

# Matrix-Valued Little $q$ -Jacobi Polynomials

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*Dedicated to Dick Askey on the occasion of his 80th birthday  
with admiration for Dick's achievements in special functions and mathematics education*

## Abstract

Matrix-valued analogues of the little  $q$ -Jacobi polynomials are introduced and studied. For the  $2 \times 2$ -matrix-valued little  $q$ -Jacobi polynomials explicit expressions for the orthogonality relations, Rodrigues formula, three-term recurrence relation and their relation to matrix-valued  $q$ -hypergeometric series and the scalar-valued little  $q$ -Jacobi polynomials are presented. The study is based on a matrix-valued  $q$ -difference operator, which is a  $q$ -analogue of Tirao's matrix-valued hypergeometric differential operator.

## 1 Introduction

Matrix-valued orthogonal polynomials were originally introduced by M.G.Kreĭn in 1949, initially studying the corresponding moment problem, see references in [3, 5], and to study differential operators and their deficiency indices, see also [25]. Since then an increasing amount of authors are contributing to build up a general theory of matrix-valued orthogonal polynomials (see for example [7, 14, 17, 23, 27], etc.).

In the study of matrix-valued orthogonal polynomials the general theory deals with obtaining appropriate analogues of classical results known for (scalar-valued) orthogonal polynomials, and many results and proofs have been generalized in this direction, see [6, 7] and the overview paper [5]. But also new features that do not hold in the scalar theory have been discovered, like the existence of different second order differential equations satisfied by a family of matrix orthogonal polynomials, see [10, 23]. The theory of matrix-valued orthogonal polynomials has also turned out to be a fruitful tool in the solution of higher order recurrence relations, see [12, 15].

For orthogonal polynomials the theory is complemented by many explicit families of orthogonal polynomials, notably the ones in the Askey scheme and its  $q$ -analogue, see [21, 22], which have turned out to be very useful in many different contexts, such as mathematical physics, representation theory, combinatorics, number theory, etc. The orthogonal polynomials in the ( $q$ -)Askey scheme are characterized by being eigenfunctions of a suitable second order differential or difference operator, so that all these families correspond to solutions of a bispectral problem. E.g., for the Jacobi polynomials this is the hypergeometric differential operator and for the little  $q$ -Jacobi polynomials this is the  $q$ -hypergeometric difference operator, see also [13, 20]. This is closely related to Bochner's 1929 classification theorem of second order differential operators having polynomial eigenfunctions, see [20] for extensions and references.

For matrix-valued orthogonal polynomials there is no classification result of such type known, so that we have to study the properties of specific examples of families of matrix-valued orthogonal polynomials. Already many examples are known, either from scratch, [11], related to representation theory

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e.g. [16, 23, 24] or motivated from spectral theory [15]. In most of these papers, the matrix-valued orthogonal polynomials are eigenfunctions of a second order matrix-valued differential operator, so that these polynomials are usually considered as matrix-valued analogues of suitable polynomials from the Askey-scheme. The matrix-valued differential operator is often of the type of the matrix-valued hypergeometric differential operator of Tirao [28] and this makes it possible to express matrix-valued orthogonal polynomials in terms of the matrix-valued hypergeometric functions, see e.g. [24] for an example.

More recently, in [2] the step has been made to use matrix-valued difference operators and consider corresponding matrix-valued orthogonal polynomials as eigenfunctions. Again these matrix-valued orthogonal polynomials can be seen as analogues of orthogonal polynomials from the Askey-scheme. In this paper, motivated by [2], we study a specific case of matrix-valued orthogonal polynomials which are analogues of the little  $q$ -Jacobi polynomials, moving from analogues of classical discrete orthogonal polynomials to orthogonal polynomials on a  $q$ -lattice. As far as we are aware, these matrix-valued orthogonal polynomials are a first example of the matrix-valued analogue of a family of polynomials in the  $q$ -Askey scheme. An essential ingredient in the study of these matrix-valued little  $q$ -Jacobi polynomials is the second order  $q$ -difference operator (3.2). In particular this gives the possibility to introduce and employ matrix-valued basic hypergeometric series in the same spirit as Tirao [28], which differs from the approach of Conflitti and Schlosser [4].

The content of the paper are as follows. In Section 2 we recall the basics of the (scalar-valued) little  $q$ -Jacobi polynomials and the general theory of matrix-valued orthogonal polynomials. In Section 3 we study the matrix-valued second order  $q$ -difference equations as well as under which conditions such an operator is symmetric for a suitable matrix-weight function. In Section 4 we study the relevant  $q$ -analogue of Tirao's [28] matrix-valued hypergeometric functions. In Section 5 the  $2 \times 2$ -matrix-valued little  $q$ -Jacobi polynomials are studied in detail. In particular, we give explicit orthogonality relations, the moments, the matrix-valued three-term recurrence relation, expressions in terms of the matrix-valued basic hypergeometric function, the link to the scalar little  $q$ -Jacobi polynomials, and the Rodrigues formula for these family of polynomials.

