BOUNDARY QUANTUM KNIZHNIK-ZAMOLODCHIKOV EQUATIONS AND FUSION

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ABSTRACT. In this paper we extend our previous results concerning Jackson integral solutions of the boundary quantum Knizhnik-Zamolodchikov equations with diagonal K-operators to higher-spin representations of quantum affine sl$_2$. First we give a systematic exposition of known results on R-operators acting in the tensor product of evaluation representations in Verma modules over quantum sl$_2$. We develop the corresponding fusion of K-operators, which we use to construct diagonal K-operators in these representations. We construct Jackson integral solutions of the associated boundary quantum Knizhnik-Zamolodchikov equations and explain how in the finite-dimensional case they can be obtained from our previous results by the fusion procedure.

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1. INTRODUCTION

The boundary q-Knizhnik-Zamolodchikov (qKZ) equations have their origins in the representation theory through works of Cherednik [6, 7] and in quantum field theory and in statistical mechanics with special ”integrable” boundary conditions, see, e.g., [1, 22, 18, 19, 37]. For detailed references see [32]. Their formulation involves solutions to the Yang-Baxter equation, the so-called R-operators or R-matrices, and solutions to the reflection equation, known as (boundary) K-operators or K-matrices.

1.1. The boundary qKZ equations. Let $M^\ell$ be the Verma module over quantum sl$_2$ with highest weight $\ell \in \mathbb{C}$. Then we will denote by $R^{kl}(x)$ the operator acting in $M^k \otimes M^\ell$ which is the evaluation of the truncated universal R-matrix for quantum affine sl$_2$ acting in the tensor product of corresponding evaluation representations. It satisfies the Yang-Baxter equation:

\begin{equation}
R^{kl}_{12}(x - y)R^{km}_{13}(x - z)R^{\ell m}_{23}(y - z) = R^{\ell m}_{23}(y - z)R^{km}_{13}(x - z)R^{kl}_{12}(x - y)
\end{equation}
This is an equation in $M^k \otimes M^l \otimes M^m$ and we are using the standard notations $R^{kl}_{12}(x) = R^{kl}(x) \otimes Id_{M^m}$, etc. For details and references see Section 2.

Given the above $R$-operator $R^{kl}(x)$, operators $K^{+,+}(x)$ and $K^{-,-}(x)$ acting in $M^l$ are called left and right $K$-operators if they satisfy the left and right reflection equations, respectively. These equations are also known as “boundary Yang Baxter equations” and were introduced in [34]. In the current setting they are given by

$$
\begin{align*}
R^{kl}(x+y)K^{+,+}_{12}(x+y)\bar{K}^{-,-}_{21}(y) &= K^{+,+}_{12}(x)\bar{K}^{-,-}_{21}(y)R^{kl}(x+y), \\
R^{kl}_{21}(x+y)K^{+,+}_{12}(x+y)\bar{K}^{-,-}_{21}(y) &= K^{+,+}_{12}(x)\bar{K}^{-,-}_{21}(y)R^{kl}_{21}(x+y). 
\end{align*}
$$

These are equations in $M^k \otimes M^l$; we are using the notations $K^{+,+}(x) = K^{+,+}(x) \otimes Id$, $K^{-,-}(y) = Id \otimes K^{-,-}(y)$ and $R^{kl}_{21}(x) := P^{kl}\bar{K}^{kl}(x)P^{kl}$, where $P^{kl} : M^k \otimes M^l \rightarrow M^l \otimes M^k$ is the permutation operator $P^{kl}(m^k \otimes m^l) = m^l \otimes m^k$ ($m^k \in M^k$, $m^l \in M^l$).

For $\xi = (\ell_1, \ldots, \ell_N) \in \mathbb{C}^N$, consider the tensor product $M^\xi = M^{\ell_1} \otimes \cdots \otimes M^{\ell_N}$.

The boundary qKZ equations [6, 7] in $M^\xi$ are given by the following compatible system of difference equations

$$
f(t + \tau e_r) = \Xi^\xi(t; \xi_+, \xi_-; \tau)f(t), \quad r = 1, \ldots, N
$$

for $M^\xi$-valued meromorphic functions $f(t)$ in $t \in \mathbb{C}^N$, where $\tau \in \mathbb{C}^x$ and $\{e_r\}_r$ is the standard orthonormal basis of $\mathbb{R}^N$. Here

$$
\Xi^\xi(t; \xi_+, \xi_-; \tau) := \prod_{r=1}^{N} (K^{+,+}_{r-1,r}(t_r - t_{r+1} + \tau) \cdots K^{+,+}_{r,N}(t_r - t_N + \tau)
\times K^{-,-}_{r-1,r}(t_r - t_{r+1} + \tau) \cdots K^{-,-}_{r,N}(t_r - t_N + \tau)
\times R^{\ell_{r-1},\ell_r}_{r-1,r}(t_r - t_{r+1} + \tau) \cdots R^{\ell_{r-1},\ell_r}_{1,r}(t_1 + t_r)K^{-,-}_{1,r}(t_r)
\times R^{\ell_{r-1},\ell_r}_{r-1,r}(t_r - t_{r-1}) \cdots R^{\ell_{r-1},\ell_r}_{r-1,r}(t_r - t_{r-1})
$$

is the (boundary) transport operator on $M^\xi$, depending meromorphically on $t \in \mathbb{C}^N$. The compatibility of the system (1.3) is guaranteed by the conditions

$$
\Xi^\xi(t + e_r \tau; \xi_+, \xi_-; \tau)\Xi^\xi(t; \xi_+, \xi_-; \tau) = \Xi^\xi(t + e_r \tau; \xi_+, \xi_-; \tau)\Xi^\xi(t; \xi_+, \xi_-; \tau),
$$

for $r, s = 1, \ldots, N$, which themselves are consequences of the quantum Yang-Baxter and reflection equations [11, 12]. In this paper we construct explicit Jackson integral solutions of (1.3) when the left and right $K$-operators $K^{+,+}(x)$ are of the form $K^{\xi,\xi}(x)$ with $K^{\xi,\xi}(x)$ ($\xi \in \mathbb{C}$) an explicit one-parameter family of $K$-operators diagonal with respect to the weight basis of $M^\xi$.

1.2. Finite-dimensional representations and fusion. When $\ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ the representation $M^\ell$ is no longer irreducible; it has an infinite-dimensional subrepresentation and an irreducible finite-dimensional quotient representation $V^\ell$. When some of the $\ell_r$’s in the boundary qKZ equations are in $\frac{1}{2}\mathbb{Z}_{\geq 0}$ the equations descend to the tensor product of corresponding quotient modules.

For $k, \ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, the tensor product of the associated evaluation modules $V^k(x) \otimes V^\ell(y)$ becomes reducible for special values of $x, y \in \mathbb{C}$. Owing to this degeneracy,
$R$-operators acting in (tensor products of) higher-dimensional evaluation modules can be obtained from corresponding objects acting in (tensor products of) lower-dimensional evaluation modules through a process called fusion [24][15]. We extend this representation-theoretic approach to fusion of $K$-operators in Section 3. Such $R$- and $K$-operators can then be generalized to $R$- and $K$-operators associated with modules $M'(x)$ for arbitrary $\ell \in \mathbb{C}$ by means of an analytical continuation. This will allow us to establish the above reflection equation (1.2) for a large class of $R$-operators than hitherto has been done. In particular, we obtain the diagonal $K$-operators $K^{\xi,\ell}(x)$ from this fusion approach applied to Cherednik's [7] diagonal $K$-matrix associated to $V^\frac{1}{2}$. The $K^{\xi,\ell}(x)$ are closely related to the family of $K$-operators constructed in [11] using the $q$-Onsager algebra.

For other approaches to fusion of $K$-operators, see e.g. [13][25][21][26][28][38].

1.3. Main result. In [32] we constructed $q$-integral solutions to (1.3) when all $\ell_s = \frac{1}{2}$. In this case the corresponding irreducible quotient spaces are two-dimensional and (1.3) reduces to an equation in $(\mathbb{C}^2)^\otimes N$. The main result of this paper is the construction of $q$-integral solutions to (1.3) for arbitrary $\ell_s \in \mathbb{C}$. For $\ell_s \in \frac{1}{2}\mathbb{Z}_{>0}$ it gives Jackson integral solutions in the tensor product of corresponding irreducible representations $V^\ell$. Our main result (Theorem 6.2) can be summarized as follows.

**Theorem 1.1.** Let $\xi_+, \xi_- \in \mathbb{C}$ and let $g_{\xi_+\xi_-}(x), h(x)$ and $F^\ell(x)$ be meromorphic functions in $x \in \mathbb{C}$ satisfying the functional equations

$$g_{\xi_+\xi_-}(x + \tau) = \frac{\sinh(\xi_+ - x - \frac{\tau}{2}) \sinh(\xi_- - x - \frac{\tau}{2})}{\sinh(\xi_+ + x + \tau - \frac{\tau}{2}) \sinh(\xi_- + x + \frac{\tau}{2})} g_{\xi_+\xi_-}(x),$$

$$h(x + \tau) = \frac{\sinh(x + \tau) \sinh(x + \eta)}{\sinh(x) \sinh(x + \tau - \eta)} h(x),$$

$$F^\ell(x + \tau) = \frac{\sinh(x + \tau - \ell \eta)}{\sinh(x + \tau + \ell \eta)} F^\ell(x).$$

Given fixed generic $x_0 \in \mathbb{C}^S$, and fixed parameters $\xi_+, \xi_-, \eta, \tau$ in a suitable parameter domain (see Section 3), the $M^\ell$-valued sum

$$f^\ell_\Omega(t) := \sum_{x \in x_0 + \tau \mathbb{Z}^S} \left( \prod_{i=1}^S g_{\xi_+\xi_-}(x_i) \right) \left( \prod_{1 \leq i < j \leq S} h(x_i + x_j) h(x_i - x_j) \right) \times \left( \prod_{r=1}^N \prod_{i=1}^S F^\ell_r(t_r + x_i) F^\ell_r(t_r - x_i) \right) \left( \prod_{i=1}^S \mathcal{F}^\ell_r(x_i; t) \right) \Omega$$

is a solution of the boundary $qKZ$ equations (1.3), meromorphic in $t \in \mathbb{C}^N$. Here, $\mathcal{F}^\ell_r(x; t)$ are matrix elements of the boundary quantum monodromy matrix and $\Omega = m_1^{\ell_1} \otimes \cdots \otimes m_1^{\ell_N}$ is the tensor product of highest-weight vectors $m_1^{\ell_s} \in M^\ell_s$ (see Section 6 for details).

Explicit formulae for functions $g_{\xi_+\xi_-}, h$ and $F^\ell$ are given in Section 6. We will discuss integral (not Jackson integral) solutions in a forthcoming paper. It yields a complete system of solutions to the boundary $qKZ$ equations.

Theorem 1.1 gives for $\ell_s \in \frac{1}{2}\mathbb{Z}_{>0}$ Jackson integral solutions of the boundary $qKZ$ equations taking values in

$$V^\ell = V^{\ell_1} \otimes \cdots \otimes V^{\ell_s}.$$
These can alternatively be obtained from a fusion procedure applied to the Jackson integral solutions when all $\ell_s = \frac{1}{2}$ derived earlier in \[32\] (see Subsection \[33\]). It seems though that the result for continuous spin $\ell_s \in \mathbb{C}$ (Theorem \[141\]) cannot be obtained from half-integer spins by analytic continuation.

1.4. Outline of the paper. In Section 2 and Section 3 we overview solutions to the quantum Yang-Baxter equation corresponding to quantum $\mathfrak{sl}_2$ and their fusion, following \[23, 24, 15\]. Reflection equations and the fusion of $K$-operators are discussed in Section 4. The boundary monodromy matrices defined in terms of these $R$- and $K$-operators are introduced in Section 5 as are the off-shell Bethe vectors $(\prod_{i=1}^{S} \mathcal{B}_{-}(x_i; t))\Omega$. In Section 3 we state and discuss the main theorem on the Jackson integral solutions of the boundary qKZ equations with continuous spins; its proof is given in Section 7. In Section 8 we show that the boundary qKZ equations \[1.3\] acting on $V^2(\ell_s \in \frac{1}{2}\mathbb{Z}_{\geq 0})$ and the associated Jackson integral solutions of the boundary qKZ equations can be obtained from the special case when all $\ell_s = \frac{1}{2}$ by fusion.

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2. Quantum affine $\mathfrak{sl}_2$ and $R$-operators

In this section we discuss basic facts on quantum affine $\mathfrak{sl}_2$ and its associated evaluation $R$- and $L$-operators, following \[17, 15\]. We use slightly different conventions compared to \[17, 15\] in order to obtain a direct match with the $R$- and $L$-operators of the 6-vertex model (see Subsection \[2.3\]).

2.1. Quantum affine algebra $\mathfrak{sl}_2$ and the universal $R$-matrix. We fix $\eta \in \mathbb{C}$ such that $p := e^{\eta}$ is not a root of unity. We write $p^x := e^{\eta x}$ for $x \in \mathbb{C}$.

Set $\mathfrak{h} = \mathcal{C} h_0 \oplus \mathcal{C} h_1$. Quantum affine $\mathfrak{sl}_2$ is the Hopf algebra $\hat{U}_\eta := U_\eta(\widehat{\mathfrak{sl}_2})$ over $\mathbb{C}$ with generators $e_i, f_i$ ($i = 0, 1$), $p^h$ ($h \in \mathfrak{h}$) and with defining relations

\[
p^0 = 1, \quad p^{h+h'} = p^h p^{h'},
\]

\[
p^h e_i p^{-h} = p^{\alpha_i(h)} e_i, \quad p^h f_i p^{-h} = p^{-\alpha_i(h)} f_i, \quad [e_i, f_j] = \delta_{ij} \frac{p^{h_i} - p^{-h_i}}{p-p^{-1}}.
\]

\[
e_i^3 e_j - (p^2 + 1 + p^{-2}) e_i^2 e_j e_i + (p^2 + 1 + p^{-2}) e_i e_j e_i^2 - e_j e_i^3 = 0, \quad i \neq j,
\]

\[
f_i^3 f_j - (p^2 + 1 + p^{-2}) f_i^2 f_j f_i + (p^2 + 1 + p^{-2}) f_i f_j f_i^2 - f_j f_i^3 = 0, \quad i \neq j
\]

for $i, j = 0, 1$ and $h, h' \in \mathfrak{h}$. Here $\alpha_i$ are linear functionals on $\mathfrak{h}$ satisfying $\alpha_j(h_i) = a_{ij}$ with Cartan matrix

\[
\begin{pmatrix}
a_{00} & a_{01} \\
a_{10} & a_{11}
\end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.
\]
The comultiplication \( \Delta \) and the counit \( \epsilon \) are determined by their action on generators:

\[
\Delta(p^h) = p^h \otimes p^h, \\
\Delta(e_i) = e_i \otimes 1 + p^{-h_i} \otimes e_i, \\
\Delta(f_i) = f_i \otimes p^{h_i} + 1 \otimes f_i
\]

and

\[
\epsilon(p^h) = 1, \quad \epsilon(e_i) = 0, \quad \epsilon(f_i) = 0.
\]

The antipode is determined by \( S(p^h) = p^{-h}, \ S(e_i) = -p^{h_i} e_i \) and \( S(f_i) = -f_i p^{-h_i} \).

The extension \( \tilde{U}_\eta \) of this algebra by generators \( p^{\lambda h} (\lambda \in \mathbb{C}) \) such that \([p^{\lambda h}, p^h] = 0\)
and \( p^{\lambda h} e_i = p^{\lambda h, a_i} e_i p^{\lambda h}, p^{\lambda h} f_i = p^{-\lambda h, a_i} f_i p^{\lambda h}\) is a quantized Kac-Moody algebra.

The corresponding Lie algebra has a non-degenerate scalar product and there is a universal \( R \)-matrix \( R \in \tilde{U}_\eta \otimes \tilde{U}_\eta \). It has the form

\[
R = \exp(\eta(c \otimes d + d \otimes c)) R
\]

where \( c = h_0 + h_1 \) and \( R \in \tilde{U}_\eta \otimes \tilde{U}_\eta \). In the category of modules where \( c \) acts by zero (zero-level representations), the element \( R \) satisfies all properties of the universal \( R \)-matrix:

\[
R \Delta(a) = \Delta^{op}(a) R, \\
(\Delta \otimes \text{Id})(R) = R_{13} R_{23}, \quad (\text{Id} \otimes \Delta)(R) = R_{13} R_{12}.
\]

Here, \( \Delta^{op} \) the opposite comultiplication. See also [15, Lecture 9] for further details (note though that we have a different convention for the comultiplication).

2.2. Evaluation representations. We write \( U_\eta \subset \tilde{U}_\eta \) for the Hopf subalgebra generated by \( e_1, f_1 \) and \( p^{\lambda h_1} (\lambda \in \mathbb{C}) \). It is the quantized universal enveloping algebra of \( \mathfrak{sl}_2 \).

Let \( \ell \in \mathbb{C} \) and \( M^{\ell} := \bigoplus_{n=1}^{\infty} \mathbb{C} m^{\ell}_n \) be a left \( U_\eta \)-module with the action given by

\[
\pi^{\ell}(p^{\lambda h_1}) m^\ell_n = p^{2\lambda(\ell+1-n)} m^{\ell}_{n-1}, \\
\pi^{\ell}(e_1) m^\ell_n = \frac{\sinh((n-1)\eta) \sinh((2\ell+2-n)\eta)}{\sinh(\eta)^2} m^{\ell}_{n-1}, \\
\pi^{\ell}(f_1) m^\ell_n = m^{\ell}_{n+1},
\]

where \( m^\ell_0 := 0 \). The \( U_\eta \)-module \( (\pi^{\ell}, M^{\ell}) \) is the Verma module with highest weight \( \ell \) and highest weight vector \( m^\ell_1 \).

