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The *q*-harmonic oscillator in a lattice model

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We give an explicit proof of the pair partitions formula for the moments of the q-harmonic oscillator, and of the claim made by Parisi that the q-deformed lattice Laplacian on the d-dimensional lattice tends to the q-harmonic oscillator in distribution for $d \to \infty$. © *1998 American Institute of Physics.*
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I. INTRODUCTION

In 1994, Parisi published his paper ''D-dimensional arrays of Josephson junctions, spin glasses and q-deformed harmonic oscillators'' (Ref. 1). It describes a lattice model (to be called the ''Parisi model'' here) that shows a connection with the q-harmonic oscillator. We prove some of the claims and conjectures made by Parisi after setting the stage for them in some detail.

We shall work within the framework of noncommutative probability theory or ''quantum probability'' (see, for example, Refs. 2–6), to be described briefly in Sec. II. A fine example of noncommutative probability is free probability theory, as discovered by Voiculescu.

Here we are concerned with an interpolation between the classical Gaussian distribution and the ''free Gaussian'' distribution: Wigner's semicircle law. This is done by q deforming the quantum mechanical harmonic oscillator. In Sec. III we show how the q-deformed Fock space, as introduced by Bożejko and Speicher in Refs. 7 and 8, can be constructed in an algebraic way, starting from the q-deformed commutation relations. It is shown that the moments of the ground state distribution of the q-harmonic oscillator can be calculated as a sum over pair partitions interpolating nicely between the well-known moment formulas of the Gauss and Wigner distributions.

In Sec. IV we prepare for the Parisi model by introducing the d-dimensional lattice, $\mathcal{T}(N,d)$, and its boundary and coboundary operators. The Parisi model itself is introduced in Sec. V. Section VI shows how we can use pair partitions to describe the walks on $\mathcal{T}(N,d)$ that turn out to be relevant for the Parisi model. Using the possibility of describing walks in terms of partitions, we show in Sec. VII that the position operator of a q-harmonic oscillator has the same distribution as the normalized q-deformed lattice Laplacian in the Parisi model if we let $d \to \infty$.

In Sec. VIII we show that, in fact, sequences of independent q-Gaussians are present in the Parisi model, although on the grounds of Ref. 9 we hesitate to call them ''q-independent.''

II. NONCOMMUTATIVE PROBABILITY AND INDEPENDENCE

In this section we shall give a brief outline of noncommutative probability theory and the free probabilistic example.

We shall first describe the transition from classical to noncommutative probability theory.[3,4] This transition follows much the same recipe as the search for ''noncommutative versions'' of mathematical objects as initiated by Connes.[10] A classical probability space will be, following the axiomatic approach outlined by Kolmogorov, a triple $(\Omega,\Sigma,\mathbb{P})$, where Ω is a state space, Σ is a σ-algebra of events, and $\mathbb{P}$ is a probability measure on Σ. The transition of classical probability to noncommutative probability theory is preceded by a replacement of functions by multiplication operators: on $(\Omega,\Sigma,\mathbb{P})$ we can construct a commutative algebra of functions together with a state that contains the same information as $(\Omega,\Sigma,\mathbb{P})$ itself. Problems defined on the space can then be translated to algebraic problems and worked out in the commutative algebra of functions. Then we

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drop the commutativity requirement on the algebra. The noncommutative algebra can no longer be identified with a space, but it still represents some kind of probability theory.

Definition 1: A general probability space is a unital von Neumann algebra, $\mathcal{A}$, together with a normal state φ.

The events in a general probability space are the projections $p=p^*=p^2$ in $\mathcal{A}$, and the probability that p occurs is $\varphi(p)$. A self-adjoint element $X\in\mathcal{A}$ will be called a random variable and $\varphi(X)$ its expectation. The distribution, μ_X, of a random variable $X\in\mathcal{A}$ will be a linear functional on the ring of complex polynomials in one variable, $\mu_X:\mathbb{C}[x]\to\mathbb{C}$, that satisfies $\mu_X(f)=\varphi(f(X))$ for every $f\in\mathbb{C}[x]$. Such a linear functional uniquely determines a probability measure with compact support on the real line, which is also denoted by μ_X.

As an illustration we shall reconstruct the classical probability space from a commutative general probability space $(\mathcal{A},\varphi)$. First, if $(\Omega,\Sigma,\mathbb{P})$ is a classical probability space, then the associated general probability space $(\mathcal{A},\varphi)$ is $\mathcal{A}:=L^\infty(\Omega,\Sigma,\mathbb{P})$ and $\varphi(f):=\int f d\mathbb{P}$, the expectation of f with respect to $\mathbb{P}$. Now, we can *re*construct the events in Σ by considering all projections $p\in\mathcal{A}$, which are of the form $p=1_S$ for some $S\in\Sigma$. Up to equivalence we can then reconstruct the concrete realization of Ω we had.

In this paper we choose to extend the classical definition of independence in the following way.

Definition 2 (independence): Random variables $X_1,\ldots,X_m$ in a general probability space $(\mathcal{A},\varphi)$ are called independent if for all $n\leqslant m$ and polynomials $f_1,\ldots,f_n$, we have

$$\varphi(f_1(X_{k(1)})\cdots f_n(X_{k(n)}))=\varphi(f_1(X_{k(1)}))\cdots\varphi(f_n(X_{k(n)})), \tag{1}$$

provided $k\in\{1,...,m\}^n$ has all components $k(i)$, $1\leqslant i\leqslant n$, different from each other.

A completely noncommutative notion paralleling the idea of independence is the concept of free independence of random variables.

Definition 3 (free independence): Random variables $X_1,\ldots,X_m$ in a general probability space $(\mathcal{A},\varphi)$ are called freely independent if for all $n\in\mathbb{N}$ and polynomials $f_1,\ldots,f_n$ we have

$$\forall i\leqslant n:\varphi(f_i(X_{k(i)}))=0\Rightarrow\varphi(f_1(X_{k(1)})\cdots f_n(X_{k(n)}))=0,$$

provided $k\in\{1,...,m\}^n$ satisfies $k(1)\neq k(2)\neq k(3)\neq\cdots\neq k(n)$.

Proposition 4: Free independence implies independence.

Proof: Suppose $X_1,\ldots,X_m$ are freely independent. Note that (1) is valid for $n=1$. Suppose that (1) is valid for all polynomials $f_1,\ldots,f_{n-1}$ and different indices $k(1),\ldots,k(n-1)$. Now let $f_1,\ldots,f_n$ be given polynomials and $k(1),\ldots,k(n)$ be different indices. By g_j we denote the difference between f_j and the constant $\varphi(f_j(X_{k(j)}))$ so that $\varphi(g_j(X_{k(j)}))=0$. Then free independence implies

$$\varphi(g_1(X_{k(1)})\cdots g_n(X_{k(n)}))=0.$$

Writing out this product, we find

$$\varphi(f_1(X_{k(1)})\cdots f_n(X_{k(n)}))=-\sum_{\gamma\subsetneq\{1,\ldots,n\}}(-1)^{n-\#\gamma}\left(\prod_{i\notin\gamma}\varphi(f_i(X_{k(i)}))\right)\varphi\left(\prod_{j\in\gamma}f_j(X_{k(j)})\right).$$

Since γ contains less than n elements, the expectation of $\Pi_{j\in\gamma}f_j(X_{k(j)})$ factorizes by the induction hypothesis. So all the terms in the sum are equal apart from their sign. Since the sum of these signs is 1, the result follows. □

III. q-HARMONIC OSCILLATORS AND THE q-DEFORMED FOCK SPACE

In this section we show how the q-deformed Fock space, as introduced in the papers of Bożejko and Speicher,[7,8] is found naturally, starting from the commutation relation. We show how the moments of the q-harmonic oscillator can be expressed in terms of pair partitions.

We start from the operators $\mathbf{1}$ and $a_1,a_2,\ldots,a_m$, generating a *-algebra $\mathcal{A}_q$ and satisfying

$$a_ia_j^*-qa_j^*a_i=\delta_{i,j}\mathbf{1},\quad q\in(-1,1). \tag{2}$$

On $\mathcal{A}_q$ we introduce a linear functional φ_q that we require to satisfy

$$\varphi_q(\mathbf{1})=1, \quad \varphi_q(a_i^* a_{\mathbf{k}}^{\epsilon})=0, \quad \varphi_q(a_{\mathbf{k}}^{\epsilon} a_i)=0, \tag{3}$$

for $i,j\in\{1,...,m\}$, $\mathbf{k}\in\{1,...,m\}^n$, $n\in\mathbb{N}$, and $\epsilon\in\{1,*\}^n$, i.e., a sequence with length n of 1's and $*$'s. So, by $a_{\mathbf{k}}^{\epsilon}$ we mean the ordered product:

$$a_{\mathbf{k}}^{\epsilon}=\prod_{i=1}^{n}\lambda_i, \quad \text{where } \lambda_i=\begin{cases} a_{\mathbf{k}(i)}, & \text{if } \epsilon(i)=1,\\ a_{\mathbf{k}(i)}^*, & \text{if } \epsilon(i)=*. \end{cases}$$

