

**STOCHASTIC CALCULUS FOR QUANTUM BROWNIAN MOTION
OF NON-MINIMAL VARIANCE**

— an approach using integral-sum kernel operators

J Martin Lindsay and Hans Maassen

Abstract

Introduction

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ABSTRACT

The stochastic calculus of non-minimal variance quantum Brownian motion is developed by means of a representation in terms of integral-sum kernels. This representation permits a direct definition of stochastic integrals; a clear view of the structure of martingales; a unified approach to linear quantum stochastic differential equations with explicit expression for their solution; and further insight into the structure of adapted cocycles with direct means of finding their generators. Quantum sde's are means by which dynamical equations for dissipative quantum systems may be solved. The stationary Markov processes resulting from their solution are characterised in terms of quantum detailed balance. Finally an elementary example is treated and the physical interpretation of its constituents is given.

Introduction

Quantum stochastic calculus has flowered from its fundamental papers [HP 1], [BSW 1] into a subject rich, both in structure and applications, with two books on the subject appearing this year ([Par], [Me 3]). The wider subject of quantum, or non-commutative, probability continues to be intensively developed, from the diverse view-points of probability, physics and analysis (see e.g. [QPI-VII]). The present paper is based on an earlier preprint ([LM]). We hope that it is more-or-less self-contained, but our intention is to complement Meyer's lecture notes, where one of the central themes is the interplay between algebraic structures on Hilbert space (especially Fock space) and probabilistic interpretations.

The Barnett–Streater–Wilde theory is based on the Clifford process, which is a precise fermionic analogue of classical Brownian motion ([LM 2]). Barnett's extension ([Bar]) of Segal's non-commutative integration theory ([Seg]) is applied to the Clifford algebra of $L^2(\mathbb{R}_+)$ with its natural trace and filtration of sub-algebras. The Hudson–Parthasarathy calculus is based on a (minimal variance) quantum Brownian motion ([CoH]). Loosely speaking, this consists of a pair of classical Brownian motions (Q, P) satisfying the canonical commutation relations (with probabilists' normalisation):

$$Q_s P_t - P_t Q_s = i 2 (s \wedge t). \quad (0.1)$$

Equivalently, quantum Brownian motion may be considered as a non-commutative complex Brownian motion $A = (Q + iP)/2$, whose real and imaginary parts satisfy (0.1).

How can classical processes fail to commute? Each process must be represented as a family of commuting self-adjoint operators on a Hilbert space \mathcal{H} , with a unit vector ψ determining the law of the process $Q + iP$. The pair (Q, P) is then a quantum Brownian motion (qBm), of variance σ^2 , if the following algebraic, probabilistic and non-degeneracy conditions are satisfied:

$$e^{i(xP_s + yQ_t)} = e^{ixP_s} e^{iyQ_t} e^{ixy(s \wedge t)} \quad (0.2)$$

$$\mathbb{E}^\psi[e^{i(xP_s + yQ_t)}] = e^{-\sigma^2(sx^2 + ty^2)/2}$$

$$\{e^{i(xP_s + yQ_t)}\psi : x, y \in \mathbb{R}, s, t \geq 0\} \text{ generates } \mathcal{H}. \quad (0.3)$$

As usual, we are working in units in which Planck's constant is 2π . The Weyl relations (0.2), which are a mathematically convenient form for the commutation relations, impose a constraint on the variance of a quantum Brownian motion: $\sigma^2 \geq 1$. This is a manifestation of Heisenberg's uncertainty principle.

There is a qualitative difference between the calculus of minimal, and non-minimal, variance qBm. The *degeneracy* of minimal variance qBm is discussed in [HL 2] from physical, probabilistic and mathematical points of view. The crucial mathematical point is that the state which determines the law of the Brownian motion is not faithful. It is therefore not sufficient to know only how operators act on the single vector ψ – one must work with a convenient dense subspace of \mathcal{H} such as the exponential domain ([HP 1]). The algebra generated by minimal variance qBm is the full algebra of all operators on \mathcal{H} . One consequence of this is that the quantum Brownian filtration admits martingales quite different in character to (quantum) Brownian motion, such as the preservation, or number, process and both classical and quantum Poisson processes ([HP 1], [FrM]).

The non-minimal variance (or quasi-free) theory was developed by Barnett, Streater and Wilde ([BSW 2]), Hudson and one of the present authors ([HL 1,2,3], [L 1,2]). Here the state is faithful, so that as well as being cyclic (0.3), the vector ψ is also separating for the algebra \mathcal{N}^σ generated by the qBm:

$$T_1, T_2 \in \mathcal{N}^\sigma, T_1\psi = T_2\psi \implies T_1 = T_2.$$

This allows operator questions to be tackled by vector considerations, and leads to a tighter theory. For example, there is a Kunita-Watanabe type representation theorem for square-integrable martingales ([HL 1], [L 2]), which follows from an orthogonal decomposition of the Hilbert space. This fails in the minimal variance case, even when the preservation process is included ([JoM]); and so far there are only partial results ([PS 1,3]).

Symmetric Fock space (over $L^2(\mathbb{R}_+)$) may be identified with *Guichardet space*, which is an L^2 -space of functions defined on the finite power set of \mathbb{R}_+ ([Gui]). This representation was used by one of us to formulate quantum Itô calculus in terms of integral-sum kernel operators ([M1,2]). An advantage of this approach is that solutions of linear quantum stochastic differential equations appear in a very explicit form. The key idea is the multiple quantum stochastic integral representation

$$X = \iint x(\sigma, \tau) dA_\sigma^* dA_\tau \quad (0.4)$$

where $\sigma, \tau \subset \mathbb{R}_+$, $A_\tau = \prod_{t \in \tau} A_t$ and $A_\sigma^* = (A_\sigma)^*$ (adjoint), for operators X on \mathcal{H} . This combines with (quantum) Itô relations to reveal x as an integral-sum *kernel* for the operator X , and also to represent the product of operators in terms of a *convolution-like* product of kernels. Meyer extended this idea to incorporate the preservation process ([Me2]). The integral-sum representation also helps to clarify the relationship between quantum stochastic and the calculus of classical, but anticipating, stochastic processes ([L3]). The idea of obtaining algebraic structure from multiple (quantum) stochastic integrals, commutation relations and Itô relations is further discussed in [Me1], [Me3], [LM1] and [LiP].

The non-minimal variance theory is *even better* suited to a development in terms of integral-sum kernels. Part of the reason for this is that in this theory *every* L^2 -operator is represented by (0.4). In this paper we develop the whole theory from the kernel point of view. New features include a direct definition of the stochastic integrals (Definition 3.2.4); a very simple proof of the stochastic integral representability of martingales (Proposition 7.5.1); a unified approach to linear quantum stochastic differential equations (Section 7.6); explicit expression for the solutions of such equations (Theorems 4.1.1 and 7.6.1); further insight into the structure of adapted cocycles, and a direct means of finding their *generators* (proofs of Propositions 4.3.2 and 8.1.2).

The original physical motivation for quantum stochastic calculus was to integrate dynamical equations describing dissipative quantum systems ([HP1,2]); in other words, to dilate quantum dynamical semigroups to

(stationary) quantum Markov processes ([Küm], [M1,2], [Fri]). This is done by solving a quantum stochastic differential equation whose coefficients are related to the generator of the semigroup, and may be done with non-minimal variance qBm if and only if the semigroup satisfies detailed balance ([Ali], [KFGV]). This is shown in Section 9, where physical parameters are re-introduced. In the final section a simple example is described, and the physical interpretation of each of the constituents is explained. For a *field test* of the theory described here see [RoM], where it is used to calculate the dynamical Stark effect. This spectacular phenomenon in quantum optics was predicted on theoretical grounds in the late 60's and observed a few years later. The approach using integral-sum kernels neatly unifies two opposing viewpoints on the phenomenon; one coming from the master equation, and the other from perturbative methods using Feynman diagrams.

The Itô-Clifford theory ([BSW1]), and the generic variance Fermi theory ([BSW2], [L2]), are amenable to a very similar treatment. The formula for the product of Clifford (respectively fermionic) random variables is obtained simply by introducing a ± 1 -valued signature function into the Wiener (resp. bose) product (see [Me3], [LM1]). The Fock Fermi theory ([ApH]), which has *to some extent* been subsumed by the Bose theory (see [HP3], [PS2]), may be described in terms of kernels too.

Before beginning with a brief heuristic discussion of classical Brownian motion from the present point of view, we mention some of the standard notations and conventions used here. \mathcal{L}^0 and \mathcal{L}^p will denote the linear spaces of measurable, respectively p -summable functions on a measure space; \mathcal{B} in (X, \mathcal{B}, μ) the Borel σ -algebra of a topological space X ; $C_0(X)$ the space of continuous functions, on a locally compact space, which vanish at infinity; and $C^1_\kappa(I)$ the space of once continuously differentiable functions of compact support on an interval I . The indicator function of a set S will be denoted χ_S ; bold symbols always denote n -tuples and inner products follow Dirac's convention of linearity in the *second* argument. A list of special symbols, and where they are introduced, is given at the end.

1. Commutative kernel calculus.

1.1 Wiener-Fock space.

Let \mathbb{P} be the Wiener measure on $\{\gamma: I \mapsto \mathbb{R} \text{ s.t. } \gamma \text{ is continuous and } \gamma(0) = 0\}$ where I is the unit interval $[0, 1]$ and B the coordinate (Wiener) process. It is well known that any complex-valued random variable $F \in L^2(\mathbb{P})$ can be expanded as an infinite sum of iterated stochastic integrals:

$$F = \sum_{n=0}^{\infty} \int_{\Omega_n} f_n(t_1, \dots, t_n) dB_{t_1} \dots dB_{t_n}, \quad (1.1)$$

where Ω_n is the n -dimensional simplex $\{t \in I: t_1 < t_2 < \dots < t_n\}$. For $n > 0$, f_n denotes a square integrable function on Ω_n , and f_0 is the constant $\mathbb{E}[F]$. The sequence $f = (f_0, f_1, \dots)$ will be called the *integral-sum kernel* of F . We have the relation

$$\mathbb{E}[F^2] = \sum_{n=0}^{\infty} \|f_n\|^2$$

indicating that the correspondence between F and f is a unitary equivalence between *Wiener space* $\mathcal{W} := L^2(\mathbb{P})$ and $\mathcal{F} := \bigoplus_{n=0}^{\infty} L^2(\Omega_n)$, called the *symmetric (or boson) Fock space* of $L^2(I)$ in the physics literature. This isomorphism invites several questions. For instance, what algebraic structure is induced on \mathcal{F} by the multiplication of random variables in \mathcal{W} ? How is this structure connected to stochastic integration?

In this section we answer these questions on a *formal level*. In the remaining sections we treat in detail the situation which arises when, in the above, the Wiener process is replaced by a *quantum Brownian motion*.

1.2 Set notation.

The n -dimensional ordered simplex Ω_n can be naturally identified with the set $\{\omega \in I: \#\omega = n\}$ and hence the infinite union $\Omega = \bigcup_{n=0}^{\infty} \Omega_n$ may be regarded as the *finite power set* of the interval I :

$$\Omega(I) = \{\omega \subset I: \omega \text{ is finite}\}$$

For this section (only) the measure on $\Omega(I)$ which on $\Omega_n(I)$ is given by the Lebesgue measure $dt_1 \dots dt_n$, and which has \emptyset as an atom of weight 1 will be

denoted by $d\omega$. We may now write the space of kernels as $L^2(\Omega, d\omega)$ and rewrite (1.1) as

$$F = \int_{\Omega} f(\omega) dB_{\omega}.$$

1.3 An algebraic structure on Fock space.

Let f and g be the kernels of F and $G \in \mathcal{W}$ respectively, we calculate formally the kernel of the product FG :

$$FG = \int_{\Omega} f(\omega) dB_{\omega} \cdot \int_{\Omega} g(v) dB_v$$

Because of the *Itô rule* $(dB)^2 = dt$, the integral over $(\omega, v) \in \Omega \times \Omega$ contains non-zero contributions from those regions where points of ω and v coalesce: $\gamma := \omega \cap v \neq \emptyset$. Performing a change of variable $\alpha = \omega \setminus \gamma$ and $\beta = v \setminus \gamma$ one obtains

$$FG = \iiint_{\alpha \cap \beta = \emptyset} f(\alpha \cup \gamma) g(\beta \cup \gamma) dB_{\alpha} dB_{\beta} d\gamma.$$

Next, the integrals over α and β may be replaced by a single integral over $\sigma = \alpha \cup \beta$, followed by a sum over $\alpha \subset \sigma$:

$$FG = \int_{\Omega} \left\{ \sum_{\alpha \subset \sigma} \int_{\Omega} f(\alpha \cup \gamma) g(\bar{\alpha} \cup \gamma) d\gamma \right\} dB_{\sigma}$$

where $\beta = \bar{\alpha}$, the complement of α in σ .

The expression in brackets is thus the integral-sum kernel of FG and (in this section only) will be denoted $f * g$. For details of the proof of this correspondence see [LM2].

1.4 Kernel calculus.

Taking the product $(f, g) \mapsto f * g$ as a starting point one may build up a stochastic calculus. After specifying a class \mathcal{K} of kernels on which this product is well defined, one introduces \mathcal{K} -valued *processes* $\{f_t\}_{t \in I}$ which are *non-anticipating* in the sense that $f_t(\omega) = 0$ as soon as $\max \omega > t$. By the nature of formula (1.1) one easily derives the form of the kernel $(If)_t$ of the stochastic

integral $\int_0^t F_s dB_s$ of an L^2 -process F :

$$(If)_t(\sigma) = \begin{cases} f_{\max\sigma}(\sigma \setminus \{\max\sigma\}) & \text{if } \sigma \neq \emptyset \text{ and } t \geq \max\sigma \\ 0 & \text{otherwise} \end{cases}$$

and similarly, the inverse operation of *stochastic differentiation*, is given by:

$$(\Delta f)_t(\sigma) = f_t(\sigma \cup \{t\})$$

Apart from these operations there are the pathwise integration and differentiation of processes:

$$(Df)_t(\sigma) = \frac{d}{dt} f_t(\sigma).$$

This is the kernel of the forward derivative ([Nel]) of the process F . The following relations hold:

$$\Delta If = f; \quad D \int_0^\cdot f_s ds = f;$$

and also,

$$f_t - f_0 = (If)_t + \int_0^t (Df_s) ds$$

which decomposes the semimartingale (f_t) into a martingale part and a bounded variation part. The connection between the algebraic and the differential structure of kernel processes is given by the Leibnitz formula for Δ and Itô formula for D :

$$\Delta(f * g) - \Delta f * g - f * \Delta g = 0$$

$$D(f * g) - Df * g - f * Dg = \Delta f * \Delta g.$$

2. Bose chaos

In the next two sections we describe the algebraic part of non-commutative kernel calculus.

2.1 $\Gamma, \mathcal{V}, \rho, \mu_{c_+, c_-}, \dagger$

For each sub-interval I of \mathbb{R} we introduce $\Gamma_I = \Gamma(I)$, the *charged finite power set* of I , consisting of finite subsets of $\text{int}(I)$, the interior of I , each point carrying a “charge” — positive or negative.

Definition 2.1.1: $\Gamma_I := \{\sigma : \text{int}(I) \rightarrow \{0, +1, -1\} \mid \text{the support of } \sigma \text{ is finite}\}$.

We shall write Γ when the interval I is understood. For $\sigma \in \Gamma$, let $\sigma^\pm = \sigma^{-1}(\{\pm 1\})$ and $|\sigma|$, $|\sigma^+|$, $|\sigma^-|$ be the cardinalities of $\text{supp } \sigma$, σ^+ and σ^- respectively. We continue to think of elements of Γ as sets, using notations like $\sigma \cup \tau$ and $\sigma \setminus \tau$ where there is no danger of confusion. To each $\sigma \in \Gamma$ there corresponds a unique element $(s, \varepsilon) \in \text{int}(I)^{|\sigma|} \times \{+1, -1\}^{|\sigma|}$ such that $s_1 < s_2 < \dots < s_n$, where $\varepsilon_j = \sigma(s_j)$ and $n = |\sigma|$. We shall frequently make these identifications, and write p_1, p_2 for the projections

$$\sigma = (s, \varepsilon) \mapsto s; \quad \sigma = (s, \varepsilon) \mapsto \varepsilon, \quad (p_i(\emptyset) = \emptyset),$$

respectively. Note the following partitions of Γ :

$$\Gamma = \bigcup_{j,k=0}^{\infty} \Gamma^{j,k} = \bigcup_{n=0}^{\infty} \Gamma^n, \quad (2.1)$$

where $\Gamma^{j,k} = \{\sigma \in \Gamma : |\sigma^+| = j, |\sigma^-| = k\}$ and $\Gamma^n = \{\sigma \in \Gamma : |\sigma| = n\}$. Let $\rho_0 : \Gamma \times \Gamma \rightarrow \mathbb{R}_+$ be the map given by

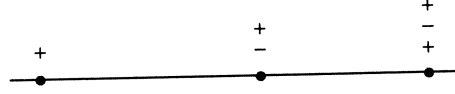
$$(\sigma, \tau) \mapsto \begin{cases} 0 & \text{if } \sigma = \tau = \emptyset \\ 1 \wedge \max_i |s_i - t_i| & \text{if } p_2(\sigma) = p_2(\tau) \\ 1 & \text{otherwise} \end{cases}$$

then (Γ, ρ_0) is a metric space, and we denote its completion by (\mathcal{V}, ρ) . $\mathcal{V} = \mathcal{V}_I$ may be identified with the set

$$\{(s, \varepsilon) \in \bigcup_{n=0}^{\infty} I^n \times \{+1, -1\}^n : s_1 \leq s_2 \leq \dots \leq s_n\}$$

and its elements considered as (*charged*) *generalised subsets* of I , in the sense

that it includes elements of the form



in which sites are occupied by more than one "particle". Under this identification, the union map

$$(\sigma, \tau) \in \mathcal{P} \times \mathcal{P} \mapsto \sigma \cup \tau \in \mathcal{P}$$

is measurable. The partitions (2.1) carry over to \mathcal{P} and we shall write $\mathcal{P}^{\leq N}$ for $\bigcup_{j+k \leq N} \mathcal{P}^{j,k}$. Now fix constants $c_+ \geq c_- > 0$. A Borel measure $\mu = \mu_{c_+, c_-}$ on (\mathcal{P}, ρ) is defined as follows. First define a Borel measure λ on (\mathcal{P}, ρ) by

$$\lambda(\emptyset) = 1; \quad \lambda|_{\mathcal{P}^{j,k}} = \lambda_{j+k} \times d_{j,k} \quad (j+k > 0)$$

where λ_n is n -dimensional Lebesgue measure and $d_{j,k}$ is the counting measure on $\{\varepsilon \in \{+, -\}^{j+k} : \text{exactly } j \text{ '+'s and } k \text{ '-'s occur}\}$. Then let μ be defined by

$$d\mu = m d\lambda \quad \text{where} \quad m(\sigma) = c_+^{|\sigma_+|} c_-^{|\sigma_-|},$$

in other words $m(s, \varepsilon) = \prod_i c_{\varepsilon_i}$. Clearly $\mathcal{P}_I \setminus \Gamma_I$ is μ -null and, if I is bounded,

$$\mu(\mathcal{P}_I) = \exp\{(c_+ + c_-)\lambda_1(I)\}. \quad (2.2)$$

The measure therefore simply counts the positive charges and the negative charges, and weights accordingly, whereas the metric is sensitive to the way in which the charges are distributed.

Now let \mathcal{A} be an involutive Banach algebra with unit I and involution $*$.

Definition 2.1.2: Let $\dagger: \mathcal{P}_I \rightarrow \mathcal{P}_I$ be the *charge changing map*

$$(s, \varepsilon) \rightarrow (s, -\varepsilon),$$

μ^\dagger the *induced measure*

$$\mu^\dagger(U) = \mu(\{\sigma : \sigma^\dagger \in U\}),$$

and, for $x: \mathcal{P} \rightarrow \mathcal{A}$, let x^\dagger be its *involute*:

$$\sigma \rightarrow x(\sigma^\dagger)^*.$$

2.2 Smooth kernels.

We now introduce the class of \mathcal{A} -valued functions on Γ which will form the basis of the present treatment of the kernel calculus. The choice of class is motivated by our requirement that it support both an *algebraic* and a *stochastic differential structure*.