If \( k \in \frac{1}{2} \mathbb{Z}_{\geq 0} \) the subspace \( N^k := \bigoplus_{n=2k+2}^{\infty} \mathbb{C} m^k_n \subset M^k \) is a \( U_\eta \)-submodule. We write \( V^k := M^k/N^k \) for the resulting quotient \( U_\eta \)-module. The cosets \( v^k_n := m^k_n + N^k \) \((1 \leq n \leq 2k+1)\) form a weight basis in \( V^k \). The associated representation map will be denoted by \( \varphi^k \) and for this representation of \( U_\eta \) we will write \((\varphi^k, V^k)\).

For each \( x \in \mathbb{C} \) there exists a unique unit-preserving algebra homomorphism \( \phi_x : \tilde{U}_\eta \to U_\eta \) satisfying

\[
\phi_x(p^{\lambda h_0}) = p^{-\lambda h_1}, \quad \phi_x(p^{\lambda h_1}) = p^{\lambda h_1}, \\
\phi_x(e_0) = e^{-x} f_1, \quad \phi_x(e_1) = e^{-x} e_1, \\
\phi_x(f_0) = e^x e_1, \quad \phi_x(f_1) = e^x f_1.
\]
Given a representation $\pi$ of $\mathcal{U}_q$ on $V$ we write $\pi_x := \pi \circ \phi_x$, which turns $V$ in a representation of $\tilde{\mathcal{U}}_q$ called the evaluation representation. Sometimes we will denote it by $V(x)$.

In what follows we will work with evaluation representation $(\pi_x^k, V^k)$ and $(\pi_x^\ell, M^\ell)$, where $k \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ and $\ell \in \mathbb{C}$.

2.3. Evaluation $R$-and $L$-operators. We follow here [15, Lecture 9]. Fix $x, y \in \mathbb{C}$ with $\Im(x - y) \ll 0$. For $k, \ell \in \mathbb{C}$ the evaluation of the truncated universal $R$-matrix

\[(\pi_x^k \otimes \pi_y^\ell)(\mathcal{R})\]

is a linear operator on $M^k \otimes M^\ell$ which only depends on the difference $x - y$ of $x$ and $y$. It acts on the tensor product of highest weight vectors as

\[(\pi_x^k \otimes \pi_y^\ell)(\mathcal{R})m_1^k \otimes m_1^\ell = \alpha^{kt}(x - y)m_1^k \otimes m_1^\ell\]

where $\alpha^{kt}(x - y)$ is invertible for generic $p$ and $x - y$. Define

\[\mathcal{R}^{kt}(x - y) := \alpha^{kt}(x - y)^{-1}(\pi_x^k \otimes \pi_y^\ell)(\mathcal{R}).\]

The operator $\mathcal{R}^{kt}(x - y)$ intertwines the action of $\tilde{\mathcal{U}}_q$ with its opposite

\[(\mathcal{R}^{kt}(x - y)(\pi_x^k \otimes \pi_y^\ell)(\Delta(X))) = (\pi_x^k \otimes \pi_y^\ell)(\Delta^{op}(X))\mathcal{R}^{kt}(x - y), \quad X \in \tilde{\mathcal{U}}_q\]

and satisfies $\mathcal{R}^{kt}(x - y)m_l^k \otimes m_1^\ell = m_1^k \otimes m_l^\ell$. These properties determine $\mathcal{R}^{kt}(x - y)$ uniquely for generic values of $x - y$.

The dependence of the operator $\mathcal{R}^{kt}(x - y)$ on $x, y, k, \ell$ is as a rational function in $e^{x - y}, p^k$ and $p^\ell$. Analytic continuation thus gives a well-defined linear operator $\mathcal{R}^{kt}(x - y)$ on $M^k \otimes M^\ell$ for generic values of $x - y$, which can be characterized by the same intertwining property (2.1) with respect to the action of $\tilde{\mathcal{U}}_q$.

Let $k \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ and write $\text{pr}^k : M_k(x) \to V_k(x)$ of $\tilde{\mathcal{U}}_q$-modules. Note that for $k \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, there exists a unique linear map

\[L^{kt}(x - y) : V^k \otimes M^\ell \to V^k \otimes M^\ell\]

depending rationally on $e^{x - y}$ and satisfying

\[(\text{pr}^k \otimes \text{Id}_{M^\ell})\mathcal{R}^{kt}(x - y) = L^{kt}(x - y)(\text{pr}^k \otimes \text{Id}_{M^\ell}).\]

Similarly, for $k, \ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, there exists a unique linear map

\[R^{kt}(x - y) : V^k \otimes V^\ell \to V^k \otimes V^\ell\]

satisfying

\[(\text{pr}^k \otimes \text{pr}^\ell)\mathcal{R}^{kt}(x - y) = R^{kt}(x - y)(\text{pr}^k \otimes \text{pr}^\ell).\]

2.4. Basic properties of evaluation $R$-and $L$-operators. We follow [17] and for details [15, Lecture 9].

The basic properties of the universal $R$-matrix give the quantum Yang-Baxter equation

\[\mathcal{R}^{kt}_{12}(x - y)\mathcal{R}^{km}_{13}(x - z)\mathcal{R}^{\ell m}_{23}(y - z) = \mathcal{R}^{\ell m}_{23}(y - z)\mathcal{R}^{km}_{13}(x - z)\mathcal{R}^{kt}_{12}(x - y)\]

as linear operators on $M^k \otimes M^\ell \otimes M^m$. In addition, the operator $\mathcal{R}^{kt}(x - y)$ satisfies unitarity:

\[\mathcal{R}^{kt}(x - y)^{-1} = \mathcal{R}^{kt}_{21}(y - x),\]
Proof. Hence it suffices to show that for generic $x$

$$R^β_1(x) := P^β_1 R^β_1(x) R^β_1 : M^k ⊗ M^ℓ → M^k ⊗ M^ℓ$$

and $P^β_1 : M^k ⊗ M^ℓ → M^ℓ ⊗ M^k$ is the permutation operator.

Both properties descend naturally to the $L$-operators and finite $R$-operators. In particular, the familiar RLL-relations

$$R^β_1(x) y_k(x-z) R^β_{13}(y-z) R^β_{23}(x-z) R^β_{12}(x-y)$$

for $k, ℓ ∈ \frac{1}{2}Z_{≥0}$ as well as the quantum Yang-Baxter equation for the $R$-operators $R^β_1(x)$ ($k, ℓ ∈ \frac{1}{2}Z_{≥0}$) follow immediately from the quantum Yang-Baxter equation for $R^β_1$.

The next property of $R^β_1(x)$ is $P$-symmetry:

**Lemma 2.1.** As linear maps on $M^k ⊗ M^ℓ$ we have for generic $x ∈ C$,

$$R^β_1(x) = R^β_1(x).$$

**Proof.** Write $T^β_1(x) for the left-hand side of (2.4). Then clearly

$$T^β_1(x)m_1^k ⊗ m_1^ℓ = m_1^k ⊗ m_1^ℓ = R^β_1(x)m_1^k ⊗ m_1^ℓ.$$ 

Hence it suffices to show that for generic $x$ and $y$,

$$T^β_1(x - y)(π_x^k ⊗ π_x^ℓ)(Δ(X)) = (π_x^k ⊗ π_x^ℓ)(Δ^{op}(X))T^β_1(x - y), \quad ∀ X ∈ \hat{U}_v.$$ 

This is clear for $X = p^h$ ($h ∈ h$). For $X = e_0, f_1$ it is a direct consequence of the identity

$$(π_x^k ⊗ π_x^ℓ)(Δ^{op}(e_0)) = (π_x^k ⊗ π_x^ℓ)(Δ(f_1))$$

and (2.1). For the algebraic generators $X = e_1, f_0$ it follows similarly from (2.1) using the fact that

$$(π_x^k ⊗ π_x^ℓ)(Δ^{op}(e_1)) = (π_x^k ⊗ π_x^ℓ)(Δ(f_0)).$$

Finally we discuss crossing symmetry. We start with crossing symmetry for $L$-operators:

**Lemma 2.2.** Let $k ∈ \frac{1}{2}Z_{≥0}$ and $ℓ ∈ C$. Let $w^k : V^k ↘ V^k$ be the linear isomorphism defined by

$$w^k(v^k_n) := c_n v^k_{2k+2-n}$$

with $c_n ∈ C^+$ determined by the recursion $c_{n+1} := -c_n p^{2k+1-2n}$ and $c_1 := 1$. Then

$$L^β_1(x) y_k(x) = α^β_1(x) α^β_1(x - η)(w^k ⊗ Id_{M^ℓ})L^β_1(x - η)w^k ⊗ Id_{M^ℓ}^{-1} \quad \text{with } T_1 \text{ the transpose in the first tensor component with respect to the weight basis.}$$

**Proof.** For an evaluation module $(π, V)$ over $\hat{U}_v$ we write $(π^*, V^*)$ for the graded dual $V^*$ of $V$ with respect to the weight grading, with $\hat{U}_v$-action $(π^*(X)ϕ)(v) := ϕ(π(S(X)v))$. If $A : V → V$ is a linear map, then we write $A^t : V^* → V^*$ for the corresponding dual linear operator.

It follows from the identity $(S ⊗ \text{Id})(R) = R^{-1}$ that

$$(\pi_x^k ⊗ π_x^ℓ)(R) = ((\pi_x^k ⊗ π_x^ℓ)(R^{-1}))^{t_1}.$$ 

Here $t_1$ means taking the dual with respect to the first component in the tensor product. Write $\{v^k_n\}$ for the basis of $(V^k)^*$ dual to the weight basis $\{v^k_n\}_n$ of $V^k$. We identify $V^k ≃ (V^k)^*$ by $v^k_n → (v^k_n)^*$ (the dual $A^t$ of a linear operator $A : V^k → V^k$ then corresponds to the transpose $A^T$ of $A$ with respect to the
weight basis \( \{ v^n \} \) of \( V^k \). Accordingly we interpret the map \( w^k \) as a linear map \( w^k : V^k \to (V^k)^* \), in which case it defines an isomorphism \( V^k(x - \eta) \xrightarrow{\sim} V^k(x)^* \) of \( \mathcal{U}_q \)-modules. Consequently

\[
L^{kt}(-x + y)^T_1 = \alpha^{kt}(x - y)(\pi^k_x \otimes \pi^\ell_y)(R^{-1})^T_1
\]

\[
= \alpha^{kt}(x - y)((\pi^k_x)^* \otimes \pi^\ell_y)(R)
\]

\[
= \alpha^{kt}(x - y)(w^k \otimes \text{Id}_M)(\pi^k_x \otimes \pi^\ell_y)(R)(w^k \otimes \text{Id}_M)^{-1},
\]

where we have used (2.6) for the second equality. This proves the desired result. \( \square \)

**Remark 2.3.** For \( k \in \mathbb{C} \) the canonical linear isomorphism \( M^k \xrightarrow{\sim} (M^k)^{**} \) defines an isomorphism \( M^k(x - 2\eta) \xrightarrow{\sim} M^k(x)^{**} \) of \( \mathcal{U}_q \)-modules (cf. Lemma 2.2). It then follows from a double application of (2.6) (for arbitrary evaluation modules) that

\[
R^{kt}(x - 2\eta) = \frac{\alpha^{kt}(x)}{\alpha^{kt}(x - 2\eta)}(((R^{kt}(x)^{-1}T_1)^{-1})^T_1.
\]

Note the difference with [15, Prop. 9.5.2], which involves an additional conjugation by a diagonal operator in the first tensor component.

### 2.5. Explicit formulae for \( L \)-operators.

It is possible to compute \( L^{kt}(x) \) explicitly using the expression of the universal \( R \)-matrix (a comprehensive survey of this can be found in [4]). This leads to the formulae

\[
L^{lt}(x)(v^n_1 \otimes m^n) = \frac{\sinh(x + (\frac{3}{2} + \ell - n)\eta)}{\sinh(x + (\frac{3}{2} + \ell)\eta)} v^n_1 \otimes m^n + e^{(\ell + \frac{1}{2} - n)\eta}\sinh((n - 1)\eta)\sinh((2\ell + 2 - n)\eta)\frac{1}{\sinh(\eta)} v^n_1 \otimes m^n_{n-1}
\]

and

\[
L^{lt}(x)(v^n_2 \otimes m^n) = e^{-(-\ell + \frac{1}{2} + n)\eta} \frac{\sinh(\eta)}{\sinh(x + (\ell + \frac{1}{2})\eta)} v^n_1 \otimes m^n_{n+1}
\]

\[
+ \frac{\sinh(x + (-\ell + \frac{1}{2} + n)\eta)}{\sinh(x + (\ell + \frac{1}{2})\eta)} v^n_2 \otimes m^n.
\]

Note that exponential factors can be removed by a similarity transformation. After this, the result coincides with the \( L \)-operator found in [23]. It follows from these formulae that the finite \( R \)-operator \( R^{\frac{1}{2}t}(x) \) is the 6-vertex \( R \)-operator:

\[
R^{\frac{1}{2}t}(x) = \frac{1}{\sinh(x + \eta)} \begin{pmatrix}
\sinh(x + \eta) & 0 & 0 \\
0 & \sinh(x) & \sinh(\eta) \\
0 & \sinh(\eta) & \sinh(x) \\
0 & 0 & \sinh(x + \eta)
\end{pmatrix}
\]

with respect to the ordered basis \( \{ v^{\frac{1}{2}}_1 \otimes v^{\frac{1}{2}}_1, v^{\frac{1}{2}}_1 \otimes v^{\frac{1}{2}}_2, v^{\frac{1}{2}}_2 \otimes v^{\frac{1}{2}}_1, v^{\frac{1}{2}}_2 \otimes v^{\frac{1}{2}}_2 \} \) of \( V^{\frac{1}{2}} \otimes V^{\frac{1}{2}} \).

The crossing symmetry of the \( L \)-operators (Lemma 2.2) becomes

\[
L^{kt}(-x)^T_1 = \vartheta^{lt}(x)\sigma^n_l L^{lt}(x - \eta)\sigma^n_l
\]

as linear operators on \( V^{\frac{1}{2}} \otimes M^{\ell} \), where \( T_1 \) is the matrix transpose with respect to the weight basis in \( V^{\frac{1}{2}} \) and

\[
\sigma^n := \begin{pmatrix}
0 & -\sqrt{-1} \\
\sqrt{-1} & 0
\end{pmatrix}, \quad \vartheta^{lt}(x) = \frac{\sinh(x - (\frac{1}{2} + \ell)\eta)}{\sinh(x - (\frac{1}{2} + \ell)\eta)}.
\]
Lemma 3.2. \( V^k(x) \otimes V^\ell(y) \) is reducible.

We need explicit expressions for its action in case that the tensor product of evaluation modules plays a crucial role. We need explicit expressions for its action in case that the tensor product of evaluation modules plays a crucial role.

3. Fusion of \( R \)-operators

3.1. Tensor products of evaluation representations. Let \( k, \ell \in \frac{1}{2} \mathbb{Z}_{\geq 0} \). By [5] Thm. 4.8 the tensor product \( \tilde{U}_\eta \)-module \( V^k(x) \otimes V^\ell(y) \) is irreducible for generic \( x, y \in \mathbb{C} \). For the fusion of \( R \)- and \( K \)-operators we need to focus on the special cases that the \( \tilde{U}_\eta \)-module \( V^k(x) \otimes V^\ell(y) \) is reducible.

For \( k, \ell \in \frac{1}{2} \mathbb{Z}_{\geq 0} \) we write \( P^{k\ell} : V^k \otimes V^\ell \to V^\ell \otimes V^k \) for the permutation operator. The following result should be compared with [5, Prop. 4.9]. The proof is by a straightforward computation.

Proposition 3.1. Let \( k \in \frac{1}{2} \mathbb{Z}_{\geq 0} \).

(i) The linear map \( \iota^k : V^{k+\frac{1}{2}} \hookrightarrow V^\frac{1}{2} \otimes V^k \), defined by

\[
\iota^k(v_n^{k+\frac{1}{2}}) = e^{\frac{2}{\pi}(n-1)}v_1^{\frac{1}{2}} \otimes v_n^k + e^{-\frac{2}{\pi}(n-2-2k)} \sinh((n-\eta)\frac{1}{2})v_2^{\frac{1}{2}} \otimes v_{n-1}^k,
\]

defines a \( \tilde{U}_\eta \)-intertwiner \( \iota^k : V^{k+\frac{1}{2}}(x) \hookrightarrow V^\frac{1}{2}(x-\eta) \otimes V^k(x+\eta) \).

(ii) The linear map \( j^k := P^{k\ell} \iota^k : V^{k+\frac{1}{2}} \hookrightarrow V^k \otimes V^\frac{1}{2} \) defines a \( \tilde{U}_\eta \)-intertwiner

\[
j^k : V^{k+\frac{1}{2}}(x) \hookrightarrow V^k(x-\eta) \otimes V^\frac{1}{2}(x+\eta).
\]

Note that the intertwiners \( \iota^k \) and \( j^k \) do not depend on \( x \) as linear maps. We add the subscript \( x \) to clarify the \( \tilde{U}_\eta \)-action we are considering.

3.2. Fusion operators. It follows from Lemma 2.1 that the \( R \)-operators \( R^{k\ell}(x) \) \((k, \ell \in \frac{1}{2} \mathbb{Z}_{\geq 0})\) are \( P \)-symmetric. In the remainder of this section we focus on the fusion of the \( R \)-operators \( R^{k\ell}(x) \) \((k, \ell \in \frac{1}{2} \mathbb{Z}_{\geq 0})\).

For the fusion of the \( R \)-operators the interpretation of \( R \)-operators as intertwiners between tensor products of evaluation modules plays a crucial role. We need explicit expressions for its action in case that the tensor product of the evaluation modules is reducible.