It would be interesting to find a group theory interpretation of these matrix-valued little  $q$ -Jacobi polynomials along the lines of [16, 23, 24] in the quantum group setting.

## 2 Preliminaries

### 2.1 Basic hypergeometric functions

We recall some of the definitions and facts about basic hypergeometric functions, see Gasper and Rahman [13]. We fix  $0 < q < 1$ . For  $a \in \mathbb{C}$  the  $q$ -Pochhammer symbol is defined recursively by  $(a; q)_0 = 1$  and

$$(a; q)_n = (1 - aq^{n-1})(a; q)_{n-1}, \quad (a; q)_{-n} = \frac{1}{(aq^{-n}; q)_n}, \quad n \in \mathbb{N} = \{0, 1, 2, \dots\}.$$

The infinite  $q$ -Pochhammer symbol is defined as

$$(a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k).$$

For  $a_1, \dots, a_\ell \in \mathbb{C}$  we use the abbreviation  $(a_1, a_2, \dots, a_\ell; q)_n = \prod_{i=1}^{\ell} (a_i; q)_n$ . The basic hypergeometric series  ${}_{r+1}\phi_r$  with parameters  $a_1, \dots, a_{r+1}, b_1, \dots, b_r \in \mathbb{C}$ , base  $q$  and variable  $z$  is defined by the series

$${}_{r+1}\phi_r \left[ \begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, b_2, \dots, b_r \end{matrix}; q, z \right] = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \dots, a_{r+1}; q)_k}{(q; q)_k (b_1, b_2, \dots, b_r; q)_k} z^k, \quad |z| < 1.$$

The  $q$ -derivative  $D_q$  of a function  $f$  at  $z \neq 0$  is defined by

$$(D_q f)(z) = \frac{f(z) - f(qz)}{(1 - q)z},$$

and  $(D_q f)(0) = f'(0)$ , provided that  $f'(0)$  exists. Two useful formulas are the  $q$ -Leibniz rule [13, Exercise 1.12.iv]

$$D_q^n(fg)(z) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q D_q^{n-k} f(q^k z) D_q^k g(z), \quad (2.1)$$

and the formula

$$(D_q^n f)(z) = \frac{1}{(1-q)^n q^{\binom{n}{2}} z^n} \sum_{j=0}^n (-1)^{n-j} \begin{bmatrix} n \\ j \end{bmatrix}_q q^{\binom{n-j}{2}} f(q^j z), \quad (2.2)$$

where the  $q$ -binomial coefficient  $\begin{bmatrix} n \\ k \end{bmatrix}_q$  is given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}.$$

The  $q$ -integral of a function  $f$  is defined as

$$\int_0^1 f(z) d_q z = (1-q) \sum_{k=0}^{\infty} f(q^k) q^k,$$

whenever the series converges. The  $q$ -analogue of the fundamental theorem of calculus states

$$\int_0^1 (D_q f)(z) d_q z = f(q^x)|_{x=\infty}^0 = f(1) - f(0), \quad (2.3)$$

whenever all the limits converge.

## 2.2 The little $q$ -Jacobi polynomials

Let  $0 < a < q^{-1}$  and  $b < q^{-1}$ . The little  $q$ -Jacobi polynomials are the polynomials defined by

$$p_n(z; a, b; q) = {}_2\phi_1 \left[ \begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix}; q, qz \right]. \quad (2.4)$$

The little  $q$ -Jacobi polynomials have been introduced by Andrews and Askey [1], see also [13, §7.3] and [21, §14.12]. These polynomials satisfy the following orthogonality relation

$$\begin{aligned} \langle p_m(z, a, b; q), p_n(z, a, b; q) \rangle &= \sum_{k=0}^{\infty} (aq)^k \frac{(bq; q)_k}{(q; q)_k} p_m(q^k; a, b; q) p_n(q^k; a, b; q) \\ &= \frac{(abq^2; q)_{\infty}}{(aq; q)_{\infty}} \frac{(1-abq)(aq)^n}{(1-abq^{2n+1})} \frac{(q, bq; q)_n}{(aq, abq; q)_n} \delta_{m,n} = h_n(a, b; q) \delta_{m,n}, \end{aligned} \quad (2.5)$$

where  $\delta_{m,n}$  is the Kronecker delta function and  $h_n(a, b; q) > 0$ . If we need to emphasize the dependence on  $a$  and  $b$  we write  $\langle \cdot, \cdot \rangle_{(a,b)}$ . The moments of the little  $q$ -Jacobi polynomials are given by

$$m_n(a, b) = \langle z^n, 1 \rangle_{(a,b)} = \frac{(abq^{n+2}; q)_{\infty}}{(aq^{n+1}; q)_{\infty}}. \quad (2.6)$$