Definition 2.2.1: For $x : \Gamma_I \rightarrow \mathcal{A}$, consider the properties

$\mathcal{K}i$: $\exists J_x$ a compact sub-interval of I such that x vanishes outside Γ_{J_x} ;

$\mathcal{K}ii$ $\exists K_x \geq 1$ such that $\|x(\sigma)\| \leq K_x^{|\sigma|+1} \forall \sigma \in \Gamma_I$;

$\mathcal{K}iii$ $\exists K_x \geq 1$ such that $\|x(\sigma) - x(\tau)\| \leq \rho_0(\sigma, \tau) K_x^{|\sigma|+1}$ as soon as $\rho_0(\sigma, \tau) < 1$.

Denote the class of functions satisfying $\mathcal{K}i$, $\mathcal{K}ii$ and $\mathcal{K}iii$ by $\mathcal{K}_0^{\mathcal{A}}(I)$, or \mathcal{K}_0 , and call the elements of \mathcal{K}_0 *smooth kernels*. Also denote the class of strongly \mathcal{A} -measurable functions ([Yos]) satisfying $\mathcal{K}i$ and $\mathcal{K}ii$ by $\mathcal{K}_b^{\mathcal{A}}(I)$. Each of the properties is clearly preserved under the involution \dagger . For subintervals J of I , $\mathcal{K}_0^{\mathcal{A}}(J)$ is naturally included in $\mathcal{K}_b^{\mathcal{A}}(I)$ but not in $\mathcal{K}_0^{\mathcal{A}}(I)$. By $\mathcal{K}iii$, any smooth kernel has a unique extension to \mathcal{V}_I , now satisfying $\mathcal{K}i$, $\mathcal{K}ii$ and $\mathcal{K}iii$ with Γ replaced by \mathcal{V} and ρ_0 by ρ , and with the same J_x and K_x . Smooth kernels will therefore frequently be defined only on Γ_I but will thereafter be *considered as functions on the whole of \mathcal{V}_I* with no notational change.

Example 2.2.2: For $f \in C_K^1(I)$, the following are smooth \mathbb{C} -valued kernels

$$\begin{aligned} \pi_f : \sigma &\mapsto \prod_{s \in \sigma^+} f(s) \cdot \prod_{t \in \sigma^-} \overline{(-f(t))}; \\ w_f &= \hat{\mu}(f) \pi_f; \\ \pi_f^{j,k} &= \chi_{\Gamma^{j,k}} \pi_f; \quad \pi_f^{(n)} = \chi_{\Gamma^n} \pi_f; \end{aligned} \tag{2.3}$$

$$a_f : \sigma \mapsto \begin{cases} \overline{f(t)} & \text{if } \sigma = \{t^-\} \in \Gamma^{0,1} \\ 0 & \text{otherwise} \end{cases}; \quad r_f = -i \pi_f^{(1)} = i^{-1} \{a_f^\dagger - a_f\}$$

where $\hat{\mu} : C_K^1(I) \rightarrow \mathbb{R}_+$ is the map $f \mapsto \exp\{-\frac{1}{2}(c_+ + c_-) \|f\|_{L^2(I)}^2\}$. The w_f 's, a_f^\dagger 's and a_f 's will be called respectively the (smooth) *Weyl*, *creation* and

annihilation kernels. For $w_0 = \pi_0$ we shall sometimes write δ_\emptyset .

2.3 The Bose product.

We next introduce the product on \mathcal{K}_b whose form is dictated by the non-commutative duality transform (see section 6). For $x \in \mathcal{K}_b$, the maps $\omega \mapsto x(\omega \cup \alpha)$ and $\omega \mapsto x(\omega^\dagger \cup \alpha)$ are strongly measurable for each subset α of σ , so the next definition is a good one.

Proposition 2.3.1: For $x, y \in \mathcal{K}_b^{\mathcal{A}}(I)$ the following map also belongs to \mathcal{K}_b :

$$z: \sigma \mapsto \int \sum_{\alpha \subset \sigma} x(\omega^\dagger \cup \alpha) y(\omega \cup \bar{\alpha}) d\omega$$

where the sum is over disjoint partitions $\alpha \cup \bar{\alpha}$ of σ . Moreover if $x, y \in \mathcal{K}_0$ then $z \in \mathcal{K}_0$ also.

Note that the sum is a finite one, and we have abbreviated $d\mu(\omega)$ to $d\omega$.

Proof: Let J be a compact sub-interval of I containing J_x and J_y . Notice that for each $\alpha \in \Gamma_I$,

$$\begin{aligned} \int_{\Gamma} \|x(\omega^\dagger \cup \alpha) y(\omega \cup \bar{\alpha})\| d\omega &\leq \int_{\Gamma_J} K_x^{|\omega|+|\alpha|+1} K_y^{|\omega|+|\bar{\alpha}|+1} d\omega \\ &\leq K_x^{|\alpha|+1} K_y^{|\bar{\alpha}|+1} \int_{\Gamma_J} (K_x K_y)^{|\omega|} d\omega \\ &= K_x^{|\alpha|+1} K_y^{|\bar{\alpha}|+1} \exp\{(c_+ + c_-) K_x K_y \lambda_1(J)\}, \end{aligned}$$

so that z is well-defined as a Bochner integral. Next, since

$$\sum_{\alpha \subset \sigma} K_x^{|\alpha|} K_y^{|\bar{\alpha}|} = (K_x + K_y)^{|\sigma|},$$

z satisfies $\mathcal{K}ii$ so that $z \in \mathcal{K}_b$. Now suppose that x and y are smooth. Then if $\rho_0(\sigma, \tau) < 1$ — $\sigma = (s, \varepsilon)$, $\tau = (t, \varepsilon)$, say — let $\pi: 2^\sigma \rightarrow 2^\tau$ be the bijective map between power sets induced by the pointwise map $s_j \mapsto t_j$ ($j = 1, 2, \dots, |\sigma|$), then

$$\|z(\sigma) - z(\tau)\| \leq \int_{\Gamma_J} \sum_{\alpha \subset \sigma} \|x(\omega^\dagger \cup \alpha) y(\omega \cup \bar{\alpha}) - x(\omega^\dagger \cup \pi(\alpha)) y(\omega \cup \pi(\bar{\alpha}))\| d\omega$$

which is bounded by $2\rho_0(\sigma, \tau)(K_x + K_y)^{|\sigma|} \exp\{(c_+ + c_-)K_x K_y \lambda_1(J)\}$ so that z also satisfies $\mathcal{K}iii$. Hence $z \in \mathcal{K}_0^d(I)$. ■

Definition 2.3.2: For $x, y \in \mathcal{K}_b^d(I)$ we denote the kernel z defined above by $x * y$.

Some immediate properties are listed next.

1. \mathcal{K}_0 is also closed under the *point-wise* product $(x, y \in \mathcal{K}_0 \Rightarrow \sigma \mapsto x(\sigma)y(\sigma) \in \mathcal{K}_0)$.

2. When $\mathcal{K} = \mathbb{C}$, $\langle x, y \rangle_{L^2(\mathcal{P}, \mu)} = (x^\dagger * y)(\emptyset)$.

3. For $f, g \in C_\kappa^1(I)$,

$$\begin{aligned} a_f * a_g^\dagger(\emptyset) - a_g^\dagger * a_f(\emptyset) &= \langle a_f^\dagger, a_g^\dagger \rangle - \langle a_g, a_f \rangle, = c_+ \langle f, g \rangle_{L^2(I)} - c_- \langle \bar{g}, \bar{f} \rangle_{L^2(I)} \\ &= (c_+ - c_-) \langle f, g \rangle_{L^2(I)}, \end{aligned}$$

whereas,

$$(a_f * a_g^\dagger)(\{t^+, s^-\}) = a_f(\{s^-\}) a_g^\dagger(\{t^+\}).$$

Since $a_f * a_g^\dagger$ and $a_g^\dagger * a_f$ are supported by $\mathcal{P}^0 \cup \mathcal{P}^{1,1}$ these kernels satisfy the *canonical commutation relations*

$$a_f * a_g^\dagger - a_g^\dagger * a_f = (c_+ - c_-) \langle f, g \rangle \delta_\emptyset. \quad (2.4)$$

We next state two combinatorial facts, the first of which will be repeatedly used in the sequel.

Lemma 2.3.3: (a) For an integrable Banach space valued function g on $\Gamma \times \Gamma$

$$\iint g(\alpha, \beta) \, d\alpha \, d\beta = \int \left\{ \sum_{\alpha \subset \omega} g(\alpha, \bar{\alpha}) \right\} d\omega.$$

(b) For a vector space valued function f on $\Gamma \times \Gamma$

$$\sum_{\alpha \subset \sigma} \sum_{\beta \subset \tau} f(\alpha \cup \beta, \bar{\alpha} \cup \bar{\beta}) = \sum_{\gamma \subset \sigma \cup \tau} f(\gamma, \bar{\gamma})$$

whenever σ and τ are disjoint.

Proof: (a) For $n \in \mathbb{N}$, let 2^n denote the power set of $n := \{1, 2, \dots, n\}$ considered as a measure space with the counting measure, and for $\omega = (t, \varepsilon) \in \Gamma^{(n)}$, $\alpha \subset \omega$, let $S(\alpha, \omega) = \{j \in n : t_j \in \text{supp}(\alpha)\}$. The mapping $\iota : (\alpha, \beta) \rightarrow (\alpha \cup \beta, S(\alpha, \alpha \cup \beta))$ defines a bijection between the set $\{(\alpha, \beta) \in \Gamma \times \Gamma : \alpha \cap \beta = \emptyset\}$, which has full measure in $\Gamma \times \Gamma$, and the set $\bigcup_{n=0}^{\infty} \Gamma^n \times 2^n$. Since ι is measure preserving, the result follows.

(b) Immediate. ■

Proposition 2.3.4: $(\mathcal{K}_b^{\mathcal{A}}(I), *)$ and $(\mathcal{K}_0^{\mathcal{A}}(I), *)$ are associative involutive algebras with unit $1_{\mathcal{A}} \delta_{\emptyset}$ and involution \dagger .

Proof: For $x, y, z \in \mathcal{K}_b$ define a function k on Γ by

$$\sigma \mapsto \iiint \sum x(\alpha \cup \omega_2^\dagger \cup \omega_3^\dagger) y(\omega_1^\dagger \cup \beta \cup \omega_3) z(\omega_1 \cup \omega_2 \cup \gamma) d\omega_1 d\omega_2 d\omega_3,$$

where the sum is over partitions of σ into a disjoint union of α, β and γ . Applying Lemma 2.3.3 (a) to (ω_2, ω_3) and then Lemma 2.3.3 (b) to $(\omega, \bar{\alpha})$, where ω is the new variable $\omega_2 \cup \omega_3$, gives the following expression for $k(\sigma)$:

$$\begin{aligned} & \int \sum_{\alpha \subset \sigma} d\omega_1 \int d\omega \sum_{\omega_3 \subset \omega} \sum_{\beta \subset \bar{\alpha}} x(\alpha \cup \omega^\dagger) y(\omega_1^\dagger \cup \beta \cup \omega_3) z(\omega_1 \cup \bar{\omega}_3 \cup \bar{\beta}) \\ &= \int d\omega \sum_{\alpha \subset \sigma} x(\alpha \cup \omega^\dagger) \int d\omega_1 \sum_{\delta \subset \omega \cup \bar{\alpha}} y(\omega_1^\dagger \cup \delta) z(\omega_1 \cup \bar{\delta}) \\ &= x * (y * z)(\sigma). \end{aligned}$$

On the other hand, applying Lemma 2.3.3 (a) to (ω_1, ω_2) and then Lemma 2.3.3 (b) to $(\omega^\dagger, \bar{\gamma})$, where ω is the new variable $\omega_1 \cup \omega_3$, yields $k(\sigma) = (x * y) * z(\sigma)$, establishing the associativity of $*$. Since the involution on \mathcal{A} is conjugate linear and isometric,

$$\begin{aligned} (x * y)^\dagger(\sigma) &= \left\{ \int \sum_{\alpha \subset \sigma^\dagger} x(\alpha \cup \omega^\dagger) y(\bar{\alpha} \cup \omega) d\omega \right\}^* \\ &= \int \sum_{\alpha \subset \sigma} y(\bar{\alpha}^\dagger \cup \omega)^* x(\alpha^\dagger \cup \omega^\dagger)^* d\omega \end{aligned}$$

$$\begin{aligned}
&= \int \sum_{\alpha \subset \sigma} y^\dagger(\bar{\alpha} \cup \omega^\dagger) x^\dagger(\alpha \cup \omega) d\omega \\
&= (y^\dagger * x^\dagger)(\sigma).
\end{aligned}$$

So \dagger is an involution. Since $1_{\mathcal{A}} \delta_\emptyset$ is obviously a unit the result follows. \blacksquare

Remark: 1. If \mathcal{A}_1 is a sub-algebra of \mathcal{A}_2 then in a natural way $\mathcal{K}_0^{\mathcal{A}_1}(I)$ is a subalgebra of $\mathcal{K}_0^{\mathcal{A}_2}(I)$.

2. \mathcal{K}_0 is non-commutative unless $c_+ = c_-$.

2.4 The Weyl Relations.

For $a, b, c \neq 0$ let $\gamma_{a,b} : \mathcal{V}_I \rightarrow \mathbb{C}$ be the map $\sigma \rightarrow a^{|\sigma^+|} b^{|\sigma^-|}$, and let $\gamma_c = \gamma_{c,c^{-1}}$.

Lemma 2.4.1: For $\alpha, \beta \in \mathcal{V}_I$, $x, y \in \mathcal{K}_0^{\mathcal{A}}(I)$

- (i) $\gamma_{a,b}(\alpha \cup \beta) = \gamma_{a,b}(\alpha) \gamma_{a,b}(\beta)$
- (ii) $\gamma_{a,b} x \in \mathcal{K}_0^{\mathcal{A}}(I)$
- (iii) $\gamma_{a,b}^\dagger = \gamma_{b,a}$
- (iv) $\gamma_c(x * y) = \gamma_c x * \gamma_c y$

Proof: The estimate $|\gamma_{a,b}(\sigma)| \leq (|a| + |b|)^{|\sigma|}$ and the identity

$$\gamma_c(\alpha \cup \omega^\dagger) \gamma_c(\bar{\alpha} \cup \omega) = \gamma_c(\sigma)$$

for $a \subset \sigma, \omega \in \mathcal{V}$, suffice to establish the lemma. \blacksquare

We shall write γ for γ_c when $c = c_-/c_+$. Thus $\gamma = m^\dagger/m$ and $m = \gamma_{c_+, c_-}$.

Definition 2.4.2: Define the following maps on $\mathcal{K}_0^{\mathbb{C}}(I)$:

$$\begin{aligned}
\Gamma_0 : x &\rightarrow \sqrt{\gamma} x & G_y : x &\mapsto y * x \\
S_0 : x &\rightarrow x^\dagger \\
J_0 : x &\rightarrow \sqrt{\gamma} x^\dagger & D_y : x &\mapsto x * y \quad (y \in \mathcal{K}_0^{\mathbb{C}}(I)).
\end{aligned}$$

Proposition 2.4.3: The following relations obtain

- (i) $J_o = \Gamma_o S_o = S_o \Gamma_o^{-1}$
- (ii) $J_o^2 (= S_o^2) = \text{id}_{\mathcal{K}_o}$
- (iii) J_o is L^2 -isometric
- (iv) $G_y G_z = G_{y * z} ; D_z D_y = D_{y * z}$
- (v) $G_y D_z = D_z G_y$
- (vi) $J_o G_y J_o = D_{J_o y}, (y, z \in \mathcal{K}_o^{\mathbb{C}}(I))$

Proof: (i) follows from Lemma 2.4.1, and (ii) follows from (i). Since

$$\int |J_o x|^2 d\mu = \int \gamma |x^\dagger|^2 m d\lambda = \int m^\dagger |x^\dagger|^2 d\lambda = \int m |x|^2 d\lambda = \int |x|^2 d\mu,$$

(iii) follows. (iv) and (v) are a consequence of the associativity of $*$, and (vi) of the identities

$$\begin{aligned} J_o G_y J_o x &= J_o (y * \sqrt{\gamma} x^\dagger) \\ &= \sqrt{\gamma} \left(\frac{1}{\sqrt{\gamma}} x * y^\dagger \right) = x * J_o y = D_{J_o y} x. \end{aligned}$$

■

Definition 2.4.4: Let $\zeta = \zeta_{c_+, c_-} : C_\kappa^1(I) \times C_\kappa^1(I) \rightarrow \mathbb{R}$ be the *symplectic form*

$$(f, g) \mapsto -(c_+ - c_-) \text{Im} \int_I \bar{f} g d\lambda_1.$$

For $f \in C_\kappa^1$ let

$$v_f = J_o w_{\bar{f}} = \sqrt{\gamma} w_{-\bar{f}}.$$

Proposition 2.4.5: For $f, g \in C_\kappa^1(I), x \in \mathcal{K}_o^{\mathbb{C}}(I)$:

- (i) $w_f * w_g = e^{i\zeta(f, g)} w_{f+g} ; w_f^\dagger = w_{-f}$

$$\begin{aligned}
\text{(ii)} \quad v_g * v_f &= e^{i\zeta(f,g)} v_{f+g} ; \quad v_f^\dagger = \gamma^{-1} v_{-f} \\
\text{(iii)} \quad \int |w_f * x|^2 d\mu &= \int |x * v_g|^2 d\mu = \int |x|^2 d\mu \\
\text{(iv)} \quad \int \bar{w}_0 w_f d\mu &= \int \bar{v}_0 v_f d\mu = \hat{\mu}(f).
\end{aligned}$$

Proof:

$$\begin{aligned}
\text{(i)} \quad \pi_f * \pi_g(\sigma) &= \sum_{\alpha \subset \sigma} \pi_f(\alpha) \pi_g(\bar{\alpha}) \int \pi_{-\bar{f}} \pi_g d\mu \\
&= \pi_{f+g}(\sigma) \exp\left\{-c_+ \int \bar{f}g d\lambda_1 - c_- \int f\bar{g} d\lambda_1\right\} \\
&= \pi_{f+g}(\sigma) \exp\left\{-(c_+ + c_-) \operatorname{Re} \int \bar{f}g d\lambda_1 - i(c_+ - c_-) \operatorname{Im} \int \bar{f}g d\lambda_1\right\} \\
&= e^{i\zeta(f,g)} \left[\frac{\hat{\mu}(f+g)}{\hat{\mu}(f)\hat{\mu}(g)} \right] \pi_{f+g}(\sigma).
\end{aligned}$$

$$\begin{aligned}
\text{(ii)} \quad v_g * v_g &= \sqrt{\gamma} w_{-\bar{g}} * \sqrt{\gamma} w_{-\bar{f}} \\
&= \sqrt{\gamma} e^{i\zeta(f,g)} w_{-\bar{f}-\bar{g}} = e^{i\zeta(f,g)} v_{f+g}
\end{aligned}$$

$$v_f^\dagger = \sqrt{\gamma} w_{-\bar{f}}^\dagger = \frac{1}{\sqrt{\gamma}} w_{\bar{f}} = \gamma^{-1} v_{-f}.$$

$$\begin{aligned}
\text{(iii)} \quad \int |w_f * x|^2 d\mu &= (x^\dagger * w_{-f} * w_f * x)(\emptyset) \\
&= (x^\dagger * x)(\emptyset) = \int |x|^2 d\mu,
\end{aligned}$$

and since $D_{v_g} = J_0 G_{w_{\bar{g}}} J_0$, the L^2 -isometry of D_{v_g} follows from that of J_0 (Proposition 2.4.3) and of $G_{w_{\bar{g}}}$.

(iv) is immediate. ■

2.5 L^2 -density of kernels

In this subsection we establish some density results for $\mathcal{K}_0^C(I)$ and the smooth Weyl kernels. Let \mathcal{W} and \mathcal{V} be respectively the linear spans of w_f and v_f with f running through $C_\kappa^1(I)$, let $\mathcal{W}_J = \{x \in \mathcal{W} : \operatorname{supp} x \subset \mathcal{V}_J\}$ and let $\mathcal{W}_J^{j,k} = \chi_{\mathcal{P}^{j,k}} \mathcal{W}_J$ where J is a compact interval.

Lemma 2.5.1: For $f \in C_\kappa^1(I)$

$$(i) \quad \pi_f^{(n)} = (n!)^{-1} \frac{d^n}{ds^n} \Big|_{s=0} \pi_{sf}$$

$$(ii) \quad \pi_f^{j,k} = (2\pi)^{-1} \int_0^{2\pi} e^{-i(j-k)\theta} \pi_{e^{i\theta}f}^{(j+k)} d\theta$$

Proof:

(i) For $s \in \mathbb{R}$,

$$\pi_{sf} = \sum_{n=0}^{\infty} s^n \pi_f^{(n)}$$

and (i) follows.

