and some $n \geq 1$ the canonical map $X_n \to M_{\Lambda^n_k} X$ is a surjective submersion. Then*

$$M_{\Lambda^n_k} P \to M_{\Lambda^n_k} X$$

is a smooth principal bundle with structure group the Lie group $M_{\Lambda^n_k} G$.

Proof of Lemma 4.28. Let Y be an object of SmthMfd. Then we can form simplicial sets $\mathrm{SmthMfd}(Y, P)$ and $\mathrm{SmthMfd}(Y, X)$ whose sets of n-simplices are given by $\mathrm{SmthMfd}(Y, P_n)$ and $\mathrm{SmthMfd}(Y, X_n)$ respectively. Since the functor $\mathrm{SmthMfd}(Y, -)$ preserves limits and the projections $P_n \to X_n$ admit sections for all $n \geq 0$, we see that the induced map

$$\mathrm{SmthMfd}(Y, P) \to \mathrm{SmthMfd}(Y, X)$$

is a strictly principal bundle in sSet with structure group $\mathrm{SmthMfd}(Y, G)$. In particular the map

$$\mathrm{SmthMfd}(Y, P_n) \to M_{\Lambda^n_k} \mathrm{SmthMfd}(Y, P) \times_{M_{\Lambda^n_k} \mathrm{SmthMfd}(Y,X)} \mathrm{SmthMfd}(Y, X_n)$$

is surjective. Taking $Y = M_{\Lambda^n_k} P \times_{M_{\Lambda^n_k} X} X_n$ we see that the map

$$P_n \to M_{\Lambda^n_k} P \times_{M_{\Lambda^n_k} X} X_n \tag{1}$$

admits a section. The map (1) is a morphism of principal bundles over X_n, covering the homomorphism of Lie groups $G_n \to M_{\Lambda^n_k} G$. Since the smooth map underlying this homomorphism admits a section it follows that we can find a section of (1) through every point of P_n. Therefore (1) is a surjective submersion. □

Proof of Lemma 4.29. The limit $M_{\Lambda^n_k} P$, if it exists, is uniquely determined by the requirement that $M_{\Lambda^n_k} P \to M_{\Lambda^n_k} X$ is a smooth $M_{\Lambda^n_k} G$ bundle, and that $P_n \to M_{\Lambda^n_k} P$ is equivariant for the homomorphism $h^n_k \colon G_n \to M_{\Lambda^n_k} G$. Since the quotient $P_n / \ker(h^n_k)$ of P_n by the free action of the normal Lie subgroup $\ker(h^n_k)$ has both of these properties, it follows that $M_{\Lambda^n_k} P$ exists and is isomorphic to $P_n / \ker(h^n_k)$. □

□

Proposition 4.30. *Let G be a simplicial Lie group which presents a smooth ∞-group in* $\mathrm{Grp}(\mathrm{SmoothGrpd}_\infty)$. *Suppose that $\overline{W}G$ is* CartSp*-acyclic (Definition 3.43). Then every G-principal ∞-bundle over a smooth manifold $X \in \mathrm{SmthMfd} \hookrightarrow \mathrm{SmoothGrpd}_\infty$ has a presentation by a weakly principal G-bundle $P \to X$ for which P is a locally fibrant simplicial smooth manifold and $P \to X$ is a submersive local fibration.*

Proof. By assumption of CartSp-acyclicity and theorem 3.46, we have that

$$\overline{W}G \in [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$$

is fibrant. It follows that any cocycle that classifies a given G-principal ∞-bundle according to Theorem 3.95 is presented by a morphism of simplicial presheaves into $\overline{W}G$ out of a cofibrant resolution of X. By Observation 4.16 we may choose this to be given by the Čech nerve $\check{C}(\{U_i\}) \to X$ of a (differentiably) good open cover $\{U_i \to X\}$ of X. By example 4.21 this is itself a submersive local acyclic fibration. By Proposition 4.26 and Proposition 4.24 the morphism $g^*WG \to \check{C}(U_i)$ in the pullback diagram of simplicial sheaves

$$\begin{tikzcd} g^*WG \arrow[r] \arrow[d] & WG \arrow[d] \\ \check{C}(U_i) \arrow[r, "g"] & \overline{W}G \end{tikzcd}$$

is a submersive local fibration between locally Kan simplicial smooth manifolds. Hence so is the composite $P := g^*WG \to \check{C}(U_i) \to X$, which, by Theorem 3.95 is the principal ∞-bundle $P \xrightarrow{p} X$ classified by g. □

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