Next, we show the connection between pair partitions and $\varphi_q(a_{\mathbf{k}}^{\epsilon})$. Let $S=\{1,...,n\}$, for some $n\in\mathbb{N}$. A sequence,

$$\Pi=\{(\alpha_1,\beta_1),(\alpha_2,\beta_2),\ldots,(\alpha_{n/2},\beta_{n/2})\}, \quad \text{for } n \text{ even},$$

such that $\cup_{i=1}^{n/2}\{\alpha_i,\beta_i\}=S$, will be called a *pair partition* of S. A pair partition will be ordered in the sense that $\alpha_i<\beta_i$ for all $i\in\{1,...,n/2\}$ and $\alpha_1<\cdots<\alpha_{n/2}$. The collection of all such pair partitions of the set S will be denoted by $\mathcal{P}_2(S)$ or $\mathcal{P}_2(n)$. For n odd this is just the empty set. A crossing in Π is a subset of Π with two elements, $\{(\alpha_i,\beta_i),(\alpha_j,\beta_j)\}$, which satisfies either $\alpha_i<\alpha_j<\beta_i<\beta_j$ or $\alpha_j<\alpha_i<\beta_j<\beta_i$. The set of all crossings of a partition Π can be conveniently labeled by

$$c(\Pi):=\{(i,j)\,|\,1\leqslant i<j\leqslant n/2,\{(\alpha_i,\beta_i),(\alpha_j,\beta_j)\} \text{ is a crossing}\}.$$

Lemma 5: For all $n\in\mathbb{N}$, $n\geqslant 1$, $\epsilon\in\{1,*\}^n$, and $\mathbf{h}\in\{1,...,m\}^n$, we have

$$\varphi_q(a_{\mathbf{h}}^{\epsilon})=\sum_{\Pi\in\mathcal{P}_2(n)} q^{\#c(\Pi)}\prod_{(i,j)\in\Pi}\varphi_q(a_{\mathbf{h}(i)}^{\epsilon(i)}a_{\mathbf{h}(j)}^{\epsilon(j)}). \tag{4}$$

Note that in this notation (i,j) can stand for an element of Π or for a pair of elements. The context will make clear, which is meant.

A pair partition $\Pi\in\mathcal{P}_2(n)$ is compatible with ϵ, denoted $\Pi\sim\epsilon$, if pairs $(\alpha_i,\beta_i)\in\Pi$ are such that $\epsilon(\alpha_i)=1$ and $\epsilon(\beta_i)=*$ for all $i\in\{1,...,n/2\}$. Compatibility of Π with $\mathbf{h}$ means that for each pair $(\alpha_i,\beta_i)\in\Pi$ we have $\mathbf{h}(\alpha_i)=\mathbf{h}(\beta_i)$. Using this notion of compatibility we can rewrite relation (4) as follows:

$$\varphi_q(a_{\mathbf{h}}^{\epsilon})=\sum_{\substack{\Pi\in\mathcal{P}_2(n)\\ \Pi\sim\epsilon,\mathbf{h}}} q^{\#c(\Pi)}. \tag{5}$$

We shall now prove Lemma 5.

Proof: Suppose that $\epsilon\in\{1,*\}^n$ can be written as $(\sigma,1,*,\varrho)$, where $\sigma\in\{1,*\}^k$ and $\varrho\in\{1,*\}^l$, with $k+l+2=n$. Suppose, furthermore, that $\mathbf{i}\in\{1,...,m\}^k$ and $\mathbf{j}\in\{1,...,m\}^l$ with $(\mathbf{h}(1),\ldots,\mathbf{h}(n))=(\mathbf{i}(1),\ldots,\mathbf{i}(k),\mathbf{h}(k+1),\mathbf{h}(k+2),\mathbf{j}(1),\ldots,\mathbf{j}(l))$. Then the following holds:

$$\varphi_q(a_{\mathbf{h}}^{\epsilon})=\varphi_q(a_{\mathbf{i}}^{\sigma}a_{\mathbf{h}(k+1)}a_{\mathbf{h}(k+2)}^*a_{\mathbf{j}}^{\varrho})=\delta_{\mathbf{h}(k+1),\mathbf{h}(k+2)}\varphi_q(a_{\mathbf{i}}^{\sigma}a_{\mathbf{j}}^{\varrho})+q\,\varphi_q(a_{\mathbf{j}}^{\sigma}a_{\mathbf{h}(k+2)}^*a_{\mathbf{h}(k+1)}a_{\mathbf{j}}^{\varrho}).$$

This relation, together with (3), determines the left-hand side of (4), and therefore φ_q, completely.

We shall refer to the right-hand side of (5) as $F(a_{\mathbf{h}}^{\epsilon})$ putting $F(\mathbf{1})=1$, and shall show that F satisfies the same recursion relation. Let $\Pi\in\mathcal{P}_2(n)$. Then, for every pair $(i,j)\in\Pi$ we have $\varphi_q(a_{\mathbf{h}(i)}^{\epsilon(i)}a_{\mathbf{h}(j)}^{\epsilon(j)})\neq 0$ if and only if $\epsilon(i)=1$ and $\epsilon(j)=*$ and $\mathbf{h}(i)=\mathbf{h}(j)$. There are now two possibilities: $(k+1,k+2)\in\Pi$ or $(k+1,k+2)\notin\Pi$. In the first case $\Pi':=\Pi\setminus\{(k+1,k+2)\}$ is a partition of $\{1,...,k,k+3,...,n\}$ with $\#c(\Pi)=\#c(\Pi')$. In the second case there must be $i<k+1$ and $j>k+2$ such that $(i,k+2)$ and $(k+1,j)$ are crossing pairs in Π. It is then possible to remove the crossing and construct

$$\Pi''=(\Pi\setminus\{(i,k+2),(k+1,j)\})\cup\{(i,k+1),(k+2,j)\},$$

for which $\#c(\Pi'')=\#c(\Pi)-1$. Now we have $\Pi''\sim(\sigma,*,1,\varrho),\mathbf{h}''$, where $\mathbf{h}''=(\mathbf{i},\mathbf{h}(k+2),\mathbf{h}(k+1),\mathbf{j})$. Note that conversely every pair partition of n, compatible with ϵ and $\mathbf{h}$ but not containing $(k+1,k+2)$, can be found in $\mathcal{P}_2(n)$ as an element with a crossing between the two pairs that contain $k+1$ and $k+2$. We find

$$
\begin{aligned}
F(a_{\mathbf{h}}^{\epsilon}) &:= \sum_{\substack{\Pi\in\mathcal{P}_2(n)\\ \Pi\sim\epsilon,\mathbf{h}}} q^{\#c(\Pi)} \\
&= \sum_{\substack{\Pi\in\mathcal{P}_2(n)\\ \Pi\sim(\sigma,1,*,\varrho)\mathbf{h}\\ (k+1,k+2)\in\Pi}} q^{\#c(\Pi)} + \sum_{\substack{\Pi\in\mathcal{P}_2(n)\\ \Pi\sim(\sigma,1,*,\varrho)h\\ (k+1,k+2)\notin\Pi}} q^{\#c(\Pi)} \\
&= \delta_{\mathbf{h}(k+1),\mathbf{h}(k+2)} \sum_{\substack{\Pi'\in\mathcal{P}_2(n-2)\\ \Pi'\sim(\sigma,\varrho),(\mathbf{i},\mathbf{j})}} q^{\#c(\Pi')} + \sum_{\substack{\Pi''\in\mathcal{P}_2(n)\\ \Pi''\sim(\sigma,*,1,\varrho),\mathbf{h}''}} q^{\#c(\Pi'')+1} \\
&= \delta_{\mathbf{h}(k+1),\mathbf{h}(k+2)} F(a_{\mathbf{i}}^{\sigma}a_{\mathbf{j}}^{\varrho}) + qF(a_{\mathbf{i}}^{\sigma}a_{\mathbf{h}(k+2)}^{*}a_{\mathbf{h}(k+1)}a_{\mathbf{j}}^{\varrho}).
\end{aligned}
$$

Relation (5) now follows because the right-hand side $F(a_{\mathbf{h}}^{\epsilon})$ satisfies the same recursion relation as $\varphi(a_{\mathbf{h}}^{\epsilon})$ with the boundary conditions (3). □

Since the position operator of a harmonic oscillator is usually represented by an operator of the form

$$X_i := a_i + a_i^{*},$$

we define X_i to be the position of a q-harmonic oscillator and show that its moments under φ_q can be calculated as a sum over partitions. The operators X_i, $i\in\{1,...,m\}$, generate the *-algebra $\mathfrak{B}_q\subset\mathcal{A}_q$. We shall refer to $\mathfrak{B}_q$ as the q-harmonic oscillator algebra. Let ρ_q denote the restriction of φ_q to $\mathfrak{B}_q$.

Theorem 6: For all $\mathbf{h}\in\{1,...,m\}^n$, $n\in\mathbb{N}$, we have

$$\rho_q(X_{\mathbf{h}}) = \sum_{\Pi\in\mathcal{P}_2(n)} q^{\#c(\Pi)} \prod_{(l,m)\in\Pi} \delta_{\mathbf{h}(l),\mathbf{h}(m)}.$$

Proof: This follows from Lemma 5 by summation over all $\epsilon\in\{1,*\}^n$. □

Corollary 7: The linear functional ρ_q satisfies the following.

(1) For all $j\in\{1,...,m\}$ and $n\in\mathbb{N}$, we have

$$\rho_q(X_j^n) = \sum_{\Pi\in\mathcal{P}_2(n)} q^{\#c(\Pi)}.$$

(2) For $\mathbf{i}\in\{1,...,m\}^n$ and a cyclic permutation $\tau\in S_n$, we have

$$\rho_q(X_{\mathbf{i}(\tau(1))}\cdots X_{\mathbf{i}(\tau(n))}) = \rho_q(X_{\mathbf{i}(1)}\cdots X_{\mathbf{i}(n)}).$$

Proof of 1: Put $\mathbf{h}(1)=\cdots=\mathbf{h}(n)=j$ in Theorem 6. □

Before proving (2) we first give some considerations concerning pair partitions.