(ii) For $c \in \mathbb{C}$,

$$\pi_{cf}^{(n)} = \pi_f^{(n)} \cdot \sum_{j+k=n} c^j \bar{c}^k \chi_{\mathcal{P}^{j,k}},$$

thus

$$\pi_{e^{i\theta}f}^{(n)} = \sum_{j+k=n} e^{i(j-k)\theta} \pi_f^{j,k},$$

from which (ii) follows. ■

Proposition 2.5.2: $\bar{\mathcal{W}}^\mu \subset C_0(\mathcal{V}_I)$ (uniform closure).

Proof: Since $\mathcal{W}_J^{j,k}$ is an algebra under pointwise multiplication the Stone-Weierstrass theorem implies that $C(\mathcal{V}_J^{j,k})$ is the uniform closure of $\mathcal{W}_J^{j,k}$. Moreover Lemma 2.5.1 yields $\mathcal{W}_J^{j,k} \subset \bar{\mathcal{W}}_J^\mu$, thus $\bar{\mathcal{W}}^\mu \supset C(\mathcal{V}_J^{\leq N})$ for each N and compact J . But any compact set in \mathcal{V} is a subset of some $\mathcal{V}_J^{\leq N}$, therefore $\bar{\mathcal{W}}^\mu \supset \overline{C_\kappa(\mathcal{V})}^\mu = C_0(\mathcal{V})$. ■

Corollary 2.5.3: \mathcal{W} and \mathcal{V} are dense in $L^2(\mathcal{V}_I, \nu)$, where $\nu = \mu + \mu^\dagger$.

For $f \in L^2(I)$ the Weyl kernel w_f given in (2.3) is still a well-defined element of $L^2(\mathcal{V}_I, \mu)$, but not necessarily an element of $\mathcal{K}_0^C(I)$, i.e. not necessarily a smooth kernel.

Proposition 2.5.4: The map $w : L^2(I, \lambda_1) \rightarrow L^2(\mathcal{V}_I, \nu)$ is continuous.

Proof: Since

$$\begin{aligned} \langle \pi_f, \pi_g \rangle_{L^2(\nu)} &= 2e^{(c_+ + c_-)\text{Re}\langle f, g \rangle} \cos\{(c_+ - c_-)\text{Im}\langle f, g \rangle\}, \\ \|\pi_f - \pi_g\|_{L^2(\nu)}^2 &= 2[e^{(c_+ + c_-)\|f\|^2} + e^{(c_+ + c_-)\|g\|^2} \\ &\quad - 2e^{(c_+ + c_-)\text{Re}\langle f, g \rangle} \cos\{(c_+ - c_-)\text{Im}\langle f, g \rangle\}] \end{aligned}$$

which tends to 0 as f approaches g . Since $\hat{\mu}$ is clearly continuous, the result follows. ■

Corollary 2.5.5: If D is dense in $L^2(I)$, then the linear span of $\{w_f : f \in D\}$ is dense in $L^2(\mathcal{V}_I, \nu)$.

3. Adapted processes

In this section subintervals I of the real line will be *assumed to have a left end point* 0. In order to discuss processes we introduce the *adapted power set*.

$$\Gamma_{\text{ad}}(I) := \{(\sigma, t) \in \Gamma_I \times I : \max \sigma < t \text{ or } \sigma = \emptyset\}.$$

Thus $(\sigma, t) \in \Gamma_{\text{ad}}(I)$ when $\text{supp}(\sigma) \subset I_t := I \cap (-\infty, t)$. Notice that the maps $\iota^\pm : (\sigma, t) \mapsto \sigma \cup \{t^\pm\}$ are injective $\Gamma_{\text{ad}} \rightarrow \Gamma$ with images

$$\Gamma^\pm := \{\sigma \in \Gamma : \sigma \neq \emptyset, \max \sigma \text{ has charge } \pm\}$$

and, that if $\Gamma^0 = \{\emptyset\}$ then

$$\Gamma^0 \cup \Gamma^+ \cup \Gamma^- \tag{3.1}$$

is a disjoint partition of Γ .

3.1 Smooth adapted processes.

Any map $x : \Gamma_{\text{ad}} \rightarrow \mathcal{A}$ determines a map $k_x : \Gamma \rightarrow \mathcal{A}$ by

$$k_x(\sigma) = \begin{cases} x(\sigma \setminus \{\max \sigma\}, \max \sigma) & \text{if } \sigma \neq \emptyset, \\ 0 & \text{if } \sigma = \emptyset. \end{cases} \tag{3.2}$$

Definition 3.1.1: $x: \Gamma_{\text{ad}}(I) \rightarrow \mathcal{A}$ is a *smooth adapted (kernel) process* if for each compact sub-interval J of I , $k_x^J := \chi_{\Gamma_J} k_x$ belongs to $\mathcal{K}_0^{\mathcal{A}}(J)$. We denote the class of smooth adapted processes by $\mathcal{P}_0^{\mathcal{A}}(I)$.

Each smooth adapted process x will be considered as a function on the whole of $\mathcal{V}_I \times I$ as follows:

$$x(\sigma, t) = \begin{cases} \lim_{n \rightarrow \infty} x(\sigma_n, t) & \text{if } \exists \sigma_n \in \Gamma(I_t) \text{ s.t. } \rho(\sigma_n, \sigma) \rightarrow 0, \\ 0 & \text{otherwise.} \end{cases} \quad (3.3)$$

For $x \in \mathcal{P}_0^{\mathcal{A}}(I)$, $t \in I$, $\sigma \in \mathcal{V}_I$, let x_t, x^σ denote the functions $x(\cdot, t), x(\sigma, \cdot)$ on \mathcal{V}_I and I respectively. Thus, for each $\sigma \in \mathcal{V}_I$, x^σ is a locally Lipschitz function on $I_{[\max \sigma]}$ and, for each $t \in I$, $x_t \in \mathcal{K}_0^{\mathcal{A}}(I_t]$. In particular, since $\Gamma(\{0\}) (= \mathcal{V}(\{0\})) = \{\emptyset\}$, $x_0 = a\delta_\emptyset$ for some $a \in \mathcal{A}$. If $x, y \in \mathcal{P}_0^{\mathcal{A}}(I)$, then for each $t \in I$, $x_t * y_t \in \mathcal{K}_0^{\mathcal{A}}(I_t]$ —in fact, $*$ extends to \mathcal{P}_0 :

$$(\sigma, t) \mapsto x_t * y_t(\sigma), \quad (\sigma, t) \in \Gamma_{\text{ad}}(I)$$

determines an element of $\mathcal{P}_0^{\mathcal{A}}(I)$, denoted $x * y$.

3.2 Kernel differential and integral operators

We are now in a position to introduce the differential operators of the kernel calculus.

Definition 3.2.1: For $x \in \mathcal{P}_0^{\mathcal{A}}(I)$ let $\Delta^+ x, \Delta^- x: \Gamma_{\text{ad}}(I) \rightarrow \mathcal{A}$ be given by

$$\Delta^+ x(\sigma, t) = x(\sigma \cup \{t^+\}, t)$$

$$\Delta^- x(\sigma, t) = x(\sigma \cup \{t^-\}, t)$$

where for each t the continuous extension of x_t to \mathcal{V}_{I_t} is invoked (see (3.3)).

Proposition 3.2.2: For $x \in \mathcal{P}_0^{\mathcal{A}}(I)$, $\Delta^\pm x \in \mathcal{P}_0^{\mathcal{A}}(I)$.

Proof:: Let $\sigma \in \Gamma_I$, $\sigma \neq \emptyset$, then

$$\begin{aligned} k_{\Delta^\pm x}(\sigma) &= \Delta^\pm x(\sigma \setminus \{\max \sigma\}, \max(\sigma)) \\ &= x(\sigma', \max \sigma) \text{ where } \sigma' = \sigma \setminus \{\max \sigma\} \cup \{\max \sigma\}^+ \end{aligned}$$

$$= k_x(\sigma' \cup \{\max \sigma\}^\pm),$$

so for $\sigma \in \Gamma_J$, J a compact subinterval of I ,

$$\|k_{\Delta^\pm x}(\sigma)\| < (K_x^J)^{|\sigma|+2} \leq \{(K_x^J)^2\}^{|\sigma|+1}.$$

If $\sigma, \tau \in \Gamma_J$ and $\rho_0(\sigma, \tau) < 1$ then

$$\begin{aligned} \|k_{\Delta^\pm x}(\sigma) - k_{\Delta^\pm x}(\tau)\| &= \|k_x(\sigma' \cup \{\max \sigma\}^\pm) - k_x(\tau' \cup \{\max \tau\}^\pm)\| < \rho_0(\sigma, \tau)(K_x^J)^{|\sigma|+2} \\ &\leq \rho_0(\sigma, \tau)([K_x^J]^2)^{|\sigma|+1} \end{aligned}$$

and the proof is complete. ■

Definition 3.2.3: Let $\mathcal{P}_d = \{x \in \mathcal{P}_0 : x^\sigma \text{ is differentiable on } I_{(\max \sigma)} \forall \sigma \in \Gamma_I\}$

and, for $x \in \mathcal{P}_d$ let $\Delta^\circ x : \Gamma_{\text{ad}} \rightarrow \mathcal{A}$ be given by

$$\Delta^\circ x(\sigma, t) = (x^\sigma)'(t).$$

Let $\mathcal{P}_1 = \{x \in \mathcal{P}_d : \Delta^\circ x \in \mathcal{P}_0\}$ — the *domain of the pathwise derivative operator* Δ° .

We next introduce the integral operators.

Definition 3.2.4: For $x \in \mathcal{P}_0^{\mathcal{A}}(I)$, $(\sigma, t) \in \Gamma_{\text{ad}}(I)$ let

$$I^\pm x(\sigma, t) = \begin{cases} x(\sigma \setminus \{\max \sigma\}, \max \sigma) & \text{if } \sigma \in \Gamma^\pm, \\ 0 & \text{otherwise;} \end{cases}$$

$$I^\circ x(\sigma, t) = \int_0^t x^\sigma(s) ds.$$

These define smooth adapted processes $I^+x, I^-x, I^\circ x$ and the relations

$$\Delta^+ I^+ = \Delta^- I^- = \Delta^\circ I^\circ = \text{id}_{\mathcal{P}_0^{\mathcal{A}}(I)} \quad (3.4)$$

are immediate. Moreover, the following *fundamental theorem* holds:

Proposition 3.2.5: For $x \in \mathcal{P}_0^{\mathcal{A}}(I)$, $t \in I$

$$x_t - x_0 = I^+ \Delta^+ x_t + I^- \Delta^- x_t + I^\circ \Delta^\circ x_t. \quad (3.5)$$

Proof: On Γ^0 (3.5) is an immediate consequence of the fundamental theorem of calculus since the first two terms on the right hand side vanish. On Γ^+ , $I^-\Delta^-x(\sigma, t) = 0$, $I^+\Delta^+x(\sigma, t) = x(\sigma, \max\sigma)$ and

$$I^0\Delta^0x(\sigma, t) = \int_0^t (x^\sigma)'(s) ds = \int_{\max\sigma}^t (x^\sigma)'(s) ds = x(\sigma, t) - x(\sigma, \max\sigma)$$

again by the fundamental theorem of calculus. Since x_0 vanishes on Γ^+ , (3.5) holds there. Similarly the identity is valid on Γ^- . ■

3.3 Itô relation.

Lemma 3.3.1: Let $x \in \mathcal{P}_0^{\mathcal{A}}(I)$, $t \in I$, then

$$\int_0^t \int_{\Gamma_t} x_s d\mu ds = c_+^{-1} \int_{\Gamma_t} (I^+x)_t d\mu = c_-^{-1} \int_{\Gamma_t} (I^-x)_t d\mu.$$

Proof:

$$\begin{aligned} \int_{\Gamma_t} (I^+x)_t d\mu &= \int_{\Gamma_{I_0}^+} (I^+x)(\sigma, t) d\sigma \\ &= \int_{\Gamma_{I_0}^+} x(\sigma \setminus \{\max\sigma\}, \max(\sigma)) d\sigma \\ &= c_+ \int_0^t \int_{\Gamma_t} x(\tau, s) d\tau ds, \end{aligned}$$

which gives the first equality. For the second replace $+$ by $-$. ■

Proposition 3.3.2: For $x \in \mathcal{P}_1^{\mathcal{A}}(I)$, $t \mapsto \int_{\Gamma_t} x_t d\mu$ is differentiable on I with derivative

$$\int_{\Gamma_t} \sum_{\kappa} c_{\kappa} \Delta^{\kappa} x_t d\mu, \quad (c_0 = 1).$$

Proof: $\int_{\Gamma} x_t d\mu - x_t(\emptyset) = \int_{\Gamma^+ \cup \Gamma^-} \sum_{\kappa} I^{\kappa} \Delta^{\kappa} x_t d\mu$

$$\begin{aligned}
&= \int_{\Gamma^+} I^+ \Delta^+ x_t \, d\mu + \int_{\Gamma^-} I^- \Delta^- x_t \, d\mu + \int_{\Gamma \setminus \Gamma^0} I^0 \Delta^0 x_t \, d\mu \\
&= \int_0^t \int_{\Gamma} \{c_+ \Delta^+ x_s + c_- \Delta^- x_s\} \, d\mu \, ds + \int_{\Gamma \setminus \Gamma^0} \left\{ \int_0^t (x^\sigma)'(s) \, ds \right\} \, d\sigma,
\end{aligned}$$

but

$$x_t(\emptyset) = \int_{\Gamma^0} x^\sigma(t) \, d\sigma = x(\emptyset, 0) + \int_{\Gamma^0} \left\{ \int_0^t (x^\sigma)'(s) \, ds \right\} \, d\sigma,$$

so that the result follows by an application of Fubini's theorem. ■

Proposition 3.3.3: The operators Δ^\pm are derivations on $(\mathcal{P}_0^d(I), *)$:

$$\Delta^\pm(x * y) = \Delta^\pm x * y + x * \Delta^\pm y. \quad (3.6)$$

Moreover, if $x, y \in \mathcal{P}_1$ then $x * y \in \mathcal{P}_d$ and Δ^0 satisfies the Itô rule

$$\Delta^0(x * y) = \Delta^0 x * y + x * \Delta^0 y + c_+ \Delta^- x * \Delta^+ y + c_- \Delta^+ x * \Delta^- y. \quad (3.7)$$

In particular $(\mathcal{P}_1, *)$ is a subalgebra of $(\mathcal{P}_0, *)$.

Proof: For $x, y \in \mathcal{P}_0$ and $(\sigma, t) \in \Gamma_{\text{ad}}$.

$$\begin{aligned}
\Delta^+(x * y)_t(\sigma) &= x * y(\sigma \cup \{t^+\}, t) \\
&= \int \sum_{\alpha \subset \sigma \cup \{t^+\}} x_t(\alpha \cup \omega^\dagger) y_t(\bar{\alpha} \cup \omega) \, d\omega \\
&= \int \sum_{\beta \subset \sigma} x_t(\beta \cup \{t^+\} \cup \omega^\dagger) y_t(\bar{\beta} \cup \omega) \, d\omega \\
&\quad + \int \sum_{\gamma \subset \sigma} x_t(\gamma \cup \omega^\dagger) y_t(\bar{\gamma} \cup \{t^+\} \cup \omega) \, d\omega \\
&= \Delta^+ x_t * y_t(\sigma) + x_t * \Delta^+ y_t(\sigma)
\end{aligned}$$

and since the same holds for Δ^- , (3.6) follows. To prove the Itô relation, let $x, y \in \mathcal{P}_1$, and $(\sigma, t_0) \in \Gamma_{\text{ad}}$. Since $t_0 > \max \sigma$ there is an interval $I_1 := [t_0 - \varepsilon, t_0 + \varepsilon)$ not containing any point of σ . Let $I_2 := [0, t_0 - \varepsilon)$ denote the remaining part of I_{t_0} and, for $\alpha, \beta \in \Gamma(I_2)$ let the process $z^{\alpha, \beta} \in \mathcal{P}_1(I_1)$ be

given by

$$z^{\alpha, \beta}: (\omega_1, t) \rightarrow x_t(\alpha \cup \omega_1^\dagger) y_t(\beta \cup \omega_1)$$

so that $\Delta^\pm z_t^{\alpha, \beta}(\omega_1) = \Delta^\mp x_t(\alpha \cup \omega_1^\dagger) \Delta^\pm y_t(\beta \cup \omega_1)$. Now for $\alpha, \beta, \omega_2 \in \Gamma(I_2)$ and $t \in I_1$, put

$$f_t^{\alpha, \beta}(\omega_2) = \int_{\Gamma_{I_1}} z_t^{\alpha \cup \omega_2^\dagger, \beta \cup \omega_2}(\omega_1) d\mu(\omega_1).$$

Then by the previous proposition, $t \rightarrow f_t^{\alpha, \beta}(\omega_2)$ is differentiable at t_0 with derivative

$$\begin{aligned} & \int_{\Gamma_{I_1}} \sum_{\kappa} c_{\kappa}(\Delta^{\kappa} z^{\alpha \cup \omega_2^\dagger, \beta \cup \omega_2})(\omega_1, t_0) d\mu(\omega_1) \\ &= \int_{\Gamma_{I_1}} \left\{ c_+ \Delta^- x_{t_0}(\alpha \cup \omega_2^\dagger \cup \omega_1^\dagger) \Delta^+ y_{t_0}(\beta \cup \omega_2 \cup \omega_1) \right. \\ & \quad + c_- \Delta^+ x_{t_0}(\alpha \cup \omega_2^\dagger \cup \omega_1^\dagger) \Delta^- y_{t_0}(\beta \cup \omega_2 \cup \omega_1) \\ & \quad + \Delta^0 x_{t_0}(\alpha \cup \omega_2^\dagger \cup \omega_1^\dagger) y_{t_0}(\beta \cup \omega_2 \cup \omega_1) \\ & \quad \left. + x_{t_0}(\alpha \cup \omega_2^\dagger \cup \omega_1^\dagger) \Delta^0 y_{t_0}(\beta \cup \omega_2 \cup \omega_1) \right\} d\omega_1. \end{aligned}$$

Since this derivative (considered as a function of ω_2) is dominated by an (integrable) function of the form $\omega_2 \rightarrow K^{|\omega_2|+1}$, we may conclude that $(x * y)^\sigma$ is differentiable at t_0 with derivative

$$\begin{aligned} & \left. \frac{d}{dt} \right|_{t=t_0} \int_{\Gamma_{I_2}} \sum_{\alpha \subset \sigma} f_t^{\alpha, \bar{\alpha}}(\omega_2) d\omega_2 \\ &= \int_{\Gamma(I_2)} \sum_{\alpha \subset \sigma} \left. \frac{d}{dt} \right|_{t=t_0} f_t^{\alpha, \bar{\alpha}}(\omega_2) d\omega_2 \\ &= c_+ \Delta^- x_{t_0}(\sigma) * \Delta^+ y_{t_0}(\sigma) + c_- \Delta^+ x_{t_0} * \Delta^- y_{t_0}(\sigma) + \Delta x_{t_0} * y_{t_0}(\sigma) + x_{t_0} * \Delta y_{t_0}(\sigma), \end{aligned}$$

in other words (3.7). Each term on the right hand side of (3.7) being \mathcal{P}_0 , the process $x * y$ must be \mathcal{P}_1 . ■

4. Kernel differential equations.

We now demonstrate the ease with which linear stochastic differential equations may be treated in this kernel calculus—moreover we obtain an explicit form for the solutions of such equations (4.3). Again let I have left end point 0. We first extend the definition of Δ^+ and Δ^- as follows. Let

$$\mathcal{V}_{\text{ad.}}(I) = \{(\sigma, t) \in \mathcal{V}_I \times I : t \geq \max \sigma \text{ or } \sigma = \emptyset\}$$

and, for a function x on $\mathcal{V}_{\text{ad.}}(I)$, let

$$\Delta^\pm x(\sigma, t) = x(\sigma \cup \{t^\pm\}, t)$$

with the convention that if for example

$$\sigma = \begin{array}{c} + \\ \bullet \\ \hline \end{array} \begin{array}{c} + \\ - \\ \bullet \\ \hline \end{array} \begin{array}{c} - \\ + \\ \bullet \\ \hline \end{array}, \text{ then } \sigma \cup \{t^+\} = \begin{array}{c} + \\ \bullet \\ \hline \end{array} \begin{array}{c} + \\ - \\ \bullet \\ \hline \end{array} \begin{array}{c} + \\ - \\ + \\ \bullet \\ \hline \end{array}$$

By an adapted (kernel) process we simply mean an \mathcal{A} -valued function on $\mathcal{V}_{\text{ad.}}(I)$.