The sequence of the little  $q$ -Jacobi polynomials satisfies the three term recurrence relation

$$-z p_n(z; a, b; q) = A_n p_{n+1}(z; a, b; q) - (A_n + C_n) p_n(z; a, b; q) + C_n p_{n-1}(z; a, b; q) \quad (2.7)$$

with

$$A_n = q^n \frac{(1-aq^{n+1})(1-abq^{n+1})}{(1-abq^{2n+1})(1-abq^{2n+2})}, \quad C_n = aq^n \frac{(1-q^n)(1-bq^n)}{(1-abq^{2n})(1-abq^{2n+1})}.$$

They are also eigenfunctions of the second order  $q$ -difference operator

$$\lambda_n p_n(z) = a(bq - z^{-1})(E_1 p_n)(z) - ((abq + 1) - (1+a)z^{-1})(E_0 p_n)(z) + (1 - z^{-1})(E_{-1} p_n)(z), \quad (2.8)$$

where  $\lambda_n = q^{-n}(1-q^n)(1-abq^{n+1})$ ,  $p_n(z) = p_n(z; a, b; q)$  and  $E_\ell$  are the  $q$ -shift operators defined by  $(E_\ell p)(z) = p(q^\ell z)$ .

### 2.3 Matrix-valued orthogonal polynomials

We review here some basic concepts of the theory of matrix-valued orthogonal polynomials, also see [5, 18, 26]. A matrix-valued polynomial of size  $N \in \mathbb{N}$  is a polynomial whose coefficients are elements of  $\text{Mat}_N(\mathbb{C})$ . If no confusion is possible we will omit the size parameter  $N$  and write  $\mathbb{P}[z]$  for the space of matrix polynomials with coefficients in  $\text{Mat}_N(\mathbb{C})$  and  $\mathbb{P}_n[z]$  for polynomials in  $\mathbb{P}[z]$  of degree at most  $n$ . The orthogonality will be with respect to a  $N \times N$  weight matrix  $W$ , that is a matrix of Borel measures supported on a common set of the real line  $\mathfrak{S}$ , such that the following is satisfied:

1. for any Borel set  $A \subseteq \mathfrak{S}$  the matrix  $W(A) = \int_A dW(z)$  is positive semi-definite,
2.  $W$  has finite moments of every order, i.e.  $\int_{\mathfrak{S}} z^n dW(z)$  is finite for all  $n \geq 0$ ,
3. if  $P$  is a matrix-valued polynomial with non-singular leading coefficient then  $\int_{\mathfrak{S}} P(z)dW(z)P^*(z)$  is also non-singular.

A weight matrix  $W$  defines a matrix-valued inner product on the space  $\mathbb{P}[z]$  by

$$\langle P, Q \rangle = \int_{\mathfrak{S}} P(z)dW(z)Q^*(z) \in \text{Mat}_N(\mathbb{C}).$$

Note that for every matrix-valued polynomial  $P$  with non-singular leading coefficient,  $\langle P, P \rangle$  is positive definite. A sequence of matrix-valued polynomials  $(P_n)_{n \geq 0}$  is called orthogonal with respect to the weight matrix  $W$  if

1. for every  $n \geq 0$  we have  $\text{dgr}(P_n) = n$  and  $P_n$  has non-singular leading coefficient,
2. for every  $m, n \geq 0$  we have  $\langle P_m, P_n \rangle = \Gamma_m \delta_{m,n}$ , where  $\Gamma_m$  is a positive definite matrix.

Given a weight matrix  $W$  there always exists a unique sequence of polynomials  $(P_n)_{n \geq 0}$  orthogonal with respect to  $W$  up to left multiplication of each  $P_n$  by a non-singular matrix, see [5, Lemma 2.2 and Lemma 2.7] or [18]. We say that a matrix-valued orthogonal polynomials sequence  $(P_n)_{n \geq 0}$  is orthonormal if  $\Gamma_n = I$  for all  $n \geq 0$ . We call  $(P_n)_{n \geq 0}$  monic if every  $P_n$  is monic, i.e. the leading coefficient of  $P_n$  is the identity matrix.

A weight matrix  $W$  with support  $\mathfrak{S}$  is said to be reducible to scalar weights if there exists a non-singular matrix  $K$ , independent of  $z$ , and a diagonal matrix  $D(z) = \text{diag}(w_1(z), w_2(z), \dots, w_N(z))$  such that for all  $z \in \mathfrak{S}$

$$W(z) = K D(z) K^*.$$

In this case the orthogonal polynomials with respect to  $W(z)$  are of the form

$$P_n(z) = \begin{pmatrix} p_{n,1}(z) & 0 & \cdots & 0 \\ 0 & p_{n,2}(z) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_{n,N}(z) \end{pmatrix} K^{-1}$$

where  $(p_{n,i})_n$  are the orthogonal polynomials with respect to  $D_{i,i}(z) = w_i(z)$  for  $i = 1, \dots, N$ . Therefore weight matrices that reduce to scalar weights can be viewed as a set of independent scalar weights, so they are not interesting for the theory of matrix orthogonal polynomials. In this paper  $\mathfrak{S}$  is countable and assuming additionally that  $W(a) = I$  for some  $a \in \mathfrak{S}$ , by [19, Theorem 4.1.6] the weight matrix  $W$  can be reduced to scalar weights if and only if  $W(x)W(y) = W(y)W(x)$  for all  $x, y \in \mathfrak{S}$ , also see [2, p. 43].