Lemma 3.2. For \( k \in \frac{1}{2} \mathbb{Z}_{\geq 0} \) the linear operators \( R^{\frac{1}{2}}(x) \) and \( R^{\frac{1}{2}}(x) \) are regular at \( x = (k+\frac{1}{2})\eta \). The resulting linear maps \( S^k := P^{k\eta} R^{\frac{1}{2}}((k+\frac{1}{2})\eta) \) and \( T^k := P^{k\eta} R^{\frac{1}{2}}((k+\frac{1}{2})\eta) \), which we will view as \( \tilde{U}_\eta \)-intertwiners

\[
S_x^k : V^\frac{1}{2}(e^{x+\eta}) \otimes V^k(e^{x-\eta}) \to V^k(e^{x-\eta}) \otimes V^\frac{1}{2}(e^{\eta+kx}),
\]

\[
T_x^k : V^k(e^{x+\eta}) \otimes V^\frac{1}{2}(e^{x-\eta}) \to V^\frac{1}{2}(e^{x-\eta}) \otimes V^k(e^{x+\eta})
\]

are explicitly given by

\[
S^k(v_1^\frac{1}{2} \otimes v_n^k) = \frac{\sinh((2k+2-n)\eta)}{\sinh((2k+1)\eta)} e^{-\frac{2}{\pi}(n-1)}j^k(v_n^{k+\frac{1}{2}}),
\]

\[
S^k(v_2^\frac{1}{2} \otimes v_n^k) = \frac{\sinh(\eta)}{\sinh((2k+1)\eta)} e^{-\frac{2}{\pi}(n-2k-1)}j^k(v_n^{k+\frac{1}{2}}).
\]
\[ T^k(v_n^k \otimes v_2^{k/2}) = \frac{\sinh((2k + 2 - n)\eta)}{\sinh((2k + 1)\eta)} e^{-\frac{\pi}{2}(n-1)} i_k^k(v_n^k \otimes v_2^{k/2}), \]

\[ T^k(v_n^k \otimes v_2^{k/2}) = \frac{\sinh(\eta)}{\sinh((2k + 1)\eta)} e^{\frac{\pi}{2}(n-2k-1)} i_k^k(v_n^k \otimes v_2^{k/2}). \]

**Proof.** By $P$-symmetry we have $R^{k+\frac{j}{2}}(x) = P^{k} R^{\frac{k+j}{2}}(x) P^{k}$, and Proposition 3.1 gives $i^k = P^{k+j} j^k$. So it suffices to prove the statement for $S^k$. Using the fact that $(Id_{V^{\frac{k}{2}} \otimes pr^k}) L^{\frac{k+j}{2}}(x) = R^{\frac{k+j}{2}}(x) (Id_{V^{\frac{j}{2}}} \otimes pr^k)$, Remark 2.5 gives explicit formulae for $S^k$. Comparing those formulae with the explicit formulae for $j^k$ (see Proposition 3.1) now leads to the desired result. \[ \square \]

### 3.3. The fusion formula for the $R$- and $L$-operators.

The fusion formulae for the $R$-operators $R^{k\ell}(x)$ ($k, \ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}$) and $L^{k\ell}(x)$ ($k \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, $\ell \in \mathbb{C}$) follow directly from the representation-theoretic considerations of the previous subsection. Recall the linear map $i^k : V^{k+\frac{j}{2}} \hookrightarrow V^{k} \otimes V$ from Proposition 3.1.

**Proposition 3.3.** For $k \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ and $\ell \in \mathbb{C}$ we have the fusion formula

\[
(i^k \otimes Id_{M^\ell}) L^{k+\frac{j}{2},\ell}(x-y) = L^{\frac{k+j}{2},\ell}_{13}(x-k\eta-y)L^{k\ell}_{23}(x + \frac{\eta}{2} - y)(i^k \otimes Id_{M^\ell})
\]

as linear maps $V^{k+\frac{j}{2}} \otimes M^\ell \rightarrow V^{\frac{k+j}{2}} \otimes V^k \otimes M^\ell$.

**Proof.** Using the fact that

\[
(\pi_{\frac{k}{2}}^\frac{j}{2} \otimes \pi_y^\ell \otimes \pi_x^\ell)(\mathcal{R}_{13} \mathcal{R}_{23}) = (\pi_{\frac{k}{2}}^\frac{j}{2} \otimes \pi_y^\ell \otimes \pi_x^\ell)((\Delta \otimes Id)(\mathcal{R}))
\]

and the intertwining property of $i^k_x$ (see Proposition 3.1), gives

\[
L^{\frac{k+j}{2},\ell}_{13}(x-k\eta-y)L^{k\ell}_{23}(x + \frac{\eta}{2} - y)(i^k \otimes Id_{M^\ell}) = (i^k \otimes Id_{M^\ell}) L^{k+\frac{j}{2},\ell}(x-y)
\]

as linear maps $V^{k+\frac{j}{2}}(x) \otimes M^\ell(y) \rightarrow V^{\frac{k+j}{2}}(x-k\eta) \otimes V^k(x + \frac{\eta}{2} \otimes M^\ell(y)$. The result follows now immediately. \[ \square \]

**Remark 3.4.** Proposition 3.3 leads for $k, \ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ to the fusion formula

\[
(i^k \otimes Id_{V^{\ell}}) R^{k+\frac{j}{2},\ell}(x-y) = R^{\frac{k+j}{2},\ell}_{13}(x-k\eta-y)R^{k\ell}_{23}(x + \frac{\eta}{2} - y)(i^k \otimes Id_{V^{\ell}})
\]

for the $R$-operators.

**Remark 3.5.** Another approach to fusion formulae for $L$-operators (originating from [24]) is by specialization of the RLL relations (2.4) at values of $x-y$ for which $R^{k\ell}_{12}(x-y)$ is not invertible. For instance, in the present setting (24) gives

\[
(T^k \otimes Id_{M^\ell}) L^{k\ell}_{13}(x + \frac{\eta}{2} - y)L^{k\ell}_{23}(x-k\eta-y) = \]

\[
= L^{\frac{k+j}{2},\ell}_{13}(x-k\eta-y)L^{k\ell}_{23}(x + \frac{\eta}{2} - y)(T^k \otimes Id_{M^\ell}),
\]

which shows directly that the operator $L^{\frac{k+j}{2},\ell}_{13}(x-k\eta-y)L^{k\ell}_{23}(x + \frac{\eta}{2} - y)$ restricts to a linear endomorphism on the image of $T^k \otimes Id_{M^\ell}$. The resulting linear operator is equivalent to the fused $L$-operator $L^{k+\frac{j}{2},\ell}(x-y)$ in view of Lemma 3.2.
4. The Reflection Equation, Fusion of $K$-operators and Diagonal $K$-operators

4.1. Reflection equations. A collection of linear maps $\mathcal{K}^k(x) : M^k \to M^k$ is called a family of higher-spin $K$-operators if they satisfy the reflection equations in $M^k \otimes M^\ell$:

\begin{equation}
\mathcal{R}^{k\ell}(x-y)\mathcal{K}^k_1(x)\mathcal{R}^{k\ell}(x+y)\mathcal{K}^\ell_2(y) = \mathcal{K}^\ell_2(y)\mathcal{R}^{k\ell}(x+y)\mathcal{K}^k_1(x)\mathcal{R}^{k\ell}(x-y).
\end{equation}

Remark 4.1. The natural representation-theoretic forms of the reflection equations \eqref{eq:reflection} involve $\mathcal{R}^{k\ell}_\pm(x) = \mathcal{P}^{k\ell}\mathcal{R}^{k\ell}(x)\mathcal{P}^{k\ell}$, cf. \cite{34}. However, the $P$-symmetry \eqref{eq:psym} of the $R$-operators has the simplifying effect that all $R$-operators can be put into the form $\mathcal{R}^{k\ell}$ and consequently the distinction between left and right versions of reflection equations disappears (cf. \cite{34}).

Suppose that for $k \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ there exists a (necessarily unique) linear map $K^k(x) : V^k \to V^k$ such that

\[ \text{pr}^k \circ K^k(x) = K^k(x) \circ \text{pr}^k. \]

Then the equations \eqref{eq:reflection} naturally give rise to (semi-)finite-dimensional versions which will also be referred to as reflection equations. More precisely, when $k \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ equation \eqref{eq:reflection} projects to the following equation in $V^k \otimes M^\ell$:

\begin{equation}
L^{k\ell}(x-y)K^k_1(x)L^{k\ell}(x+y)K^\ell_2(y) = K^\ell_2(y)L^{k\ell}(x+y)K^k_1(x)L^{k\ell}(x-y);
\end{equation}

Furthermore, when $k, \ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ equation \eqref{eq:reflection} then projects to the following equation in $V^k \otimes V^\ell$:

\begin{equation}
R^{k\ell}(x-y)K^k_1(x)R^{k\ell}(x+y)K^\ell_2(y) = K^\ell_2(y)R^{k\ell}(x+y)K^k_1(x)R^{k\ell}(x-y).
\end{equation}

Just as solutions to the quantum Yang-Baxter equation are related to the representation theory of quantized universal enveloping algebras, solutions to the reflection equation ($K$-operators) are related to co-ideal subalgebras of quantized universal enveloping algebras. We will discuss it briefly in Subsection 4.4.

4.2. $K$-matrices for spin-$\frac{1}{2}$. With respect to the 6-vertex $R$-operator $R^{\frac{1}{2}\frac{1}{2}}(x)$ (see \eqref{eq:6vertex}), the general diagonal solution of \eqref{eq:reflection} (for $k = \ell = \frac{1}{2}$) is given by Cherednik’s \cite{7} one-parameter family

\[ K^{\xi,\frac{1}{2}}(x) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\sinh(\xi-x)}{\sinh(\xi+x)} \end{pmatrix} \]

written with respect to the basis $(v_1^\frac{1}{2}, v_2^\frac{1}{2})$ of $V^\frac{1}{2}$. To simplify notations we will use $R(x)$ for $R^{\frac{1}{2}\frac{1}{2}}(x)$ and $K^{\xi}(x)$ for $K^{\xi,\frac{1}{2}}(x)$. In other words, this matrix acts on the weight basis as

\[ K^{\xi}(x)v_1^\frac{1}{2} = v_1^\frac{1}{2}, \quad K^{\xi}(x)v_2^\frac{1}{2} = \frac{\sinh(\xi - x)}{\sinh(\xi + x)}v_2^\frac{1}{2}. \]

Remark 4.2. The proof that $K^{\xi}(x)$ satisfies \eqref{eq:reflection} for $k = \ell = 1/2$ reduces to the identity

\[ \sum_{\epsilon_1, \epsilon_2 \in \{\pm 1\}} \epsilon_1 \epsilon_2 \frac{\sinh(\xi + \epsilon_1 x) \sinh(\xi + \epsilon_2 y)}{\sinh(\epsilon_1 x + \epsilon_2 y)} = 0 \]

cf. \cite{32}. 


The reflection operator $K^\xi(x)$ satisfies the boundary crossing symmetry:

\begin{equation}
(4.4) \quad \text{Tr}_2\left(R_{12}(2x - 2\eta)P_{12}K^\xi_2(x)\right) = \frac{\sinh(\xi + x - \eta)\sinh(2x)}{\sinh(\xi + x)\sinh(2x - \eta)} K^\xi_1(x - \eta),
\end{equation}

where $\text{Tr}_2$ is the partial trace over the second tensor component of $V^{\frac{1}{2}} \otimes V^{\frac{1}{2}}$ and $P = P^{\frac{1}{2}}$. The identity (4.4) is equivalent to the trigonometric identity

\begin{equation}
(4.5) \quad \sinh(\xi + x)\sinh(x - z) + \sinh(\xi - x)\sinh(x + z) = \sinh(\xi - z)\sinh(2x).
\end{equation}

In Lemma 7.8 we prove a multivariate extension of (4.5), which plays an important role in the proof of the main result (Theorem 6.2).

A three-parameter family of solutions $K^{\frac{1}{2}}_2(x)$ of (4.3) (with $k = \ell = \frac{1}{2}$) is known, see [12, 29].

4.3. Fusion formula for $K$-operators when $k, \ell \in \mathbb{Z}_{\geq 0}$. Notwithstanding Remark 4.1, in order to put formulas in the natural representation-theoretic form, we will sometimes use the notation $R_{21}^{k\ell}(x)$. The intertwining property of the $R$-operator $R^{k\ell}(x)$ gives

\[ R_{21}^{k\ell}(x - y)(\pi_k^\ell \otimes \pi_k^\ell)\left(\Delta_{op}(X)\right) = (\pi_{-x} \otimes \pi_{-y})\left(\Delta(X)\right) R_{21}^{k\ell}(x - y), \quad \forall X \in \hat{\mathcal{U}}_n. \]

**Proposition 4.3.** Suppose that the $K^{\frac{1}{2}}_2(x)$ are complex-linear operators on $V^{\frac{1}{2}}$ depending meromorphically on $x \in \mathbb{C}$ and satisfying the reflection equation

\begin{equation}
(4.6) \quad R_{21}^{k\ell}(x - y)K^\xi_1(x)R_{21}^{k\ell}(x + y)K^\xi_2(y) = K^\xi_2(y)R_{21}^{k\ell}(x + y)K^\xi_1(x)R_{21}^{k\ell}(x - y)
\end{equation}

as linear operators on $V^{\frac{1}{2}} \otimes V^{\frac{1}{2}}$. Then there exist unique complex-linear operators $K^k(x)$ on $V^k$ for $k \in \frac{1}{2}\mathbb{Z}_{\geq 2}$ satisfying

\begin{equation}
(4.7) \quad j^k K^{k+\frac{1}{2}}(x) = P^{\frac{1}{2}} K^k_1(x - k\eta)R_{21}^{k\ell}(2x - (k - \frac{1}{2})\eta)K^k_2(x + \frac{\eta}{2})j^k
\end{equation}

for all $k \in \frac{1}{2}\mathbb{Z}_{\geq 2}$. Furthermore,

\begin{equation}
(4.8) \quad R_{21}^{k\ell}(x - y)K^k_1(x)R_{21}^{k\ell}(x + y)K^k_2(y) = K^k_2(y)R_{21}^{k\ell}(x + y)K^k_1(x)R_{21}^{k\ell}(x - y)
\end{equation}

as linear operators on $V^k \otimes V^\ell$ for all $k, \ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}$.

**Remark 4.4.** We will always set $K^0(x) := \text{Id}_{V^0}$. Then formulæ (4.7) and (4.8) are trivially satisfied for $k = 0$ and/or $\ell = 0$.

**Remark 4.5.** Fusion of $K$-operators has been studied before in various different contexts, see, e.g., [13, 25, 21, 28, 20, 27, 38].

**Proof of Proposition 4.3.** Let $m \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ and suppose that the $K$-operators $K^k(x)$ have been constructed for $k \leq m$ satisfying (4.7) for $m < k$ and satisfying (4.8) for $k, l \leq m$.

Consider (4.8) for $\ell = \frac{1}{2}$ and $k = m$, and replace $x$ by $x + \frac{1}{2}$ and $y$ by $x - m\eta$. Then we obtain

\begin{equation}
S^m K^m_2(x + \frac{\eta}{2})R_{21}^{m\ell}(2x - (m - \frac{1}{2})\eta)K^m_2(x - m\eta) = P^{\frac{1}{2}} K^m_1(x - m\eta)R_{21}^{m\ell}(2x - (m - \frac{1}{2})\eta)K^m_2(x + \frac{\eta}{2})T^m
\end{equation}
with $R^{k\ell}(x) := P^{k\ell} R^{k\ell}(x)$ (see Lemma 3.2 for the definition of $S^m$ and $T^m$). Since the images of the linear maps $T^m$ and $\iota^m$ coincide by Lemma 3.2 it follows that the image of the linear map

$$P^{\frac{1}{2}m} K_1^{\frac{1}{2}}(x - m\eta) R^{\frac{1}{2}m}(2x - (m - \frac{1}{2})\eta) K_2^{\frac{1}{2}}(x + \frac{\eta}{2}) \iota^m$$

is contained in the image of $S^m$. By Lemma 3.2 again, the image of $S^m$ coincides with the image of $j^m$, hence there exists a unique linear operator $K^{m+\frac{1}{2}}(x)$ on $V^{m+\frac{1}{2}}$ such that

$$j^m K^{m+\frac{1}{2}}(x) = P^{\frac{1}{2}m} K_1^{\frac{1}{2}}(x - m\eta) R^{\frac{1}{2}m}(2x - (m - \frac{1}{2})\eta) K_2^{\frac{1}{2}}(x + \frac{\eta}{2}) \iota^m.$$

It remains to show that (4.8) is valid for three cases:

1. $(k, \ell) = (m + \frac{1}{2}, \ell)$ with $\ell \leq m$.
2. $(k, \ell) = (k, m + \frac{1}{2})$ with $k \leq m$.
3. $(k, \ell) = (m + \frac{1}{2}, m + \frac{1}{2})$.

If the reflection equation (4.8) is proved for case (1), then (2) follows from (1) using the unitarity of the $R$-operator, and (3) follows from (1) and (2) by taking $\ell = m + \frac{1}{2}$ in the following proof of (1).