4.1 Existence and uniqueness.

A linear kernel differential equation is a system

$$\Delta x_t = L(t)x_t, \quad (4.1)$$

where $L: t \rightarrow (L^+(t), L^-(t), L^0(t)) \in \mathcal{L}(\mathcal{A}) \times \mathcal{L}(\mathcal{A}) \times \mathcal{L}(\mathcal{A})$, and a solution of (4.1) is an adapted process x for which the left-hand side is defined (i.e. each path x^σ is differentiable on $I_{[\max \sigma]}$ and $\Delta^k x(\sigma, t) = L^k(t)[x(\sigma, t)]$ for all $(\sigma, t) \in \mathcal{V}_{\text{ad.}}(I)$). Thus an adapted process x satisfies (4.1) if and only if

$$(i) \quad x^\sigma(\max \sigma) = L^\pm(\max \sigma)[x(\sigma \setminus \{\max \sigma\}, \max \sigma)], \quad \sigma \in \mathcal{V}^\pm; \quad (4.2)$$

$$(ii) \quad \frac{d}{dt} x^\sigma(t) = L^0(t)[x^\sigma(t)] \text{ for } t \geq \max \sigma, \quad \sigma \in \mathcal{V}.$$

Note that x satisfies (4.1) if and only if x^\dagger satisfies the *adjoint* k.d.e.:

$$\Delta y_t = L^\dagger(t)y_t,$$

where $(L^\dagger)^\pm(t)[b] = (L^\mp(t)[b^*])^*$ and $(L^\dagger)^\circ(t)[b] = (L^\circ(t)[b^*])^*$ ($b \in \mathcal{A}$).

Theorem 4.1.1: Let $L^\circ: I \rightarrow \mathcal{L}(\mathcal{A})$ be strongly continuous and locally uniformly bounded. Then for each $b \in \mathcal{A}$ there is a unique solution to (4.1) for which $x(\emptyset, 0) = b$.

Proof: Let $y: \mathcal{P}_{\text{ad}}(I) \rightarrow \mathcal{A}$ be given by

$$y(\sigma, t) = V(t, s_n) L^{\varepsilon_n}(s_n) V(s_n, s_{n-1}) \dots L^{\varepsilon_1} V(s, 0)[b] \quad (4.3)$$

if $\sigma = (s, \varepsilon) \in \mathcal{P}^n$, where $V: I \times I \rightarrow \mathcal{L}(\mathcal{A})$ is the solution of the ordinary differential equation

$$\frac{d}{dt} V(t, s) = L(t) V(t, s) \quad V(s, s) = \text{id}_{\mathcal{A}}, \quad (s, t \in I).$$

y defines a pathwise differentiable process satisfying (4.2) and $y(\emptyset, 0) = b$. Moreover it is clearly the unique such process. ■

One could define and solve non-linear kernel differential equations, however, these would appear to be uninteresting from the point of view of corresponding operator stochastic differential equations. We next isolate sufficient conditions for the solution to be a smooth adapted process.

Theorem 4.1.2: Let $L: I \rightarrow \mathcal{L}(\mathcal{A})^3$ be locally Lipschitz. Then the unique solution to (4.1) belongs to $\mathcal{P}_1^{\mathcal{A}}(I)$.

Proof: For a compact subinterval J of I let

$$L_J = \max_{\kappa} \sup_{t \in J} \|L^\kappa(t)\|_{\mathcal{L}(\mathcal{A})}; \quad V_J = \exp\{\lambda_1(J) L_J\}$$

and let M_J be a Lipschitz constant for L on J , so that

$$\|V(t, s)\| \leq V_J; \quad \|V(t, s) - V(t', s')\| \leq 2\eta L_J V_J^2, \quad (s, t, s', t' \in J),$$

as soon as $|t - t'|, |s - s'| < \eta$. The estimate

$$\|k_v(\tau)\| \leq V_J^{|\tau|} L_J^{|\tau|-1} \|b\|, \quad (\tau \in \mathcal{P}_J \setminus \{\emptyset\}),$$

ensures that k_v^J satisfies $\mathcal{K}ii$. Now suppose that $\rho(\tau, \tau') < \eta \leq 1$ and let

$$z_{2j} = L^{\varepsilon_j}(t_j); \quad z'_{2j} = L^{\varepsilon'_j}(t'_j); \quad z_{2j-1} = V(t_j, t_{j-1}); \quad z'_{2j-1} = V(t'_j, t'_{j-1}),$$

where $j = 1, \dots, n$ and $\tau = (t, \varepsilon)$, $\tau' = (t', \varepsilon') \in \mathcal{P}^n(J)$. Then

$$\begin{aligned} \|k_v(\tau) - k_v(\tau')\| &= \|(z_{2n-1} \dots z_1 - z'_{2n-1} \dots z'_1)b\| \\ &= \left| \sum_{j=1}^{2n-1} z_{2n-1} \dots z_{j+1} (z_j - z'_j) z'_{j-1} \dots z'_1 b \right| \\ &\leq \{2\eta n L_J^n V_J^{n+1} + \eta(n-1) L_J^{n-1} V_J^n M_J\} \|b\|, \end{aligned}$$

so that k_v^J also satisfies $\mathcal{K}iii$. Thus $v \in \mathcal{P}_0^{\mathcal{A}}(I)$, but since L° is locally Lipschitz it maps \mathcal{P}_0 into itself and $\Delta^\circ x = L^\circ x \in \mathcal{P}_0$, that is $x \in \mathcal{P}_1$. ■

4.2 Unitarity.

Now consider the following important special case. Let $q^K: I \rightarrow \mathcal{A}$ be locally Lipschitz and let $x \in \mathcal{P}_1$ satisfy the k.d.e.

$$\Delta x_t = q(t)x_t. \quad (4.4)$$

Then one calculates, using the Itô rule (3.7) that for all $b \in \mathcal{A}$,

$$\Delta(x^\dagger * bx)_t = x_t^\dagger * L_q(t)[b]x_t \quad (4.5)$$

and

$$\Delta(x * bx^\dagger)_t = M_q(t)(x * bx^\dagger)_t \quad (4.6)$$

where L_q and M_q are given by

$$L_q^\pm(t)[b] = q^\mp(t)^* b + b q^\pm(t),$$

$$L_q^\circ(t)[b] = q^\circ(t)^* b + b q^\circ(t) + c_+ q^+(t)^* b q^+(t) + c_- q^-(t)^* b q^-(t); \quad (4.7)$$

$$M_q^\pm(t)[b] = q^\pm(t)b + b q^\mp(t)^*,$$

$$M_q^\circ(t)[b] = q^\circ(t)b + b q^\circ(t)^* + c_+ q^-(t)b q^-(t)^* + c_- q^+(t)b q^+(t)^*. \quad (4.8)$$

Proposition 4.2.1: Let $x \in \mathcal{P}_1$ satisfy the k.d.e. (4.4) with initial conditions $x_0 = u_0 \delta_\emptyset$ for some unitary $u_0 \in \mathcal{A}$. Then the following are equivalent:

- (i) $x * x^\dagger = \mathbb{1}_{\mathcal{A}} \delta_\emptyset$;
- (ii) $L_q(\cdot)[\mathbb{1}_{\mathcal{A}}] = 0$;
- (iii) $x * x^\dagger = x^\dagger * x = \mathbb{1}_{\mathcal{A}} \delta_\emptyset$.

Moreover, if $t \rightarrow q(t)$ is constant, then these are equivalent to

- (iv) $x^\dagger * x = \mathbb{1}_{\mathcal{A}} \delta_\emptyset$.

Proof: First note that for all t : $L(t)[\mathbb{1}] = 0 \Leftrightarrow M(t)[\mathbb{1}] = 0$.

(i) \Rightarrow (ii): If $x * x^\dagger = \mathbb{1}_{\mathcal{A}} \delta_\emptyset$ then $M(t)[\mathbb{1}] = 0 \forall t$ by (4.6).

(ii) \Rightarrow (iii): If $L(\cdot)[\mathbb{1}] = 0$ then, by (4.5), $x^\dagger * x - \mathbb{1} \delta_\emptyset$ satisfies $\Delta y = 0$; $y_0 = 0$ and, by (4.6), $x * x^\dagger - \mathbb{1} \delta_\emptyset$ satisfies $\Delta y_t = M(t)y_t$; $y_0 = 0$, but the unique solution of these is 0.

(iv) \Rightarrow (ii): If $x^\dagger * x = \mathbb{1} \delta_\emptyset$ then, by (4.5), $x^\dagger * L(\cdot)[\mathbb{1}]x = 0$. If q is constant then, since x_0 is unitary, $L[\mathbb{1}] = x_0 * (x_0^\dagger * L[\mathbb{1}]x_0) * x_0^\dagger = 0$.

Since (iii) obviously implies (i) and (iv) the proof is complete. ■

Remark: Under the equivalent conditions of the above proposition

$$q(t) = \begin{pmatrix} v(t) \\ -v(t)^* \\ -\frac{1}{2}[c_+ v(t)^* v(t) + c_- v(t) v(t)^*] + i h(t) \end{pmatrix}$$

for certain Lipschitz functions $h, v : I \rightarrow \mathcal{A}$ such that $h(t) = h(t)^*$. In particular, if q is constant, L_q° is given by

$$L_q^\circ(b) = L_v(b) - i[h, b], \quad (4.9)$$

where

$$L_v(b) = c_+ \{v^* b v - \frac{1}{2}(v^* v b + b v^* v)\} + c_- \{v b v^* - \frac{1}{2}(v v^* b + b v v^*)\} \quad (4.10)$$

4.3 Adapted kernel cocycles

Let \mathbb{I}_t ($t \in \mathbb{R}$) denote the right shift on functions defined on $\mathcal{V}(\mathbb{R})$:

$$(\mathbb{I}_t f)(\omega) = f(\omega - t),$$

where $(s, \varepsilon) - t := (s - t, \varepsilon)$. The following class of kernel processes will play a role in the construction of quantum Markov processes.

Definition 4.3.1: $x = \{x_t : t \geq 0\}$ is an *adapted kernel cocycle* if

$$(akc \text{ i}) \quad x \in \mathcal{P}_0^{\mathcal{A}}(\mathbb{R}_+);$$

$$(akc \text{ ii}) \quad x_{s+t} = \mathbb{I}_s(x_t) * x_s; \quad x_0 = \mathbb{I}_{\mathcal{A}} \delta_{\emptyset}, \quad (s, t \geq 0).$$

Remark: If x is an adapted kernel cocycle then the two parameter family $\{x_{s,t} := \mathbb{I}_s(x_{t-s}), s \leq t\}$ satisfies

$$(i)' \quad \text{supp } x_{r,t} \in \mathcal{V}_{[r,t]},$$

$$(ii)' \quad x_{r,t} = x_{s,t} * x_{r,s}; \quad x_{0,0} = \mathbb{I}_{\mathcal{A}} \delta_{\emptyset};$$

$$(iii)' \quad \mathbb{I}_u(x_{s,t}) = x_{s+u,t+u};$$

$$(iv)' \quad t \rightarrow x_{0,t} \in \mathcal{P}_0^{\mathcal{A}}(\mathbb{R}_+), \text{ for all } (r \leq s \leq t, u \in \mathbb{R}).$$

Conversely a two parameter family of kernels $\{x_{s,t} : s \leq t\}$ satisfying (i)'-(iv)' determines an adapted cocycle: $\{x_{0,t} : t \geq 0\}$.

Proposition 4.3.2: For a family of kernels $x := \{x_t : t \geq 0\}$ the following are equivalent:

(a) x is an adapted kernel cocycle;

(b) x satisfies a kernel differential equation of the form

$$\Delta x = qx, \quad (q \in \mathcal{A}^3);$$

$$x_0 = \mathbb{I}_{\mathcal{A}} \delta_{\emptyset}; \tag{4.11}$$

(c)

$$x_t(\sigma) = \begin{cases} e^{(t-s_n)q} q^{\varepsilon_n} e^{(s_n-s_{n-1})q} \dots q^{\varepsilon_1} e^{s_1 q} & \text{if } \sigma = (s, \varepsilon) \in \mathcal{V}_{[0,t]}(4.12) \\ 0 & \text{otherwise.} \end{cases}$$

Proof: The equivalence of (b) and (c) is contained in the proof of Theorem 4.1.1 while if x satisfies (4.11) then, by Theorem 4.1.2, $x \in \mathcal{P}_1$, and (akc ii) follows from the explicit expression for x (4.12); it therefore remains to

establish the implication (a) \Rightarrow (c).

Let x be an adapted kernel cocycle, and for $s \leq t$ let $x_{s,t} = \text{III}_s(x_{t-s})$. Then $t \rightarrow x_t(\emptyset)$ is continuous, by (akc i), and a semigroup since

$$\begin{aligned} x_{s+t}(\emptyset) &= x_{t,s+t} * x_{0,t}(\emptyset) \quad \text{by (ii)'} \\ &= x_{t,s+t}(\emptyset) x_{0,t}(\emptyset) \quad \text{by (i)'} \\ &= x_s(\emptyset) x_t(\emptyset) \quad (\text{since } \emptyset - t = \emptyset). \end{aligned}$$

Let q be its generator. For $0 < \eta < t$,

$$x_{t-\eta,t+\eta}(\{t^\pm\}) = (\text{III}_{t-\eta} x_{2\eta})(\{t^\pm\}).$$

So, by (iv)', $q^\pm = \lim_{\eta \downarrow 0} x_{t-\eta,t+\eta}(\{t^\pm\})$ exists and is independent of t . By repeated application of (ii)', if $(\sigma, t) \in \Gamma_{\text{ad}}(\mathbb{R}_+)$ then

$$x_t(\sigma) = p_{(t-s_n-\eta)} x_{s_n-\eta, s_n+\eta}(\{(s_n, \varepsilon_n)\}) p_{(s_n-s_{n-1}-2\eta)} \cdots x_{s_1-\eta, s_1+\eta}(\{(s_1, \varepsilon_1)\}) p_{s_1-\eta}$$

for $0 < \eta < \min_{i,j} |s_i - s_j|$, where $\sigma = (s, \varepsilon)$. Finally, letting $\eta \downarrow 0$ we obtain (4.12) and the proof is complete. \blacksquare

4.4 Generator:

The *generator* of an adapted kernel cocycle is the $q \in \mathcal{A}^3$ which determines its explicit expression (4.12).

Proposition 4.4.1: Let x be an adapted kernel cocycle with generator q , then $T_t: b \rightarrow x_t^\dagger * b x_t(\emptyset)$ ($t \in \mathbb{R}_+$) is a one parameter semigroup on \mathcal{A} with generator L_q° .

Proof: The third component of (4.5), evaluated at \emptyset , reads

$$\frac{d}{dt} T_t(b) = T_t(L_q^\circ[b]).$$

Since $T_0(b) = b$ the result follows. \blacksquare

5. Quantum probability.

We now describe some theory from operator algebras, associated unbounded operators and quantum dynamical semigroups. Takesaki's books [Ta 1,3] are standard references for the operator algebra theory. We also present Kümmerer's formulation of quantum dynamical semigroups and their dilations (in which invariance of the state is incorporated in the definitions) [Küm]. The material of this section will be used to construct a stochastic calculus for operator valued processes from the kernel calculus described above, thereby streamlining the existing constructions [BSW 2], [HL 1,2], [L 1].

5.1 Some generalities

A *von Neumann algebra* acting on a Hilbert space \mathfrak{h} is a unital $*$ -subalgebra of $\mathcal{L}(\mathfrak{h})$, the algebra of bounded linear operators on \mathfrak{h} , which is closed in the strong operator topology. For a subset \mathcal{X} of $\mathcal{L}(\mathfrak{h})$, \mathcal{X}' denotes its *commutant*: $\{T \in \mathcal{L}(\mathfrak{h}) : TX = XT \ \forall X \in \mathcal{X}\}$. If \mathcal{X} is self-adjoint then \mathcal{X}' is a von Neumann algebra, and the von Neumann algebra *generated* by \mathcal{X} is $(\mathcal{X}')'$. The tensor product $\mathfrak{B}_1 \otimes \mathfrak{B}_2$ of two von Neumann algebras $\mathfrak{B}_1, \mathfrak{B}_2$ acting on $\mathfrak{h}_1, \mathfrak{h}_2$ respectively, is the von Neumann subalgebra $\{T_1 \otimes T_2 : T_i \in \mathfrak{B}_i\}$ of $\mathcal{L}(\mathfrak{h}_1 \otimes \mathfrak{h}_2)$. The non-trivial relation $(\mathfrak{B}_1 \otimes \mathfrak{B}_2)' = \mathfrak{B}_1' \otimes \mathfrak{B}_2'$ holds.

An unbounded operator cannot *belong* to a von Neumann algebra \mathfrak{B} , however we say that an operator T is *affiliated* to \mathfrak{B} (written $T\eta\mathfrak{B}$) if $\mathfrak{B}'\text{Dom}(T) \subset \text{Dom}(T)$ and $TB'\varphi = B'T\varphi \ \forall \varphi \in \text{Dom}(T)$, $B' \in \mathfrak{B}'$ where $\text{Dom}(T)$ denotes the *domain* of T . Equivalently if $\text{Gr}(T)$, the graph of T , is considered as a subspace of $\mathfrak{h} \otimes \mathbb{C}^2 = \mathfrak{h} \oplus \mathfrak{h}$, T is affiliated to \mathfrak{B} if and only if $(\mathfrak{B}' \otimes I)\text{Gr}(T) \subset \text{Gr}(T)$. When T is closed this is equivalent to $P_{\text{Gr}(T)} \in \mathfrak{B} \otimes M_2(\mathbb{C}) = M_2(\mathfrak{B})$, where $P_{\text{Gr}(T)}$ is the orthogonal projection onto the *graph* of T . If $T\eta\mathfrak{B}$ then $T^*\eta\mathfrak{B}$, in particular, if $T\eta\mathfrak{B}$ is closable then its closure \bar{T} is also affiliated. For operators X and Y , Y is an *extension* of X (or X is a restriction of Y), written $X \subset Y$, means that $\text{Gr}(X)$ is a subspace of $\text{Gr}(Y)$.

The following result will be useful later.

Proposition 5.1.1: Let \mathfrak{B} be a von Neumann algebra and $X\eta\mathfrak{B}$ be closed. If $T \subset X$ and $\mathcal{E}_0\text{Dom}(T) \subset \text{Dom}(T)$ for some strongly dense $*$ -subalgebra \mathcal{E}_0 of

\mathfrak{B}' , then $\bar{T}\eta\mathfrak{B}$.

Proof: Since $\text{Dom}(T)$ is invariant under \mathcal{C}_0 and T has an affiliated extension, $\text{Gr}(T)$ is invariant under $\mathcal{C}_0 \otimes I$. Hence $P_{\text{Gr}(\bar{T})} \in (\mathcal{C}_0 \otimes I)' = (\mathfrak{B}' \otimes CI)' = \mathfrak{B} \otimes M_2(\mathbb{C})$. ■

5.2 Quantum probability spaces.

A *quantum probability space* Q is a triple $(\mathfrak{h}, \mathfrak{B}, \xi)$ where \mathfrak{h} is a Hilbert space, $\mathfrak{B} \subset \mathcal{L}(\mathfrak{h})$ is a von Neumann algebra and $\xi \in \mathfrak{h}$ is a vector which is both

cyclic : $\mathfrak{B}\xi$ is dense in \mathfrak{h}

and *separating* : $T \in \mathfrak{B}, T\xi = 0 \Rightarrow T = 0$ (5.1)

for \mathfrak{B} . Associated to a quantum probability space Q are three operators: S_Q , J_Q and Δ_Q . S is the closure of the conjugate linear operator with domain $\mathfrak{B}\xi$ which maps $T\xi$ to $T^*\xi$ and $S = J\Delta^{1/2}$ is its polar decomposition— J being a conjugate linear isometric involution and Δ a positive self-adjoint operator. Tomita's fundamental lemma states that

$$J\mathfrak{B}J = \mathfrak{B}', \quad (5.2)$$

$$\Delta^{it}\mathfrak{B}\Delta^{-it} = \mathfrak{B}, \quad (t \in \mathbb{R}). \quad (5.3)$$

The map $\sigma^Q : t \rightarrow \Delta^{it} \cdot \Delta^{-it}$ on \mathfrak{B} is called the *modular automorphism group*.

We shall denote $\text{Dom}(S)$, considered as a Hilbert space with the graph norm $x \rightarrow \{\|x\|^2 + \|Sx\|^2\}^{1/2}$, by Σ_Q .

5.3 *-affiliation.