The usual way to visualize a partition, $\Pi\in\mathcal{P}_2(S)$, would be to draw the elements of S on a straight line and connect every two points belonging to the same pair in Π with an arc above the

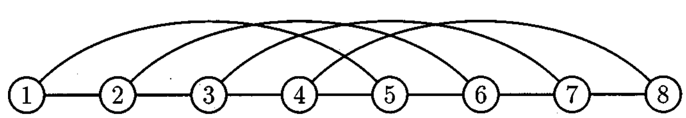

FIG. 1. The line representation of the partition $\Pi=\{(1,5),(2,6),(3,7),(4,8)\}$.

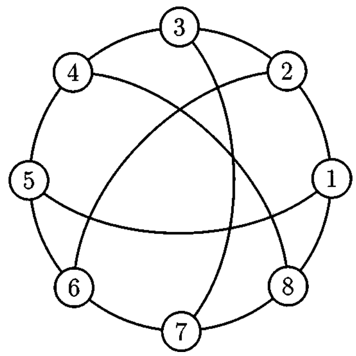

FIG. 2. A circle representation of the partition $\Pi=\{(1,5),(2,6),(3,7),(4,8)\}$.

line in such a way that two different lines cross at most once and no three lines intersect in one point. As an example, in Fig. 1 we draw the partition $\Pi=\{(1,5),(2,6),(3,7),(4,8)\}$. We will refer to this method of visualization as the line representation of a pair partition.

Another way to visualize a pair partition is its circle representation. This consists in drawing the points of S on a circle and connecting them by lines inside the circle, subject to the same restrictions. An example is given in Fig. 2. We note that this can be done in more than one way.

To make the circle representation of pair partitions explicit, we regard the circle as the unit circle in $\mathbb{C}$, and we make the map

$$\mathfrak{T}:S\to\mathbb{C}:s\mapsto e^{2i(s-1)\pi/n}.$$

This map converts every pair $(\alpha,\beta)\in\Pi$ to a pair (f,g) on the unit circle in $\mathbb{C}$.

Definition 8: Two pairs (f,g) and (f',g') of different points on the unit circle in $\mathbb{C}$ are said to be separated if and only if the straight line from f to g crosses the straight line from f' to g' inside the unit circle.

(See Fig. 3.) It is obvious that two pairs (α_i,β_i) and (α_j,β_j) in a partition Π are crossing if and only if the pairs $(\mathfrak{T}(\alpha_i),\mathfrak{T}(\beta_i))$ and $(\mathfrak{T}(\alpha_j),\mathfrak{T}(\beta_j))$ are separated. Let $\tau\in S_n$ be a cyclic permutation and $\Pi\in\mathcal{P}_2(n)$ any pair partition. By $\tau(\Pi)$ we denote the rotated partition:

$$\tau(\Pi):=\{(\tau(\alpha_1),\tau(\beta_1)),\ldots,(\tau(\alpha_{n/2}),\tau(\beta_{n/2}))\}.$$

Since separated pairs on the unit circle remain separated under rotation, we have

$$\#c(\Pi)=\#c(\tau(\Pi)). \tag{6}$$

We now turn to the proof of the second part of Corollary 7.

Proof of (2): The essence of the proof that ρ_q has this cyclic property is the fact that the circle representation of a partition $\Pi\in\mathcal{P}_2(n)$ also has this cyclic property. This is shown in Fig. 4.

More formally, for $\mathbf{i}\in\{1,...,m\}^n$ and $\tau\in S_n$ cyclic, we have the following:

FIG. 3. Two separated and two nonseparated pairs.

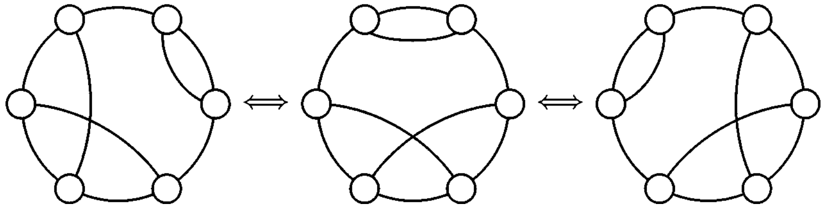

FIG. 4. The linear functional ρ_q satisfies a cyclic property.

$$\rho_q(X_{\mathbf{i}(\tau(1))}\cdots X_{\mathbf{i}(\tau(n))}) = \sum_{\Pi\in\mathcal{P}_2(n)} q^{\#c(\Pi)} \prod_{(l,m)\in\Pi} \delta_{\mathbf{i}(\tau(l)),\mathbf{i}(\tau(m))}$$

$$= \sum_{\Pi\in\mathcal{P}_2(n)} q^{\#c(\Pi)} \prod_{(l,m)\in\tau(\Pi)} \delta_{\mathbf{i}(l),\mathbf{i}(m)}$$

$$= \sum_{\Pi\in\mathcal{P}_2(n)} q^{\#c(\tau^{-1}(\Pi))} \prod_{(l,m)\in\Pi} \delta_{\mathbf{i}(l),\mathbf{i}(m)} = \rho_q(X_{\mathbf{i}(1)}\cdots X_{\mathbf{i}(n)}).$$

We shall represent the pair $(\mathcal{A}_q,\varphi_q)$ on a Hilbert space $\mathcal{K}$ with inner product $\langle\cdot,\cdot\rangle_q$ in which a unit vector Ψ, from now on referred to as vacuum vector, is singled out. To make this representation explicit, let $\mathfrak{h}$ be a Hilbert space with dim $\mathfrak{h}=m$ and orthonormal basis $\{e_1,\ldots,e_m\}$. Because of (3) we require that for all $i\in\{1,...,m\}$:

$$a_i\Psi = 0.$$

If now we put for all $\mathbf{k}\in\{1,...,m\}^n$:

$$a^*_{\mathbf{k}(1)}a^*_{\mathbf{k}(2)}\cdots a^*_{\mathbf{k}(n)}\Psi =: e_{\mathbf{k}(1)}\otimes\cdots\otimes e_{\mathbf{k}(n)},$$

then we see that for $\mathcal{K}$ we can take the full Fock space over $\mathfrak{h}$, denoted $\mathcal{F}(\mathfrak{h})$:

$$\mathcal{F}(\mathfrak{h}) := \bigoplus_{i=1}^{\infty} \mathfrak{h}^{\otimes i}, \quad \mathfrak{h}^{\otimes 0} := \mathbb{C},$$

with vacuum vector $\Psi = 1\oplus 0\oplus 0\oplus\cdots$. We define a bilinear form $\langle\cdot,\cdot\rangle_q$ on $\mathcal{K}$ as follows:

$$\langle e_{\mathbf{k}(1)}\otimes\cdots\otimes e_{\mathbf{k}(n)}, e_{\mathbf{l}(1)}\otimes\cdots\otimes e_{\mathbf{l}(n')}\rangle_q = \langle a^*_{\mathbf{k}(1)}\cdots a^*_{\mathbf{k}(n)}\Psi, a^*_{\mathbf{l}(1)}\cdots a^*_{\mathbf{l}(n')}\Psi\rangle_q$$

$$:= \varphi_q(a_{\mathbf{k}(n)}\cdots a_{\mathbf{k}(1)}a^*_{\mathbf{l}(1)}\cdots a^*_{\mathbf{l}(n')})$$

$$= \sum_{\Pi\in\mathcal{P}_2(n+n')} q^{\#c(\Pi)} \prod_{(p,q)\in\Pi} \varphi_q(a^{\epsilon(p)}_{\mathbf{h}(p)}a^{\epsilon(q)}_{\mathbf{h}(q)}),$$

for $l\in\{1,...,m\}^{n'}$, $n'\in\mathbb{N}$, $\mathbf{h}=(\mathbf{k}(n),\ldots,\mathbf{k}(1),\mathbf{l}(1),\ldots,l(n))\in\{1,...,m\}^{n+n'}$, and $\epsilon\in\{1,*\}^{n+n'}$, such that $\epsilon(1)=\cdots=\epsilon(n)$ and $\epsilon(n+1)=\cdots=\epsilon(n+n')=*$. Bożejko and Speicher in Ref. 7 show positivity of the bilinear form $\langle\cdot,\cdot\rangle_q$ by proving positive definiteness of the function $S_n\to\mathbb{C}$: $\sigma\mapsto q^{\#i(\sigma)}$. Here the set $i(\sigma)$ is the set of inversions of the permutation $\sigma\in S_n$:

$$i(\sigma) := \{(l,m) \mid 1\leq l<m\leq n, \sigma(l)>\sigma(m)\}.$$

Because of its positivity, the bilinear form $\langle\cdot,\cdot\rangle_q$ can be regarded as an inner product on $\mathcal{K}$, so we conclude that the linear functional φ_q is, in fact, a state. Therefore the second part of Corollary 7 implies that ρ_q is a trace state.