In the rest of this paper we only consider weight matrices such that  $dW(z) = \frac{1}{1-q} W(z) d_q z$  and we assume that  $W(q^n) > 0$  for all  $n \in \mathbb{N}$ . These weight matrices are called  $q$ -weight matrices or just  $q$ -weights. The matrix-valued inner product defined by such a  $q$ -weight is of the form

$$\langle P, Q \rangle_W = \frac{1}{1-q} \int_0^1 P(z)W(z)Q^*(z)d_q z = \sum_{n=0}^{\infty} q^n P(q^n)W(q^n)(Q(q^n))^*, \quad (2.9)$$

whenever the series converges termwise.

### 3 $q$ -Difference operators

In order to study matrix-valued analogues of the little  $q$ -Jacobi polynomials appearing in the  $q$ -Askey scheme we focus our attention on operators of the form

$$D = E_{-1}F_{-1} + E_0F_0 + E_1F_1, \quad (3.1)$$

where  $F_\ell(z)$  are matrix-valued polynomials in  $z^{-1}$  satisfying certain degree conditions assuring the preservation of the polynomials, cf (2.8). In particular we are interested in operators having families of matrix-valued polynomials as eigenfunctions,

$$(DP_n)(z) = P_n(q^{-1}z)F_{-1}(z) + P_n(z)F_0(z) + P(qz)F_1(z) = \Lambda_n P_n(z). \quad (3.2)$$

It is important to notice that the coefficients  $F_\ell$  appear on the right whereas the eigenvalue matrix  $\Lambda_n$  appears on the left, cf. [8].

#### 3.1 $q$ -Difference operators preserving polynomials

Suppose that there is a family of solutions of matrix-valued orthogonal polynomials to (3.2), then  $D$  preserves polynomials and does not raise the degree of a polynomial. Theorem 3.1 characterizes the  $q$ -difference operators with polynomial coefficients in  $z^{-1}$  preserving polynomials of degree  $n$  for all  $n$ . Theorem 3.1 is an analogue of [2, Lemma 2.2] and [9, Lemma 3.2], where the proof is a slight adaptation of [2, Lemma 2.2] and [9, Lemma 3.2].

**Theorem 3.1.** Let

$$D = \sum_{\ell=s}^r E_\ell F_\ell, \quad F_\ell \in \mathbb{P}_n[z^{-1}]$$

with  $r, s$  integers such that  $s \leq r$ . The following conditions are equivalent:

1.  $D: \mathbb{P}_n[z] \rightarrow \mathbb{P}_n[z]$  for all  $n \geq 0$ .
2.  $F_\ell(z) \in \mathbb{P}_{r-s}[z^{-1}]$  for  $\ell = s, \dots, r$  and  $\sum_{\ell=s}^r q^{\ell k} F_\ell(z) \in \mathbb{P}_k[z^{-1}]$  for  $k = 0, \dots, r - s$ .

To prove Theorem 3.1 we use Lemma 3.2.

**Lemma 3.2.** Let  $r, s$  and  $n$  be integers such that  $s \leq r$  and  $0 \leq n$ . Let  $G_k(z)$  be matrix-valued polynomials in  $z^{-1}$  of degree at most  $n$  for  $k = 0, \dots, r - s$ . The system of linear equations

$$\sum_{\ell=s}^r q^{\ell k} F_\ell(z) = G_k(z), \quad 0 \leq k \leq r - s,$$

determines the functions  $F_\ell(z)$ ,  $\ell = s, \dots, r$ , uniquely as polynomials in  $z^{-1}$  of degree at most  $n$ .

The proof is straightforward, see [9, Lemma 2.1], using the Vandermonde matrix.

*Proof of Theorem 3.1.* First we prove  $1 \Rightarrow 2$ . For  $k = 0, \dots, r - s$ , let  $G_k(z) = \sum_{\ell=s}^r q^{\ell k} F_\ell(z)$ . If  $0 \leq n \leq r - s$ , write  $P(z) = z^n I$ . Then

$$(DP)(z) = \sum_{\ell=s}^r (q^\ell z)^n F_\ell(z) = z^n \sum_{\ell=s}^r q^{\ell n} F_\ell(z) = z^n G_n(z) \in \mathbb{P}_n[z].$$

Since  $DP$  is a polynomial with  $\text{dgr}(DP) \leq \text{dgr}(P)$ ,  $G_n$  is a polynomial in  $z^{-1}$  of degree at most  $n$  for  $0 \leq n \leq r - s$ . By Lemma 3.2  $F_\ell$  are actually polynomials in  $z^{-1}$  of degree at most  $r - s$ .