Since ξ is cyclic for \mathfrak{B}' if (and only if) it is separating for \mathfrak{B} , the prescription

$$\begin{aligned} \text{Dom}(\dot{X}) &= \mathfrak{B}'\xi, \\ \dot{X}T'\xi &= T'x, \quad (T' \in \mathfrak{B}'), \end{aligned} \quad (5.4)$$

associates to each vector $x \in \mathfrak{h}$, a densely defined operator \dot{X} affiliated to \mathfrak{B} . In general \dot{X} will fail to be closable, however if $\xi \in \text{Dom}(\dot{X}^*)$ then, since $\dot{X}^*\eta\mathfrak{B}, \dot{X}^*$ will be densely defined and so \dot{X} will be closable. Now

$\xi \in \text{Dom}(\dot{X}^*)$ if and only if $x \in \Sigma_Q$ in which case $\dot{X}^*\xi = Sx$. This is the class of operators with which we shall be dealing.

Definition 5.3.1: Let $Q = (\mathfrak{h}, \mathfrak{B}, \xi)$ be a quantum probability space. A closed operator T is **-affiliated* to Q if

- (i) $T\eta\mathfrak{B}$
- (ii) $\xi \in \text{Dom}(T) \cap \text{Dom}(T^*)$
- (iii) $B'\xi$ is a core for T .

The set of *-affiliated operators will be denoted $\eta^*(Q)$.

Let $Q = (\mathfrak{h}, \mathfrak{B}, \xi)$ be a quantum probability space. There is a bijective correspondence between vectors x in Σ_Q and operators X *-affiliated to Q which is determined by the relation

$$X\xi = x.$$

$\eta^*(Q)$ is a linear space under the strong sum $X+Y := \overline{(X+Y)}|_{\mathfrak{B}'\xi}$ with a conjugation given by

$$X^+ = \overline{X^*}|_{\mathfrak{B}'\xi}.$$

The strong product $X \cdot Y = \overline{XY}|_{\mathfrak{B}'\xi}$ is also defined for pairs of *-affiliated operators X, Y for which $\xi \in \text{Dom}(S_Q XY)$. We shall drop the dots with the understanding that sums, differences and products (when defined) are in the strong sense. By Proposition 5.1.1 *-affiliated operators have common core $\mathfrak{B}'_0\xi$ whenever B'_0 is a strongly dense *-subalgebra \mathfrak{B}'_0 of \mathfrak{B}' .

5.4 Quantum dynamical semigroups.

If $\mathfrak{B}_1, \mathfrak{B}_2$ are von Neumann algebras then $T \in \mathcal{L}(\mathfrak{B}_1, \mathfrak{B}_2)$ is *completely positive* if

$$T \otimes \text{id}_{M_n} : M_n(\mathfrak{B}_1) \rightarrow M_n(\mathfrak{B}_2), \quad [x_{ij}] \rightarrow [T(x_{ij})]$$

is positivity preserving for each n , equivalently if

$$\sum_{i,j=1}^n y_i^* T(x_i^* x_j) y_j \geq 0$$

for each $n \in \mathbb{N}$, $x_1, \dots, x_n \in \mathcal{B}_1$, and $y_1, \dots, y_n \in \mathcal{B}_2$. A *morphism* between quantum probability spaces $Q_i = (\mathfrak{h}_i, \mathcal{B}_i, \xi_i)$ ($i = 1, 2$) will be an element T of $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ satisfying

- (i) T is completely positive;
- (ii) $T(\mathbb{1}_{\mathcal{B}_1}) = \mathbb{1}_{\mathcal{B}_2}$;
- (iii) $\langle \xi_2, T(x)\xi_2 \rangle = \langle \xi_1, x\xi_1 \rangle$, ($x \in \mathcal{B}_1$).

If $u \in \mathcal{B}$ is unitary then $\text{Ad } u: b \rightarrow u^*bu$ is a morphism of Q if and only if

$$\sigma_t^Q(u) = u, \quad \forall t \in \mathbb{R}. \quad (5.5)$$

A morphism T on Q satisfying

$$T(bc) = T(b)c \quad (b \in \mathcal{B}, c \in \mathcal{C}),$$

where $\mathcal{C} = \text{Range}(T)$, is called a *conditional expectation onto* \mathcal{C} . Clearly the range of a conditional expectation is an algebra. Given a subalgebra \mathcal{C} of \mathcal{B} , a conditional expectation onto \mathcal{C} exists if and only if $\sigma_t^Q(\mathcal{C}) = \mathcal{C} \quad \forall t \in \mathbb{R}$ ([Tak 2]). In particular, contrary to classical probability, conditional expectation onto a subalgebra does not always exist. This is a fundamental distinction between classical and quantum probability.

A *quantum dynamical semigroup* is a one parameter semigroup of morphisms of a quantum probability space which is continuous in the pointwise weak-* topology, that is $t \mapsto \langle \psi, T_t(b)\psi \rangle$ is continuous on \mathbb{R}_+ for each $b \in \mathcal{B}$, $\psi \in \mathfrak{h}$. If T is a quantum dynamical semigroup on $Q = (\mathfrak{h}, \mathcal{B}, \xi)$, and each T_t is an automorphism of Q (equivalently, if each T_t is an automorphism of \mathcal{B} preserving the state $\langle \xi, \cdot \xi \rangle$) then T extends to a *quantum dynamical group*: $T_t = (T_{-t})^{-1}$, ($t < 0$).

A *dilation* of a dynamical semigroup $\{T_t^0: t \geq 0\}$ on Q_0 consists of a dynamical group $\{T_t: t \in \mathbb{R}\}$ on a quantum probability space Q together with morphisms $j: Q_0 \rightarrow Q$ and $\mathbb{P}: Q \rightarrow Q_0$ such that

$$\mathbb{P} \circ T_t \circ j = T_t^0, \quad (t \geq 0).$$

j will then be an injective $*$ -homomorphism $\mathcal{B}_0 \rightarrow \mathcal{B}$ and $j \circ \mathbb{P}$ a conditional expectation. Conditional expectations $\mathbb{E}_I: \mathcal{B} \rightarrow \mathcal{B}_I = \{T_t \circ j(\mathcal{B}_0): t \in I\}$ exist for each subinterval I of \mathbb{R} and a dilation is called *Markov* if

$$\mathbb{E}_{(-\infty, 0]} \circ T_t \circ j = \mathbb{E}_{\{0\}} \circ T_t \circ j, \quad (t \geq 0). \quad (5.6)$$

6. From kernels to operators.

In this section we construct operators from kernels via the $*$ product, using the results from Section 5, thereby defining a (non-commutative) duality transform between vectors and $*$ -affiliated operators of the quantum probability space of interest to us here.

6.1 Canonical commutation relations

In view of Propositions 2.4.5 and 2.4.3 we may define operators $W(f)$, $W'(f)$ and J_1 ($f \in C_\kappa^1(I)$) to be the unique isometric extensions to $L^2(\mathcal{V}_I, \mu)$ of G_{w_f} , D_{v_f} and J_0 respectively (see Definition 2.4.2). The following relations are immediate.

Proposition 6.1.1: For $f, g \in C_\kappa^1(I)$,

- (i) $W(f)W(g) = e^{i\zeta(f, g)}W(f+g)$; $W(0) = \mathbf{1}$;
- (ii) $W'(f)W'(g) = e^{i\zeta(f, g)}W'(f+g)$; $W'(0) = \mathbf{1}$;
- (iii) $W'(f)W(g) = W(g)W'(f)$;
- (iv) $W'(f) = J_1 W(\bar{f}) J_1$;
- (v) $\langle w_0, W(f)w_0 \rangle = \langle w_0, W'(f)w_0 \rangle$

Let \mathcal{N}_0 and \mathcal{M}_0 be the linear spans of the sets $\{W(f): f \in C_\kappa^1(I)\}$ and $\{W'(f): f \in C_\kappa^1(I)\}$, and let $\mathcal{N} = \mathcal{N}_0''$ and $\mathcal{M} = \mathcal{M}_0''$ be the respective von Neumann algebras they generate. In view of Corollary 2.5.3, w_0 is a cyclic vector for both \mathcal{N} and \mathcal{M} , and since $\mathcal{N} \subset \mathcal{M}'$ by 6.1.1 (iii) w_0 is also a separating vector for both \mathcal{N} and \mathcal{M} . Let S , J and Δ be the Tomita operators for (\mathcal{N}, w_0) (see Section 5). We now relate these to the operators S_0 , J_0 and Γ_0

introduced in Definition 2.4.2.

Proposition 6.1.2:

$$S = \bar{S}_0; \quad J = J_1 = \bar{J}_0; \quad \Delta^{1/2} = \bar{\Gamma}_0.$$

Proof: $\mathcal{W} = \mathcal{N}_0 w_0$ is a core for $\Gamma := M_{\sqrt{\gamma}}$ by Corollary 2.5.3 and also a core for S and $\Delta^{1/2}$ by Kaplansky's density theorem [Tak 3]. But $S|_{\mathcal{W}} = S_0$, so $J\Delta^{1/2} = S = \bar{S}_0 = \overline{J_1 \Gamma_0} = J_1 \Gamma$ and the result follows by the uniqueness of polar decompositions. ■

Corollary 6.1.3:

$$\mathcal{N} = \mathcal{M}'$$

Proof: $\mathcal{M} = \text{strong closure of } J\mathcal{N}_0 J$ (by Propositions 6.1.1, 6.1.2)

$$= J\mathcal{N}J \quad (\text{by the isometry of } J)$$

$$= \mathcal{N}' \quad (\text{by Tomita's relation (5.1)}).$$

■

W and W' are therefore a pair of commuting (cyclic) representations of the canonical commutation relations over the symplectic space $(C_k^1(I), \zeta_{c_+, c_-})$ with generating functional $\hat{\mu}_{c_+, c_-}$ [BrR] which may justifiably be called *com-mutant representations*.

Notice that the algebras \mathcal{N} and \mathcal{M} are equally the von Neumann algebras generated by *bounded* left and right multiplication operators (in the $*$ -sense) respectively:

$$\left\{ \overline{G_x} : x \in \mathcal{K}_0^C(I), \sup_{y \neq 0} \frac{\|x * y\|}{\|y\|} < \infty \right\},$$

$$\left\{ \overline{D_z} : z \in \mathcal{K}_0^C(I), \sup_{y \neq 0} \frac{\|y * z\|}{\|y\|} < \infty \right\}.$$

We shall denote the quantum probability space $(L^2(\mathcal{V}_I, \mu), \mathcal{N}, w_0)$ by Q_1^I . It follows from Proposition 6.1.2 that $\Sigma_{Q_1^I} = L^2(\mathcal{V}_I, \nu)$.

6.2 Initial space.

Now let $Q_0 = (\mathfrak{h}_0, \mathfrak{U}_0, \xi_0)$ be a quantum probability space and $Q = Q^I = Q_0 \otimes Q_1^I$, in other words $Q^I = (\mathfrak{h}, \mathfrak{U}, \xi)$ where $\mathfrak{h} = \mathfrak{h}_0 \otimes L^2(\mathcal{V}_I, \mu) = L^2(\mathcal{V}_I, \mu; \mathfrak{h}_0)$, $\mathfrak{U} = \mathfrak{U}_0 \otimes \mathcal{N}$ and $\xi = \xi_0 \otimes w_0$. Then Q^I is a quantum probability space, $\Sigma_{Q^I} = L^2(\mathcal{V}_I, \nu; \Sigma_{Q_0})$ and, (for ν -almost all σ),

$$(\Delta_{Q^I}^{1/2} f)(\sigma) = \sqrt{\gamma(\sigma)} \Delta_{Q_0}^{1/2} [f(\sigma)]$$

$$(S_{Q^I} f)(\sigma) = S_{Q_0} [f(\sigma^\dagger)].$$

6.3 Non-commutative duality transform

We next introduce left multiplication operators for \mathfrak{U}_0 -valued smooth kernels and show that they are $*$ -affiliated to Q^I .

Lemma 6.3.1: For $x, y \in \mathcal{K}_b^{\mathcal{L}(\mathfrak{h}_0)}(I)$, $u, v \in \mathfrak{h}_0$,

$$\langle x(\cdot)u, y(\cdot)v \rangle_{\mathfrak{h}} = \langle u, x^\dagger * y(\emptyset)v \rangle_{\mathfrak{h}_0}.$$

Proof:

$$\int \langle x(\sigma)u, y(\sigma)v \rangle_{\mathfrak{h}_0} d\sigma = \langle u, \int_{\mathcal{V}_I} x^\dagger(\sigma^\dagger)y(\sigma) d\sigma v \rangle_{\mathfrak{h}_0}.$$

■

Definition 6.3.2: For $x \in \mathcal{K}_b^{\mathcal{L}(\mathfrak{h}_0)}(I)$ let G_x be the map

$$y(\cdot)v \rightarrow (x * y)(\cdot)v, \quad v \in \mathfrak{h}_0, y \in \mathcal{K}_b^{\mathcal{L}(\mathfrak{h}_0)}(I).$$

Lemma 6.3.3: (i) G_x is a densely defined closable operator on \mathfrak{h}

$$(ii) G_x^* \supset G_x^\dagger$$

$$(iii) G_{x_1} G_{x_2} = G_{x_1 * x_2}$$

Proof: Since, by an application of Fubini's theorem, $y(\cdot)v = 0$ almost everywhere implies that $x * y(\cdot)v = 0$ almost everywhere, the map is well-defined. Let $x, y, z \in \mathcal{K}_b^{\mathcal{L}(\mathfrak{h}_0)}(I)$, then by Lemma 6.3.1 and the associativity of $*$,

$$\langle y(\cdot)u, (x * z)(\cdot)v \rangle_{\mathfrak{h}} = \langle u, (y^\dagger * x * z)(\emptyset)v \rangle_{\mathfrak{h}_0}$$

$$= \langle x^\dagger * y(\cdot)u, z(\cdot)v \rangle_{\mathfrak{H}}, \quad (u, v \in \mathfrak{H}_0).$$

So $G_x^* \supset G_x^\dagger$ which is densely defined—this proves (i) and (ii). (iii) follows from the associativity of $*$. ■

Definition 6.3.4: Now let $x \in \mathcal{K}_b^{\mathfrak{H}_0}(I)$ and G_x^0 be the restriction of G_x to $(\mathcal{U}'_0 \otimes_{\text{alg}} \mathcal{M}_0)^\xi$.

Proposition 6.3.5: If $X\eta^*Q^I$ is the $*$ -affiliated operator corresponding to the vector $x(\cdot)\xi_0$, where $x \in \mathcal{K}_b^{\mathfrak{H}_0}(I)$, then $\overline{G_x^0} = X \subset \overline{G_x}$.

Proof: $\overline{G_x^0} \subset X$ and, by Proposition 5.1.1, $(\mathcal{U}'_0 \otimes_{\text{alg}} \mathcal{M})^\xi$ is a core for X . ■

Notation 6.3.6: For $x \in \Sigma_{Q^I}$ let \hat{x} be the corresponding $*$ -affiliated operator and for $X\eta^*Q^I$, X^\vee will denote the corresponding vector. Thus

$$\hat{x} \text{ is the closure of the operator } T'\xi \mapsto T'x \quad (T' \in \mathcal{U}')$$

and

$$X^\vee \text{ is the vector } X\xi.$$

The next result justifies the name *non-commutative duality transform* for the map $^\wedge$.

Proposition 6.3.7: Let $x, y \in \mathcal{K}_b^{\mathfrak{H}_0}(I)$ be such that $\xi \in \text{Dom}(\hat{x}\hat{y})$, then

$$(x * y)^\wedge \subset \hat{x}\hat{y}, \quad (6.1)$$

(where, as in future, we abbreviate $z(\cdot)\xi_0$ to \hat{z} when $z \in \mathcal{K}_b^{\mathfrak{H}_0}$).

Proof: Suppose that $\xi \in \text{Dom}(\hat{x}\hat{y})$, in other words $y \in \text{Dom}(\hat{x})$, then $\hat{x}\hat{y}\xi = \hat{x}y = G_x y = x * y = (x * y)^\wedge \xi$ and (6.1) follows. ■

Corollary 6.3.8: Let $x \in \mathcal{K}_b^{\mathfrak{H}_0}(I)$

- (i) $\hat{x} = \mathbb{1}_x \Leftrightarrow x = \mathbb{1}_{x_0} \delta_\emptyset$;
(ii) \hat{x} is isometric $\Leftrightarrow x^\dagger * x = \mathbb{1}_{x_0} \delta_\emptyset$.

Proof: (i) is immediate and so (ii) follows from the proposition above. ■

Notation 6.3.9: For a function f on \mathcal{V} and interval I , let f^I denote the function on \mathcal{V} :

$$\sigma \rightarrow f(\sigma \cap I).$$

Proposition 6.3.10: Let $x \in \mathcal{K}_b^{\mathcal{X}_0}(\mathbb{R})$ have support in \mathcal{V}_I , then $\text{Dom}(\hat{x}) \supset L^2(\mathcal{V}_{\mathbb{R} \setminus I}; \mathfrak{h}_0)$ and for $\varphi \in L^2(\mathcal{V}_{\mathbb{R} \setminus I}; \mathfrak{h}_0)$

$$\hat{x} \varphi = x^I(\cdot) \varphi^{\mathbb{R} \setminus I}(\cdot). \quad (6.2)$$

Proof: Let $\varphi \in L^2(\mathcal{V}_{\mathbb{R} \setminus I}; \mathfrak{h}_0)$ and choose a sequence $\{T'_n : n = 1, 2, \dots\}$ in $\mathcal{X}'_0 \otimes_{\text{alg}} \mathcal{M}_0^{\mathbb{R} \setminus I}$ such that $\varphi_n := T'_n \xi \rightarrow \varphi$. Then $\varphi_n \in \text{Dom}(\hat{x})$ and

$$\hat{x} \varphi_n(\sigma) = \sum_{\alpha \subset \sigma} x(\alpha) \varphi_n(\bar{\alpha}) = x(\sigma \cap I) \varphi_n(\sigma \cap (\mathbb{R} \setminus I)),$$

so that, by the combinatorial Lemma 2.3.3(a):

$$\begin{aligned} \|\hat{x}(\varphi_n - \varphi_m)\|_h^2 &= \int_{\mathcal{V}} \|x(\sigma \cap I)(\varphi_n - \varphi_m)(\sigma \cap (\mathbb{R} \setminus I))\|_{\mathfrak{h}_0}^2 d\sigma \\ &= \int_{\mathcal{V}} \sum_{\alpha \subset \sigma} \|x(\alpha)(\varphi_n - \varphi_m)(\bar{\alpha})\|_{\mathfrak{h}_0}^2 d\sigma \\ &= \int_{\mathcal{V}} \int_{\mathcal{V}} \|x(\alpha)(\varphi_n - \varphi_m)(\beta)\|_{\mathfrak{h}_0}^2 d\alpha d\beta \\ &\leq \int_{\mathcal{V}} \|x(\alpha)\|_{\mathcal{X}_0}^2 d\alpha \cdot \|\varphi_n - \varphi_m\|_{\mathfrak{h}}^2 \\ &\rightarrow 0 \quad \text{as } n, m \rightarrow \infty. \end{aligned}$$

In other words, since \hat{x} is closed, $\varphi = \lim \varphi_n \in \text{Dom}(\hat{x})$ and (6.2) holds. ■

Corollary 6.3.11: Let $x \in \mathcal{K}_b^{\mathcal{H}_0}(\mathbb{R})$. Then $\text{Dom}(\hat{x}) \supset \mathfrak{h}_0 \delta \emptyset$

$$\hat{x} v \delta \emptyset = x(\cdot) v, \quad (v \in \mathfrak{h}_0).$$

The modular automorphism group σ^Q is expressed most conveniently with the use of the non-commutative duality transform:

$$\sigma_t^Q(\hat{x}) = \left(\gamma^{it}(\cdot) \sigma_t^{Q_0}(x(\cdot)) \right)^\wedge \quad (x \in \Sigma_Q). \quad (6.3)$$

7. Operator stochastic calculus.

In this section we again take I to have left end point 0, and we abbreviate Σ_{Q_0} and Σ_Q to Σ_0 and Σ respectively, where Q_0 is a fixed initial quantum probability space and $Q = Q_0 \otimes Q_1^I$ (Section 6.2).