Since $\varphi_q(a^{\epsilon(p)}_{\mathbf{h}(p)}a^{\epsilon(q)}_{\mathbf{h}(q)})$ only yields something different from 0 for $\epsilon(p)=1$ and $\epsilon(q)=*$, we know that $\langle\cdot,\cdot\rangle_q\neq 0$ only in case $n=n'$, and that the partitions $\Pi\in\mathcal{P}_2(2n)$ contributing to this inner product have to be compatible with ϵ. This only happens for partitions where, for all $i \in \{1,...,n\}$, we have $1 \leqslant \alpha_i \leqslant n$ and $n+1 \leqslant \beta_i \leqslant 2n$. There are exactly n! of these partitions in $\mathcal{P}_2(2n)$, to be labeled by permutations σ of $\{1,...,n\}$. We find

$$\varphi_q(a_{\mathbf{h}}^{\epsilon}) = \delta_{n,n'} \sum_{\substack{\Pi \in \mathcal{P}_2(2n) \\ \Pi \sim \epsilon}} q^{\#c(\Pi)} \prod_{(p,q) \in \Pi} \delta_{\mathbf{h}(p),\mathbf{h}(q)} = \delta_{n,n'} \sum_{\sigma \in S_n} q^{\#i(\sigma)} \delta_{\mathbf{h}(1),\mathbf{h}(\sigma(1))} \cdots \delta_{\mathbf{h}(n),\mathbf{h}(\sigma(n))}.$$

Repeated use of the commutation relations (2) and the fact that $a_i \Psi = 0$ for all $i \in \{1,...,m\}$ yields, for all $\mathbf{h} \in \{1,...,m\}^n$, the following action of a_i:

$$a_i e_{\mathbf{h}(1)} \otimes \cdots \otimes e_{\mathbf{h}(n)} := \sum_{j=1}^{n} q^{j-1} \delta_{i,\mathbf{h}(j)} e_{\mathbf{h}(1)} \otimes \cdots \check{e}_{\mathbf{h}(j)} \cdots \otimes e_{\mathbf{h}(n)},$$

where by $e_{\mathbf{h}(1)} \otimes \cdots \check{e}_{\mathbf{h}(j)} \cdots \otimes e_{\mathbf{h}(n)}$ we mean the tensor product $e_{\mathbf{h}(1)} \otimes \cdots \otimes e_{\mathbf{h}(j-1)} \otimes e_{\mathbf{h}(j+1)} \otimes \ldots \otimes e_{\mathbf{h}(n)}$.

IV. THE DISCRETE LATTICE $\mathcal{T}(N,d)$

For d, $N \in \mathbb{N}$ we will define the Parisi model on a discrete d-dimensional lattice:

$$\mathcal{T}(N,d) := \{-N,\ldots,0,...,N\}^d.$$

Define the set of h-dimensional, $h \in \mathbb{N}$, elementary facets (i.e., points, edges, planes, etc.) in $\mathcal{T}(N,d)$ as follows:

$$\mathcal{X}_h := \{(v,\gamma) \in \mathcal{T}(N,d) \times \mathcal{P}_h(\{1,...,d\}) \mid j \in \gamma \Rightarrow v(j) \neq N\},$$

where $\mathcal{P}_h(\{1,...,d\})$ is the collection of all subsets of $\{1,...,d\}$ that contain exactly h elements. In this notation (v,γ) stands for the elementary facet, which has $v \in \mathcal{T}(N,d)$ as its lowest vertex, and whose spatial orientation is defined by the set of spatial directions $\gamma \subset \{1,2,...,d\}$. Then $\mathcal{X}_0$ stands for the set $\mathcal{T}(N,d)$ of vertices itself, $\mathcal{X}_1$ is the set of edges in $\mathcal{T}(N,d)$, and $\mathcal{X}_2$ is the set of two-dimensional planes in $\mathcal{T}(N,d)$, from now on referred to as plaquettes. By $\mathcal{G}_h$ we shall mean the set of all functions from $\mathcal{X}_h$ to $\mathbb{Z}$:

$$\mathcal{G}_h := \mathbb{Z}^{\mathcal{X}_h},$$

and by Ω_h the set of all functions from $\mathcal{X}_h$ to $\mathbb{C}$:

$$\Omega_h := \mathbb{C}^{\mathcal{X}_h}.$$

Then Ω_h is the set of ''forms'' on $\mathcal{X}_h$. They are a discrete analog of the differential forms of Cartan. The set Ω of all differential forms will be denoted by $\Omega := \oplus_{h=0}^{\infty} \Omega_h$. Similarly, we define $\mathcal{G} := \oplus_{h=0}^{\infty} \mathcal{G}_h$. Note that $\#(\mathcal{X}_h) = \dim \mathcal{G}_h = \dim \Omega_h = \binom{d}{h}(2N)^h(2N+1)^{d-h}$. An element of $\mathcal{G}_1$ will be called a curve $[$on $\mathcal{T}(N,d)]$ and an element of $\mathcal{G}_2$ will be called a surface $[$on $\mathcal{T}(N,d)]$. The mapping,

$$L: \mathcal{G}_1 \rightarrow \mathbb{N}: l \mapsto \sum_{j \in \mathcal{X}_1} |l(j)|,$$

associates to a curve its length, and the mapping,

$$A: \mathcal{G}_2 \rightarrow \mathbb{N}: k \mapsto \sum_{j \in \mathcal{X}_2} |k(j)|,$$

sends a surface to its area. By a *walk* of length $n \in \mathbb{N}$ on $\mathcal{T}(N,d)$ we shall mean a series of consecutively neighboring points $(\alpha_1,\ldots,\alpha_n)$ in $\mathcal{T}(N,d)$ that trace out a curve $l \in \mathcal{G}_1$. This curve l assigns to every edge $x \in \mathcal{X}_1$, the number of times the walk $(\alpha_1,\ldots,\alpha_n)$ runs through x in positive direction minus the number of times this walk runs through x in the negative direction. So a curve is thought of as just any configuration of edges equipped with a direction and a multiplicity. A surface is conceived of analogously. A walk will be called *closed* if $\alpha_1 = \alpha_n$.

We define the integral of an h-form $\omega \in \Omega_h$ over some element $k \in \mathcal{G}_h$ as follows.

Definition 9:

$$\int_k \omega := \langle k, \omega \rangle = \sum_{x \in \mathcal{X}_h} k(x)\omega(x).$$

We define the boundary and coboundary operators $\partial_h : \mathcal{G}_h \to \mathcal{G}_{h-1}$ and $\delta_h : \Omega_h \to \Omega_{h+1}$ as follows:

$$(\partial_h g)(v,\gamma) := \sum_{j \notin \gamma} \epsilon(j,\gamma)(g(v-e_j, \gamma \cup \{j\}) - g(v, \gamma \cup \{j\})) \quad h \geqslant 1,$$

$$(\delta_h f)(v,\gamma) := \sum_{j \in \gamma} \epsilon(j, \gamma \backslash \{j\})(f(v+e_j, \gamma \backslash \{j\}) - f(v, \gamma \backslash \{j\})), \quad h \geqslant 0,$$

where $\epsilon(j,\varrho) := (-1)^{\#\{i \in \varrho | i < j\}}$ and e_j, $j \in \{1,...,d\}$, denotes the unit vector in the jth direction in $\mathcal{T}(N,d)$. We define the operator δ on Ω as δ_h for a differential form in Ω_h, so $\delta_h = \delta|_{\Omega_h}$. The operator δ can be considered the discrete version of the derivative operator on h forms. The operator ∂ on $\mathcal{G}$, defined as $\partial_h = \partial|_{\mathcal{G}_h}$, will be referred to as the boundary operator, since it yields the $(h-1)$-dimensional boundary of an h-dimensional object for $h \geqslant 1$. Stokes' theorem is the statement that δ and ∂ are each other's adjoints.

Theorem 10 (Stokes): For $\omega \in \Omega_h$ and $k \in \mathcal{G}_{h+1}$, we have

$$\int_k \delta\omega = \int_{\partial k} \omega.$$

Proof: By $1_{(v,\gamma)}$ we denote the characteristic function of $(v,\gamma) \in \mathcal{X}_1$. Then to prove this theorem it suffices to show that $\langle \partial 1_{(v,\gamma)}, \omega \rangle = \langle 1_{(v,\gamma)}, \delta\omega \rangle$ for every $(v,\gamma) \in \mathcal{X}_{h+1}$ and $\omega \in \Omega_h$:

$$\begin{aligned}\langle \partial 1_{(v,\gamma)}, \omega \rangle &= \sum_{(w,\varrho) \in \mathcal{X}_h} (\partial 1_{(v,\gamma)})(w,\varrho)\omega(w,\varrho) \\ &= \sum_{(w,\varrho) \in \mathcal{X}_h} \sum_{j \notin \varrho} \epsilon(j,\varrho)(1_{(v,\gamma)}(w-e_j, \varrho \cup \{j\}) - 1_{(v,\gamma)}(w, \varrho \cup \{j\}))\omega(w,\varrho) \\ &= \sum_{j \in \gamma} \epsilon(j, \gamma \backslash \{j\})(\omega(v+e_j, \gamma \backslash \{j\}) - \omega(v, \gamma \backslash \{j\})) = (\delta\omega)(v,\gamma) = \langle 1_{(v,\gamma)}, \delta\omega \rangle. \qquad \square\end{aligned}$$

Theorem 11: $\delta^2 = 0$.

Proof: For $\omega \in \Omega_h$ and $i < j$ we have, for all $(v,\gamma) \in \mathcal{X}_{h+2}$:

$$(\delta_{h+1}\delta_h \omega)(v,\gamma) = \sum_{i \in \gamma} \epsilon(i,\gamma)(\delta_h \omega(v+e_i, \gamma \backslash \{i\}) - \delta_h \omega(v, \gamma \backslash \{i\})) = \sum_{\substack{i,j \in \gamma \\ i \neq j}} F(i,j) G(i,j),$$

where

$$F(i,j) = \epsilon(i,\gamma)\epsilon(j, \gamma \backslash \{i\})$$

and

$$G(i,j) = \omega(v+e_i+e_j, \gamma \backslash \{i,j\}) - \omega(v+e_i, \gamma \backslash \{i,j\}) - \omega(v+e_j, \gamma \backslash \{i,j\}) + \omega(v, \gamma \backslash \{i,j\}).$$

Note that G is symmetric and F is antisymmetric:

$$F(i,j)F(j,i)=\epsilon(i,\gamma)\epsilon(j,\gamma\backslash\{i\})\epsilon(j,\gamma)\epsilon(i,\gamma\backslash\{j\})$$
$$=(-1)^{\#\{k\in\gamma|k<i\}}(-1)^{\#\{k\in\gamma\backslash\{i\}|k<j\}}(-1)^{\#\{k\in\gamma|k<j\}}(-1)^{\#\{k\in\gamma\backslash\{j\}|k<i\}}=-1.$$

Therefore $\delta_{h+1}\delta_h\omega=\Sigma_{i,j\in\gamma}F(i,j)G(i,j)=0$ for all $\omega\in\Omega_h$. □

Because $\partial=\delta^*$, viewed as $(\#\mathcal{X}_h)\times(\#\mathcal{X}_{h+1})$ matrices, Corollary 12 immediately follows.