To prove  $2 \Rightarrow 1$ , consider  $G_k(z) = \sum_{\ell=s}^r q^{\ell k} F_\ell(z)$ , for  $k \geq 0$ . Now  $G_k$  is a matrix-valued polynomial in  $z^{-1}$ . If  $0 \leq k \leq r - s$  then, by 2,  $\text{dgr}(G_k) \leq k$  and if  $k \geq r - s + 1$  then  $\text{dgr}(G_k) \leq r - s$ . Now put  $P(z) = z^n I$  so

$$DP(z) = \sum_{\ell=s}^r (q^\ell z)^n F_\ell(z) = z^n \sum_{\ell=s}^r q^{\ell n} F_\ell(z).$$

If  $0 \leq n \leq r - s$  we have that  $\sum_{\ell=s}^r q^{\ell n} F_\ell(z)$  is a polynomial in  $z^{-1}$  of degree at most  $n$ . Hence  $DP$  is a polynomial of degree at most  $n$ . On the other hand if  $n \geq r - s$  then  $\sum_{\ell=s}^r q^{\ell n} F_\ell(z)$  is a matrix-valued polynomial in  $z^{-1}$  of degree at most  $r - s$ . Hence  $DP$  is also a polynomial in  $z$  of degree at most  $n$ .  $\square$

### 3.2 The symmetry equations

Symmetry is a key concept when looking for weight matrices having matrix-valued orthogonal polynomials as eigenfunctions of a suitable  $q$ -difference operator. In Definition 3.3 we use the notation (2.9).

**Definition 3.3.** An operator  $D: \mathbb{P}[z] \rightarrow \mathbb{P}[z]$  is symmetric with respect to a weight matrix  $W$  if  $\langle DP, Q \rangle_W = \langle P, DQ \rangle_W$  for all  $P, Q \in \mathbb{P}[z]$ .

Theorem 3.4 is a well-known result relating symmetric operators and matrix-valued orthogonal polynomials, see [8, Lemma 2.1] and [18, Proposition 2.10 and Corollary 4.5].

**Theorem 3.4.** Let  $D$  be a  $q$ -difference operator preserving  $\mathbb{P}[z]$ , so that  $\text{dgr}(DP) \leq \text{dgr}(P)$  for any  $P \in \mathbb{P}[z]$ . If  $D$  is symmetric with respect to a weight matrix  $W$  then there exists a sequence of matrix-valued orthonormal polynomials  $(P_n)_{n \geq 0}$  and a sequence of Hermitian matrices  $(\Lambda_n)_{n \geq 0}$  such that

$$DP_n = \Lambda_n P_n, \quad \forall n \geq 0. \quad (3.3)$$

Conversely if  $W(z)$  is a weight matrix and  $(P_n)_{n \geq 0}$  a sequence of matrix-valued orthonormal polynomials such that there exists a sequence of Hermitian matrices  $(\Lambda_n)_{n \geq 0}$  satisfying (3.3), then  $D$  is symmetric with respect to  $W$ .

It should be observed that Theorem 3.5 is an analogue of a similar statement for differential operators in [8, 18]. Also note that the  $q$ -difference operator  $D$  has polynomial coefficients in  $z^{-1}$  (instead of  $z$ ), so that the essential condition is the preservation of the space of polynomials instead of the degree condition, which is the essential condition in [8, 18].

Theorem 3.5 is an analogue of [11, Theorem 3.1] and [17, Section 4] for symmetric second order differential operators.

**Theorem 3.5.** Let  $D$  be a  $q$ -difference operator preserving  $\mathbb{P}[z]$  of the form (3.1) with  $F_\ell$  matrix-valued polynomials in  $z^{-1}$ . Let  $W$  be a  $q$ -weight matrix as in (2.9). Suppose that the coefficients  $F_\ell$  and the weight matrix  $W$  satisfy the following equations

$$F_0(q^x)W(q^x) = W(q^x)F_0(q^x)^*, \quad x \in \mathbb{N}, \quad (3.4)$$

$$F_1(q^{x-1})W(q^{x-1}) = qW(q^x)F_{-1}(q^x)^*, \quad x \in \mathbb{N} \setminus \{0\}, \quad (3.5)$$

and the boundary conditions

$$W(1)F_{-1}(1)^* = 0, \quad (3.6)$$

$$q^{2x}F_1(q^x)W(q^x) \rightarrow 0, \quad \text{as } x \rightarrow \infty,$$

$$q^x(F_1(q^x)W(q^x) - W(q^x)F_1(q^x)^*) \rightarrow 0, \quad \text{as } x \rightarrow \infty.$$

Then the  $q$ -difference operator  $D$  is symmetric with respect to  $W$ .