7.1 Extension of the differential and integral operators

First embed \mathcal{P}_0 and \mathcal{P}_1 into the set of *adapted kernel processes*

$$\mathfrak{p} := \{f \in \mathcal{L}^0(I, \mathcal{B}; \Sigma) : f(t) \in \Sigma_t \quad \forall t\}$$

by the prescription

$$x \rightarrow x(\cdot) \xi_0,$$

and denote the resulting subspaces of \mathfrak{p} by \mathfrak{p}_0 and \mathfrak{p}_1 respectively. In this way smooth kernel processes may be thought of as Hilbert space-valued rather than algebra valued. Now consider the *locally square integrable kernel processes* and the *martingale kernels*:

$$\mathfrak{l}^2 := \{f \in L_{\text{loc}}^2(I, \mathcal{B}, \lambda; \Sigma) : f_t \in \Sigma_t \text{ for almost all } t\}$$

$$\mathfrak{m} := \{f \in \mathfrak{p} : \chi_{\Gamma_{[0,s]}} f_t = f_s \quad \forall s \leq t\}$$

For $T \in I$, $x \in \mathfrak{l}^2$ let

$$\|x\|_T = \left\{ \int_0^T \int_{\Gamma_t} \|x(\sigma, t)\|_{\mathfrak{h}_0}^2 d\sigma dt \right\}^{1/2} = \left\{ \int_0^T \|x_t\|_{\mathfrak{h}}^2 dt \right\}^{1/2}$$

—the seminorms $\{\|\cdot\|_T : T \in I\}$ clearly separate \mathfrak{l}^2 .

The operators I^κ , Δ^\pm and Δ° extend to \mathfrak{l}^2 , $\mathfrak{p}_0 + \mathfrak{m}$ and $\mathfrak{p}_1 + \mathfrak{m}$ respectively with the same definitions as in Section 3. In particular, if $x \in \mathfrak{m}$ then $\Delta^\circ x = 0$ and, for $(\sigma, t) \in \Gamma_{\text{ad}}(I)$:

$$\Delta^\pm x(\sigma, t) = x(\sigma \cup \{t^\pm\}, T), \quad (T > t).$$

Moreover, if $x \in \mathfrak{p}$ and $x_t = 0$ for almost all t , then $I^\pm x_t = 0$ for all t , so these integrals do not distinguish versions. Kernel stochastic integrals of locally square integrable processes are *martingales*, kernel stochastic derivatives of martingale kernels are \mathfrak{l}^2 (as is seen by an application of the combinational Lemma 2.3.3(a)), and the fundamental theorem (3.2.5) continues to hold for $\mathfrak{p}_1 + \mathfrak{m}$ (Proposition 7.5.1 establishes this). The new element here is the isometry/orthogonality relation.

Proposition 7.1.1: Let $x, y \in \mathfrak{l}^2$, then for $T \in I$

$$\|I^+ x_T + I^- y_T\|_{\mathfrak{g}}^2 = c_+ \|x\|_T^2 + c_- \|y\|_T^2. \quad (7.1)$$

Proof: For $z \in \mathfrak{l}^2$,

$$\begin{aligned} \|I^\pm z_T\|^2 &= \int_{\mathcal{P}_{[0,T]}^\pm} \|z(\sigma \setminus \{\max \sigma\}, \max \sigma)\|^2 d\sigma \\ &= c_\pm \int_0^T \int_{\mathcal{P}} \|z(\tau, t)\|^2 d\tau dt, \end{aligned}$$

and, since $I^+ x_T$ and $I^- y_T$ are supported by the disjoint sets \mathcal{P}^+ and \mathcal{P}^- respectively, (7.1) follows. \blacksquare

The following extension of the previous result is useful.

Proposition 7.1.2: Let $f^\kappa \in \mathfrak{l}^2$ and $x = I^+ f^+ + I^- f^- + I^\circ f^\circ$, then

$$\|x_t\|_{\mathfrak{g}}^2 = \int_0^t \{c_+ \|f_s^+\|_{\mathfrak{g}}^2 + c_- \|f_s^-\|_{\mathfrak{g}}^2 + 2\text{Re}\langle f_s^\circ, x_s \rangle_{\mathfrak{g}}\} ds. \quad (7.2)$$

Proof: By (7.1)

$$\begin{aligned}\|x_t\|^2 &= c_+ \|f^+\|_t^2 + c_- \|f^-\|_t^2 + \int_{\mathcal{P}} \left\| \int_0^t f_s^\circ(\sigma) \, ds \right\|^2 d\sigma \\ &\quad + 2 \operatorname{Re} \int_{\mathcal{P}} \left\langle \int_0^t f_s^\circ(\sigma) \, ds, I^+ f_t^+(\sigma) + I^- f_t^-(\sigma) \right\rangle d\sigma,\end{aligned}$$

but (for almost all σ)

$$\langle f_s^\circ(\sigma), I^\pm f_t^\pm(\sigma) \rangle = \langle f_s^\circ(\sigma), I^\pm f_s^\pm(\sigma) \rangle \quad (s \leq t),$$

since $I^\pm f^\pm$ are martingales, and

$$\left\| \int_0^t f_s^\circ(\sigma) \, ds \right\|^2 = 2 \operatorname{Re} \int_0^t \left\langle f_s^\circ(\sigma), \int_0^s f_r^\circ(\sigma) \, dr \right\rangle ds$$

so (7.2) follows. ■

7.2 Simple approximation.

The *simple* and *continuous kernel processes* are defined by:

$$\mathfrak{s} = \{f \in \mathfrak{p} : f \text{ is a step function and } f(t) \in \mathcal{K}_0^{\mathfrak{X}_0}(I) \xi_\sigma \cap \mathfrak{X} \xi \quad \forall t\}.$$

$$\mathfrak{c} = \mathfrak{p} \cap \mathcal{C}(I; \Sigma).$$

Thus martingale kernels and kernel stochastic integrals are continuous, and continuous kernel processes are locally square integrable. The next result establishes the density of \mathfrak{s} in \mathfrak{t}^2 .

Proposition 7.2.1: Let $f \in \mathfrak{t}^2$, then there is a sequence $f^{(n)} \in \mathfrak{s}$ such that for all $t \in I$,

$$\int_0^t \|f_s - f_s^{(n)}\|_\Sigma^2 \, ds \rightarrow 0 \quad (7.3)$$

as $n \rightarrow \infty$.

Proof: For each $N > 0$,

$$f\chi_{[0,N]} \in L^2(\Gamma_{\text{ad}}(I), \mathfrak{B}, \mu \times \lambda_1; \Sigma_\sigma) = L^2(\Gamma_{\text{ad}}(I), \mathfrak{B}, \mu \times \lambda_1) \otimes \Sigma_\sigma.$$

Now

$$\{\chi_{B \times [b, c]} : B \subset \Gamma_{[0, a]} \text{ compact, } a < b < c\}$$

is total in $L^2(\Gamma_{\text{ad}}(I), \mathcal{B}, \mu \times \lambda_1)$ moreover, by Kaplansky's density theorem, each $\chi_B \otimes v$ ($v \in \Sigma_0$) may be approximated, in Σ , by elements from $\mathcal{U}\xi$ (or $\mathcal{R}\xi$ for any strongly dense $*$ -subalgebra \mathcal{R} of \mathcal{U}). Combining these facts, the result follows. ■

We write $f^{(n)} \rightarrow f$ when $f^{(n)}, f \in \mathfrak{l}^2$ satisfy (7.3) for each t . In view of (7.1) we therefore have

Corollary 7.2.2: Let $f \in \mathfrak{l}^2$. Then there is a sequence $f^{(n)} \in \mathfrak{s}$ such that, for each $t \in I$

$$(I^\pm f)_t = \Sigma\text{-}\lim_{n \rightarrow \infty} (I^\pm f^{(n)})_t. \quad (7.4)$$

7.3 Conditional expectations.

For each subinterval J of I let P_J be the orthogonal projection of multiplication by $\chi_{\mathcal{P}_J}$, $\Sigma_J = P_J \Sigma$ and $\mathcal{U}_J = \mathcal{U} \cap (\Sigma_J)^\wedge$.

Proposition 7.3.1: Let $x \in \Sigma$, then

$$x \in \Sigma_J \Leftrightarrow \hat{x} \eta \mathcal{U}_J.$$

Proof: If $x \in \Sigma_J$ then it follows from Theorem 2.2 of [LiW] that $\hat{x} = \bar{Y}$ where $\text{Dom}(Y) = \mathcal{U}'_J \xi$, $YR\xi = Rx$ ($R \in \mathcal{U}'_J$). In particular, $\hat{x} \eta \mathcal{U}_J$. Conversely if $\hat{x} \eta \mathcal{U}_J$ then, since $P_J \in \mathcal{U}'_J$, $\chi_{\mathcal{P}_J} x = x$, in other words $x \in \Sigma_J$. ■

Remark 7.3.2: The duality transform establishes a bijection between operator valued maps $F: I \rightarrow \eta^*(Q)$ for which $F_t \eta \mathcal{U}_{[t]}$ —called *adapted operator processes*—and maps $f: I \rightarrow \Sigma$ for which $f(t) \in \Sigma_{[t]}$. In particular, to each of the classes $\mathfrak{p}_0, \mathfrak{p}_1, \mathfrak{l}^2, \mathfrak{m}, \mathfrak{s}$ and \mathfrak{c} corresponds a class of adapted operator processes $\hat{\mathfrak{p}}_0, \dots, \hat{\mathfrak{c}}$.

Definition 7.3.3: For each subinterval J of I , the conditional expectation \mathbb{E}_J on $\eta^*(Q)$ is given by

$$\mathbb{E}_J[\hat{x}] = (P_J x)^\wedge.$$

By Proposition 7.3.1, $\mathbb{E}_J[X] \in \mathfrak{U}_J$ for each $*$ -affiliated operator X . \mathbb{E}_J extends a conditional expectation in the sense of Section 5 to $*$ -affiliated operators. We shall list some of the properties enjoyed by these maps. Let $A, B \in \mathfrak{U}$ and $X \in \eta^*(Q)$ be such that XA and $\overline{BX} \in \eta^*(Q)$, then

- (i) $\mathbb{E}_J[X^+] = \mathbb{E}_J[X]^+$ adjoint preserving
- (ii) if $J_1 \subset J_2$ then $\mathbb{E}_{J_1} \circ \mathbb{E}_{J_2} = \mathbb{E}_{J_1}$ projectivity
- (iii) if $X \in \hat{\Sigma}_J$ then $\mathbb{E}_J[XA] \subset X\mathbb{E}_J[A]$
if $A \in \mathfrak{U}_J$ then $\mathbb{E}_J[XA] \subset \mathbb{E}_J[X]A$
- (iv) if $B \in \mathfrak{U}_J$ then $\mathbb{E}_J[\overline{BX}] = \overline{B\mathbb{E}_J[X]}$
if $X \in \hat{\Sigma}_J$ then $\mathbb{E}_J[\overline{BX}] = \overline{\mathbb{E}_J[B]X}$
- (v) if $J_1 \cap J_2 = \emptyset$ then \mathbb{E}_{J_1} agrees with $\mathbb{E}_{\{0\}}$ on $\hat{\Sigma}_{J_2}$ (7.5)
- (vi) $\mathbb{E}_{\{0\}}[\hat{x}] = (\delta_\emptyset(\cdot)x(\emptyset)\xi_0)^\wedge.$

These properties are straightforward to verify, for instance (i) follows from the fact that P_J commutes with S_Q and (iv) from the fact that P_J commutes with \mathfrak{U}_J .

Remarks: 1. If $T \in \eta^*(Q)$ then $t \rightarrow \mathbb{E}_t[T]$ is a *martingale* (i.e. belongs to $\hat{\mathfrak{m}}$).

2. An adapted operator process X is a martingale if and only if $\mathbb{E}_s[X_t] = X_s \forall s \leq t$.

7.4 Quantum stochastic integration ([BSW 2], [L 1,2], [HL 1]).

We now introduce *operator* creation and annihilation processes which together constitute a *quantum Brownian motion* ([CoH]) and define the stochastic integral with respect to these processes. We shall then argue that our present definition agrees with previous definitions.

Let $a_t := \mathbf{1}_{\mathcal{X}_0} a_{\mathcal{X}_{[0,t]}}$ ($t \in I$) where a_f, a_f^\dagger are the kernels of (2.3), then $a, a^\dagger \in p_1$. The operator process $A^- := \hat{a}$, $A^+ := (a^\dagger)^\wedge$ are called the *annihilation* and *creation processes*; they are mutually adjoint martingales $((A_t^\pm)^\dagger = A_t^\mp \quad \forall t)$.

Definition 7.4.1: We define (operator) quantum stochastic integrals as follows. For $F \in (\mathcal{I}^2)^\wedge$,

$$\int_0^\cdot F dA^\pm := (I^\pm f)^\wedge$$

where $f = F^\vee$.

Lemma 7.4.2: Let $f = x\chi_{(u,v]}$ where $x \in (\mathcal{X}_b^{\mathcal{X}_0} \xi_0) \cap (\mathcal{X}_u \xi)$ and $u < v$. Then for $t \geq v$,

$$\int_0^t \hat{f} dA^\pm = (A_v^\pm - A_u^\pm) \hat{x}. \quad (7.6)$$

Proof: First note that

$$(A_v^+ - A_u^+) \hat{x} = a_{\mathcal{X}_{(u,v]}}^\dagger * x$$

But, since the supports of x and $a_{\mathcal{X}_{(u,v]}}^\dagger$ only have \emptyset in common,

$$\begin{aligned} a_{\mathcal{X}_{(u,v]}}^\dagger * x(\sigma) &= \sum_{\alpha \subset \sigma} a_{\mathcal{X}_{(u,v]}}^\dagger(\alpha) x(\bar{\alpha}) \\ &= \begin{cases} x(\sigma \setminus \{\max \sigma\}) & \text{if } \sigma \in \Gamma^+, \max \sigma \in (u, v] \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} f(\sigma \setminus \{\max \sigma\}, \max \sigma) & \text{if } \sigma \in \Gamma^+ \\ 0 & \text{otherwise} \end{cases} \\ &= I^+ f(\sigma, t) \quad \text{for } t \geq v. \end{aligned}$$

By a similar argument for A^- the result now follows. ■

When these stochastic integrals have been defined in the past, the procedure has been the familiar one of first defining them for elementary processes by (7.6), then extending by linearity to simple processes, then invoking the density result (Lemma 7.2.1) and the isometry property (Proposition 7.1.1)—

proved for simple processes—to extend the integral to (locally) square integrable processes by the *prescription* (7.4) preserving isometry. Our approach has been to both define the stochastic integrals and establish their isometry/orthogonality properties *directly*—always invoking the duality transform (6.9) to pass back and forth between vectors and *-affiliated operators. The previous lemma, together with the density result establish the equivalence of our definition with the previous ones, modulo variations in domain which, in view of Proposition 5.1.1 and [LiW], have no significance.

7.5 Martingale representation theorem.

Since I^\pm map \mathfrak{l}^2 into \mathfrak{m} , stochastic integrals of adapted operator processes are a source of martingales. In this section it is shown that all martingales arise in this way ([HL 1,2], [L 2]).

Proposition 7.5.1: Let $X \in \hat{\mathfrak{m}}$ then there are $F^\pm \in \hat{\mathfrak{l}}^2$ such that

$$X_t = X_0 + \int_0^t F^+ dA^+ + \int_0^t F^- dA^- \quad (t \in I) \quad (7.7)$$

the processes F^\pm being unique up to a Lebesgue null set.

Proof: Let $F^\pm = (\Delta^\pm x)^\wedge$ where $x = X^\vee$. Then

$$\int_0^t F^+ dA^+ + \int_0^t F^- dA^- = (I^+ \Delta^+ x_t)^\wedge + (I^- \Delta^- x_t)^\wedge = (\chi_{\mathcal{P}^+} x_t)^\wedge + (\chi_{\mathcal{P}^-} x_t)^\wedge$$

so $X_t - \int_0^t F^+ dA^+ - \int_0^t F^- dA^- = (\chi_{\mathcal{P}^0} x_t)^\wedge = \hat{x}_0$, and (7.7) follows. Uniqueness follows from the isometry/orthogonality result (Proposition 7.1.1). ■

7.6 Stochastic differential equations.

The kernel formalism allows a unified approach to the existence problem for linear (operator) stochastic differential equations, and, since the solution of corresponding kernel differential equations may be given explicitly, this approach gives more information than a purely operator approach [L 1], [HL 2].

Theorem 7.6.1: Let F_i^K, G_i^K be locally Lipschitz $\mathfrak{U}_{\{0\}}$ -valued functions on I . Then the operator sde

$$\begin{cases} dX = \sum_{i=1}^l F_i^+ X G_i^+ dA^+ + \sum_{i=1}^m F_i^- X G_i^- dA^- + \sum_{i=1}^n F_i^\circ X G_i^\circ dt; \\ X(0) = X_0, \quad (X_0 \in \mathfrak{U}_{\{0\}}) \end{cases} \quad (7.8)$$

has a solution. In other words, there is an adapted process X such that $G_i^K(s)\xi \in \text{Dom}(X(s)) \forall s, \kappa, i$, and

$$X(t) - X(0) = \int_0^t Y^+(s) dA^+ + \int_0^t Y^-(s) dA^- + \int_0^t Y^\circ(s) ds, \quad (7.9)$$

where Y^K is the process $s \mapsto$ closure of $\sum_i F_i^K(s)X(s)G_i^K(s)$. Moreover X^+ satisfies the *conjugate equation* to (7.8).

Proof: Let x be the \mathfrak{p}_1 solution of the corresponding kde, that is the solution of (4.1) in which $L^K(t)[b] = \sum_i f_i^K(t) b g_i^K(t)$ where $f_i^K(s) \otimes \mathbf{1}_\eta = F_i^K(s)$ and $g_i^K(s) \otimes \mathbf{1}_\eta = G_i^K(s)$. Now $G_i^K(s)\xi = g_i^K(s)\xi_0 \delta_\emptyset \in D(\hat{x}_s)$ by Corollary 6.3.11. By the fundamental theorem (3.2.5)

$$x_t - x_0 = \sum_\kappa I^\kappa \left[\sum_i g_i^K(\cdot) x(\cdot, \cdot) f_i^K(\cdot) \right]_t$$

which implies (7.9) for $X = \hat{x}$. By symmetry $X^+ = (x^\dagger)^\wedge$ satisfies the conjugate equation. ■

The proof of uniqueness does not use kernels in any essential way.

Theorem 7.6.2: ([L 1], [HL 2]) Let F^K, G^K be locally bounded maps $I \rightarrow \mathfrak{U}_{\{0\}}$ which are strongly measurable. There is at most one solution to each of the operator sde's

$$dX = F^+ X dA^+ + F^- X dA^- + F^\circ X dt; \quad X(0) = X_0 \quad (X_0 \in \mathfrak{U}_{\{0\}}) \quad (7.10)$$

$$dY = YG^+ dA^+ + YG^- dA^- + YG^\circ dt; \quad Y(0) = Y_0 \quad (Y_0 \in \mathfrak{U}_{\{0\}}) \quad (7.11)$$

Proof: (i) Let X_1 and X_2 be processes satisfying (7.10). Putting $Z = X_1 - X_2$ and applying Proposition 7.1.2 we have

$$\begin{aligned}\|z(t)\|^2 &= \int_0^t \{c_+ \|f^+ z\|^2 + c_- \|f^- z\|^2 + 2 \operatorname{Re} \langle z, f^0 z \rangle\} ds \\ &\leq c_T \int_0^t \|z(s)\|^2 ds \quad \text{for } t \in [0, T],\end{aligned}$$

where $c_T = 2 \sum_{\kappa} \sup_{[0, T]} c_{\kappa} \|F^{\kappa}(\cdot)\|^2$. Iterating this inequality yields $z = 0$, i.e. $X_1 = X_2$.

(ii) Let $Z = Y_1 - Y_2$ where, Y_1 and Y_2 satisfy (7.11), then Z^+ satisfies (7.10) with $X_0 = 0$, $F^+ = G^{-*}$, $F^- = G^{+*}$, $F = G^*$, but so does $X \equiv 0$, so $Z^+ = 0$ by (i), and therefore $Y_1 = Y_2$. ■

Combining these results we have

Theorem 7.6.3: Let F^{κ} be locally Lipschitz $\mathfrak{U}_{(0)}$ -valued maps on I . Then there is a unique solution to the operator sde

$$dX = F^+ X dA^+ + F^- X dA^- + F^0 X dt; \quad X(0) = X_0 \quad (X_0 \in \mathfrak{U}_{(0)})$$

Its adjoint is the unique solution to the operator sde

$$dY = Y F^{-*} dA^+ + Y F^{+*} dA^- + Y F^{0*} dt; \quad Y(0) = (X_0)^*$$

Moreover the operator processes possess smooth kernels which satisfy corresponding kde's and may be written explicitly (4.3).

Remark: A class of operator stochastic differential equations, not covered by the results here, were considered by Barnett, Streater and Wilde ([BSW2]). They established existence and uniqueness for equations of the form

$$dY = f(Y, t) dB + g(Y, t) dt$$

where B is a linear combination of the creation and annihilation process and f and g are adapted (in the obvious sense) and satisfy Lipschitz and continuity conditions.