Corollary 12: $\partial^2=0$.

Definition 13: An element $p\in\mathcal{G}_h$ is called closed if $\partial_h(p)=0$.

We shall call a closed curve a *loop*. The boundary of a surface k is always a loop since $\partial^2 k=0$. We shall say that $\partial_2 k$ *spans* k. With every loop l we can associate a class of surfaces $\Gamma_l:=\{k\in\mathcal{G}_2|\partial_2 k=l\}$, the class of surfaces spanned by l. A surface $p\in\Gamma_l$ will be called *minimal*, with respect to l, if its area $A(p)$ is minimal in $A(\Gamma_l)$.

Definition 14: By the area $A(l)$ of a loop we shall mean the area of a surface that is minimal with respect to l.

A closed walk traces out a loop. By the area of a closed walk $(\alpha_1,\dots,\alpha_n=\alpha_1)$ we shall mean the area of the loop traced out by this walk, and denote it by $A((\alpha_1,\dots,\alpha_n))$.

V. THE PARISI MODEL

In the description of the Parisi model we try to stay as close as possible to the notation used in Ref. 1.

Consider $\mathcal{T}(N,d)$ and put on every plaquette a magnetic field, i.e., we define a 2-form $B\in\Omega_2$ with strength $\mathcal{B}\in[0,\pi]$, the sign of which will be chosen independently for every pair of spatial directions:

$$B(v,\{i,j\})=S_{\{i,j\}}\mathcal{B},$$

where $S_{\{i,j\}}$ is a random variable depending on the pair $\{i,j\}$. The random variable $S_{\{i,j\}}$ is a coin toss, i.e., the value of $S_{\{i,j\}}$ is chosen from $\{-1,1\}$ with distribution $\{1/2,1/2\}$. It is obvious that, for any of the $2^{d(d-1)/2}$ choices for S, the constant field B on $\mathcal{T}(N,d)$ is divergence-free: $\delta B=0$. We put $q:=\cos\mathcal{B}$; then $q\in[-1,1]$.

By $\Phi(k)$ we denote the magnetic flux through some surface $k\in\mathcal{G}_2$. This flux is simply the sum of the fluxes through the plaquettes in k, which, by Definition 9, equals

$$\Phi(k)=\sum_{x\in\mathcal{X}_2}k(x)B(x)=\int_k B.$$

We define the magnetic flux enclosed by a loop $l\in\mathcal{G}_1$ as the flux $\Phi(k)$ for some k spanned by l, and denote it by $\Phi(l)$. Since the field B is divergence-free there must exist some $A\in\Omega_1$ for which $\delta_1 A=B$. Indeed, if, for $k\in\{1,...,d\}$ we choose any $\mathcal{C}_0,\dots,\mathcal{C}_d\in\mathbb{R}$, and put

$$A(w,\{k\})=\mathcal{C}_0+\sum_{j=1}^{k-1}(\mathcal{C}_j+w(j)S_{\{j,k\}}\mathcal{B});$$

then the requirement $\delta_1 A=B$ is fulfilled, as can be checked easily by calculating $(\delta_1 A)$ at $(v,\{i,j\})$ explicitly. By Stokes' theorem we find, for $\Phi(l)$,

$$\Phi(l)=\int_k \delta A=\int_l A=\sum_{x\in\mathcal{X}_1}l(x)A(x),$$

for all $k\in\Gamma_l$. So $\Phi(l)$ is well defined and does not depend on the choice of k.

Let the magnetic field B on $\mathcal{T}(N,d)$ induce a deformation of the nearest-neighbor $(2N+1)^d\times(2N+1)^d$-interaction matrix or lattice Laplacian, leading to the deformed lattice Laplacian Δ_q. This matrix $(\Delta_q)_{v,w\in\mathbb{N}^d}$ is defined by

$$|(\Delta_q)_{v,w}|=\begin{cases}1, & \text{if } |v-w|=1, \quad \text{for } v,w\in\mathcal{X}_0,\\ 0, & \text{otherwise,}\end{cases}$$

where the phases are determined by the field

$$(\Delta_q)_{v,v+e_k} := e^{iA(v,\{k\})},$$

$$(\Delta_q)_{v+e_k,v} := e^{-iA(v,\{k\})},$$

for all $(v,\{k\}) \in \mathcal{X}_1$. From this definition it is clear that $(\Delta_q)_{v,w} = \overline{(\Delta_q)_{w,v}}$. Now let

$$W(l) := e^{i\Phi(l)} = \prod_{x \in \mathcal{X}_1} e^{il(x)A(x)}$$

be the product of the Δ_q's along a loop l. $W(l)$ is known as the ''Wilson loop,'' although $W(l)$ itself is not a loop but a complex number assigned to the loop l. For a closed walk l' that traces out a loop $l \in \mathcal{G}_1$ we have that the product of the Δ_q's along l' equals the product of the Δ_q's along l, hence $W(l') = W(l)$.

We define an operator $\hat{X}$ as follows:

$$\hat{X} = \frac{1}{\sqrt{2d}}\, \Delta_q\,, \tag{7}$$

where the overcaret symbolizes the dependence on N and d. Then $\hat{X}$ is an element in the algebra $\mathcal{A}$ of matrix-valued functions,

$$\{-1,+1\}^{\binom{d}{2}} \to M_{(2N+1)^d}, \tag{8}$$

where $\{-1,+1\}^{\binom{d}{2}}$ is the space of outcomes of the coin tosses and $M_{(2N+1)^d}$ denotes the $(2N+1)^d \times (2N+1)^d$ matrices with complex entries. On $M_{(2N+1)^d}$ we have a normalized trace tr $= (1/(2N+1)^d)$ Tr, where Tr is the standard trace on $M_{(2N+1)^d}$. Since we have here a trace that satisfies $\mathrm{tr}(\mathbf{1}) = 1$, its expectation can serve as a generalized probability measure (a state) $\hat{\varphi}$ on $\mathcal{A}$, and it is therefore possible to calculate generalized expectations of elements of $\mathcal{A}$:

$$\hat{\varphi}(\hat{X}^n) := \hat{\mathbb{E}}(\mathrm{tr}(\hat{X}^n)) = \frac{1}{(2d)^{n/2}(2N+1)^d}\, \hat{\mathbb{E}}(\mathrm{Tr}(\Delta_q^n)), \quad \text{for all} \quad n \in \mathbb{N}.$$

Here $\hat{\mathbb{E}}$ yields the expectation value with respect to the $d(d-1)/2$ coin tosses. The standardization factor $1/\sqrt{2d}$ in (7) ensures that $\hat{X}$ has variance 1 in the limit $N \to \infty$. It remains to show that indeed $\hat{\varphi}(\hat{X}^n)$ can be interpreted as a sum over walks on $\mathcal{T}(N,d)$:

$$\hat{\varphi}(\hat{X}^n) = \frac{1}{(2N+1)^d}\, \hat{\mathbb{E}}(\mathrm{Tr}(\hat{X}^n)) = \frac{1}{(2N+1)^d}\, \hat{\mathbb{E}}\left(\sum_{i_1 \in \mathcal{T}(N,d)} \cdots \sum_{i_n \in \mathcal{T}(N,d)} \hat{X}_{i_1,i_2}\hat{X}_{i_2,i_3}\cdots\hat{X}_{i_n,i_1} \right). \tag{9}$$

The product $\hat{X}_{i_1,i_2}\cdots\hat{X}_{i_n,i_1}$ on the right-hand side of this equation yields something different from 0 if and only if

$$|i_1 - i_2| = |i_2 - i_3| = \cdots = |i_{n-1} - i_n| = |i_n - i_1| = 1;$$

$(i_1, i_2, i_3, \ldots, i_n, i_1)$ is a closed walk on $\mathcal{T}(N,d)$ that starts from i_1 and returns to i_1 and hence describes a loop in $\mathcal{G}_1$. If the walk crosses some point in $\mathcal{T}(N,d)$ more than once, then there is more than one walk yielding the same loop. Therefore the sum over walks in (9) cannot be reduced to a sum over loops easily. However, we shall see that in the limit $d \to \infty$ only a narrow class of walks survive.

VI. PAIR PARTITIONS, WALKS, AND LOOPS

In this section we shall describe the connection between pair partitions, as described in the Introduction, closed walks on $\mathcal{T}(N,d)$, and loops in $\mathcal{G}_1$.

Suppose we have some $\Pi \in P_2(S)$ consisting of n pairs $(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n)$ for $S=\{1,...,2n\}$. To every pair (α_1,β_1), $i\in\{1,...,n\}$, we assign the unit vector in the ith direction, e_i. Now for every element $s\in S$, starting with 1, we make a step in the lattice in the direction assigned to the pair in Π to which this number belongs. If s is the first element of such a pair, then the step will be taken in the positive direction. If s is the second element of such a pair, then the step will be taken in the negative direction. If we choose the origin as a starting point for our walks, then we always get a loop on $\{0,1\}^n$, the corners of the unit cube in n dimensions.