Proof of (1): Suppose $\ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ and $\ell \leq m$. Using the fusion formulae of the $R$- and $K$-operators we obtain

$$R^{\frac{1}{2}m + \frac{1}{2}}(x - y) K_1^{m + \frac{1}{2}}(x) R^{\frac{1}{2}m + \frac{1}{2}}(x + y) K_2^\ell(y) =$$

$$= (\iota^m \otimes \text{Id}_{V^\ell})^{-1} R^{\frac{1}{2}m}(x - m\eta - y) R^{\frac{1}{2}m}(x + \frac{\eta}{2} - y) (\iota^m \otimes \text{Id}_{V^\ell})$$

$$\times (j^m \otimes \text{Id}_{V^\ell})^{-1} P^{\frac{1}{2}m} K_1^{\frac{1}{2}}(x - m\eta) R^{\frac{1}{2}m}(2x - (m - \frac{1}{2})\eta) K_2^{m}(x + \frac{\eta}{2})$$

$$\times R^{\frac{1}{2}m}(x - m\eta + y) R_2^{\frac{1}{2}m}(x + \frac{\eta}{2} + y) K_3(y) (\iota^m \otimes \text{Id}_{V^\ell}),$$

where the sublabels 1, 2, 3 in the right-hand side stand for the first, second and third tensor component in $V^{\frac{1}{2}} \otimes V^m \otimes V^\ell$ and the sublabels 1, 2 in the left-hand side stand for the first and second tensor component in $V^{m + \frac{1}{2}} \otimes V^\ell$. Using $P^{\frac{1}{2}m} j^m = \iota^m$ the expression simplifies to

$$(\iota^m \otimes \text{Id}_{V^\ell})^{-1} R^{\frac{1}{2}m}(x - m\eta - y) K_1^{\frac{1}{2}}(x - m\eta)$$

$$\times R^{\frac{1}{2}m}(x + \frac{\eta}{2} - y) R^{\frac{1}{2}m}(2x - (m - \frac{1}{2})\eta) R^{\frac{1}{2}m}(x - m\eta + y)$$

$$\times K_2^{m}(x + \frac{\eta}{2}) R^{\frac{1}{2}m}(x + \frac{\eta}{2} + y) K_3(y) (\iota^m \otimes \text{Id}_{V^\ell}) \iota^m \otimes \text{Id}_{V^\ell}).$$

Using the quantum Yang-Baxter equation in the second line the expression can be rewritten as

$$(\iota^m \otimes \text{Id}_{V^\ell})^{-1} R^{\frac{1}{2}m}(x - m\eta - y) K_1^{\frac{1}{2}}(x - m\eta)$$

$$\times R^{\frac{1}{2}m}(x - m\eta + y) R^{\frac{1}{2}m}(2x - (m - \frac{1}{2})\eta)$$

$$\times R^{\frac{1}{2}m}(x + \frac{\eta}{2} - y) K_2(x + \frac{\eta}{2}) R^{\frac{1}{2}m}(x + \frac{\eta}{2} + y) K_3(y) (\iota^m \otimes \text{Id}_{V^\ell}).$$
Applying the reflection equation to the last line leads to the expression
\[
(t^m \otimes \text{Id}_{V^\ell})^{-1} R^{\ell}_{21} \left( x - m\eta - y \right) K_1^\ell(x - m\eta) R^{m\ell}_{13} \left( x - m\eta + y \right) K_3^\ell(y) \\
\times R^{m\ell}_{12} \left( 2x - (m - \frac{1}{2})\eta \right) R^{m\ell}_{32} \left( x + \frac{\eta}{2} + y \right) \\
\times K_2^{m\ell} \left( x + \frac{\eta}{2} \right) R^{m\ell}_{23} \left( x + \frac{\eta}{2} - y \right) (\ell^m \otimes \text{Id}_{V^\ell}).
\]
Now applying the reflection equation to the first line gives
\[
(t^m \otimes \text{Id}_{V^\ell})^{-1} K_3^\ell(y) R^{\ell}_{31} \left( x - m\eta + y \right) K_1^\ell(x - m\eta) \\
\times R^{\ell}_{12} \left( x - m\eta - y \right) R^{m\ell}_{13} \left( 2x - (m - \frac{1}{2})\eta \right) R^{m\ell}_{32} \left( x + \frac{\eta}{2} + y \right) \\
\times K_2^{m\ell} \left( x + \frac{\eta}{2} \right) R^{m\ell}_{23} \left( x + \frac{\eta}{2} - y \right) (\ell^m \otimes \text{Id}_{V^\ell}).
\]
Applying the quantum Yang-Baxter equation to the second line leads to
\[
(t^m \otimes \text{Id}_{V^\ell})^{-1} K_3^\ell(y) R^{\ell}_{31} \left( x - m\eta + y \right) R^{m\ell}_{32} \left( x + \frac{\eta}{2} + y \right) \\
\times K_1^{\ell}(x - m\eta) R^{\ell}_{12} \left( x - m\eta - y \right) R^{m\ell}_{13} \left( 2x - (m - \frac{1}{2})\eta \right) K_2^{m\ell}(x + \frac{\eta}{2}) \\
\times R^{m\ell}_{12} \left( x - m\eta - y \right) R^{m\ell}_{23} \left( x + \frac{\eta}{2} - y \right) (\ell^m \otimes \text{Id}_{V^\ell}).
\]
The fusion formulae for the $R$- and $K$-operators and the fact that $P^{m\frac{\ell}{2}} = \ell^m$ show that the last expression equals
\[
K_3^\ell(y) R^{m\ell}_{12} \left( x + y \right) K_1^{m\ell}(x) R^{m\ell}_{13} \left( x - y \right),
\]
where the sublabels 1 and 2 stand for the first and second tensor component in $V^{m\frac{\ell}{2}} \otimes V^\ell$. This completes the proof of the reflection equation for case (1). \[\square\]

4.4. Reflection equation and coideal subalgebras. Here we briefly discuss the representation-theoretical meaning of reflection equations, cf., e.g., \cite{9, 10, 8}. Let $\mathcal{A} \subseteq \hat{U}_q$ be a left coideal subalgebra, i.e. it is a unital subalgebra of $\hat{U}_q$ satisfying $\Delta(\mathcal{A}) \subseteq \hat{U}_q \otimes \mathcal{A}$. If $M$ is a $\hat{U}_q$-module, we write $M|\mathcal{A}$ for the $\mathcal{A}$-module obtained by restricting the action of $\hat{U}_q$ on $M$ to $\mathcal{A}$.

Suppose that for $k, \ell \in \frac{1}{2} \mathbb{Z}_{>0}$ we have $\mathcal{A}$-intertwiners
\[
(4.9) \quad K^k(x) : V^k(x)|\mathcal{A} \rightarrow V^k(-x)|\mathcal{A}, \quad K^\ell(x) : V^\ell(x)|\mathcal{A} \rightarrow V^\ell(-x)|\mathcal{A}.
\]
Then the left and right sides of the reflection equation \eqref{4.8} are $\mathcal{A}$-intertwiners $\left( V^k(x) \otimes V^\ell(y) \right)|\mathcal{A} \rightarrow \left( V^k(-x) \otimes V^\ell(-y) \right)|\mathcal{A}$. Consequently, if $\left( V^k(x) \otimes V^\ell(y) \right)|\mathcal{A}$ is an irreducible $\mathcal{A}$-module for generic $x$ and $y$, then Schur’s lemma implies the reflection equation \eqref{4.8} up to a constant. Such examples of $K$-operators have been constructed with $\mathcal{A}$ the $q$-Onsager algebra, cf., e.g., \cite{8, 9, 10, 11}.

The fusion formula \eqref{4.7} is compatible with this representation-theoretic perspective in the following sense. Assume that $K^\frac{k}{2}(x) : V^{\frac{k}{2}}(x)|\mathcal{A} \rightarrow V^{\frac{k}{2}}(-x)|\mathcal{A}$ and $R^{\frac{k}{2}}(x) : V^{\frac{k}{2}}(x)|\mathcal{A} \rightarrow V^{\frac{k}{2}}(-x)|\mathcal{A}$ are $\mathcal{A}$-intertwiners. Then the right-hand side of \eqref{4.7}, which can be written as
\[
K_2^\frac{k}{2}(x - k\eta) R^{\frac{k}{2}} \left( 2x - \left( k - \frac{1}{2} \right)\eta \right) K_2^\frac{k}{2}(x + \frac{\eta}{2}) \ell^k
\]
with $\hat{R}^{kl}(x) := P^{kl} R^{kl}(x)$, is an $\mathcal{A}$-intertwiner

$$V^{k+\frac{1}{2}}(x)|_{\mathcal{A}} \to (V^k(-x - \frac{n}{2}) \otimes V^\frac{1}{2}(-x + k\eta))|_{\mathcal{A}}.$$  

It follows that the corresponding fused $K$-operator $K^{k+\frac{1}{2}}(x) : V^{k+\frac{1}{2}} \to V^{k+\frac{1}{2}}$, characterized by

$$j^k_{-x} K^{k+\frac{1}{2}}(x) = K^\frac{1}{2}_2(x - k\eta) \hat{R}^{k\ell}(2x - (k - \frac{1}{2})\eta) K^\frac{1}{2}_2(x + \frac{\eta}{2})^\ell_k,$$

becomes an intertwiner

$$K^{k+\frac{1}{2}}(x) : V^{k+\frac{1}{2}}(x)|_{\mathcal{A}} \to V^{k+\frac{1}{2}}(-x)|_{\mathcal{A}}$$

of $\mathcal{A}$-modules.

### 4.5. Diagonal K-operators.

**Proposition 4.6.** The $K$-operator $K^{\xi,\ell}(x) : V^\ell \to V^\ell$ ($\ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}$) obtained by recursively fusing $K^\xi(x) = K^{\xi,\frac{1}{2}}(x)$ using (4.7) acts on the weight basis as

$$K^{\xi,\ell}(x) v^\ell_n = C^\ell_n(x; \xi) v^\ell_n, \quad 1 \leq n \leq 2\ell + 1,$$

where

$$C^\ell_n(x; \xi) := \prod_{j=1}^{n-1} \frac{\sinh(\xi - x + (\ell + \frac{j}{2} - j)\eta)}{\sinh(\xi + x + (\ell + \frac{j}{2} - j)\eta)}$$

for $n \in \mathbb{Z}_{> 1}$ and $C^\ell_1(x; \xi) = 1$.

**Remark 4.7.** The $K$-operators $K^{\xi,\ell}(x)$ coincide with an appropriate limit of the explicit $\mathcal{A}$-intertwiner $V^\ell(x)|_{\mathcal{A}} \to V^\ell(-x)|_{\mathcal{A}}$ for the $q$-Onsager coideal subalgebra $\mathcal{A} \subset \hat{U}_q$ derived in [11]. This is to be expected from the representation-theoretic context of the fusion procedure of $K$-operators, cf. Section 4.4.

**Proof of Proposition 4.6.** By induction with respect to $\ell$. By the fusion formula (4.7) for $K$-operators it suffices to show that

$$C^{\ell+\frac{1}{2}}_n(x; \xi) j^\ell(v^\ell_n) = P^{\ell+\frac{1}{2}} K^{\xi,\ell+\frac{1}{2}}(x - \ell\eta) \hat{R}^{\ell+\frac{1}{2}}(2x - (\ell - \frac{1}{2})\eta) K^{\xi,\ell+\frac{1}{2}}(x + \frac{\eta}{2})^\ell(x; \xi; \eta),$$

with $K^{\xi,\ell}(x)$ satisfying (4.10). Both sides can be computed using the explicit actions of the maps on the standard bases. It follows that the desired identity (4.12) is equivalent to

$$C^{\ell+\frac{1}{2}}_n(x; \xi) = \frac{\sinh(2x + (2 - n)\eta)}{\sinh(2x + \eta)} C^\ell_n(x + \frac{\eta}{2}; \xi) + \frac{\sinh((n - 1)\eta)}{\sinh(2x + \eta)} C^{\ell+\frac{1}{2}}_{n-1}(x + \frac{\eta}{2}; \xi),$$

and

$$C^{\ell+\frac{1}{2}}_n(x; \xi) = \frac{\sinh(\xi - x + \ell\eta)}{\sinh(\xi + x - \ell\eta)} \left( \frac{\sinh((2\ell + 2 - n)\eta)}{\sinh(2x + \eta)} C^\ell_n(x + \frac{\eta}{2}; \xi) + \frac{\sinh(2x + (n - 1 - 2\ell)\eta)}{\sinh(2x + \eta)} C^{\ell+\frac{1}{2}}_{n-1}(x + \frac{\eta}{2}; \xi) \right)$$

for $1 \leq n \leq 2\ell + 1$. These follow easily from the trigonometric identity (4.15). □
Definition 4.8. For $\ell \in \mathbb{C}$ define the linear operator $K^{\xi,\ell}(x)$ on $M^\ell$ by

$$K^{\xi,\ell}(x)m^\ell_n = C^\ell_n(x; \xi)m^\ell_n, \quad n \geq 1.$$ 

Here functions $C^\ell_n(x; \xi)$ are defined in (4.17).

Note that if $k \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ and $pr^k : M^k \to V^k$ is the projection from the Verma module to the corresponding finite-dimensional irreducible quotient $V^k$, then

$$pr^k \circ K^{\xi,k}(x) = K^{\xi,k}(x) \circ pr^k,$$

where $K^{\xi,k}(x) : V^k \to V^k$ is the $K$-operator obtained by fusion in the previous subsection.

Proposition 4.9. Let $\xi \in \mathbb{C}$ then the operators $K^{\xi,k}(x)$ satisfy the reflection equation:

$$R^{\xi}(x-y)K^{\xi-k}(x)e^{\xi}(x+y)K^{\xi}(y) = K^{\xi}(y)e^{\xi}(x+y)K^{\xi-k}(x)R^{\xi}(x-y)$$

for all $k, \ell \in \mathbb{C}$.

Remark 4.10. From the observations in Subsection 4.4 it follows from Proposition 4.9 that for $k \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ and $\ell \in \mathbb{C}$, the $K$-operators $K^{\xi,k}(x)$ and $K^{\xi,\ell}(x)$ satisfy (4.12).

Proof of Proposition 4.9. For $k, \ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ denote by $d^k_{n,r,s}(x^\ell)$ the matrix elements of $R^k(x)$ in the weight basis:

$$R^k(x)v^k_n \otimes v^\ell_r = \sum_s d^k_{n,r,s}(x^\ell)v^k_{n-s} \otimes v^\ell_{r+s}$$

for $1 \leq n \leq 2k+1$, $1 \leq r \leq 2\ell+1$ and $s \in \mathbb{Z}$ such that $1 \leq n-s \leq 2k+1$ and $1 \leq r+s \leq 2\ell+1$. Similarly, we write for $k, \ell \in \mathbb{C}$

$$R^k(x)m^k_n \otimes m^\ell_r = \sum_s c^k_{n,r,s}(x^\ell;m^k,p^{2k},p^{2\ell})m^k_{n-s} \otimes m^\ell_{r+s}, \quad n, r \in \mathbb{Z}_{>0}$$

with the sum running over the integers $s$ such that $n-s, r+s \geq 1$. The coefficients $c^k_{n,r,s}(x^\ell;m^k,p^{2k},p^{2\ell})$ are rational functions in $e^{x^\ell}, p^{2k}$ and $p^{2\ell}$.

Let $n, r \in \mathbb{Z}_{>0}$ such that $n-s, r+s \in \mathbb{Z}_{>0}$. Then we have

$$c^k_{n,r,s}(x^\ell;e^{2nk},e^{2n\ell}) = d^k_{n,r,s}(x^\ell)$$

for sufficiently large $k, \ell \in \frac{1}{2}\mathbb{Z}_{>0}$ by (4.12).

Note furthermore that the dependence of $C^k_n(x; \xi)$ on $k$ is by a rational dependence on $p^{2k}$. To emphasize it, we write $C^\ell_n(x; \xi ; p^{2k}) := C^k_n(x; \xi)$ for the remainder of the proof.

The equation (4.14) we want to prove is equivalent to the following identities: for all $n, r \in \mathbb{Z}_{>0}$ and $t \in \mathbb{Z}$ satisfying $1-r \leq t \leq n-1$,

$$\sum_{s=1}^{n-1} C_{n-s,r+s,t-s}(e^{x^\ell}; p^{2k}, p^{2\ell}) C_{n-s}(x; \xi ; p^{2k})$$

$$\times C_{n,r,s}(e^{x^\ell}; p^{2k}, p^{2\ell}) C_{n}(y; \xi ; p^{2\ell}) =$$

$$= \sum_{s=1}^{n-1} C_{r+s}(y; \xi ; p^{2\ell}) C_{n-s,r+s,t-s}(e^{x^\ell}; p^{2k}, p^{2\ell})$$

$$\times C_{n-s}(x; \xi ; p^{2k}) C_{n,r,s}(e^{x^\ell}; p^{2k}, p^{2\ell}).$$
Since these identities depend rationally on $p^{2k}$ and $p^{2\ell}$, it suffices to prove them for $k, \ell \in \frac{1}{2} \mathbb{Z}_{\geq 0}$ sufficiently large. But then they follow from (1.17) and the "finite" reflection equations

$$R^{k\ell}(x-y)K_1^{\xi \cdot k}(x)R^{k\ell}(x_y)K_2^{\xi \cdot \ell}(y) = K_2^{\xi \cdot \ell}(y)R^{k\ell}(x+y)K_1^{\xi \cdot k}(x)R^{k\ell}(x-y)$$

for $k, \ell \in \frac{1}{2} \mathbb{Z}_{\geq 0}$.

5. Boundary monodromy operators and Bethe vectors

5.1. Monodromy matrices. In order to formulate our (Jackson integral) solutions to the boundary qKZ equations in $M_\ell = M_{\ell_1} \otimes \cdots \otimes M_{\ell_N}$ we need to introduce (off-shell) Bethe vectors for the reflecting chain, which in turn are defined using boundary monodromy operators. Boundary monodromy operators are linear operators acting on the extended tensor product $V^+ \otimes M_\ell$; the component $V^+$ is called auxiliary space and the component $M_\ell$ state space. From now on we restrict our attention to the case that the $\mathcal{K}$-matrices are diagonal (cf. Subsection 4.5).

The definition of the boundary monodromy operators involves the $L$-operators

$$L^\ell(x) := L^\ell_{\xi}(x) : V^+ \otimes M_\ell \to V^+ \otimes M_\ell$$

for $\ell \in \mathbb{C}$. They provide the link between the integrable structure on the auxiliary space and the integrable structure on the state space and satisfy the RLL commutation relations (2.3) (with $k = \ell = \frac{1}{2}$) and $R^{\xi \cdot \ell}(x)$ the 6-vertex $R$-operator) as well as the "mixed" reflection equations (4.2) (with $k = \frac{1}{2}$, $R^{\xi \cdot \ell}(x) = K^{\xi}(x)$ and $\mathcal{K}^\ell(x) = K^{\xi \cdot \ell}(x)$). In addition,

$$(5.1) \quad L^k(x)L^\ell(x+y)R^{k\ell}(y) = R^{k\ell}(x)L^\ell(x+y)L^k(x)$$

as linear operators on $V^+ \otimes M^k \otimes M^\ell$. The $L$-operators $L^\ell(x)$, together with the integrable data $K^{\xi}(x)$ and $R(x)$ on the auxiliary space, define an integrable quantum spin chain with diagonal reflecting ends (see 34). It is the inhomogeneous Heisenberg XXZ spin chain with continuous spins.