8. Adapted cocycles

In this section we characterize adapted operator cocycles (cf. Section 4) as groundwork for the construction of quantum Markov processes in the next section. Q_0 will denote a fixed initial quantum probability space and \mathbb{M}_t ($t \in \mathbb{R}$) will be the right shift on $\eta^*(Q)$, (where $Q = Q_0 \otimes Q_1$, as in Section 6.2 and $I = \mathbb{R}$):

$$\begin{aligned}\mathbb{M}_t(\hat{x}) &:= (\mathbb{M}_t x)^\wedge, \quad t \in \mathbb{R} \\ &= \mathbb{M}_t \hat{x} \mathbb{M}_{-t}\end{aligned}$$

(see Section 4.3). For an early paper on Markovian cocycles, see [AcF].

Definition 8.0.1: An \mathcal{U} -valued process $\{X_t : t \geq 0\}$ is an *adapted (operator) cocycle* if

(aci) $X \in \hat{\mathcal{C}}$.

(acii) $\mathbb{M}_t(X_s)X_t = X_{s+t}; \quad X_0 = \mathbf{1} \quad (s, t \geq 0)$.

Remark: If X is an adapted cocycle then the two parameter family $\{X_{s,t} : \mathbb{M}_s(X_{t-s}), s \leq t\}$ satisfies:

(i)' $X_{s,t} \in (\Sigma_{[s,t]})^\wedge$

(ii)' $X_{s,t}X_{r,s} = X_{r,t}; \quad X_{0,0} = \mathbf{1}_{\mathcal{U}}$

(iii)' $\mathbb{M}_u(X_{s,t}) = X_{s+u,t+u}$

(iv)' $t \mapsto (X_{u,t})^\vee$ is continuous on $[u, \infty)$, ($u \in \mathbb{R}, r \leq s \leq t$).

Conversely a two parameter family $\{X_{s,t} : s \leq t\}$ from \mathcal{U} satisfying (i)' - (iv)' determines an adapted cocycle: $\{X_{0,t} : t \geq 0\}$.

It is now a simple matter (easy part of Proposition 8.1.2) to show that bounded operator valued solutions of stochastic differential equations of the form

$$dX = Q^+ X dA^+ + Q^- X dA^- + Q^0 X dt; \quad X(0) = \mathbf{1}_{\mathcal{U}}, \quad (Q^\kappa \in \mathcal{U}_{(0)}), \quad (8.1)$$

provide adapted cocycles. In the converse direction, it was established in [HL2] that *unitary valued* adapted cocycles (there called covariant adapted evolutions) are necessarily solutions of equations of the form (8.1). Here we apply the kernel formalism to establish this for adapted cocycles over a *finite dimensional* quantum probability space. The idea of the proof is again to apply the duality transform to Proposition 4.3.2, but since the continuity condition (iv)' is considerably weaker than the corresponding one for kernel cocycles we have to work a little harder.

8.1 Characterization.

For a function f on \mathcal{V} and an interval I , let f^I denote the map $\sigma \rightarrow f(\sigma \cap I)$.

Lemma 8.1.1: Let $x \in \Sigma_I$ be (almost everywhere) $\mathcal{U}_0\xi$ -valued and such that $\hat{x} \in \mathcal{U}$. Then for $y \in \Sigma_{\mathbf{R} \setminus I}$

$$\hat{x}y = x^I(\cdot)y^{\mathbf{R} \setminus I}(\cdot) \quad (\text{a.e.})$$

(using notation 6.3.9).

Proof: First choose a countable dense set D' from $(\mathcal{U}'_0 \otimes_{\text{alg}} \mathcal{M}_0)\xi$. Let $\{X_m \in \mathcal{U}_0 \otimes_{\text{alg}} \mathcal{N}_0^I : m = 1, 2, \dots\}$ be a sequence which strongly approximates \hat{x} . By taking subsequences we may assume that $X_m d'$ is pointwise convergent (outside a null set Ξ_1) to $\hat{x} d'$ for each $d' \in D'$. Now let $\{y_n = Y_n \xi : n = 1, 2, \dots\}$ be a sequence in D' converging to y but also, again by taking subsequences, such that $\hat{x} y_n$ converges pointwise (outside a null set Ξ_2) to $\hat{x} y$. Then, since x is \mathcal{U}_0 -valued and the y_n are \mathcal{U}'_0 -valued, Proposition 6.3.10 gives, for $\sigma \notin \Xi_1 \cup \Xi_2$,

$$\begin{aligned} \hat{x}y(\sigma) &= \lim_{n \rightarrow \infty} (\hat{x}y_n)(\sigma) \\ &= \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} x_m^I(\sigma) y_n^{\mathbf{R} \setminus I}(\sigma) \\ &= \lim_{n \rightarrow \infty} y_n^{\mathbf{R} \setminus I}(\sigma) x^I(\sigma) \\ &= \lim_{n \rightarrow \infty} x^I(\sigma) y_n^{\mathbf{R} \setminus I}(\sigma) \\ &= x^I(\sigma) y^{\mathbf{R} \setminus I}(\sigma), \end{aligned}$$

and the result is proved. ■

Proposition 8.1.2: Let $\{Y_t : t \geq 0\}$ be a family of operators in \mathfrak{U} with \mathfrak{U}_0 finite dimensional, then the following are equivalent:

- (a) Y is an adapted cocycle;
- (b) Y satisfies a stochastic differential equation of the type (8.1);
- (c) Y^\vee has a version which satisfies the equivalent conditions of Proposition 4.3.2.

Proof: (c) \Leftrightarrow (b): By the fundamental Theorem (3.2.5) the duality transform maps the unique solution of $\Delta z = qz$, $z_0(\emptyset) = \mathbb{1}_{\mathfrak{U}_0}$ to the unique solution of

$$dZ = Q^+ Z dA^+ + Q^- Z dA^- + Q^0 Z dt, \quad Z_0 = \mathbb{1}.$$

(c) \Rightarrow (a): By Proposition 6.3.7 the duality transform maps adapted kernel cocycles to adapted operator cocycles.

(a) \Rightarrow (c): Let Y be an adapted cocycle. Define $y_{s,t} \in \Sigma$ ($s \leq t$) by $y_{s,t} = \mathbb{W}_s(Y_{t-s})$, then we immediately have, for $r \leq s \leq t$, $a \in \mathbb{R}$

$$y_{s,t}(\omega) = y_{s-a,t-a}(\omega-a) \text{ for a.a. } \omega \quad (8.3)$$

in particular,

$$y_{s,t}(\emptyset) = y_{t-s}(\emptyset) \quad (8.4)$$

and, by Lemma 8.1.1,

$$y_{r,t}^{[r,t]} = y_{s,t}^{[s,t]} y_{r,s}^{[r,s]} \text{ a.e.} \quad (8.5)$$

which may be extended to

$$y_{r_0, r_n}^{[r_0, r_n]} = y_{r_{n-1}, r_n}^{[r_{n-1}, r_n]} y_{r_{n-2}, r_{n-1}}^{[r_{n-2}, r_{n-1}]} \dots y_{r_0, r_1}^{[r_0, r_1]} \text{ a.e.,} \quad (8.6)$$

where $r \in \mathbb{R}_{\leq}^{n+1} := \{r \in \mathbb{R}^{n+1} : r_0 \leq r_1 \leq \dots \leq r_n\}$. By (8.4) and (8.5) the map

$$p : \mathbb{R}_+ \rightarrow \mathfrak{U}_0, \quad t \rightarrow y_t(\emptyset)$$

is a semigroup which, by (aci) is continuous. Let q be its generator. Let

$(s_1, s_5) \in \mathbb{R}_{\leq}^2$, then by (8.6), for almost all $\{s_3^+\}$, $s_1 < s_3 < s_5$

$$y_{s_2, s_4}(\{s_3^+\}) = p_{s_5-s_4}^{-1} y_{s_1, s_5}(\{s_3^+\}) p_{s_2-s_1}^{-1}$$

for (s_2, s_4) outside a null set $N_{(s_1, s_3, s_5)}$, $s \in \mathbb{R}_{\leq}^5$. So

$$\lim_{\substack{s_2 \uparrow s_3, s_4 \downarrow s_3 \\ (s_2, s_4) \notin N}} y_{s_2, s_4}(\{s_3^+\})$$

exists, by the continuity of p , equals

$$p_{s_5-s_3}^{-1} y_{s_1, s_5}(\{s_3^+\}) p_{s_3-s_1}^{-1}$$

and moreover is clearly independent of (s_1, s_5) . Thus there is a map $q^+ : \mathbb{R} \rightarrow \mathcal{U}_0$ and a subset V of \mathbb{R}_{\leq}^3 of full Lebesgue measure such that

$$F(s) := p_{s_3-s_2}^{-1} y_{s_1, s_3}(\{s_2^+\}) p_{s_2-s_1}^{-1} = q^+(s_2) \quad (s \in V)$$

By covariance (8.3), $\{(s, a) \in \mathbb{R}_{\leq}^3 \times \mathbb{R} : F(s) = F(s-a)\}$ has full Lebesgue measure. Moreover,

$$\{(s, a) \in \mathbb{R}_{\geq}^3 \times \mathbb{R} : q^+(s_2-a) = F(s-a)\} = T(V \times \mathbb{R})$$

where T is the measure preserving map $(x, a) \rightarrow (x-a, a)$. Hence for almost all $(s, a) \in \mathbb{R}_{\leq}^3 \times \mathbb{R}$,

$$q^+(s_2-a) = F(s-a) = F(s) = q^+(s_2)$$

In particular, q^+ is almost everywhere constant. Applying the same argument to $\mathcal{P}^{0,1}$ we obtain $q^\pm \in \mathcal{U}_0$ satisfying

$$y_{s_1, s_3}^{[s_1, s_3]}(\{s_2^\pm\}) = p_{s_3-s_2} q^\pm p_{s_2-s_1} \quad \text{for a.a. } s \in \mathbb{R}_{\geq}^3.$$

Combining this with (8.6) gives

$$\begin{aligned} y_{r_0, r_n}((s, \varepsilon)) &= (p_{r_n-s_n} q^{\varepsilon_n} p_{s_n-r_{n-1}}) \cdots (p_{r_1-s_1} q^{\varepsilon_1} p_{s_1-r_0}) \\ &= p_{r_n-s_n} (q^{\varepsilon_n} p_{s_n-s_{n-1}} \cdots p_{s_2-s_1} q^{\varepsilon_1}) p_{s_1-r_0} \\ &\quad \text{for a.a. } (r_0, s_1, s_1, \dots, s_n, r_n) \quad \mathbb{R}_{\geq}^{2n+1}. \end{aligned}$$

and, using (8.6) again,

$$\begin{aligned}
y_t(\sigma) &= p_{t-v}\{y_{u,v}(\sigma)\}p_u \text{ for a.a. } \sigma \in \mathcal{P}_{[u,v]}, \quad 0 \leq u < v \leq t \\
&= p_{t-s_n}q^{\varepsilon_n}p_{s_n-s_{n-1}}\cdots p_{s_2-s_1}q^{\varepsilon_1}p_{s_1} \text{ for a.a. } (u,s,v) \in \mathbb{R}_{\geq}^{n+2}
\end{aligned}$$

Letting $u_n \downarrow 0$ and $v_n \downarrow 0$ with care, we see that y has the form (4.12) and the proof is complete. ■

8.2 The reduced semigroup.

Proposition 8.2.1: If X is an adapted cocycle, then the family of operators $\{R_t := \mathbb{I}\!\!\mathbb{I}_{-t}X_t, t \geq 0\}$ satisfies

$$R_{s+t} = R_s R_t \quad (8.7)$$

Proof:

$$\begin{aligned}
R_{s+t} &= \mathbb{I}\!\!\mathbb{I}_{-s-t}\mathbb{I}\!\!\mathbb{I}_t(X_s)X_t \\
&= \mathbb{I}\!\!\mathbb{I}_{-s-t}\mathbb{I}\!\!\mathbb{I}_{s+t}\mathbb{I}\!\!\mathbb{I}_{-s}X_s\mathbb{I}\!\!\mathbb{I}_{-t}X_t \\
&= R_s R_t
\end{aligned}$$
■

In view of Propositions 4.2.1 and 4.4.1 and Corollary 6.3.8 the following is immediate.

Proposition 8.2.2: An adapted cocycle X is unitary valued if and only if its generator has the form $(V, -V^*, -\frac{1}{2}(c_+V^*V + c_-VV^*) + iH)$ where $H = H^*$. In this case the one parameter semigroup T^0 on \mathcal{U}_0 given by

$$j \circ T_t^0 = E_{\{0\}} \circ \text{Ad } X_t \circ j$$

(consisting of completely positive maps) has generator $L_v - i[H, \cdot]$.

T^0 is called the *reduced semigroup* of the adapted cocycle X .

9. Quantum Markov Processes

In this section we shall be concerned with the construction of dilations using Bose noise, and the characterization of the class of quantum dynamical

semigroups which admit such a dilation (cf. [ApF]).

9.1 Multidimensional Bose Noise

Let $\Gamma(I; l)$ denote the *charged and coloured finite power set of I with l colours*:

$$\Gamma(I; l) = \bigcup_{N=0}^{\infty} \Gamma^N(I) \times \{1, \dots, l\}^N.$$

let ρ_0 be the metric on $\Gamma(I; l)$ given by

$$\rho_0((s, \varepsilon, k), (s', \varepsilon', k')) = \begin{cases} 0 & \text{if both equal } \emptyset \\ 1 \wedge \max_i |s_i - s'_i| & \text{if } (\varepsilon, k) = (\varepsilon', k') \\ 1 & \text{otherwise} \end{cases}$$

and let $(\mathcal{V}(I; l), \rho)$ be the completion of $(\Gamma(I; l), \rho_0)$. The constants c_{\pm} will now be *colour dependent*:

$$c_{\pm, j} = c_{\pm}(\beta_j, h_j), \quad j = 1, \dots, l,$$

where $(\beta, h) \in (0, \infty)^l \times [0, \infty)^l$ is fixed and the functions c_{\pm} are determined by continuity and the relations

$$c_+(\beta, h) - c_-(\beta, h) = h; \quad c_+(\beta, h)/c_-(\beta, h) = e^{\beta h}.$$

Explicitly:

$$c_-(\beta, h) = \begin{cases} h(e^{\beta h} - 1)^{-1} \\ \beta^{-1} \end{cases}; \quad c_+(\beta, h) = \begin{cases} h(1 - e^{-\beta h})^{-1} & (h > 0) \\ \beta^{-1} & (h = 0). \end{cases}$$

The origin of this parametrisation of the pair (c_+, c_-) by (β, h) lies in physics: When a small quantum-mechanical system, such as an atom, is coupled to the electromagnetic field (a Bose field) it is usually sensitive only to certain small spectral regions of this field around frequencies ν_j ($j = 1, \dots, l$), say. When the field is at a temperature T , these regions are described to a good approximation by copies of Bose noise with parameters

$$\beta_j = 1/kT \quad \text{and} \quad h_j = \hbar \nu_j,$$

where \hbar and k are the constants of Planck (divided by 2π) and Boltzmann

respectively. The j -dependence of β_j leaves open the possibility of coupling to fields of different temperatures. A classical field corresponds to Bose noise with $\hbar = 0$.

Let $\mu = \mu^{\beta, \hbar}$ be the Borel measure on $\mathcal{V}(I; l)$ given by $d\mu = m d\lambda$ where

$$m((s, \varepsilon, k)) = \prod_j c_{\varepsilon_j, k_j}. \quad (\text{cf. 2.1})$$

A kernel calculus may be constructed on $(\Gamma(I; l) \mu^{\beta, \hbar})$ as before, involving operators $I^{\varepsilon, k}, I^\circ, \Delta^{\varepsilon, k}$ and Δ° ($\varepsilon = \pm, k = 1, \dots, l$), and a quantum probability space $\mathcal{Q}_1^{\beta, \hbar} = (L^2(d\mu^{\beta, \hbar}), \mathcal{N}^{\beta, \hbar}, \delta_{\mathcal{Q}}^{\beta, \hbar})$. Corresponding to Propositions 8.1.2 and 8.2.2 we have:

Proposition 9.1.1: For a family $\{Y_t : t \geq 0\}$ of operators in $\mathfrak{U}^{\beta, \hbar}$, the following are equivalent:

(a) Y is an $\mathcal{N}^{\beta, \hbar}$ -adapted cocycle

(b) Y satisfies an s.d.e. of the type

$$dY = \sum_{i=1}^l (Q^{+, i} dA_i^+ + Q^{-, i} dA_i^-) + Q^\circ dt; \quad Y_0 = I \quad (Q^\lambda \in \mathfrak{U}_{(0)})$$

(c) $Y = \hat{y}$ where y satisfies a k.d.e. of the type

$$\Delta y = qy; \quad y_0(\emptyset) = \mathbf{1}_{\mathfrak{X}_0} \quad (q \in \mathfrak{U}_0^{2l+1})$$

(d) $Y = \hat{y}$ where y is an $\mathcal{N}^{\beta, \hbar}$ -adapted kernel cocycle

(e) $Y = \hat{y}$ where

$$y_t(s, \varepsilon, k) = e^{(t-s_N)q} q^{\varepsilon_N, k_N} e^{(s_N-s_{N-1})q} \dots q^{\varepsilon_1, k_1} e^{s_1 q} \quad \text{for } t \geq s_N$$

as long as *either* \mathcal{Q}_0 is finite dimensional, *or* Y is unitary valued.

Proposition 9.1.2: The $\mathcal{N}^{\beta, \hbar}$ -adapted *unitary* cocycles are those with generator Q of the form

$$Q^{+, k} = V_k, \quad Q^{-, k} = -V_k^*, \quad k = 1, \dots, l,$$

$$Q = iE - \frac{1}{2} \sum_{k=1}^l \{c_+(\beta_k, h_k) V_k^* V_k + c_-(\beta_k, h_k) V_k V_k^*\}, \quad (E = E^*).$$

The generator of the corresponding reduced semigroup on \mathfrak{U}_0 is then

$$i[\cdot, e] + \sum_{k=1}^l L_{v_k}^{\beta_k, h_k} \quad (\text{where } j(e) = E, \quad j(v_k) = V_k).$$

We write $L_v^{\beta, h}(b)$ for $c_+(\beta, h)[v^*v - \frac{1}{2}\{v^*v, b\}] + c_-(\beta, h)[v b v^* - \frac{1}{2}\{v b v^*, b\}]$ where $\{.,.\}$ is the anti-commutator $a, b \mapsto ab + ba$.

9.2 Bose dilations

Lemma 9.2.1: Let Y be an $\mathcal{N}^{\beta, h}$ -adapted cocycle with generator $q \otimes \mathbf{1}_{\mathcal{N}}$. If either Y is unitary valued, or Q_0 is finite dimensional, then for each $t \in \mathbb{R}$,

$$\sigma_t^{\beta, h}(Y_s) = Y_s \quad \forall s \geq 0 \quad \Leftrightarrow \quad \begin{cases} \sigma_t^0(q) = q \\ \sigma_t^0(q^{\varepsilon, j}) = e^{it\beta_j h_j} q^{\varepsilon, j} \end{cases} \quad (\varepsilon = \pm, j = 1, \dots, l).$$

Proof: The modular automorphism group $\sigma^{\beta, h}$ is given by (6.3) where $\gamma := m^\dagger/m$ is the map

$$(r, \varepsilon, k) \rightarrow e^{-\sum_j \varepsilon_j \beta_j h_j}$$

so that

$$\begin{aligned} \sigma_t(Y_s) &= Y_s \quad \forall s \geq 0 \\ &\Leftrightarrow e^{-it \sum_j \varepsilon_j \beta_j h_j} \sigma_t^0(e^{(s-r_N)q} q^{\varepsilon_N, k_N} \dots q^{\varepsilon_1, k_1} e^{r_1 q}) \\ &= e^{(s-r_N)q} q^{\varepsilon_N, k_N} \dots q^{\varepsilon_1, k_1} e^{r_1 q} \quad \forall \varepsilon, k \text{ and } s \geq r_N \geq \dots \geq r_1 \\ &\Leftrightarrow \sigma_t^0(q) = q, \quad \sigma_t^0(q^{\varepsilon, j}) \quad (\varepsilon = \pm, j = 1, \dots, l). \end{aligned}$$

■

Proposition 9.2.2: Let U be an $\mathcal{N}^{\beta, h}$ -adapted unitary cocycle with generator $Q = q \otimes \mathbf{1}_{\mathcal{N}}$. Then

$$t \mapsto (\text{Ad } U_t) \circ \omega_t \quad (t \geq 0)$$

determines a quantum dynamical group T on $Q^{\beta, h}$ if and only if

$$\sigma_t^0(q^0) = q^0, \quad \sigma_t^0(q^{\varepsilon, j}) = e^{it\beta_j h_j} q^{\varepsilon, j}, \quad (\varepsilon = \pm, j = 1, \dots, l).$$

In this case the reduced semigroup T^0 of U is a quantum dynamical semigroup on Q_0 admitting the Markov dilation (Q, j, \mathbb{P}, T) where $j: b \rightarrow b \otimes \mathbf{1}_{\mathcal{H}}$ and \mathbb{P} is characterized by $\mathbb{P}(X) \otimes \mathbf{1} = \mathbb{E}_{\{0\}}^{\beta, h}[X]$.