*Proof.* We assume that the operator  $D$  and the weight matrix  $W$  satisfy the symmetry equations (3.4), (3.5) and the boundary conditions (3.6). For an integer  $M > 0$ , we consider the truncated inner product

$$\langle P, Q \rangle_W^M = \sum_{x=0}^M q^x P(q^x)W(q^x)Q^*(q^x).$$

It is clear that  $\langle P, Q \rangle_W^M \rightarrow \langle P, Q \rangle_W$  as  $M \rightarrow \infty$  for  $P, Q \in \mathbb{P}[z]$ . Then

$$\begin{aligned} \langle DP, Q \rangle_W^M &= \sum_{x=0}^M q^x (DP)(q^x)W(q^x)Q^*(q^x) \\ &= \sum_{x=0}^M q^x (P(q^{x+1})F_1(q^x) + P(q^x)F_0(q^x) + P(q^{x-1})F_{-1}(q^x))W(q^x)Q^*(q^x) \end{aligned}$$

By a straightforward and careful computation using (3.4), (3.5) and the first boundary condition of (3.6) we have

$$\langle DP, Q \rangle_W^M - \langle P, DQ \rangle_W^M = q^M P(q^{M+1})F_1(q^M)W(q^M)Q^*(q^M) - q^M P(q^M)W(q^M)F_1^*(q^M)Q^*(q^{M+1}).$$

Write  $P(z) = P_0 + zP_1(z)$  and  $Q(z) = Q_0 + zQ_1(z)$  so that

$$\langle DP, Q \rangle_W^M - \langle P, DQ \rangle_W^M = q^M P_0 (F_1(q^M)W(q^M) - W(q^M)F_1(q^M)^*) Q_0^* + \text{remainder}$$

where the remainder consists of terms of the form  $q^{2M}R(q^M)F_1(q^M)W(q^M)S(q^M)$  or its adjoint for suitable matrix-valued polynomials  $R$  and  $S$ . Taking  $M \rightarrow \infty$  and using the last two boundary conditions of (3.6) we get the result.  $\square$

## 4 A matrix-valued $q$ -hypergeometric equation

Motivated by Tirao [28] we define a matrix-valued analogue of the basic hypergeometric series. This definition is different from that given by Conflitti and Schlosser [4], where some additional factorization is assumed.

Consider the following  $q$ -difference equation on row-vector-valued functions  $F : \mathbb{C} \rightarrow \mathbb{C}^N$ .

$$F(q^{-1}z)(R_1 + zR_2) + F(z)(S_1 + zS_2) + F(qz)(T_1 + zT_2) = 0. \quad (4.1)$$

where  $R_1, R_2, S_1, S_2, T_1, T_2 \in \text{Mat}_N(\mathbb{C})$ . The case  $N = 1$  is the scalar hypergeometric  $q$ -difference equation, see [13, Exercise 1.13].

Let  $F$  be a solution of (4.1) of the form  $F(z) = z^\mu G(z)$  where

$$G(z) = \sum_{k=0}^{\infty} G^k z^k, \quad G^0 \neq 0, \quad G^k \in \mathbb{C}^N$$

The Frobenius method gives the recursions

$$\begin{aligned} 0 &= G^0 (q^{-\mu} R_1 + S_1 + q^\mu T_1), \\ 0 &= G^k (q^{-k-\mu} R_1 + S_1 + q^{k+\mu} T_1) + G^{k-1} (q^{-k-\mu+1} R_2 + S_2 + q^{k+\mu-1} T_2), \quad k \geq 1. \end{aligned}$$

The first equation implies  $\det(q^{-\mu} R_1 + S_1 + q^\mu T_1) = 0$  and  $(G^0)^* \in \ker(q^{-\mu} R_1^* + S_1^* + q^\mu T_1^*)$ . The solution of the indicial equation  $\det(q^{-\mu} R_1 + S_1 + q^\mu T_1) = 0$  is the set of exponents  $E$ . For each  $\mu \in E$  we write  $d_\mu = \dim(\ker(q^{-\mu} R_1^* + S_1^* + q^\mu T_1^*))$  for the multiplicity of the exponent  $\mu$ . In order to have analytic solutions of (4.1) we require that  $0 \in E$ . Moreover we assume that the multiplicity for 0 is maximal,  $d_0 = N$ , which implies  $S_1 = -R_1 - T_1$ . Under this assumption  $E = \{\mu : \det(q^{-\mu} R_1 - T_1) = 0\} \cup \{0\}$ . Since we are only interested in polynomial solutions, we only consider expansions around  $z = 0$ , but we can also study solutions at  $\infty$  in a similar fashion.