For fixed Π, define a mapping $\sigma_\Pi : S\to\{1,...,n\}$ that indicates to what pair $s\in S$ belongs: $\sigma_\Pi(s)=i$ if $s\in(\alpha_i,\beta_i)$. Furthermore, define a mapping $\vartheta_\Pi : S\to\{-1,+1\}$ that indicates whether it is the first or the second element of this pair: $\vartheta_\Pi(s)=1$ if $s=\alpha_i$ and $\vartheta_\Pi(s)=-1$ if $s=\beta_i$. To keep track of our walk we define a mapping $v:S\to\mathcal{X}_0$ as follows:

$$v(s):=\sum_{j=1}^{s}\vartheta_\Pi(j)e_{\sigma_\Pi(j)},\quad \text{with } v(0):=0.$$

To fix the starting point and direction of every edge in the walk, we need the following mapping:

$$w:S\to\mathcal{X}_0 : i\mapsto \min(v(i-1),v(i)),$$

where $\min(v(i-1),v(i))$ is the component-wise minimum of $v(i-1)$ and $v(i)$. Now, let $\gamma_1:\mathcal{P}_2(S)\to\mathcal{G}_1$ associate to Π the loop in $\mathcal{G}_1$ traced out by the walk $(0,v(1),v(2),\ldots,v(2n-1),0)$:

$$\gamma_1(\Pi):=\sum_{j=1}^{2n}\vartheta_\Pi(j)1_{(w(j),\{\sigma_\Pi(j)\})}.$$

Then one easily checks that $\gamma_1(\Pi)$ is indeed a loop, i.e., $\partial_1\gamma_1(\Pi)=0$.

Apart from γ_1 we shall also need an injective mapping $\gamma_2:\mathcal{P}_2(S)\to\mathcal{G}_2$ that maps a partition to a surface spanned by $\gamma_1(\Pi)$. To this end let us regard the pair partition in the circle representation as a planar graph with a closed outer edge. For every planar graph we can construct a dual by regarding every sector inside the outer edge of the graph as a vertex in the dual graph. The vertices in the dual graph are then connected by edges when the corresponding sectors have an edge in common.

Definition 15: The dual graph of a pair partition is the dual of the planar graph generated by its circle representation.

Figure 5 shows the construction of the dual graph of the partition $\Pi=\{(1,5),(2,6),(3,7),(4,8)\}$. To construct γ_2, consider a planar graph, H, given by a circle representation of a pair partition $\Pi\in\mathcal{P}_2(S)$. The disk sectors in this graph represent elements of $\mathcal{X}_0$, as is clear from the construction of the circle representation. Namely, we assign to sectors in

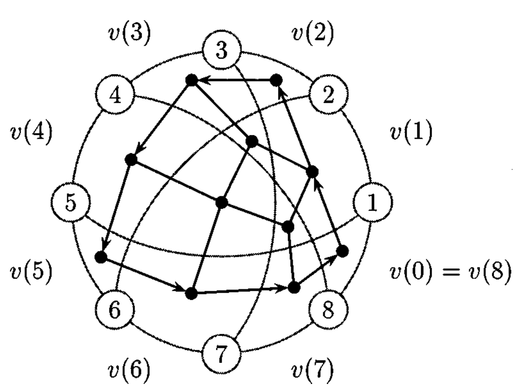

FIG. 5. The dual graph of the pair partition $\Pi=\{(1,5),(2,6),(3,7),(4,8)\}$. The orientation of the walk that traces out $\gamma_1(\Pi)$ is indicated by arrows.

H a vertex in space as follows: denote by A_0 the sector in H that has an edge in common with the circle segment between 1 and $2n$. Then to every sector A in H we associate a vertex $y = (y(1),\dots,y(n))$ according to the rule.

$$y(i) = \begin{cases} 0, & \text{if } A \text{ and } A_0 \text{ lie on the same side of the connecting line with index } i, \\ 1, & \text{otherwise.} \end{cases}$$

As a consequence to A_0 is associated $0 \in \{0,1\}^n$. This procedure is illustrated in Fig. 6, which shows a picture of the partition $\Pi = \{(1,5),(2,4),(3,6)\}$ with the corner points in every sector as an element of $\{0,1\}^3$. Since two sectors in the graph that have an edge in common can be associated to points in $\mathcal{X}_0$ that differ only one step in the direction corresponding to the shared edge, we can connect two such points by an edge in the dual graph, H', of Π, corresponding to an edge in $\mathcal{X}_1$. This way the sectors in H' point at plaquettes in $\mathcal{X}_2$, since every crossing of connecting lines in H has four sectors in H that share that point. Since any two lines cross at most once, the number of plaquettes enclosed by H' is equal to $\#c(\Pi)$, the number of crossings of Π. In fact, H' contains a plaquette, based upon the minimum of its four corner points, u_{ij}, for every pair $(i,j) \in c(\Pi)$. We can now define

$$\gamma_2(\Pi) := \sum_{(i,j)\in c(\Pi)} 1_{(u_{ij},\{i,j\})}.$$

Note that every plaquette in $\gamma_2(\Pi)$ receives positive orientation in this definition.

Theorem 16: *For every $\Pi \in \mathcal{P}_2(2n)$ we have $\partial_2 \gamma_2(\Pi) = \gamma_1(\Pi)$.*

The content of this theorem becomes apparent if one realizes that the outer edge of the dual graph of the pair partition Π represents the walk $(v(0),v(1),\dots,v(2n))$, as shown in Fig. 5. It is immediately clear that noncrossing partitions $\Pi \in \mathcal{P}_2(2n)$ yield $\gamma_2(\Pi) = 0$, so we have $\partial_2 \gamma_2(\Pi) = 0$. This agrees with $\gamma_1(\Pi) = 0$: every step that is taken is retraced later. This, however, will be a special case of the proof we shall give.

Proof: Let H be the planar graph given by the circle representation of Π. Fix an edge $(v,\{i\})$ in the dual graph, H', of Π. This edge crosses a segment, κ, of the ith connecting line in H. There are now three possibilities.

(1) The segment κ is the ith connecting line itself, i.e., the ith connecting line crosses no other connecting lines in H.

(2) The segment κ connects the edge of the circle to a crossing with the jth connecting line in H.

(3) The segment k connects 2 crossings, say with the j_1th and the j_2th connecting line.

In the first case the connecting line in the circle representation corresponding to e_i splits the circle into two separate parts connected by one edge, ζ, in H'. The ith connecting line therefore crosses no other connecting lines. This means that there exist no $k \in \{1,\dots,n\}$ such that $(i,k) \in c(\Pi)$; therefore $\gamma_2(\Pi)(v,\{i\}) = 0$. Since the walk that traces out $\gamma_1(\Pi)$ is closed, it has to visit ζ twice; once in the positive direction and once in the negative direction, so $\gamma_1(\Pi)(\zeta)$ also vanishes.

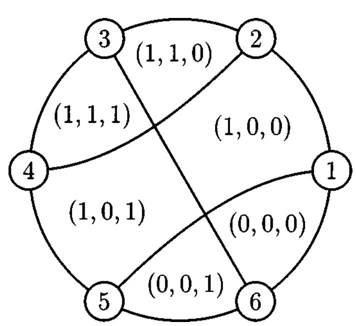

FIG. 6. The corner points of the surface generated by the partition $\Pi = \{(1,5),(2,4),(3,6)\}$.

In the second case the edge $(v,\{i\})$ is in the boundary of the plaquette $(u_{ij},\{i,j\})$, so

$$(\partial_2\gamma_2(\Pi))(v,\{i\})=\epsilon(i,\{j\})(1_{(u_{ij}+e_j,\{i\})}(v,\{i\})-1_{(u_{ij},\{i\})}(v,\{i\})). \tag{10}$$

First suppose $i<j$; then the crossing pairs (α_i,β_i) and (α_j,β_j) in Π satisfy $\alpha_i<\alpha_j<\beta_i<\beta_j$. Now if $v=u_{ij}$, meaning that v and u_{ij} lie on the same side of the ith and jth connecting line as 0, then the walk that traces out $\gamma_1(\Pi)$ reaches v before crossing the jth connecting line. Therefore the step this walk takes from v onward corresponds to α_i, so $\gamma_1(\Pi)(v,\{i\})=1$. From (10) we see that also $(\partial_2\gamma_2(\Pi))(v,\{i\})=1$. If, on the other hand, $v=u_{ij}+e_j$, then v and u_{ij} lie on the same side of the ith connecting line, but not on the same side of the jth connecting line. This means that the walk that traces out $\gamma_1(\Pi)$ has to cross the jth connecting line before it can reach v. But this means that a step in the ith direction has also been taken. Therefore we know that the step in the ith direction corresponds to β_i, so $\gamma_1(\Pi)(v,\{i\})=-1$. From (10) we see that also $(\partial_2\gamma_2(\Pi))(v,\{i\})=-1$. Now suppose $i>j$, then the crossing pairs (α_i,β_i) and (α_j,β_j) in Π satisfy $\alpha_j<\alpha_i<\beta_j<\beta_i$. The same type of argument as used for the case $i<j$ yields

$$v=u_{ij}\Rightarrow(\partial_2\gamma_2(\Pi))(v,\{i\})=\gamma_1(\Pi)(v,\{i\})=-1,$$

$$v=u_{ij}+e_j\Rightarrow(\partial_2\gamma_2(\Pi))(v,\{i\})=\gamma_1(\Pi)(v,\{i\})=1.$$

In the third case $(v,\{i\})$ is in the boundary of two plaquettes; $p_1=1_{(u_{ij_1},\{i,j_1\})}$ and $p_2=1_{(u_{ij_2},\{i,j_2\})}$. Since we are free to choose $j_1<j_2$, we can distinguish three cases:

(a) $i<j_1<j_2$,
(b) $j_1<i_1<j_2$, and
(c) $j_1<j_2<i$.