Let $S_N$ be the symmetric group in $N$ letters. For $\sigma \in S_N$ define the linear operator $T_\sigma(x; t) = T_{\sigma}^{\xi}(x; t)$ on $V^+ \otimes M_\ell$ by

$$(5.2) \quad T_\sigma(x; t) := L_{\sigma^{(1)}}^{\ell_1}(x - t_{\sigma^{(1)}}) \cdots L_{\sigma^{(N)}}^{\ell_N}(x - t_{\sigma^{(N)}})$$

where in the last equality we have written $T_\sigma(x; t)$ as a $\text{End}(M_\ell)$-valued matrix with respect to the ordered basis $(v_1^+, v_2^+)$ of $V^+$. The special case $T(x; t) := T_e(x; t)$ with $e \in S_N$ the neutral element is the (A-type) monodomy operator. We write the corresponding matrix coefficients as $A(x; t) = A_e(x; t), \ldots, D(x; t) = D_e(x; t)$.

The operators $T_\sigma(x; t)$ satisfy the commutation relations

$$(5.3) \quad R_{00'}(x-y)T_{\sigma,0}(x; t)T_{\sigma,0'}(y; t) = T_{\sigma,0'}(y; t)T_{\sigma,0}(x; t)R_{00'}(x-y)$$

as linear operators on $V^+ \otimes V^+ \otimes M_\ell$, where $T_{\sigma,0}(x; t)$ is the operator $T_\sigma(x; t)$ acting on the first and third tensor leg and $T_{\sigma,0'}(y; t)$ the operator $T_\sigma(y; t)$ on the second and third tensor leg, while $R_{00'}(x-y)$ is the action of $R(x-y)$ on the tensor product $V^+ \otimes V^+$ of the auxiliary spaces only.
Similarly, for $\sigma \in S_N$ we define $U^\xi_\sigma(x; t) = U^\xi_{\sigma, 0}(x; t)$ by
\begin{equation}
U^\xi_\sigma(x; t) := T_\sigma(x; t)^{-1} K^\xi(x)^{-1} T_\sigma(-x; t) \\
= \left( \begin{array}{cc} A^\xi_\sigma(x; t) & B^\xi_\sigma(x; t) \\ C^\xi_\sigma(x; t) & D^\xi_\sigma(x; t) \end{array} \right)
\end{equation}
as a linear operator on $V^\frac{1}{2} \otimes M^\xi$ (here $K^\xi(x)^{-1}$ only acts on the auxiliary space component of the tensor product). Then $U^\xi(x; t) := U^\xi_{\sigma, 0}(x; t)$ is the boundary monodromy operator \cite{34} associated to the $K$-operator $K^\xi$. The operators $U^\xi_\sigma(x; t)$ satisfy the commutation relations
\begin{equation}
R_{00'}(y - x) U^\xi_{\sigma, 0}(x; t) R_{00'}(-x - y) U^\xi_{\sigma, 0'}(y; t) = \nonumber
= U^\xi_{\sigma, 0'}(y; t) R_{00'}(-x - y) U^\xi_{\sigma, 0}(x; t) R_{00'}(y - x)
\end{equation}
as linear operators on $V^\frac{1}{2} \otimes V^\frac{1}{2} \otimes M^\xi$ with the same notational conventions as for (5.3). One of the consequences of these commutation relations is the commutativity of the operators $B^\xi_\sigma$:
\begin{equation}
[B^\xi_{\sigma}(x; t), B^\xi_{\sigma}(y; t)] = 0.
\end{equation}

Remark 5.1. Boundary transfer operators were constructed in \cite{34} in the context of quantum integrable models with boundaries. In the present context the boundary transfer operator is the linear operator on $M^\xi$ defined as
\begin{equation}
T^{\xi+,\xi-}(x; t) := Tr_{V^\frac{1}{2}} \left( K^{\xi+}(x - \eta) U^{\xi-}(x; t) \right) \\
= A^{\xi-}(x; t) + \frac{\sinh(\xi_+ - x + \eta)}{\sinh(\xi_+ + x - \eta)} D^{\xi-}(x; t),
\end{equation}
where $\xi_+, \xi_- \in \mathbb{C}$. It is a commuting family of operators:
\begin{equation}
[T^{\xi+,\xi-}(x; t), T^{\xi+,\xi-}(y; t)] = 0.
\end{equation}
In a similar way one can define boundary transfer operators acting on the same state space $M^\xi$ but involving higher-spin representations $V^k$ ($k \in \frac{1}{2} \mathbb{Z}_{\geq 0}$) in the auxiliary space, similar to the situation for periodic boundary conditions (see for example, the lectures \cite{31}). We will describe their properties in a separate publication.

5.2. The pseudo-vacuum and the Bethe vectors. We write
\begin{equation}
L^\ell(x) = \left( \begin{array}{cc} A^\ell(x) & B^\ell(x) \\ C^\ell(x) & D^\ell(x) \end{array} \right)
\end{equation}
with respect to the ordered basis $(v^\frac{1}{2}, v^\frac{1}{2})$ of the auxiliary space. The matrix coefficients are linear operators on $M^\ell$. Explicitly they are given by
\begin{equation}
A^\ell(x)m^\ell_n = \frac{\sinh(x + (\frac{3}{2} + \ell - n)\eta)}{\sinh(x + (\frac{1}{2} + \ell)\eta)} m^\ell_n, \\
B^\ell(x)m^\ell_n = \frac{\sinh(\eta)}{\sinh(x + (\frac{1}{2} + \ell)\eta)} e^{(-\ell + \frac{1}{2} + n)\eta} m^\ell_{n+1}, \\
C^\ell(x)m^\ell_n = \frac{\sinh((n - 1)\eta) \sinh((2\ell + 2 - n)\eta)}{\sinh(\eta) \sinh(x + (\frac{1}{2} + \ell)\eta)} e^{(\ell + \frac{1}{2} - n)\eta} m^\ell_{n-1}, \\
D^\ell(x)m^\ell_n = \frac{\sinh(x + (\frac{1}{2} - \ell + n)\eta)}{\sinh(x + (\frac{1}{2} + \ell)\eta)} m^\ell_n,
\end{equation}
where $m^\ell_0$ should be read as zero. Note that
\[ A^\ell(x)m^\ell_1 = m^\ell_1, \quad D^\ell(x)m^\ell_1 = \vartheta^\ell(-x)m^\ell_1, \quad C^\ell(x)m^\ell_1 = 0, \quad \mathcal{R}^{k\ell}(x)(m^k_1 \otimes m^\ell_1) = m^k_1 \otimes m^\ell_1, \quad \mathcal{K}^{\ell,\ell}(x)m^\ell_1 = m^\ell_1. \]
Here $\vartheta^\ell(x)$ is defined in (2.9). Set
\[ (5.7) \quad \Omega := m^\ell_1 \otimes m^{\ell_2} \otimes \cdots \otimes m^{\ell_N} \in M^\ell. \]

Note that
\[ (5.8) \quad A_\sigma(x; t)\Omega = \Omega, \quad D_\sigma(x; t)\Omega = \left( \prod_{r=1}^{N} \vartheta^{\ell_r}(t_r - x) \right) \Omega \]
for all $\sigma \in S_N$. The vector $\Omega$ will play the role of the pseudo-vacuum vector, from which off-shell Bethe vectors are generated by repeatedly applying operators $B^\ell(x; t)$, cf. [34].

For convenience, to construct our solutions to the boundary qKZ equations we will use a different normalization for $B^\ell(x; t)$:
\[ \overline{B}^\ell_\sigma(x; t) := \left( \prod_{r=1}^{N} \frac{\sinh(x - t_r - \ell_r \eta_r)}{\sinh(x - t_r + \ell_r \eta_r)} \right) \frac{\sinh(\xi - x - \eta)}{\sinh(2x + \eta)} B^\ell_\sigma(x + \eta/2; t). \]
The change from $B$ to $\overline{B}$ does not affect the commutativity:
\[ [\overline{B}^\ell_\sigma(x; t), \overline{B}^\ell_\tau(y; t)] = 0. \]
Hence, the following operator is well-defined for all $x = (x_1, \ldots, x_S)$ with $S \in \mathbb{Z}_{\geq 0}$:
\[ \overline{B}^\ell_\sigma(x; t) := \prod_{j=1}^{S} \overline{B}^\ell_{\sigma_j}(x_j; t). \]

We will write $\overline{B}^\ell(x; t) := \overline{B}^\ell_\emptyset(x; t)$ and $\overline{B}^\ell_{\{S\}}(x; t) := \overline{B}^\ell_{\{S\}}(x; t)$ when $\sigma = e$ is the identity element of $S_N$. The associated off-shell Bethe vectors are the vectors $\overline{B}^\ell_{\{S\}}(x; t)\Omega \in M^\ell$.

6. Jackson Integral Solutions of the Boundary qKZ Equations

We recall the notion of mero-uniformly convergent sums for scalar-valued functions (cf. [33]), which can be extended to $M^\ell$-valued functions in an obvious manner.

**Definition 6.1.** Let $C \subset \mathbb{C}^M$ be a discrete subset and $w(x; t)$ ($x \in C$) a weight function with values depending meromorphically on $t \in \mathbb{C}^N$. Suppose that for all $t_0 \in \mathbb{C}^N$, there exists an open neighbourhood $U_{t_0} \subset \mathbb{C}^N$ of $t_0$ and a nonzero holomorphic function $v_{t_0}$ on $U_{t_0}$ such that

1. $v_{t_0}(t)w(x; t)$ is holomorphic in $t \in U_{t_0}$ for all $x \in C$,
2. the sum $\sum_{x \in C} v_{t_0}(t)w(x; t)$ is absolutely and uniformly convergent for $t \in U_{t_0}$.

Then there exists a unique meromorphic function $f(t)$ in $t \in \mathbb{C}^N$ satisfying
\[ v_{t_0}(t)f(t) = \sum_{x \in C} v_{t_0}(t)w(x; t) \]
for \( t \in U_{t_0} \) and \( t_0 \in \mathbb{C}^N \). We will write

\[
    f(t) = \sum_{x \in \mathbb{C}} w(x; t)
\]

and we will say that the sum converges mero-uniformly.

We are now in a position to present our main theorem. For a meromorphic function \( h \) of one variable, write \( h(x \pm y) = h(x+y)h(x-y) \).

**Theorem 6.2.** Let \( \xi_+, \xi_- \in \mathbb{C} \) and let \( g_{\xi_+, \xi_-}(x) \), \( h(x) \) and \( F^\ell(x) \) be meromorphic functions in \( x \in \mathbb{C} \) satisfying the functional equations

\[
    g_{\xi_+, \xi_-}(x + \tau) = \frac{\sinh(\xi_- - x - \frac{\eta}{2}) \sinh(\xi_+ - x - \frac{\tau}{2} - \frac{\eta}{2})}{\sinh(\xi_- + x + \tau - \frac{\eta}{2}) \sinh(\xi_+ + x + \frac{\tau}{2} - \frac{\eta}{2})} g_{\xi_+, \xi_-}(x),
\]

\[
    h(x + \tau) = \frac{\sinh(x + \tau) \sinh(x + \eta)}{\sinh(x) \sinh(x + \tau - \eta)} h(x),
\]

\[
    F^\ell(x + \tau) = \frac{\sinh(x + \tau - \ell \eta)}{\sinh(x + \tau + \ell \eta)} F^\ell(x).
\]

Fix generic \( x_0 \in \mathbb{C}^S \) and suppose that the \( M^L \)-valued sum

\[
    f^\ell_S(t) := \sum_{x \in x_0 + \tau \mathbb{Z}^S} \left( \prod_{i=1}^S g_{\xi_+, \xi_-}(x_i) \right) \left( \prod_{1 \leq i < j \leq S} h(x_i \pm x_j) \right) \times \left( \prod_{r=1}^N \prod_{i=1}^S F^\ell_r(t_r \pm x_i) \right) \Omega
\]

converges mero-uniformly in \( t \in \mathbb{C}^N \). Then \( f^\ell_S \) is a solution of the boundary qKZ equations.

Theorem 6.2 generalizes the main result of [32] from 2-dimensional representations of quantum \( \mathfrak{sl}_2 \) to arbitrary Verma modules. The proof of Theorem 6.2 follows roughly the line of reasoning of the spin-\( \frac{1}{2} \) case [32], although considerably more technical difficulties need to be overcome. The proof is given in Section 7.

We now make the solutions concrete. We set \( q := e^\tau \) and we assume that \( \Re(\tau) < 0 \), so that \( |q| < 1 \). Solutions \( g_{\xi_+, \xi_-}, h \) and \( F^\ell \) of the resulting functional relations can now be expressed in terms of \( q \)-Gamma functions or, equivalently, in terms of \( q \)-shifted factorials

\[
    (x; q)_\infty := \prod_{i=0}^{\infty} (1 - q^i x).
\]

We write \( (x_1, \ldots, x_s; q)_\infty := \prod_{i=1}^s (x_i; q)_\infty \) for products of \( q \)-shifted factorials. As solutions of the functional equations we take

\[
    g_{\xi_+, \xi_-}(x) = e^{2(\xi_+ - \xi_- - \eta)x} \left( \frac{q^2 e^{2(x+\xi_-) - \eta} q e^{2(x+\xi_+ - \eta)} q^2}{(q e^{2(x-\xi_-) + \eta} q^2 e^{2(x-\xi_+) + \eta} q^2)_\infty} \right)_\infty,
\]

\[
    h(x) = e^{-2\eta x} \left( 1 - e^{2x} \right) \left( \frac{q^2 e^{2(x+\eta)} q^2}{(q^2 e^{2(x+\eta)} q^2)_\infty} \right)_\infty,
\]

\[
    F^\ell(x) = e^{2\eta x} \left( \frac{q^2 e^{2(x+\ell \eta)} q^2}{(q^2 e^{2(x+\ell \eta)} q^2)_\infty} \right)_\infty.
\]
With these choices for the solutions of the functional equations and the assumption that \( R(\tau) < 0 \), it is readily established (cf. Subsections 3.4 and 3.5) that the solution \( f_2^{\ell}(t) \) defined in Theorem 6.2 converges mero-uniformly in \( t \in \mathbb{C}^N \) for generic \( x_0 \in \mathbb{C}^N \) when \( R(\eta) \geq 0 \) and

\[
R(2\xi_+ + 2\xi_- + 2\sum_{r=1}^{N} \ell_r - 1)\eta + \tau < 0.
\]

7. PROOF OF THE MAIN RESULT

7.1. Preliminary steps. Let \( S_N \) be the symmetric group in \( N \) letters and \( \sigma \in S_N \). We view

\[
L^{\ell(x)}(x - t_{\sigma(1)})L^{\ell(x)}(x - t_{\sigma(2)}) \cdots L^{\ell(x)}(x - t_{\sigma(N-1)})
\]

as a linear operator on \( V^\frac{1}{2} \otimes M^\frac{1}{2} \) acting trivially on the tensor component of \( M^\frac{1}{2} \) labelled by \( \sigma(N) \). Write

\[
\begin{pmatrix}
\hat{A}_\sigma(x; t) \\
\hat{B}_\sigma(x; t) \\
\hat{C}_\sigma(x; t) \\
\hat{D}_\sigma(x; t)
\end{pmatrix}
\]

for the operator \( (7.1) \), written as a matrix with respect to the ordered basis \( (v^\frac{1}{2}_1, v^\frac{1}{2}_2) \) of \( V^\frac{1}{2} \). The operators \( \hat{A}_\sigma(x; t), \hat{B}_\sigma(x; t) \) act on \( M^\frac{1}{2} \). They act trivially on the \( \sigma(N) \)-th tensor component of \( M^\frac{1}{2} \) and do not depend on \( t_{\sigma(N)} \).