We specialize to the case  $R_1 = -R_2 = I$ ;

$$F(q^{-1}z)(1-z) + F(z)(-I - T_1 + zS_2) + F(qz)(T_1 + zT_2) = 0. \quad (4.2)$$

For any  $G^0 \in \mathbb{C}^N$ ,  $G(z) = \sum_{k=0}^{\infty} G^k z^k$  is a solution of (4.2) if and only if

$$0 = G^k ((q^{-k} - 1)I + (q^k - 1)T_1) + G^{k-1} (-q^{-k+1}I + S_2 + q^{k-1}T_2), \quad k \geq 1.$$

Assuming  $\sigma(T_1) \cap q^{-\mathbb{N}} = \emptyset$ , the coefficients are

$$G^k = \frac{q^k}{(q; q)_k} G^0 \prod_{i=1}^k (I - q^{i-1} S_2 - q^{2i-2} T_2) (I - q^i T_1)^{-1}, \quad k \geq 1,$$

where  $\prod_{i=1}^k A_i = A_1 A_2 \dots A_k$  is an ordered product. We summarize this discussion with Definition 4.1 and Theorem 4.2.

**Definition 4.1.** Let  $A, B, C \in \text{Mat}_N(\mathbb{C})$  where  $\sigma(C) \cap q^{-\mathbb{N} \setminus \{0\}} = \emptyset$ . Define

$$\begin{aligned} (A, B; C; q)_0 &= I, \\ (A, B; C; q)_k &= (A, B; C; q)_{k-1} (I - q^{k-1}A - q^{2k-2}B) (I - q^k C)^{-1}, \quad k \geq 1. \end{aligned}$$

Define the function  ${}_2\eta_1$  by

$${}_2\eta_1 \left[ \begin{matrix} A, B \\ C \end{matrix}; q, z \right] = \sum_{n=0}^{\infty} (A, B; C; q)_n \frac{z^n}{(q; q)_n}. \quad (4.3)$$

Now (4.3) converges for  $|z| < 1$  in the norm of  $\text{Mat}_N(\mathbb{C})$ .

**Theorem 4.2.** Let  $A, B, C \in \text{Mat}_N(\mathbb{C})$  such that  $\sigma(C) \cap q^{-\mathbb{N} \setminus \{0\}} = \emptyset$ .

$$F(z) = F^0 {}_2\eta_1 \left[ \begin{matrix} A, B \\ C \end{matrix}; q, qz \right], \quad F^0 \in \mathbb{C}^N \text{ as row-vector}, \quad (4.4)$$

is a solution of the matrix-valued  $q$ -difference equation

$$F(q^{-1}z)(1-z) + F(z)(-I - C + zA) + F(qz)(C + zB) = 0, \quad (4.5)$$

with condition  $F(0) = F^0$ . Conversely, any analytic solution  $F$  around  $z = 0$  of (4.5) with initial condition  $F(0) \neq 0$  is of the form (4.4).

## 5 The $2 \times 2$ matrix-valued little $q$ -Jacobi polynomials

Based on [2, Theorem 4.2] we present a method to construct  $q$ -difference operators  $D$  and weight matrices  $W$  satisfying the symmetry equations (3.4). By applying this method we construct a matrix analogue of the scalar little  $q$ -Jacobi polynomials, and we give explicit expressions of the matrix-valued little  $q$ -Jacobi polynomials in terms of the scalar ones. We also show how these matrix polynomials can be written as a matrix-valued  $q$ -hypergeometric function, motivated by the work of Tirao [28].

### 5.1 The construction

Lemma 5.1 is an adapted version of [2, Theorem 4.2]. We omit the proof here because it is completely analogous to that in [2].

**Lemma 5.1.** Let  $s$  be a scalar function satisfying  $s(q^x) \neq 0$  for  $x \in \mathbb{N} \setminus \{0\}$ . Assume that  $F_1$  and  $F_{-1}$  are matrix-valued polynomials satisfying

$$F_1(q^{x-1})F_{-1}(q^x) = q|s(q^x)|^2 I, \quad \forall x \in \mathbb{N} \setminus \{0\}. \quad (5.1)$$

Let  $T$  be a solution of the  $q$ -difference equation

$$T(q^{x-1}) = s(q^x)^{-1} F_{-1}(q^x) T(q^x), \quad x \in \mathbb{N} \setminus \{0\}, \quad T(1) = I. \quad (5.2)$$

Then the  $q$ -weight defined by  $W(q^x) = T(q^x)T(q^x)^*$  satisfies the symmetry equation

$$F_1(q^{x-1})W(q^{x-1}) = qW(q^x)F_{-1}(q^x)^*, \quad x \in \mathbb{N} \setminus \{0\}.$$

We are now ready to introduce  $2 \times 2$  matrix-valued orthogonal polynomials related to a specific  $q$ -difference operator.