To prove case (a) we note that in this case we have $u_{ij_2}=u_{ij_1}+e_{j_1}$ and $v=u_{ij_2}$. Now calculate

$$(\partial_2 p_1)(v,\{i\})=\epsilon(j_1,\{i\})(1_{(u_{ij_1}+e_{j_1},\{i\})}(v,\{i\})-1_{(u_{ij_1},\{i\})}(v,\{i\}))=-1,$$

$$(\partial_2 p_2)(v,\{i\})=\epsilon(j_2,\{i\})(1_{(u_{ij_2}+e_{j_2},\{i\})}(v,\{i\})-1_{(u_{ij_2},\{i\})}(v,\{i\}))=1.$$

Therefore we have that $(\partial_2(p_1+p_2))(v,\{i\})=(\partial_2\gamma_2(\Pi))(v,\{i\})=0$. Since $(v,\{i\})$ is not an outer edge of H', we also have that $\gamma_1(\Pi)(v,\{i\})=0$. The reader can now easily verify cases (b) and (c), since the method of proof for these cases is the same as for case (a).

Finally, there are the edges in $\mathcal{X}_1$ that are not part of the dual graph of Π. We have $\gamma_2(\Pi)(u_1)=0$ for every $u_1\in\mathcal{X}_2$ not corresponding to a plaquette in H'. For every such u_1, we have $\partial_2(\gamma_2(\Pi)(u_1))=0$. Furthermore, we have $\gamma_1(\Pi)(u_2)=0$ for every u_2 not in the walk $\{(w(1),\{\sigma_\Pi(1)\}),\ldots,(w(2n),\{\sigma_\Pi(2n)\})\}$, so on the edges that are not in the dual graph of Π we have that $\partial_2\gamma_2(\Pi)=\gamma_1(\Pi)=0$, since every $u_2\in\{(w(1),\{\sigma_\Pi(1)\}),\ldots,(w(2n),\{\sigma_\Pi(2n)\})\}$ is represented in the dual graph of Π. □

Theorem 17: For every $\Pi\in\mathcal{P}_2(2n)$ the surface $\gamma_2(\Pi)$ is minimal in $\Gamma_{\gamma_1(\Pi)}$.

Proof: In case $n=1$ the loop $\gamma_1(\Pi)$ cannot enclose a plaquette. It follows from the definition of $\gamma_2(\Pi)$ that in this case $\gamma_2(\Pi)=0$, so the theorem holds for $n=1$. In the following we assume $n\geqslant 2$ and $i<j$ for $i,j\in\{1,\ldots,n\}$.

On the unit vectors $\{e_1,\ldots,e_n\}$ we introduce the following projection operator:

$$Q_{ij}e_k:=1_{\{i,j\}}(k)e_k.$$

Then the element $\gamma_2(\Pi)$ will be minimal in $\Gamma_{\gamma_1(\Pi)}$ if for every pair $i,j\in\{1,\ldots,n\}$ the projection of the closed walk that traces out $\gamma_1(\Pi)$ is a closed walk around $(0,\{i,j\})\in\mathcal{X}_2$, provided that $(i,j)\in c(\Pi)$. We now introduce

$$\Upsilon_{ij}:=\bigcup_{s=1}^{2n}\{Q_{ij}v(s)\},$$

the set of elements in $\mathcal{X}_0$ the projection to the $\{i,j\}$ plane of the walk that traces out $\gamma_1(\Pi)$ visits, and note that the proof is finished if we show $(i,j)\in c(\Pi)\Rightarrow\#\Upsilon_{ij}=4$.

Suppose $(i,j)\in c(\Pi)$, then (α_i,β_i) and (α_j,β_j) are crossing pairs in Π, so $\alpha_i<\alpha_j<\beta_i<\beta_j$. This means that $\Upsilon_{ij}=\{0,e_i,e_i+e_j,e_j\}$, so $\#\Upsilon_{ij}=4$. $\square$

Corollary 18: For all $\Pi\in\mathcal{P}_2(2n)$ we have that $A(\gamma_2(\Pi))=A(\gamma_1(\Pi))=\#c(\Pi)$.

VII. THE PARISI MODEL AND THE q-HARMONIC OSCILLATOR

To show the connection between the q-harmonic oscillator and the Parisi model, we prove that, for $d\to\infty$ and $N\to\infty$, the moments of $\hat{X}$ converge to the moments of $X=a+a^*$. In short, $\hat{X}$ converges in distribution to X.

Theorem 19: *Let φ_q be the vacuum state on the q-harmonic oscillator algebra generated by $X=a+a^*$. Then for all $n\in\mathbb{N}$ the following holds:*

$$\varphi_q(X^n)=\lim_{d\to\infty}\lim_{N\to\infty}\hat{\varphi}(\hat{X}^n).$$

Corollary 7 states how the moments of the q-harmonic oscillator can be calculated as a sum over pair partitions. Therefore, here it suffices to show that

$$\lim_{d\to\infty}\lim_{N\to\infty}\hat{\varphi}(\hat{X}^n)=\sum_{\Pi\in\mathcal{P}_2(n)}q^{\#c(\Pi)}.$$

Proof: From Eq. (9) we know that $\hat{\varphi}(\hat{X}^n)$ can be interpreted as a sum over closed walks in $\mathcal{T}(N,d)$. The theorem then is trivial for n odd since an odd number has no pair partitions and a closed walk cannot return to its starting point in an odd number of steps. So we may assume $n=2r$, $r\in\mathbb{N}$, in the following.

We define a sublattice $\mathcal{T}'(N,d)$ of $\mathcal{T}(N,d)$ as follows:

$$\mathcal{T}'(N,d):=\{-N+(r+1),\ldots,N-(r+1)\}^d.$$

The set of all walks that start from some $x\in\mathcal{T}'(N,d)$ and have length n will be defined as

$$\mathcal{W}(x,n):=\{(x,j_1,\ldots,j_{n-1},x)|j_1,\ldots,j_{n-1}\in\mathcal{T}(N,d),\quad |x-j_1|=|j_1-j_2|=\cdots=|j_{n-1}-x|=1\}.$$

Note that we have

$$\#\mathcal{W}(x,n)=2^r d(d-1)\cdots(d-r+1).$$

To every walk $w\in\mathcal{W}(x,n)$ we can assign some surface $s\in\Gamma_w$, the set of minimal surfaces s that have the closed walk w as a boundary. We can then choose a set $\mathcal{S}(x,n)$ that contains exactly one surface $s\in\Gamma_w$ for every $w\in\mathcal{W}(x,n)$, i.e., $\#\mathcal{S}(x,n)=\#\mathcal{W}(x,n)$. Every surface $s\in\mathcal{S}(x,n)$ has to be minimal with respect to its corresponding walk $w\in\mathcal{W}(x,n)$ in order for the surface to have the corner points of the plaquettes in this surface in $\mathcal{T}(N,d)$.

With the use of these definitions we can rewrite the sum over i_1 in Eq. (9) as follows:

$$\hat{\varphi}(\hat{X}^n)=\frac{1}{(2N+1)^d}\sum_{i_1\in\mathcal{T}'(N,d)}\hat{\mathbb{E}}\left(\sum_{i_2\in\mathcal{T}(N,d)}\cdots\sum_{i_n\in\mathcal{T}(N,d)}\hat{X}_{i_1,i_2}\hat{X}_{i_2,i_3}\cdots\hat{X}_{i_n,i_1}\right)$$
$$+\frac{1}{(2N+1)^d}\sum_{\substack{i_1\in\mathcal{T}(N,d)\\ i_1\notin\mathcal{T}'(N,d)}}\hat{\mathbb{E}}\left(\sum_{i_2\in\mathcal{T}(N,d)}\cdots\sum_{i_n\in\mathcal{T}(N,d)}\hat{X}_{i_1,i_2}\hat{X}_{i_2,i_3}\cdots\hat{X}_{i_n,i_1}\right),\qquad(11)$$

and calculate

$$\hat{\mathbb{E}}(\hat{X}_{i_1,i_2}\cdots\hat{X}_{i_n,i_1})=\frac{1}{(2d)^{n/2}}\sum_{w\in\mathcal{W}(x,n)}\hat{\mathbb{E}}(W(w))=\frac{1}{(2d)^{n/2}}\sum_{w\in\mathcal{W}(x,n)}\hat{\mathbb{E}}(e^{i\Sigma_{p\in\mathcal{X}_1}w(p)A(p)})$$
$$=\frac{1}{(2d)^{n/2}}\sum_{w\in\mathcal{W}(x,n)}\hat{\mathbb{E}}(e^{i\int_w A})=\frac{1}{(2d)^{n/2}}\sum_{s\in\mathcal{S}(x,n)}\hat{\mathbb{E}}(e^{i\int_s B}),$$

where we used Stokes' theorem. A walk that starts in $\mathcal{T}'(N,d)$ can never reach the boundary of $\mathcal{T}(N,d)$, so we can identify every walk $w\in\mathcal{W}(x,n)$ with a walk $w'\in\mathcal{W}(y,n)$, for $x,y\in\mathcal{T}'(N,d)$, via

$$w=(x,j_2,\ldots,j_n,x)\mapsto(y,j_2-(x-y),\ldots,j_n-(x-y),y)=w'.$$

From the fact that B is constant, it follows that the magnetic flux through $s\in\mathcal{S}(x,n)$, corresponding to w, is equal to the magnetic flux through $s'\in\mathcal{S}(y,n)$, corresponding to w', for $x,y\in\mathcal{T}'(N,d)$. This implies that we have

$$\sum_{s\in\mathcal{S}(x,n)}\hat{\mathbb{E}}(e^{i\int_s B})=\sum_{s\in\mathcal{S}(y,n)}\hat{\mathbb{E}}(e^{i\int_s B}).$$