For \( \sigma \in S_N, J \subseteq \{1, \ldots, S\} \) and \( \epsilon \in \{\pm\}^S \) we write

\[
Y^{\epsilon, J}_\sigma(x; t) := \left( \prod_{i=1}^{S} \frac{\epsilon_i \sinh(\xi_i - x_i - \frac{\eta_i}{2}) \prod_{r=1}^{\frac{N}{2}} \sinh(\epsilon_i x_i - t_r - \ell_r \eta) \prod_{1 \leq i < j \leq S} \sinh(\epsilon_i x_i + \epsilon_j x_j + \eta)}{\sinh(\epsilon_i x_i - t_r + \ell_r \eta) \sinh(\xi_i x_i + \epsilon_j x_j)} \right) Y^J_\sigma((-\epsilon_1 x_1 - \frac{\eta_1}{2}, \ldots, -\epsilon_s x_s - \frac{\eta_s}{2}); t)
\]

with

\[
Y^J_\sigma(x; t) := \left( \prod_{i \in J} \frac{\sinh(x_i - t_{\sigma(N)} + \frac{1}{2} - \ell_{\sigma(N)} \eta)}{\sinh(x_i - t_{\sigma(N)} + \frac{1}{2} + \ell_{\sigma(N)} \eta)} \prod_{(i,j) \in J \times J} \frac{\sinh(x_i - x_j + \eta)}{\sinh(x_i - x_j)} \right)
\]

and \( J^c := \{1, \ldots, S\} \setminus J \) (empty products are equal to one). Similarly to the spin-\( \frac{1}{2} \) case (see [32 Cor. 4.3]) we have the explicit expression

\[
\overline{B}^{\epsilon,(S)}_\sigma(x; t) \Omega = \sum_{\epsilon \in \{\pm\}^S} \sum_{J \subseteq \{1, \ldots, S\}} Y^{\epsilon, J}_\sigma(x; t)
\]

\[
\times \left( \prod_{j \in J^c} B^J_\sigma(-\epsilon_j x_j - \frac{\eta_j}{2} - t_{\sigma(N)}) \right) \left( \prod_{i \in J} \hat{B}_\sigma(-\epsilon_i x_i - \frac{\eta_i}{2}; t) \right) \Omega
\]

of the Bethe vector (see [32 Cor. 4.3]). For \( r \in \{1, \ldots, N-1\} \) write \( s_r \in S_N \) for the simple neighbouring transposition \( r \leftrightarrow r + 1 \). In [32 Lemma 5.4] the condition that the function \( f_2^{\ell}(t) \) with \( \ell = (\frac{1}{2}, \ldots, \frac{1}{2}) \) satisfies the boundary qKZ equations is re-written as a system of equations involving the weight functions \( Y^{\epsilon, J}_\sigma \) where \( \sigma = s_r \cdots s_{N-1} \) for some \( r \in \{1, \ldots, N\} \). This directly generalizes to the following result in the current higher-spin context.
Lemma 7.1. Provided mero-uniform convergence,
\[ f_2^{(S)}(t) := \sum_{x \in x_0 + \tau \mathbb{Z}^S} w^{(S)}(x; t; \xi_+, \xi_-) \hat{B}_2^{(S)}(x; t) \Omega \]
satisfies the boundary qKZ equations (1.3) iff
\begin{equation}
\sum_{x, \epsilon, J} w^{(S)}(x; t; \xi_+, \xi_-) \left( \prod_{i=1}^{S} \frac{\sinh(x_i + t_r + \ell_r \eta)}{\sinh(x_i + t_r - \ell_r \eta)} \right) \tilde{\chi}_{\xi_-, \epsilon, J}^{x_{\epsilon, J}}(x; \epsilon, t) \times \mathcal{K}_{\xi_+, \ell_r}(t_r + \frac{\tau}{2}) \times \left( \prod_{j \in J} B^{(t_r)}(-\epsilon_j x_j - \eta \frac{t_r}{2} + t_r) \right) \left( \prod_{i \in J} \hat{B}_{s_i, \cdots, s_{N-1}}(-\epsilon_i x_i - \eta \frac{t_r}{2}, t) \right) \Omega
\end{equation}
equals
\begin{equation}
\sum_{x, \epsilon, J} w^{(S)}(x; t + \tau \epsilon_r; \xi_+, \xi_-) \tilde{\chi}_{\xi_-, \epsilon, J}^{x_{\epsilon, J}}(x; t + \tau \epsilon_r) \times \left( \prod_{j \in J} B^{(t_r)}(-\epsilon_j x_j - \eta \frac{t_r}{2} - t_r - \tau) \right) \left( \prod_{i \in J} \hat{B}_{s_i, \cdots, s_{N-1}}(-\epsilon_i x_i - \eta \frac{t_r}{2}, t) \right) \Omega
\end{equation}
for \( r = 1, \ldots, N \), where the summations are over \( x \in x_0 + \tau \mathbb{Z}^S \), \( \epsilon \in \{\pm\}^S \) and over subsets \( J \subseteq \{1, \ldots, S\} \) (recall that \( J^c = \{1, \ldots, S\} \setminus J \)).

We fix \( S \geq 1 \) and suppress it from the notations. For \( d \in \{0, \ldots, S\} \) set \( \mathcal{L}_r^{(d)}(t) \) and \( \mathcal{R}_r^{(d)}(t) \) for (7.2) and (7.3) respectively, with the sums running over \( x \in x_0 + \tau \mathbb{Z}^S \), \( \epsilon \in \{\pm\}^S \) and over subsets \( J \subseteq \{1, \ldots, S\} \) of cardinality \( S - d \). The strategy of the proof of Theorem 5.1 is to determine sufficient conditions on the weight function \( w^{(S)}(x; t; \xi_+, \xi_-) \) so that
\begin{equation}
\mathcal{L}_r^{(d)}(t) = \mathcal{R}_r^{(d)}(t)
\end{equation}
for all \( d \in \{0, \ldots, S\} \) and \( r \in \{1, \ldots, N\} \). We will call \( d \) the depth.

Remark 7.2. In the study [32] of Jackson integral solutions for the spin-\( \frac{1}{2} \) representations the terms \( \mathcal{L}_r^{(d)}(t) \) and \( \mathcal{R}_r^{(d)}(t) \) are automatically zero if \( d \geq 2 \), cf. [32] Rem. 5.5. When \( M^{(r)} \) are highest weight modules with \( \ell_r \in \mathbb{C} \) we have to deal with the terms \( \mathcal{L}_r^{(0)}(t) \) and \( \mathcal{R}_r^{(d)}(t) \) for any depth \( d \in \{0, \ldots, S\} \).

7.2. Depth zero. Completely analogous to the spin-\( \frac{1}{2} \) case (see [32] §5.1) we have the following result.

Lemma 7.3. Suppose that
\[ w^{(S)}(x; t; \xi_+, \xi_-) = \left( \prod_{r=1}^{N} \prod_{i=1}^{S} F^{(r)}(t_r \pm x_i) \right) G_{\xi_+, \xi_-}(x) \]
with \( G_{\xi_+, \xi_-}(x) \) independent of \( t \). If
\[ F^{(r)}(x + \tau) = \frac{\sinh(x + \tau - \ell_r \eta)}{\sinh(x + \tau + \ell_r \eta)} F^{(r)}(x) \]
for \( r = 1, \ldots, N \) then, provided mero-uniform convergence,
\begin{equation}
\mathcal{L}_r^{(0)}(t; \xi_+, \xi_-) = \mathcal{R}_r^{(0)}(t; \xi_+, \xi_-)
\end{equation}
for \( r = 1, \ldots, N \).
In the remainder of the section we assume that the weight function \(w(S)(x; t; \xi_+, \xi_-)\) is of the form as specified in Lemma 7.3.

**7.3. The remaining depths.** We have the setup that

\[
f_S^\text{\(S\)}(t) = \sum_{x \in x_0 + \tau Z^S} w(S)(x; t; \xi_+, \xi_-) \xi(S)(x) \Omega \]

with the sum converging mero-uniformly in \(t \in \mathbb{C}^N\) and with weight function of the form

\[
w(S)(x; t; \xi_+, \xi_-) = \left( \prod_{r=1}^N \prod_{i=1}^S F^r(t_r \pm x_i) \right) G^{S}(x) \]

with \(G^{S}(x)\) independent of \(t\) and with the \(F^r\) satisfying

\[
F^r(x + \tau) = \frac{\sinh(x + \tau - \ell \eta)}{\sinh(x + \tau + \ell \eta)} F^r(x). \tag{7.7}
\]

We are now going to show that conditions on the weight factor \(G^{S}(x)\) as stated in Theorem 6.2 imply that (7.4) is valid for \(d \in \{1, \ldots, S\}\) and \(r \in \{1, \ldots, N\}\). Combined with Lemma 7.3 and Lemma 7.4 this will complete the proof of Theorem 6.2.

Since the \(\xi_{\pm}\) are fixed throughout this subsection, we will suppress \(\xi_{\pm}\) from the notations; in particular, we write \(w(S)(x; t)\) for \(w(S)(x; t; \xi_+, \xi_-)\). We also suppress \(S \in \mathbb{Z}_{\geq 1}\) from the notations.

If \(J \subseteq \{1, \ldots, S\}\), \(\epsilon \in \{\pm\}^S\) and \(x \in x_0 + \tau Z^S\) then we write \(x_J := (x_j)_{j \in J}\) and \(\epsilon_J := (\epsilon_j)_{j \in J}\). Conversely, for given \(\epsilon_J\) and \(\epsilon_J\), the associated \(S\)-tuple of signs will be denoted by \(\epsilon\) (and similarly for \(x\)).

It is convenient to define the following weights.

**Definition 7.4.** For \(r \in \{1, \ldots, N\}\), \(\epsilon \in \{\pm\}^S\) and a subset \(J \subseteq \{1, \ldots, S\}\) we write

\[
m^{\epsilon, J}_r(x; t) := \left( \prod_{i=1}^S \frac{\sinh(\pm x_i - t_r + \ell_r \eta)}{\sinh(\pm x_i - t_r - \ell_r \eta)} \right) \prod_{j \in J} \frac{\sinh(-\epsilon_j x_j - t_r + \ell_r \eta)}{\sinh(-\epsilon_j x_j - t_r - \ell_r \eta)} \]

for \(x \in x_0 + \tau Z^S\).

It follows by a straightforward computation that

\[
m^{\epsilon, J}_r(x; t) = \left( \prod_{j \in J} \frac{\sinh(\xi_+ - \epsilon_j x_j - \frac{\eta}{2})}{\sinh(-t_r - \epsilon_j x_j + \ell_r \eta)} \prod_{s \neq r} \frac{\sinh(t_s - \epsilon_j x_j + \ell_s \eta)}{\sinh(t_s - \epsilon_j x_j - \ell_s \eta)} \right) \]

\[
\times \left( \prod_{i,j \in J \times J} \frac{\sinh(\epsilon_j x_j + \epsilon_i x_i - \eta)}{\sinh(\epsilon_j x_j + \epsilon_i x_i + \eta)} \right) \prod_{i,j \in J, j < i'} \frac{\sinh(\epsilon_i x_i + \epsilon_j x_{i'})}{\sinh(\epsilon_i x_i + \epsilon_j x_{i'})} \]

\[
\times \prod_{i \in J} \left( \epsilon_i \sinh(\xi_- - \epsilon_i x_i - \frac{\eta}{2}) \prod_{s \neq r} \frac{\sinh(t_s - \epsilon_i x_i + \ell_s \eta)}{\sinh(t_s - \epsilon_i x_i - \ell_s \eta)} \right). \tag{7.8}
\]
Lemma 7.5. Fix \(d \in \{1, \ldots, S\}\) and \(r \in \{1, \ldots, N\}\). Suppose that for all subsets \(J \subseteq \{1, \ldots, S\}\) of cardinality \(S - d\) and for all \(x_j\) and \(e_j\),
\[
C_{d+1}^{\ell_r}(t_r + \frac{r}{2}; \xi_+) \sum_{x_j, e_j} w^{(S)}(x; t; \xi_+, \xi_-) m^{\ell_r}_{d+1}(x; e, t) = \sum_{x_j, e_j} w^{(S)}(x; t; \xi_+, \xi_-) m^{\ell_r}_{d+1}(x; t + \tau e_r).
\]
Then \(L_r^{(d)}(t) = R_r^{(d)}(t)\).

Proof. Recall that
\[
\mathcal{K}_{\xi_+}^{\ell_r}(t_r + \frac{r}{2})m^{\ell_r}_{d+1} = C_{d+1}^{\ell_r}(t_r + \frac{r}{2}; \xi_+)m^{\ell_r}_{d+1},
\]
see Definition 4.8. Since
\[
\left( \prod_{j \in J} B^{\ell_r}(-\epsilon_j x_j - \frac{\eta}{2} + u) \right) m^{\ell_r}_1 = \frac{\sinh^d(\eta)e^{\eta(\frac{d^2}{4} - \ell_r d)}}{\prod_{j \in J} \sinh(-\epsilon_j x_j + u + \ell_r \eta)} m^{\ell_r}_{d+1}
\]
by (5.3) we thus have
\[
\mathcal{K}_{\xi_+}^{\ell_r}(t_r + \frac{r}{2}) \left( \prod_{j \in J} B^{\ell_r}(-\epsilon_j x_j - \frac{\eta}{2} + t_r) \right) m^{\ell_r}_1 = \frac{\sinh^d(\eta)C_{d+1}^{\ell_r}(t_r + \frac{r}{2}; \xi_+)e^{\eta(\frac{d^2}{4} - \ell_r d)}}{\prod_{j \in J} \sinh(-\epsilon_j x_j + t_r + \ell_r \eta)} m^{\ell_r}_{d+1};
\]
\[
\left( \prod_{j \in J} B^{\ell_r}(-\epsilon_j x_j - \frac{\eta}{2} - \tau - t_r) \right) m^{\ell_r}_1 = \frac{\sinh^d(\eta)e^{\eta(\frac{d^2}{4} - \ell_r d)}}{\prod_{j \in J} \sinh(-\epsilon_j x_j - \tau - t_r + \ell_r \eta)} m^{\ell_r}_{d+1}.
\]
Taking the expressions (7.2) and (7.3) for \(L_r^{(m)}(t)\) and \(R_r^{(m)}(t)\) into account we conclude that \(L_r^{(d)}(t) = R_r^{(d)}(t)\) if for all subsets \(J \subseteq \{1, \ldots, S\}\) of cardinality \(S - d\) and for all \(x_j\) and \(e_j\),
\[
C_{d+1}^{\ell_r}(t_r + \frac{r}{2}; \xi_+) \sum_{x_j, e_j} w^{(S)}(x; t; \xi_+, \xi_-) m^{\ell_r}_{d+1}(x; e, t) = \sum_{x_j, e_j} w^{(S)}(x; t + \tau e_r; \xi_+, \xi_-) \times \left( \prod_{i=1}^S \frac{\sinh(\pm x_i - t_r - \tau - \ell_r \eta)}{\sinh(\pm x_i - t_r - \tau - \ell_r \eta)} \right) m^{\ell_r}_{d+1}(x; t + \tau e_r).
\]
The lemma now follows from the fact that
\[
w^{(S)}(x; t + \tau e_r; \xi_+, \xi_-) = \left( \prod_{i=1}^S \frac{\sinh(\pm x_i - t_r + \ell_r \eta)}{\sinh(\pm x_i - t_r - \ell_r \eta)} \right) w^{(S)}(x; t; \xi_+, \xi_-),
\]
which is a direct consequence of the specific form (7.6), (7.7) of the weight function \(w^{(S)}(x; t)\). \qed
In view of the previous lemma, the desired identity (7.4) follows if suppress from the notations. Set for $\epsilon J$ subset $J$ of cardinality $S - d$, as well as $x_j$ and $\epsilon_j$, which we all suppress from the notations. Set for $\epsilon J \in \{\pm\}^d$,

$$\Lambda_{r, \epsilon J}(t) := \sum_{x J} w^{(S)}(x; t; x_+ + \epsilon J) m^x_J(x; \epsilon J, t),$$

$$\Upsilon_{r, \epsilon J}(t) := \sum_{x J} w^{(S)}(x; t; x_+ + \epsilon J) m^x_J(x; t + \epsilon J t).$$

In view of the previous lemma, the desired identity (7.4) follows if

$$C_{d+1}^L(t_r + \frac{r}{2}; \xi_+) \sum_{\epsilon J \in \{\pm\}^d} \Lambda_{r, \epsilon J}(t) = \sum_{\epsilon J \in \{\pm\}^d} \Upsilon_{r, \epsilon J}(t). \tag{7.9}$$

We write $m_r(x J; t)$ for $m^x_J(x; t)$ with $\epsilon J$ the $d$-tuple $(-, -, \cdots, -)$ of minus signs,

$$m_r(x J; t) = (-1)^d \left( \prod_{j \in J} \left( \sinh(\xi_+ + x_j - \frac{\eta}{2}) \prod_{s=1}^N \sinh(t_s + x_j + \tau J_\epsilon) \right) \prod_{s \neq r} \sinh(t_s + x_j - \tau J_\epsilon) \right)$$

$$\times \left( \prod_{i \in J} \frac{\sinh(x_j + x_i - \eta)}{\sinh(x_j + x_i)} \right) \left( \prod_{i < i'} \sinh(\epsilon_i x_i + \epsilon_i' x_i') \right)$$

$$\times \left( \prod_{i \in J} \left( \epsilon_i \sinh(\xi_+ + \epsilon_j x_i - \eta) \prod_{s \neq r} \sinh(t_s + \epsilon_i x_i + \tau J_\epsilon) \right) \right). \tag{7.10}$$

Lemma 7.6. Suppose that for all $i \in \{1, \ldots, S\}$,

$$G_{\xi_+ \xi_-}(x - \tau e_i) = \frac{\sinh(\xi_+ + x_i - \frac{\eta}{2}) \sinh(\xi_+ + x_i - \tau - \frac{\eta}{2})}{\sinh(\xi_+ + x_i + \tau - \frac{\eta}{2}) \sinh(\xi_+ + x_i + \frac{\eta}{2})} \prod_{i' \neq i} \sinh(x_i + x_i' - \tau) \sinh(x_i + x_i' + \eta) \sinh(x_i + x_i') \right) G_{\xi_+ \xi_-}(x). \tag{7.11}$$

Then

$$\Lambda_{r, \epsilon J}(t) = (-1)^{\# J_\epsilon} \sum_{x J} w^{(S)}(x; t; \xi_+, \xi J) q_{J_\epsilon}^x(x J; t_r) m_r(x J; \epsilon J, t),$$

$$\Upsilon_{r, \epsilon J}(t) = (-1)^{\# J_\epsilon} \sum_{x J} w^{(S)}(x; t; \xi_+, \xi J) q_{J_\epsilon}^x(x J; -t_r - \tau) m_r(x J; t + \epsilon J t).$$
with $J^c_+ := \{ j \in J^c \mid \epsilon_j = + \}$, $J^c_- := J^c \setminus J^c_+$ and
\[
q_{J^c_+}(x_{J^c_+}; t_r) := \prod_{j, j' \in J^c_+ \setminus \{ k \}} \sinh(x_{j} + x_{j'} - \tau - \eta) \\
\times \prod_{j \in J^c_+} \sinh(x_{j} - x_{j'} - \eta) \sinh(x_{j'} + x_{j} - \tau - \eta) \\
\times \frac{\sinh(\xi_+ + x_j - \frac{\tau}{2} - \frac{\eta}{2})}{\sinh(\xi_+ - x_j + \frac{\tau}{2} - \frac{\eta}{2})} \sinh(t_r - x_j + \tau + \ell_r \eta).}
\]

**Proof.** The formula for $\Lambda_{r,\epsilon_{J^c}}(t)$ is correct if $\epsilon_{J^c}$ is the $d$-tuple $(-, - \cdots, -)$ of minus signs since $q_0(x_{J^c}; t_r) = 1$ (empty products are equal to one by convention).