Proof: Since ξ is invariant under the shift on $L^2(\Gamma(I; l), \mathfrak{h}_0)$, $\{\mathbb{W}_t: t \in \mathbb{R}\}$ is a quantum dynamical group on Q . By Proposition 8.2.1, T satisfies the semigroup property, so the equivalence follows from (5.5) and the lemma. Each T_t^0 is then a composition of morphisms:

$$T_t^0 = \mathbb{P} \circ \text{Ad } U_t \circ j = \mathbb{P} \circ T_t \circ j \quad (\text{since } \mathbb{W}_t \circ j = j),$$

so T^0 is a quantum dynamical semigroup and T a dilation of T^0 . The Markov property follows from (7.5). ■

Definition 9.2.3: A (Markov) dilation (Q, j, \mathbb{P}, T) of a quantum dynamical semigroup T^0 on Q^0 is a *Bose dilation* if for some (β, h)

$$Q = Q^0 \otimes Q_1^{\beta, h}; \quad j(b) = b \otimes \mathbf{1}; \quad T_t = \text{Ad } U_t \circ \mathbb{W}_t$$

where U is an $\mathcal{N}^{\beta, h}$ -adapted unitary cocycle.

Proposition 9.2.4: Let T^0 be a quantum dynamical semigroup on Q^0 with generator L . The following are equivalent:

(a) T^0 admits a Bose dilation

$$(b) \quad L = i[e, \cdot] + \sum_{k=1}^l L_{v_k}^{\beta_k, h_k} \quad (9.1)$$

for some $(l, \beta, h, e = e^*, v)$ satisfying $\sigma_t^0(e) = e$,

$$\sigma_t^0(v_k) = e^{it\beta_k h_k} v_k \quad (k = 1, \dots, l). \quad (9.2)$$

Proof: (a) \Rightarrow (b): This is immediate from Propositions 9.1.2 and 9.2.2.

(b) \Rightarrow (a): Let L be determined by $(l, \beta, h, d = d^*, v)$ as in (9.1) then the $\mathcal{N}^{\beta, h}$ -adapted unitary cocycle with generator q where

$$q^\circ = \text{id.} - \frac{1}{2} \sum_{k=1}^l \{c_+(\beta_k, h_k) v_k^* v_k + c_-(\beta_k, h_k) v_k v_k^*\}$$

$$q^{+,j} = v_j, \quad q^{-,j} = -v_j^* \quad (j = 1, \dots, m).$$

leads to a Bose dilation, since $v_k^* v_k$ and $v_k v_k^*$ are fixed under σ_t° , by Proposition 9.2.2. ■

9.3 Detailed Balance

In the physics literature one finds the condition of *detailed balance* for the transition probabilities between the energy levels of a quantum-mechanical system. This condition says that the transition probabilities between any pair of levels balance each other in the equilibrium state of the whole system. For a long time it was believed that detailed balance was a necessary condition for the dynamical semigroup determined by these transition probabilities to be physically realisable, i.e. to possess a dilation. Although this belief is now known to be erroneous in general ([KüM], [FrM]), we shall prove it correct for the case of dilations using Bose noise.

Definition 9.3.1: A norm continuous quantum dynamical semigroup T° on Q° with generator L satisfies *detailed balance* if there is a quantum dynamical semigroup S° on Q° satisfying

$$(db\ i) \quad \langle S_t^\circ(a) \xi_0, b \xi_0 \rangle = \langle a \xi_0, T_t^\circ(b) \xi_0 \rangle \quad \forall a, b \in \mathfrak{U}_0, t \geq 0$$

(T° has a Q° -adjoint) and whose generator M satisfies

$$(db\ ii) \quad L - M = i[e, \cdot] \text{ for some } e = e^* \text{ in } \mathfrak{U}_0.$$

Remarks:

1. T° has a Q° -adjoint if and only if it commutes with the modular automorphism group σ^{Q° .
2. If T° has a dilation (Q, j, \mathbb{P}, T) then it has a Q° -adjoint, namely $\mathbb{P} \circ T_{-t} \circ j$ ($t \geq 0$).

Proposition 9.3.2: Let T^0 be a quantum dynamical semigroup on Q^0 . Suppose that \mathfrak{U}_0 is finite dimensional, and a factor (i.e. has trivial centre). Then T^0 admits a Bose dilation if and only if T^0 satisfies detailed balance.

Proof: Let T be a Bose dilation of T_t^0 given by the cocycle U with generator $q \otimes \mathbf{1}$. The generator of $\mathbb{P} \circ T_{-t} \circ j = \mathbb{P} \circ \text{Ad} U_t^* \circ j$ is M_q (see 4.7) and since $L_q - M_q = [q - q^*, \cdot]$, T^0 satisfies detailed balance.

Conversely ([Ali]) suppose T^0 satisfies detailed balance. \mathfrak{U}_0 , being a finite dimensional matrix algebra, which is also a factor, is isomorphic to $M_n(\mathbb{C}) \times I_m$ for some n, m , where I_m is the $m \times m$ identity matrix. The generator L of the completely positive semigroup T_0 is therefore expressible in the form ([Lin], [GKS])

$$i[u, \cdot] + \sum_{k=1}^p (v_k^* \cdot v_k - \frac{1}{2}\{v_k^* v_k, \cdot\})$$

for some $p \in \mathbb{N}$, $u = u^*$, $v_k \in \mathfrak{U}$, $k = 1, \dots, p$. Let $\sum_{\chi \in \text{sp}(\ln \Delta_0)} e^{\chi} P_\chi$ be the spectral decomposition of Δ_{Q^0} and define x_χ and $v_{\chi,k}$ ($k = 1, \dots, p$) in \mathfrak{U}_0 by

$$u_\chi \xi_0 = P_\chi u \xi_0$$

$$v_{\chi,k} \xi_0 = P_\chi v_k \xi_0$$

then,

$$\sigma_t^0(u_\chi) = e^{i\chi t} u_\chi$$

$$\sigma_t^0(v_{\chi,k}) = e^{i\chi t} v_{\chi,k} \quad (9.3)$$

for $\chi \in \text{sp}(\ln \Delta_0)$, and, since L commutes with σ^0 , we have

$$\begin{aligned} L &= \lim_{T \rightarrow \infty} (2T)^{-1} \int_{-T}^T \sigma_t^0 \circ L \circ \sigma_{-t}^0 dt \\ &= \lim_{T \rightarrow \infty} (2T)^{-1} \int_{-T}^T \left[i \sum_{\chi} e^{i\chi t} [u_\chi, \cdot] \right. \\ &\quad \left. + \sum_{k=1}^p \sum_{\chi, \chi'} e^{i(\chi - \chi')t} (v_{\chi',k}^* \cdot v_{\chi,k} - \frac{1}{2}\{v_{\chi',k}^* v_{\chi,k}, \cdot\}) \right] dt \\ &= i[u_0, \cdot] + \sum_{k,\chi} (v_{\chi,k}^* \cdot v_{\chi,k} - \frac{1}{2}\{v_{\chi,k}^* v_{\chi,k}, \cdot\}) \end{aligned} \quad (9.4)$$

Elementary manipulations with the Tomita operators, such as

$$v_{\chi,k}\xi_0 = e^{-\chi}\Delta_0 v_{\chi,k}\xi_0, \quad v_{\chi,k}^* v_{\chi,k}\xi_0 = \Delta v_{\chi,k}^* v_{\chi,k}\xi_0$$

yield the identities

$$\langle a\xi_0, i[u_0, b]\xi_0 \rangle = \langle -i[u_0, a]\xi_0, b\xi_0 \rangle$$

$$\langle a\xi_0, v_{\chi,k}^* b v_{\chi,k}\xi_0 \rangle = \langle e^{-\chi} v_{\chi,k} a v_{\chi,k}^* \xi_0, b\xi_0 \rangle$$

$$\langle a\xi_0, \{v_{\chi,k}^* v_{\chi,k}, b\}\xi_0 \rangle = \langle \{v_{\chi,k}^* v_{\chi,k}, a\}\xi_0, b\xi_0 \rangle$$

for $a, b \in \mathfrak{U}_0$. Thus the generator L^* of the Q^0 -adjoint semigroup equals

$$-i[u_0, \cdot] + \sum_{k,\chi} [e^{-\chi} v_{\chi,k} \cdot v_{\chi,k}^* - \frac{1}{2}\{v_{\chi,k}^* v_{\chi,k}, \cdot\}].$$

Since $L^*(1)$ must vanish, we have

$$L^* = -i[u_0, \cdot] + \sum_{k,\chi} e^{-\chi} [v_{\chi,k} \cdot v_{\chi,k}^* - \frac{1}{2}\{v_{\chi,k} v_{\chi,k}^*, \cdot\}] \quad (9.5)$$

By (db ii)

$$L - L^* = i[d, \cdot] \text{ for some } d = d^* \in \mathfrak{U}_0 \quad (9.6)$$

and, since L and L^* commute with σ^0 , $\sigma_t^0(d) - d$ lies in the centre of \mathfrak{U}_0 , hence

$$\sigma_t^0(d) = d \quad (t \in \mathbb{R}) \quad (9.7)$$

Since $L = \frac{1}{2}(L + L^*) + \frac{1}{2}(L - L^*)$ and $c_-(\beta, h)/c_+(\beta, h) = e^{-\beta h}$, we see, by combining (9.3–9.7) with Proposition 9.6, that T^0 admits a Bose dilation. ■

10. The Wigner-Weisskopf Atom

10.1 Description

Let $M_2(\mathbb{C})$ be the algebra of observables of an atom possessing two energy levels: a higher level 1 and a lower level 2. The equilibrium state on $M_2(\mathbb{C})$ at inverse temperature β is given by

$$\varphi_\beta : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{e^{-\beta h} a + d}{e^{-\beta h} + 1}$$

where $\hbar = h/2\pi$ and h is Plank's constant. Working in units for which $\hbar = 1$ the appropriate quantum probability space \mathcal{Q}_0 is $(\mathfrak{h}_0, \mathcal{U}_0, \xi_0)$ where

$$\mathfrak{h}_0 = M_2(\mathbb{C}) \text{ with inner product } (x, y) \rightarrow \varphi_\beta(x^*y)$$

$$\xi_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\mathcal{U}_0 = \pi(M_2(\mathbb{C}))$ where $\pi: M_2(\mathbb{C}) \rightarrow \mathcal{L}(\mathfrak{h}_0)$ is the representation given by

$$a_\pi = \pi(a): x \rightarrow ax.$$

Note that \mathfrak{h}_0 is isomorphic to $\mathbb{C}^2 \oplus \mathbb{C}^2$ with the inner product

$$((\psi_1, \psi_2), (\chi_1, \chi_2)) \rightarrow \frac{e^{-\beta}}{1+e^{-\beta}} \langle \psi_1, \chi_1 \rangle + \frac{1}{1+e^{-\beta}} \langle \psi_2, \chi_2 \rangle,$$

ξ_0 corresponding to $\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$ under this isomorphism. The modular group $\sigma_t^0 := \sigma_t^{\mathcal{Q}_0}$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}_\pi \mapsto \begin{pmatrix} a & e^{-i\beta t} b \\ e^{i\beta t} c & d \end{pmatrix}_\pi$$

The *lowering operator* of the atom $v := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_\pi$ clearly satisfies (9.2) (with $l = 1$) and $e^{\mathcal{U}_v}$ is given by

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix}_\pi &\mapsto \frac{e^{-\beta} a + d}{e^{-\beta} + 1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_\pi + e^{-t \coth \beta/2} \frac{a-d}{e^{-\beta} + 1} \begin{pmatrix} 1 & 0 \\ 0 & e^{-\beta} \end{pmatrix}_\pi \\ &+ e^{(-t/2) \coth(\beta/2)} \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix}_\pi \end{aligned}$$

This quantum dynamical semigroup is known as the approach to thermal equilibrium of the two-level atom coupled to a quantum field at inverse temperature β , a system first described by Wigner and Weisskopf ([WeW]). The construction of Proposition 9.2.2 immerses this atom in the flow of a Bose noise $\mathcal{Q}_1(\mathbb{R})$ governed by the shift \mathcal{M}_t . The development in time of the

coupled system

$$T_t : X \mapsto U_t^* \mathbb{M}_t(X) U_t = U_t^* \mathbb{M}_t X \mathbb{M}_{-t} U_t$$

can be described, in the *Schrodinger picture*, by the unitary group $R_t = \mathbb{M}_{-t} U_t$ for $t \geq 0$ and $U_{-t}^* \mathbb{M}_{-t}$ for $t < 0$.

10.2 Interpretation of \mathfrak{h}

The natural configuration space for the two-level atom is $\{1, 2\}$ and a pure state of the atom is described by a *wave function* $\psi \in L^2(\{1, 2\}) = \mathbb{C}^2$, associating the expectation $\langle \psi, b\psi \rangle$ to the observable $b \in M_2(\mathbb{C})$. However, the atom may be found in a mixed state, i.e. a convex combination of pure states, such as $\phi_\beta = (1 + e^{-\beta})^{-1}(e^{-\beta}\langle e_1, \cdot e_1 \rangle + \langle e_2, \cdot e_2 \rangle)$. One may then think of it as residing in one of these pure states, each with a probability given by its coefficient. This may be substantiated by representing the observable algebra of the atom on $\mathbb{C}^2 \oplus \mathbb{C}^2 \cong \mathfrak{h}_0$ as was done above. Since in quantum mechanics one is free to represent the Hilbert space of a system as $L^2(\Omega, \mu)$ for different choices of (Ω, μ) , thereby obtaining equally valid *configuration spaces* (Ω, μ) , we may now call $\{1, 2\} \times \{1, 2\}$ a configuration space of the atom. The state ϕ_β is then given by the wave function ξ_0 on $\{1, 2\}^2$. When in this state the atom may be *excited* by the operator v^* to $v^*\xi_0 = \pi(e_{12})$, or *de-excited* by v to $v\xi_0 = \pi(e_{21})$. Further excitation of $v^*\xi_0$ or de-excitation of $v\xi_0$ is not possible, but $vv^*\xi_0 = \pi(e_{22})$ and $v^*v\xi_0 = \pi(e_{11})$ are permissible wave functions (when normalised). We recover ξ_0 as the superposition $v^*v\xi_0 + vv^*\xi_0$.

On the other hand Γ is a configuration space for the noise in thermal equilibrium at inverse temperature β . Its interpretation resembles that for the atom: excitations added to the thermal background are described as points with positive charge, the negative charges standing for de-excitations or *particles* removed from the background. In this way we view $\mathfrak{h} = L^2(\{1, 2\}^2 \times \Gamma)$ as the space of wave functions of atom and noise.

10.3 The Schrodinger Evolution

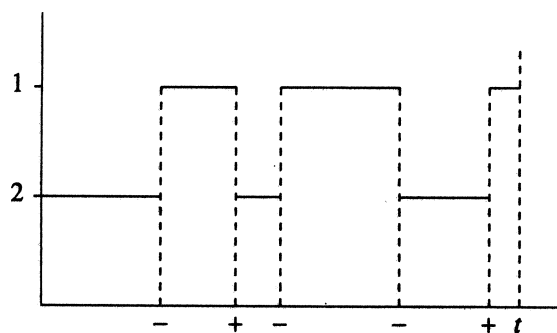
If at time 0 the system of atom and noise has the wave function ψ , then at a later time t its wave function will be $R_t\psi$, given by

$$(i, j, \sigma) \mapsto \sum_{\alpha \subset \sigma + t} \sum_{k \in \{1, 2\}} \int_{\Gamma} u_i(\alpha \cup \omega^\dagger)_{ik} \psi(\bar{\alpha} \cup \omega)_{kj} \, d\omega \quad (10.1)$$

where $u \in \mathcal{P}_1([0, \infty))$ is the adapted kernel cocycle with generator q given by

$$q^+ = v, \quad q^- = -v^*, \quad q^0 = -\frac{1}{2}(1 + e^{-\beta})^{-1} \begin{pmatrix} e^{-\beta} & 0 \\ 0 & 1 \end{pmatrix}_\tau$$

The cocycle may be represented diagrammatically: for $\sigma = (s, \varepsilon)$, $u_t(\sigma)_{ij} = 0$ unless the charges $\varepsilon_1, \dots, \varepsilon_n$ alternate and have sum $(i - j)$, in which case $u_t(\sigma)_{ij} = \exp\{-\frac{1}{2}(1 + e^{-\beta})^{-1}[l_1 + e^{-\beta}l_2]\}$, where l_1 (*respectively* l_2) is the total length of the higher (lower) plateaux in the following diagram.



The case $(i, j) = (1, 2)$

The Schrödinger evolution R_t has the following interpretation: (10.1) expresses $R_t\psi$ in terms of ψ by summation over all configurations $(k, j, \bar{\alpha} \cup \omega)$ which may lead to (i, j, σ) by the combined effect of a left shift by t and an interaction with the atom. The atom, located at the origin, can emit bosons (here written as α) and absorb others (written ω), leaving a part $\bar{\alpha} = \sigma + t \setminus \alpha$ of the initial configuration intact. The shift then takes the result $\sigma + t$ to σ . The adaptedness of u ensures that only those particles which pass the origin during the time interval $[0, t]$ may be absorbed, and only such are emitted which end up in the *space* interval $[-t, 0]$.

The value $u_i(\alpha \cup \beta^\dagger)_{ik}$ of the coefficient in the summation (10.1) is understood by considering (10.2) as a picture of a possible history of excitations and de-excitations during the time interval $[0, t]$.

Symbol	Description	Subsection
\mathcal{L}^0		0
\mathcal{B}	Borel σ -algebra	0
Γ	charged finite power set	2.1.1
\mathcal{V}	completion of above	2.1
μ	measure on \mathcal{V}	2.1
$c_+ \geq c_- > 0$	fixed constants	2.1
$(\mathcal{A}, *)$	fixed involutive Banach algebra	2.1
\dagger	involution on \mathcal{V} , μ and $\mathcal{F}(\mathcal{V}; \mathcal{A})$	2.1.2
$\mathcal{K}_b^{\mathcal{A}}, \mathcal{K}_o^{\mathcal{A}}$	bounded, smooth kernels	2.2
π_f, w_f, a_f	various kernels	2.2.2
$\hat{\mu}$	quasi-free characteristic function	2.2.2
$*$	bose convolution product	2.3.2
$\gamma_{a,b}, \gamma_c$		2.4.1
Γ_o, J_o, S_o	modular operators	2.4.1
G_y, D_x	left, right multiplication operators	2.4.2
ζ	symplectic form	2.4.4
v_f	commutant Weyl kernel	2.4.4
\mathcal{W}, \mathcal{V}	Weyl, commutant Weyl algebras	2.5.0
ν	$\mu + \mu^\dagger$	2.5.3
I_t	$I \cap (-\infty, t)$	
$\Gamma_{\text{ad.}}$	adapted simplex	3.0
Γ^+	$\{\max \sigma \text{ has } + \text{ charge}\}$	3.0
k_x		3.1
$\mathcal{P}_o^{\mathcal{A}}$	smooth adapted kernel process	3.1.1
$\Delta^\kappa, \kappa = 0, +, -$	differential operators	3.2.1
I^κ	integral operators	3.2.4
Δ	$(\Delta^+, \Delta^-, \Delta^o)$	4.1
III_t	Shift	4.3.0
\mathcal{Q}	quantum probability space	5.2
$\sigma^{\mathcal{Q}}$	modular automorphism group	5.2
$S_{\mathcal{Q}}, J_{\mathcal{Q}}, \Delta_{\mathcal{Q}}$	modular operators	5.2
$\eta^*(\mathcal{Q})$	*-affiliated operators	5.3.1
X^+	conjugation on $\eta^*(\mathcal{Q})$	5.3

\hat{x}, X^\vee	non-commutative duality	6.3.6
p_0, p_1, l^2, m	kernel processes	6.7.1
s, c	more kernel processes	7.2
E	conditional expectation	7.3.3
\mathbb{M}_t	shift	8.0
\hbar, K	Planck, Boltzmann constants	9.1.0

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Mathematics Department
University Park
GB-Nottingham NG7 2RD

Mathematisch Instituut
Katholieke Universiteit
NL-6525 ED Nijmegen