We can therefore perform the first sum over i_1 in (11), yielding

$$\frac{(2(N-(r+1))+1)^d}{(2N+1)^d}\,\hat{\mathbb{E}}\left(\sum_{i_2\in\mathcal{T}(N,d)}\cdots\sum_{i_n\in\mathcal{T}(N,d)}\hat{X}_{0,i_2}\hat{X}_{i_2,i_3}\cdots\hat{X}_{i_n,0}\right).$$

Since for every point in $\mathcal{T}(N,d)\setminus\mathcal{T}'(N,d)$ we have, at most, as many loops as for points in $\mathcal{T}'(N,d)$ we can estimate an upper bound for the second sum over i_1 in (11):

$$\frac{(2(N-(r+1))+1)^d-(2N+1)^d}{(2N+1)^d}\,\#\mathcal{W}(0,n).$$

This term tends to 0 for $N\to\infty$, so we find

$$\lim_{N\to\infty}\hat{\varphi}(\hat{X}^n)=\hat{\mathbb{E}}\left(\sum_{i_2\in\mathbb{Z}^d}\cdots\sum_{i_n\in\mathbb{Z}^d}\hat{X}_{0,i_2}\hat{X}_{i_2,i_3}\cdots\hat{X}_{i_n,0}\right).$$

We now turn to the limit $d\to\infty$. There are exactly $\#\mathcal{P}_2(n)\binom{d}{r}2^r$ walks that start in 0, have length n, and go through r different spatial directions. The number of closed walks that go through less than r, say r', spatial directions is less than $(r')^n\binom{d}{r'}$, so the contribution of these walks vanishes in the limit $d\to\infty$ due to the standardization of $\hat{X}$. The closed walks that go through r spatial directions take a step in each direction exactly twice, and therefore correspond to a pair partition $\Pi\in\mathcal{P}_2(2r)$. The expectation value of the magnetic flux through the area of the loop $\gamma_1(\Pi)$ such a walk traces out, is given by

$$\hat{\mathbb{E}}(W(\gamma_2(\Pi)))=\hat{\mathbb{E}}\left(\prod_{x\in\mathcal{X}_2'}e^{i\gamma_2(\Pi)(x)B(x)}\right)=\prod_{x\in\mathcal{X}_2'}\hat{\mathbb{E}}(e^{i\gamma_2(\Pi)(x)B(x)})=(\cos\mathcal{B})^{A(\gamma_2(\Pi))}=q^{A(\gamma_2(\Pi))},$$

where $\mathcal{X}_2':=\{(v,\gamma)\in\mathbb{Z}^d\times\mathfrak{P}_2(\{1,\ldots,d\})\}$ with the same conventions we have for $\mathcal{X}_2$. In the above calculation the expectation of the product over $\mathcal{X}_2$ is interpreted as a product of expectations. To justify this we note that $\gamma_2(\Pi)$ has, at most, one plaquette for every pair of spatial directions and that the sign of the magnetic field B for a pair of spatial directions is independent of that for every other pair of directions. This implies that the fluxes through different plaquettes in $\gamma_2(\Pi)$ are independent. It is now possible to write $\hat{\varphi}(\hat{X}^{2r})$ as a sum over partitions in the limit $d,N\to\infty$:

$$\lim_{d\to\infty}\lim_{N\to\infty}\hat{\varphi}(\hat{X}^{2r})=\lim_{d\to\infty}\frac{1}{(2d)^r}\sum_{i_2\in\mathbb{Z}^d}\cdots\sum_{i_{2r}\in\mathbb{Z}^d}\hat{\mathbb{E}}((\Delta_q)_{0,i_2}\cdots(\Delta_q)_{i_{2r},0})$$

$$=\sum_{\Pi\in\mathcal{P}_2(2r)}q^{A(\gamma_1(\Pi))}=\sum_{\Pi\in\mathcal{P}_2(2r)}q^{\#c(\Pi)}, \qquad \square$$

where we used Corollary 18.

VIII. NONCOMMUTATIVE INDEPENDENCE IN THE PARISI MODEL

Until now we concerned ourselves with only one q-Gaussian random variable. In order to illustrate the concept of q independence in this model, we shall have to define more, say $\mathcal{N}\in\mathbb{N}$, q-Gaussian random variables. The Parisi model allows for such an extension in a straightforward way.

If d is divisible by $\mathcal{N}$, the lattice $\mathcal{T}(N,d)$ can be decomposed as a product of $\mathcal{N}$ sublattices as follows:

$$\mathcal{T}(N,d)=\mathcal{M}_1\times\mathcal{M}_2\times\cdots\times\mathcal{M}_{\mathcal{N}},$$

where dim $\mathcal{M}_j=d/\mathcal{N}$ for $j\in\{1,...,\mathcal{N}\}$. We can now use the q-deformed lattice Laplacian Δ_q defined in Sec. V to define $\mathcal{N}$ q-Gaussian random variables $\hat{X}_1,\ldots,\hat{X}_{\mathcal{N}}\in\mathcal{A}$ as follows:

$$(\hat{X}_i)_{u,w}=\begin{cases}\sqrt{\dfrac{\mathcal{N}}{2d}}\,(\Delta_q)_{v,w}, & \text{if } v-w\in\mathcal{M}_i,\\ 0, & \text{otherwise.}\end{cases}$$

In this way $\hat{X}_i, i\in\{1,...,\mathcal{N}\}$, is the standardized deformed lattice Laplacian on M_i. From the previous section we can deduce that the operators $\hat{X}_1,\ldots,\hat{X}_{\mathcal{N}}$ defined in this way converge, in distribution, to the q-Gaussian random variables $X_i=a_i+a_i^*$ for $d\to\infty$ and $N\to\infty$. Now fix $\mathcal{N}=2$ and let $\hat{X}=\hat{X}_1$ and $\hat{Y}=\hat{X}_2$.

Theorem 20 (independence): *For $n,m\in\mathbb{N}$ we have that*

$$\lim_{N\to\infty}\hat{\varphi}(\hat{X}^n\hat{Y}^m)=\lim_{N\to\infty}\hat{\varphi}(\hat{Y}^m\hat{X}^n)=\lim_{N\to\infty}\hat{\varphi}(\hat{X}^n)\hat{\varphi}(\hat{Y}^m).$$

Proof: Write out $\hat{\varphi}(\hat{X}^n\hat{Y}^m)$ to find

$$\hat{\varphi}(\hat{X}^n\hat{Y}^m)=\frac{1}{(2N+1)^d}\hat{\mathbb{E}}\left(\sum_{\substack{i_1,\ldots,i_n\\ \in\mathcal{T}(N,d)}}\sum_{\substack{j_1,\ldots,j_m\\ \in\mathcal{T}(N,d)}}\hat{X}_{i_1,i_2}\cdots\hat{X}_{i_n,j_1}\hat{Y}_{j_1,j_2}\cdots\hat{Y}_{j_m,i_1}\right).$$

Now, obviously the nonvanishing terms all satisfy

$$(i_1-i_2)+\cdots+(i_n-j_1)+(j_1-j_2)+\cdots+(j_m-i_1)=0.$$

The first n differences all lie in $\mathcal{M}_1$ and the last m differences all lie in $\mathcal{M}_2$ therefore,

$$(i_1-i_2)+\cdots+(i_n-j_1)=(j_1-j_2)+\cdots+(j_m-i_1)=0,$$

and it follows that $i_1=j_1$. With reference to the proof of Theorem 19, we can choose $i_1=j_1=0$, so we find

$$\lim_{N\to\infty}\hat{\varphi}(\hat{X}^n\hat{Y}^m)=\hat{\mathbb{E}}\left(\sum_{\substack{i_2,\ldots,i_n\\ \in\mathbb{Z}^d}}\sum_{\substack{j_2,\ldots,j_m\\ \in\mathbb{Z}^d}}\hat{X}_{0,i_2}\cdots\hat{X}_{i_n,0}\hat{Y}_{0,j_2}\cdots\hat{Y}_{j_m,0}\right)$$

$$=\hat{\mathbb{E}}\left(\sum_{\substack{i_2,\ldots,i_n\\ \in\mathbb{Z}^d}}\hat{X}_{0,i_2}\cdots\hat{X}_{i_n,0}\right)\hat{\mathbb{E}}\left(\sum_{\substack{j_2,\ldots,j_m\\ \in\mathbb{Z}^d}}\hat{Y}_{0,j_2}\cdots\hat{Y}_{j_m,0}\right)=\lim_{N\to\infty}\hat{\varphi}(\hat{X}^n)\hat{\varphi}(\hat{Y}^m).$$

Since $\hat{\varphi}$ is a trace state, we have that $\hat{\varphi}(\hat{X}^n\hat{Y}^m)=\hat{\varphi}(\hat{Y}^m\hat{X}^n)$. □

The same type of proof shows that, for general $\mathcal{N}$, we have

$$\hat{\varphi}(\hat{X}_{k(1)}^{n_1}\cdots\hat{X}_{k(\mathcal{N})}^{n_{\mathcal{N}}})=\hat{\varphi}(\hat{X}_{k(1)}^{n_1})\cdots\hat{\varphi}(\hat{X}_{k(\mathcal{N})}^{n_{\mathcal{N}}}),$$

provided the values $k(1),\ldots,k(\mathcal{N})$ are all different.

We conclude that in the sense of Definition 2, the $\hat{X}_i$ tend to independent random variables in the limit $N\to\infty$, and to independent q-Gaussians in the limit $N\to\infty$, $d\to\infty$.

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