Fix $\epsilon_{J^c} \in \{\pm\}^d$ and $I \subset J^c_+$. Write $\epsilon_{J^c}^I$ for the $d$-tuple of signs obtained from $\epsilon_{J^c}$ by replacing $\epsilon_i = +$ by $-$ for all $i \in I$. Similarly, we write $\epsilon_{J^c}^{I^-}$ for the $S$-tuple of signs obtained from $\epsilon$ by replacing $\epsilon_i = +$ by $-$ for all $i \in I$.

Fix $k \in J^c_+$ and rewrite $\Lambda_{r,\epsilon_{J^c}}(t)$ as
\[
\Lambda_{r,\epsilon_{J^c}}(t) = \sum_{x_{J^c}} w^{(S)}(x; t) \beta_k(x; t) w^{(S)}(x; \xi_+, \xi_-) m_{r}^{\epsilon_i}(x; \tau e_k; \epsilon_t).
\]

By the assumptions on $w^{(S)}(x; t)$ we have
\[
w^{(S)}(x - \tau e_k; \xi_+, \xi_-) = \beta_k(x; t) w^{(S)}(x; \xi_+, \xi_-)
\]
with
\[
\beta_k(x; t) := \prod_{s=1}^{N} \frac{\sinh(t_s + x_k + \ell_s \eta) \sinh(t_s - x_k + \tau + \ell_s \eta)}{\sinh(t_s + x_k - \ell_s \eta) \sinh(t_s - x_k + \tau + \ell_s \eta)} \\
\times \frac{\sinh(\xi_+ + x_k - \frac{\tau}{2} - \frac{\eta}{2}) \sinh(\xi_+ - x_k - \frac{\tau}{2} - \frac{\eta}{2})}{\sinh(\xi_+ + x_k - \tau - \frac{\eta}{2}) \sinh(\xi_+ - x_k + \frac{\tau}{2} + \frac{\eta}{2})} \sinh(t_r + x_k + \ell_r \eta) \sinh(t_r - x_k + \tau + \ell_r \eta).}
\]

In addition, by a direct computation using [28],
\[
\beta_k(x; t) m_{r}^{\epsilon_i}(x - \tau e_k; \epsilon_t) = -\gamma_k^{\epsilon_{J^c}}(x_{J^c}; t_r) m_{r}^{\epsilon_i \cdot \epsilon_{J^c}^I}(x; \epsilon_t)
\]
with
\[
\gamma_k^{\epsilon_{J^c}}(x_{J^c}; t_r) := \prod_{j \in J^c \setminus \{ k \}} \frac{\sinh(x_k + \epsilon_j x_j - \eta) \sinh(x_k - \epsilon_j x_j - \tau)}{\sinh(x_k + \epsilon_j x_j) \sinh(x_k - \epsilon_j x_j + \tau + \eta)} \\
\times \frac{\sinh(\xi_+ + x_k + \frac{\tau}{2} - \frac{\eta}{2}) \sinh(t_r + x_k + \ell_r \eta)}{\sinh(\xi_+ - x_k - \frac{\tau}{2} - \frac{\eta}{2}) \sinh(t_r - x_k + \tau + \ell_r \eta)}.}
\]

Hence
\[
\Lambda_{r,\epsilon_{J^c}}(t) = -\sum_{x_{J^c}} w^{(S)}(x; t; \xi_+, \xi_-) \gamma_k^{\epsilon_{J^c}}(x_{J^c}; t_r) m_{r}^{\epsilon_i \cdot \epsilon_{J^c}^I}(x; \epsilon_t).
\]

This in particular proves the desired expression of $\Lambda_{r,\epsilon_{J^c}}(t)$ if $\epsilon_k = +$ and $\epsilon_j = -$ for $j \in J^c \setminus \{ k \}$. 
The formula for arbitrary $\epsilon_{J^c} \in \{\pm\}^d$ follows by an induction argument with respect to $\#J^c_+$. For a subset $I \subset J^c_+$, set

$$\tilde{q}_I(x_{J^c};t_r) := \left( \prod_{i,j \in I} \frac{\sinh(x_i + x_j - \tau - \eta)}{\sinh(x_i + x_j - \tau + \eta)} \right) \times \left( \prod_{i \in I} \frac{\sinh(\xi_i + x_i - \frac{\tau}{2} - \frac{\eta}{2}) \sinh(t_r + x_i + \ell_r \eta)}{\sinh(\xi_i + x_i - \frac{\tau}{2} + \frac{\eta}{2}) \sinh(t_r - x_i + \tau + \ell_r \eta)} \right) \times \left( \prod_{(i,j) \in I \times J^c \setminus I} \frac{\sinh(x_i + \epsilon_j x_j - \eta)}{\sinh(x_i + \epsilon_j x_j)} \sinh(x_i - \epsilon_j x_j - \tau) \right) \sinh(x_i - \epsilon_j x_j + \eta)$$

Then $\tilde{q}_0(x_{J^c};t_r) = 1$, $\tilde{q}_{J^c}(x_{J^c};t_r) = q_{J^c}(x_{J^c};t_r)$ and for a subset $I \subset J^c_+$ and $k \in J^c_+ \setminus I$,

$$\frac{\tilde{q}_{I \cup \{k\}}(x_{J^c};t_r)}{\tilde{q}_I(x_{J^c};t_r - \tau e_k; t_r)} = q_{J^c}^{}(x_{J^c}; t_r).$$

The alternative expression for $\Upsilon_{\tau, \epsilon_{J^c}}(t)$ follows from a similar computation, now using the observation that for $k \in J^c_+$,

$$\beta_k(x; t) m_{r; \epsilon_{J^c}}(x; t - \tau e_k; t + \tau e_r) = -\gamma_k(x; t) m_{r; \epsilon_{J^c}}(x; t + \tau e_r).$$

Note that (7.11) is satisfied if

$$G_{\xi_+, \xi_-}(x) = \left( \prod_{i=1}^S \frac{\sinh(\xi_i + \epsilon x_i)}{\sinh(\xi_i + x_i)} \right) \prod_{1 \leq i < j \leq S} h(x_i + x_j)$$

with $g_{\xi_+, \xi_-}$ and $h$ as in Theorem 6.2.

By the explicit expression (7.10) of $m_{r; \epsilon_{J^c}}(x; t)$ we have

$$\tilde{m}_{r; \epsilon_{J^c}}(x_{J^c}; t) := m_{r; \epsilon_{J^c}}(x_{J^c}; t) \prod_{j \in J^c} \sinh(t_r + x_j - \ell_r \eta)$$

$$= m_{r; \epsilon_{J^c}}(x_{J^c}; t - \tau e_r) \prod_{j \in J^c} \sinh(-t_r - \tau + x_j - \ell_r \eta).$$

Combined with Lemma 6.6, it follows that (7.4) is equivalent to

$$\sum_{x_{J^c}} w^{(S)}(x; t; \xi_+, \xi_-) \tilde{m}_{r; \epsilon_{J^c}}(x_{J^c}; t) C^d_{d+1}(t_r + \frac{\tau}{2}; \xi_+)$$

$$\times \sum_{\epsilon_{J^c} \in \{\pm\}^d} \frac{(-1)^{\#J^c_+} q_{J^c}(x_{J^c}; t_r)}{\prod_{j \in J^c} \sinh(t_r + x_j - \ell_r \eta)} =$$

$$\sum_{x_{J^c}} w^{(S)}(x; t; \xi_+, \xi_-) \tilde{m}_{r; \epsilon_{J^c}}(x_{J^c}; t) \sum_{\epsilon_{J^c} \in \{\pm\}^d} \frac{(-1)^{\#J^c_+} q_{J^c}(x_{J^c}; -t_r - \tau)}{\prod_{j \in J^c} \sinh(-t_r - \tau + x_j - \ell_r \eta)}.$$

Substituting the explicit expression (7.11) of $C^d_{n}(x; \xi)$, this is a direct consequence of the following lemma.

**Lemma 7.7.** Let $J \subseteq \{1, \ldots, S\}$ be a subset of cardinality $S - d$ and $\epsilon_{J^c} \in \{\pm\}^d$. Then the finite sum

$$\left( \prod_{n=1}^d \frac{\sinh(\xi_+ - t_r - \frac{\tau}{2} + (\ell_r + \frac{1}{2} - n) \eta)}{\sinh(\xi_+ - t_r - \frac{\tau}{2} + (\ell_r + \frac{1}{2} - n) \eta)} \right) \sum_{\epsilon_{J^c} \in \{\pm\}^d} \frac{(-1)^{\#J^c_+} q_{J^c}(x_{J^c}; t_r)}{\prod_{j \in J^c} \sinh(t_r + x_j - \ell_r \eta)}$$
is invariant under the exchange of \( t_r \) by \(-t_r - \tau\).

The proof of the lemma is given in the next subsection. It completes the proof of the main theorem (Theorem 6.2).

7.4. **Proof of Lemma 7.7.** Let \( J \subseteq \{1, \ldots, S\} \) be a subset of cardinality \( S - d \) and \( \epsilon_{Jc} \in \{\pm\}^d \). Choose an identification of the fixed subset \( J^c \) of cardinality \( d \) with \( \{1, \ldots, d\} \). The choice of signs \( \epsilon_{Jc} \in \{\pm\}^d \) then is identified with choosing a subset \( I \subseteq \{1, \ldots, d\} \) by the rule

\[
I := \{ i \in \{1, \ldots, d\} \mid \epsilon_i = + \}.
\]

Write \( \xi = \xi_+ - \frac{j}{2} \) and \( x = (x_1, \ldots, x_d) \). Then the statement in Lemma 7.7 is easily seen to be equivalent to the claim that

\[
F(x; t) := \left( \prod_{i=1}^d \frac{\sinh(\xi - t - \frac{j}{2} + (\ell + 1 - i)\eta)}{\sinh(t + x_i - \ell\eta)} \right) \times \sum_{I \subseteq \{1, \ldots, d\}} \left\{ (-1)^{\#I} \left( \prod_{i,j \in I} \frac{\sinh(x_i + x_j - \tau - \eta)}{\sinh(x_i + x_j - \tau + \eta)} \right) \times \left( \prod_{i \in I} \sinh(\xi_i + x_i - \frac{j}{2}) \sinh(t + x_i + \ell\eta) \right) \sinh(\xi - x_i + \frac{j}{2}) \sinh(t - x_i + \tau + \ell\eta) \right\}
\]

(7.13)

satisfies

\[
F(x; -t - \tau) = F(x; t).
\]

By substituting \( x_i \to x_i + \frac{j}{2} \) (\( i = 1, \ldots, d \)) and \( t \to t - \frac{j}{2} \) and clearing denominators in (7.14), we obtain a trigonometric polynomial identity independent of \( \tau \). More precisely, for \( i \in \{1, \ldots, d\} \) and \( I \subseteq \{1, \ldots, d\} \) write \( \epsilon_i^{(I)} = + \) if \( i \in I \) and \( \epsilon_i^{(I)} = - \) if \( i \not\in I \); also, write \( x_i^{(I)} = x_i - \epsilon_i^{(I)} \frac{j}{2} \). For \( I \subseteq \{1, \ldots, d\} \) we define

\[
Q_I(x; t) := (-1)^{\#I} \left( \prod_{i=1}^d \sinh(\xi + \epsilon_i^{(I)} x_i) \sinh(t + \epsilon_i^{(I)} x_i + \ell\eta) \right) \prod_{1 \leq i < j \leq d} \sinh(x_i^{(I)} \pm x_j^{(I)})
\]

and write

\[
V(x; t) := \left( \prod_{i=1}^d \sinh(\xi - t + (\ell - i + 1)\eta) \right) \sum_{I \subseteq \{1, \ldots, d\}} Q_I(x; t).
\]

Then (7.14) is equivalent to

\[
V(x; t) = V(x; -t).
\]

(7.15)

The identity (7.15) is a direct consequence of the following multivariate generalization of the trigonometric identity (4.10).
Lemma 7.8. We have

\[ \sum_{I \subseteq \{1, \ldots, d\}} Q_I(x; t) = \left( \prod_{1 \leq i < j \leq d} \sinh(x_i \pm x_j) \right) \prod_{i=1}^{d} \sinh(2x_i) \]

(7.16)

\[ \times (-1)^d \prod_{i=1}^{d} \sinh(\xi + t + (\ell - i + 1)\eta). \]

Proof. Write \( V(x; t) \) for the left-hand side of (7.16). It is easy to see that

\[ V(x; t) \in \mathbb{C}[e^\pm 2x_1, \ldots, e^\pm 2x_d], \]

since each term \( Q_I(x; t) \) is a Laurent polynomial in \( e^{2x_1}, \ldots, e^{2x_d} \). We now first show that \( V(x; t) \) is anti-invariant with respect to the natural action of the Weyl group \( W \) of type \( C_d \) on \( \mathbb{C}[e^\pm 2x_1, \ldots, e^\pm 2x_d] \).

Let \( W = \langle s_1, \ldots, s_d \rangle \) be the Weyl group of type \( C_d \), with the simple reflections \( s_i \) \((i = 1, \ldots, d)\) acting on \( \mathbb{C}^{d} \) by permutations and sign flips: for \( 1 \leq i < d \) the simple reflection \( s_i \) acts on \( (x_1, \ldots, x_d) \in \mathbb{C}^{d} \) by permuting \( x_i \) and \( x_{i+1} \), and \( s_d \) acts by sending \( x_d \) to \(-x_d \). The Weyl group \( W \) also acts on the power set of \( \{1, \ldots, d\} \) by

\[ s_iI = \begin{cases} (I \setminus \{i\}) \cup \{i+1\}, & \text{if } i \in I, i+1 \notin I, \\ (I \setminus \{i+1\}) \cup \{i\}, & \text{if } i \notin I, i+1 \in I, \\ I, & \text{otherwise} \end{cases} \]

for \( 1 \leq i < d \), and

\[ s_dI = \begin{cases} I \setminus \{d\}, & \text{if } d \in I, \\ I \cup \{d\}, & \text{if } d \notin I. \end{cases} \]

Note that the action of \( W \) on the power set of \( \{1, \ldots, d\} \) is transitive, and that the stabilizer subgroup of the empty set \( \emptyset \) is equal to the symmetric group \( S_d := \langle s_1, \ldots, s_{d-1} \rangle \) in \( d \) letters.

By a direct computation we obtain the invariance property

\[ Q_I(wx; t) = (-1)^{l(w)} Q_{w^{-1}I}(x; t), \quad w \in W, \]

where \( l(w) \) is the length of \( w \in W \). It follows that

\[ V(x; t) = \frac{1}{d!} \sum_{w \in W} (-1)^{l(w)} Q\emptyset(w^{-1}x; t), \]

in particular \( V(x; t) \in \mathbb{C}[e^\pm 2x_1, \ldots, e^\pm 2x_d] \) is \( W \)-anti-invariant. Thus

\[ V(x; t) = Z(x; t) \delta(x) \]

(7.18)

with the Weyl denominator

\[ \delta(x) := \left( \prod_{1 \leq i < j \leq d} \sinh(x_i \pm x_j) \right) \prod_{i=1}^{d} \sinh(2x_i) \]

and with \( Z(x; t) \in \mathbb{C}[e^\pm 2x_1, \ldots, e^2x_d] \) \( W \)-invariant. A standard argument comparing degrees on both sides of (7.18) shows that \( Z(x; t) \) is independent of \( x \). So

\[ V(x; t) = Z(t) \delta(x) \]

(7.19)
for some constant $Z(t)$. We compute $Z(t)$ by evaluating both sides of \(7.19\) in

$$y := (-\xi + (d-1)\eta, -\xi + (d-2)\eta, \ldots, -\xi).$$

By the explicit expression

$$Q_0(x; t) = \left( \prod_{i=1}^{d} \sinh(\xi - x_i) \sinh(t - x_i + \ell \eta) \right) \prod_{1 \leq i < j \leq d} \sinh(x_i - x_j) \sinh(x_i + x_j + \eta)$$

it follows that $Q_0(w^{-1}y; t) = 0$ for $w \in W$ unless $w \in S_d$. Hence

$$\nabla(y; t) = \frac{1}{dt} \sum_{w \in S_d} (-1)^{(w)}Q_0(w^{-1}y; t) = \frac{1}{dt} \sum_{w \in S_d} Q_{w0}(y; t) = Q_0(y; t),$$

and consequently

$$Z(t) = \frac{Q_0(y; t)}{\delta(y)} = (-1)^d \prod_{i=1}^{d} \sinh(\xi + t + (\ell - i + 1)\eta),$$

where the last equality follows from a straightforward computation. \(\square\)

8. Fusion for the boundary qKZ equations and their solutions.

In this section we will show that, for $\ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, the solutions $f_{\xi/2}(t)$ exhibited in Theorem $6.2$ can be directly obtained using a fusion process from the spin-half solution $\left( \text{pr}^{+}_{\ell} \right)^{\otimes N} \left( f_{\frac{1}{2}}^{(}\xi, \ldots, \frac{1}{2}\right)(t) )$ constructed before in \cite{22}. Moreover, as we will see, arbitrary solutions of the boundary qKZ equations \cite{13} in $M^{(\ell_1, \ldots, \ell_{s-1})} \otimes V^k \otimes V^{\frac{1}{2}} \otimes M^{(\ell_{s+1}, \ldots, \ell_N)}$ can be naturally fused to obtain solutions in $M^{(\ell_1, \ldots, \ell_{s-1})} \otimes V^k \otimes V^{\frac{1}{2}} \otimes M^{(\ell_{s+1}, \ldots, \ell_N)}$.

8.1. Notations. In this section, we will slightly abuse notation when considering operators acting on a “mixed” $N$-fold tensor product made up of finite- and infinite-dimensional modules $V^k$ ($k \in \frac{1}{2}\mathbb{Z}_{\geq 0}$) and $M^{\ell}$ ($\ell \in \mathbb{C}$). For example, if $\ell_s \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, there is a unique linear operator $\Xi_{\ell}(t; \xi_+, \xi_-; \tau)$ on $M^{(\ell_1, \ldots, \ell_{s-1})} \otimes V^k \otimes M^{(\ell_{s+1}, \ldots, \ell_N)}$ determined by

$$\Xi_{\ell}(t; \xi_+, \xi_-; \tau) \left( \text{Id}_{M^{(\ell_1, \ldots, \ell_{s-1})}} \otimes \text{pr}^{\ell_s} \otimes \text{Id}_{M^{(\ell_{s+1}, \ldots, \ell_N)}} \right) =$$

$$= \left( \text{Id}_{M^{(\ell_1, \ldots, \ell_{s-1})}} \otimes \text{pr}^{\ell_s} \otimes \text{Id}_{M^{(\ell_{s+1}, \ldots, \ell_N)}} \right) \Xi(t; \xi_+, \xi_-; \tau);$$

we will denote the resulting operator $\Xi_{\ell}(t; \xi_+, \xi_-; \tau)$ on $M^{(\ell_1, \ldots, \ell_{s-1})} \otimes V^k \otimes M^{(\ell_{s+1}, \ldots, \ell_N)}$ simply by $\Xi_{\ell}(t; \xi_+, \xi_-; \tau)$ as long as it is clear from context which tensor component we have projected onto its finite-dimensional quotient.

We will use this mild abuse of notation also when discussing the operators $T^{\xi}(x; t)$, $U^{\xi}(x; t)$, $B^{\xi}(x; t)$, $E^{S}(x; t)$ and $E^{(S)}(x; t)$. Similarly, we will use the notations $\Omega^{\xi}$ and $f^{\xi/2}(t)$ for those elements of $M^{(\ell_1, \ldots, \ell_{s-1})} \otimes V^k \otimes M^{(\ell_{s+1}, \ldots, \ell_N)}$ that are actually given by $\text{pr}^{\ell_s}\Omega^{\xi}$ and $\text{pr}^{\ell_s}f_{\xi/2}(t)$, respectively.

To fuse the boundary qKZ transport operators $\Xi_{\ell}(t) := \Xi_{\ell}(t; \xi_+, \xi_-; \tau)$, it is convenient to use the injection $j^{\ell} = P^{\frac{1}{2}\ell}k^{\ell} : V^k \otimes V^{\frac{1}{2}} \hookrightarrow V^k \otimes V^{\frac{1}{2}}$ instead of $k^{\ell}$. Let
$k \in \frac{1}{2} \mathbb{Z}_{\geq 0}$ and $\ell \in \mathbb{C}$. The following “local” fusion relations in terms of $j^k$ follow straightforwardly from Proposition 3.3 and (4.7) respectively,

\[(8.1) \quad (j^k \otimes \text{Id}_{M'}) L^{k+\frac{1}{2}}(x-y) = L^{k+\frac{1}{2}}_{\mathbb{Z}^2}(x-y) L^{k+\frac{1}{2}}_{\mathbb{Z}^2}(x+y) (j^k \otimes \text{Id}_{M'}),\]

\[(8.2) \quad j^k R^{k+\frac{1}{2}}(x) = K^{\frac{1}{2}}_{\mathbb{Z}^2}(x) R^{k+\frac{1}{2}}(x) K^{\frac{1}{2}}_{\mathbb{Z}^2}(x+y) j^k.\]

Furthermore, in a similar way as we derived Proposition 3.3 and (8.1),

\[(8.3) \quad (j^k \otimes \text{Id}_{M'}) L^{k+\frac{1}{2}}(x-y) = L^{k+\frac{1}{2}}_{\mathbb{Z}^2}(x-y) L^{k+\frac{1}{2}}_{\mathbb{Z}^2}(x+k-y) (j^k \otimes \text{Id}_{M'}).\]

Given $s = 1, \ldots, N$ and $k \in \frac{1}{2} \mathbb{Z}_{\geq 0}$, denote

$$j_s^k := \text{Id}_{M(t_1, \ldots, t_{s-1})} \otimes j^k \otimes \text{Id}_{M(t_s+1, \ldots, t_N)},$$

an injective map from $M(t_1, \ldots, t_{s-1}) \otimes V^k \otimes M(t_{s+1}, \ldots, t_N)$ to $M(t_1, \ldots, t_{s-1}) \otimes V^k \otimes V^k \otimes M(t_{s+1}, \ldots, t_N)$.

For the rest of this section, given $1 \leq s \leq N$ and $\underline{t} \in \mathbb{C}^N$ such that $\ell_s = k + \frac{1}{2}$ for $k \in \frac{1}{2} \mathbb{Z}_{\geq 0}$, we write

\[(8.4) \quad \underline{t'} = (\ell_1, \ldots, \ell_{s-1}, k, \frac{1}{2}, \ell_{s+1}, \ldots, \ell_N) \in \mathbb{C}^{N+1},
\]

$$t' = (t_1, \ldots, t_{s-1}, t_s + \frac{\eta}{2}, t_s - k \eta, t_{s+1}, \ldots, t_N),$$

while $t = (t_1, \ldots, t_{s-1}, t_s, t_{s+1}, \ldots, t_N)$ and $\underline{t} = (t_1, \ldots, t_N)$ with $\ell_s = k + \frac{1}{2}$.

8.2. Fusion of transport operators.

**Proposition 8.1.** Let $1 \leq s \leq N$ and $\underline{t} \in \mathbb{C}^N$ such that $\ell_s = k + \frac{1}{2}$ for $k \in \frac{1}{2} \mathbb{Z}_{\geq 0}$. For $1 \leq r \leq N$ we have

$$j_s^k \Xi^r_{\underline{t}}(t) = \begin{cases} 
\Xi^r_{\underline{t}}(t') j_s^k, & r < s, \\
\Xi^r_{\underline{t}+1}(t'+e_s \tau) \Xi^r_{\underline{t}}(t') j_s^k, & r = s, \\
\Xi^r_{\underline{t}+1}(t') j_s^k, & r > s,
\end{cases}$$

as linear operators $M(t_1, \ldots, t_{s-1}) \otimes V^k \otimes M(t_{s+1}, \ldots, t_N) \rightarrow M(t_1, \ldots, t_{s-1}) \otimes V^k \otimes V^k \otimes M(t_{s+1}, \ldots, t_N)$.

**Proof:** For the cases where $r \neq s$, simply by judiciously applying (8.1)-(8.3) to the right-hand side of (8.4) (see (1.3) for the definition of the transport operators). For $r = s$, the product of factors in $\Xi^r_{\underline{t}+1}(t'+e_s \tau) \Xi^r_{\underline{t}}(t')$ can first be simplified using...
unitarity of the $R$-operator and the RLL-relations \([2, 4]\), yielding

$$
\Xi^{t'}_{s+1}(t' + \mathbf{e}_s \tau)\Xi^t_s(t') = \left( \prod_{j=s+1}^{N} L_{t,j}^j (t_s - t_j + \tau - k\eta)L_{t,j}^j (t_s - t_j + \tau + \frac{\eta}{2}) \right)
\times K^{r_k + \frac{1}{2}r}(t_s + \frac{\tau}{2} - k\eta)K^{r_{-k}}(2t_s + \frac{\tau}{2} - (k - \frac{1}{2})\eta)K^{r_{-k}}(t_s + \frac{\eta}{2})
\times \left( \prod_{j=s}^{N-1} L_{t,j}^j (t_s - t_j - k\eta)L_{t,j}^j (t_s + t_j + \frac{\eta}{2}) \right)
\times \left( \prod_{j=s+1}^{N} L_{t,j}^j (t_s - t_j - k\eta)L_{t,j}^j (t_s - t_j + \frac{\eta}{2}) \right),
$$

where the ordering of the products over $j$ is as prescribed. Now applying \([31, 32]\) yields \((8.5)\) for the case $r = s$. \qed

8.3. Fusion of solutions.

**Proposition 8.2.** Let $1 \leq s \leq N$ and $\ell \in \mathbb{C}^N$ such that $\ell_s = k + \frac{1}{2}$ for $k \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. Suppose that $f : \mathbb{C}^{N+1} \to M^{(\ell_1, \ldots, \ell_{s-1})} \otimes V^k \otimes V^k \otimes M^{(\ell_{s+1}, \ldots, \ell_N)}$ is a meromorphic solution of the boundary qKZ equations,

$$
\Xi^{t'}_s(z) f(z) = f(z + \tau \mathbf{e}_s), \quad 1 \leq r \leq N + 1,
$$

where $t'$ is given by \([-3, 4]\). Suppose that $f$ restricts to a meromorphic function on the hyperplane

$$
H := \{ z \in \mathbb{C}^{N+1} \mid z_s - z_{s+1} = (k + \frac{1}{2})\eta \}.
$$

Then there exists a unique meromorphic function

$$
\text{Fus}^{k\ell}_s(f) : \mathbb{C}^N \to M^{(\ell_1, \ldots, \ell_{s-1})} \otimes V^k \otimes V^k \otimes M^{(\ell_{s+1}, \ldots, \ell_N)}
$$
satisfying

$$
\left( \ell' \right)_s \text{Fus}^{k\ell}_s(f)(\mathbf{t}) = f(\mathbf{t}'),
$$

with $\mathbf{t}'$ given by \([-3, 4]\). Furthermore, $\text{Fus}^{k\ell}_s(f)$ is a meromorphic solution of the boundary qKZ equations \([-1, 3]\) with values in $M^{(\ell_1, \ldots, \ell_{s-1})} \otimes V^k \otimes V^k \otimes M^{(\ell_{s+1}, \ldots, \ell_N)}$,

$$
\Xi^t_s(\mathbf{t})\text{Fus}^{k\ell}_s(f)(\mathbf{t}) = \text{Fus}^{k\ell}_s(f)(\mathbf{t} + \tau \mathbf{e}_r), \quad 1 \leq r \leq N.
$$

**Proof.** It follows from \((8.6)\) with $r = s$ that $f(z) = \Xi^t_s(z - \tau \mathbf{e}_s) f(z - \tau \mathbf{e}_s)$. By assumption the left-hand side restricts to a meromorphic vector valued function on $H$. By the explicit expressions \([-1, 3]\) for the transport operators, the operator $\Xi^t_s(z - \tau \mathbf{e}_s)$ restricts to a meromorphic operator valued function on $H$, and

$$
\Xi^t_s(z - \tau \mathbf{e}_s)|_H = R^{k\frac{1}{2}}((k + \frac{1}{2})\eta)Z(z)
$$

for some meromorphic operator valued function $Z$ on $H$. Hence $f|_H$ takes its values in the subspace $\text{Im}(R^{k\frac{1}{2}}((k + \frac{1}{2})\eta))$ of $M^{(\ell_1, \ldots, \ell_{s-1})} \otimes V^k \otimes V^k \otimes M^{(\ell_{s+1}, \ldots, \ell_N)}$. By
Lemma 3.2 we have \( \text{Im}(R^k \ell_s^f((k + \frac{1}{2})\eta)) \subseteq \text{Im}(j_s^k) \). Since \( j_s^k \) is injective, we conclude that there exists a unique meromorphic function

\[
\text{Fus}_s^k(f) : \mathbb{C}^N \rightarrow M^{(\ell_1, \ldots, \ell_{s-1})} \otimes V^{k+\frac{1}{2}} \otimes M^{(\ell_{s-1+1}, \ldots, \ell_N)}
\]

satisfying (8.4).

It remains to show that \( \text{Fus}_s^k(f) \) satisfies the boundary qKZ equations (8.8). Since \( j_s^k \) is an injection, it suffices to prove that, for \( r = 1, \ldots, N \),

\[
(8.9) \quad j_s^k \Xi_r^k(t) \text{Fus}_s^k(f)(t) = j_s^k \text{Fus}_s^k(f)(t + r e_r).
\]

For \( r < s \) we have

\[
\Xi_r^k(t) \text{Fus}_s^k(f)(t) = \Xi_r^k(t' + r e_s) f(t') = f(t' + r e_r) = j_s^k \text{Fus}_s^k(f)(t + r e_r),
\]

owing to (8.5), (8.7), the boundary qKZ equations (8.6) and (8.7) again. The case \( r > s \) is proven similarly. Finally, for \( r = s \) we have

\[
\Xi_r^k(t) \text{Fus}_s^k(f)(t) = \Xi_r^k(t' + r e_s) f(t')
\]

\[
= \Xi_r^k(t' + r e_s) f(t' + r e_s)
\]

\[
= f(t' + r e_s + r e_{s+1}) = j_s^k \text{Fus}_s^k(f)(t + r e_s),
\]

where we have applied (8.5), (8.7), (8.6) twice, and finally (8.7) again.

\[
\square
\]

8.4. Fusion of the Jackson integral solutions. The special Jackson integral solutions of the boundary qKZ equations (see Theorem 6.2) are compatible with fusion in the following sense.

**Proposition 8.3.** Let \( 1 \leq s \leq N \) and \( \ell \in \mathbb{C}^N \) such that \( \ell_s = k + \frac{1}{2} \) with \( k \in \frac{1}{2} \mathbb{Z}_{\geq 0} \). Let \( \ell' \in \mathbb{C}^{N+1} \) be given by (8.11). Let

\[
f_s^\ell : \mathbb{C}^N \rightarrow M^{(\ell_1, \ldots, \ell_{s-1})} \otimes V^{k+\frac{1}{2}} \otimes M^{(\ell_{s-1+1}, \ldots, \ell_N)}
\]

and

\[
f_s^{\ell'} : \mathbb{C}^{N+1} \rightarrow M^{(\ell_1, \ldots, \ell_{s-1})} \otimes V^k \otimes V^{\frac{1}{2}} \otimes M^{(\ell_{s-1+1}, \ldots, \ell_N)}
\]

be the Jackson integral solutions of the boundary qKZ equations as given in Theorem 6.2, with \( f_s^\ell \) and \( f_s^{\ell'} \) having the same base point \( x_0 \in \mathbb{C}^S \), the same weight factors \( g_{s}, h, F_{\ell} \) \((j \in \{1, \ldots, N\} \setminus \{s\})\) and with the remaining weight factors \( F^{k+\frac{1}{2}}, F^k \) and \( F^{\frac{1}{2}} \) satisfying the compatibility condition

\[
(8.10) \quad F^{k+\frac{1}{2}}(x) = F^k(x + \frac{\eta}{2}) F^{\frac{1}{2}}(x - k\eta).
\]

Then

\[
f_s^\ell = \text{Fus}_s^k(f_s^{\ell'}).
\]

**Remark 8.4.** Note that (8.10) is compatible with the difference equations that \( F^\ell(x) \) satisfies (see Theorem 6.2). Note furthermore that the explicit choice (6.1) of \( F^\ell(x) \) \((\ell \in \mathbb{C})\) satisfies (8.10).

**Proof.** By virtue of the fusion formulae (8.3) and (8.1), we have (cf. 6.2)

\[
j_s^k T^\xi_s^l(x; t) = T^\xi_s^l(x; t') j_s^k, \quad j_s^k T^\xi_s^l(x; t)^{-1} = T^\xi_s^l(x; t')^{-1} j_s^k,
\]

where we use the notations (8.4). Hence, owing to (5.4) we also have

\[
(8.11) \quad j_s^k U^{\ell; \xi}_s^l(x; t) = U^{\ell'; \xi}_s^l(x; t') j_s^k.
\]
The above three identities are as operators \( V^j \odot M^{(l_1, \ldots, \ell_{s-1})} \odot V^{k} \odot V^\frac{j}{2} \odot M^{(l_{s+1}, \ldots, l_N)} \rightarrow V^j \odot M^{(l_1, \ldots, \ell_{s-1})} \odot V^k \odot V^\frac{j}{2} \odot M^{(l_{s+1}, \ldots, l_N)} \). Taking the appropriate matrix coefficients in (8.11) with respect to the auxiliary space, we obtain

\[
j^j_s B^j \xi_s(x; t) = B^j \xi_s(x; t') j^j_s
\]
as operators \( M^{(l_1, \ldots, \ell_{s-1})} \odot V^{k} \odot V^\frac{j}{2} \odot M^{(l_{s+1}, \ldots, l_N)} \rightarrow M^{(l_1, \ldots, \ell_{s-1})} \odot V^k \odot V^\frac{j}{2} \odot M^{(l_{s+1}, \ldots, l_N)} \).

Writing

\[
\frac{\sinh(x - t_s - (k + \frac{1}{2})\eta)}{\sinh(x - t_s + (k + \frac{1}{2})\eta)} = \frac{\sinh(x - (t_s + \frac{\ell}{2}) - k\eta) \sinh(x - (t_s - k\eta) - \frac{\ell}{2})}{\sinh(x - (t_s + \frac{\ell}{2}) + k\eta) \sinh(x - (t_s - k\eta) + \frac{\ell}{2})}
\]
it follows that

\[
j^j_s B^j \xi_s(x; t) = B^j \xi_s(x; t') j^j_s
\]
and hence

\[
(8.12) \quad j^j_s \Omega^j = \Omega^j
\]
(see Proposition 3.1) it now follows from (8.10) that

\[
j^j_s f^j_S(t) = f^j_S(t') = j^j_s \text{Fun}_S^j(f^j_S)(t)
\]
as meromorphic \( M^{(l_1, \ldots, \ell_{s-1})} \odot V^k \odot V^\frac{j}{2} \odot M^{(l_{s+1}, \ldots, l_N)} \) valued functions in \( t \in \mathbb{C}^N \), which proves the result.

\[\Box\]

Remark 8.5. Note that \( \sum_{r=1}^N \ell_r = \sum_{r=1}^{N+1} \ell_r' \) for \( \mathbf{c} \in \mathbb{C}^N \) with \( \ell_s = k \) and with \( \ell' \) given by (8.4). Hence the region of meromorphic convergence (6.2) for the solutions \( f^j_S \) and \( f^j_S' \) with weight factors (6.1) is compatible with fusion.

REFERENCES