**Theorem 5.2.** Assume  $a$  and  $b$  satisfy  $0 < a < q^{-1}$  and  $b < q^{-1}$ . For  $v \in \mathbb{C}$  define matrices

$$K = \begin{pmatrix} 0 & v(1-q)(q^{-1}-a) \\ 0 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & v \\ 0 & 0 \end{pmatrix}, \quad A = e^{\log(q)M} = \begin{pmatrix} q & -v(1-q) \\ 0 & 1 \end{pmatrix}.$$



The  $q$ -difference operator given by

$$\begin{aligned} D &= E_{-1}F_{-1}(z) + E_0F_0(z) + E_1F_1(z), \\ F_{-1}(z) &= (z^{-1} - 1)A^{-1}, \quad F_0(z) = K - z^{-1}(A^{-1} + aA), \quad F_1(z) = (az^{-1} - abq)A, \end{aligned} \quad (5.3)$$

is symmetric with respect to the matrix-valued inner product (2.9) where the weight matrix is given by

$$W(q^x) = a^x \frac{(bq; q)_x}{(q; q)_x} A^x (A^*)^x, \quad (5.4)$$

Moreover, the monic orthogonal polynomials  $(P_n)_{n \geq 0}$  with respect to  $W$  satisfy  $DP_n = \Lambda_n P_n$ , with eigenvalues

$$\Lambda_n = \begin{pmatrix} -q^{-n-1} - abq^{n+2} & v(1-q)(abq^{n+1} - q^{-1-n} + q^{-1} - a) \\ 0 & -q^{-n} - abq^{n+1} \end{pmatrix}. \quad (5.5)$$

**Remark 5.3.** These polynomials are a matrix-valued analogues of the little  $q$ -Jacobi polynomials and for  $v \neq 0$  they cannot be reduced to scalars. This follows from  $W(q^0) = I$  and for  $v \neq 0$  and  $x, y \in \mathbb{N} \setminus \{0\}$  with  $x \neq y$  we have  $W(q^x)W(q^y) \neq W(q^y)W(q^x)$ , as can be checked by substituting (5.9) in (5.4).

*Proof.* To prove the theorem we first prove that  $D$  preserves polynomials and then apply Theorem 3.5 to see that the operator is symmetric with respect to  $W$ . We proceed in three steps.

*Step 1.  $D$  preserves polynomials and degree.*

As polynomials in  $z^{-1}$ ,  $\text{dgr}(F_i) = 1 < 2$ , so that condition 2 for  $k = 1, 2$  in Theorem 3.1 is satisfied. Because  $\text{dgr}(F_0 + F_1 + F_{-1}) = 0$  condition 2 is also satisfied for  $k = 0$  from which we conclude from Theorem 3.1 that  $D: \mathbb{P}_n[z] \rightarrow \mathbb{P}_n[z]$ .

*Step 2. Symmetry equations.*  $F_1(q^{x-1})W(q^{x-1}) = qW(q^x)F_{-1}(q^x)$  and  $F_0(q^x)W(q^x) = W(q^x)F_0(q^x)^*$ . Consider the function  $s(q^x) = q^{-x} \sqrt{a(1-q^x)(1-bq^x)}$ . The function  $s$  satisfies  $s(q^x) \neq 0$  because  $0 < a < q^{-1}$  and  $b < q^{-1}$ , and by a direct computation we see that with this choice of  $s$ , (5.1) is satisfied. The solution of (5.2) in this case is given by

$$T(q^x) = \sqrt{\frac{a^x (bq; q)_x}{(q; q)_x}} A^x,$$

By Lemma 5.1 we can conclude that the symmetry equation,  $F_1(q^{x-1})W(q^{x-1}) = qW(q^x)F_{-1}(q^x)$  holds for  $W(q^x) = T(q^x)T(q^x)^*$  and  $x = 1, 2, \dots$ .

Note that  $T(q^x)$  is invertible for all  $x \in \mathbb{N}$ . The symmetry equation (3.4) is equivalent to showing that the matrix  $T(q^x)^{-1}F_0(q^x)T(q^x)$  is Hermitian. Note that

$$T(q^x)^{-1}F_0(q^x)T(q^x) = A^{-x} (K - q^{-x} (A^{-1} + aA)) A^x = A^{-x} K A^x - q^{-x} (A^{-1} + aA). \quad (5.6)$$

Taking into account that  $A = e^{\log(q)M}$  and  $e^{-Nx} R e^{Nx} = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!} \text{ad}_N^k R$ , we see that (3.4) holds if and only if

$$A^{-x} K A^x - q^{-x} (A^{-1} + aA) = \sum_{k=0}^{\infty} \frac{(-1)^k \log(q)^k x^k}{k!} (\text{ad}_M^k K - A^{-1} - aA)$$

is Hermitian for all  $x \in \mathbb{N}$ , i.e. if and only if all coefficients  $\text{ad}_M^k K - (A^{-1} + aA)$  are Hermitian. For  $k = 0$

$$V = K - aA - A^{-1} = \begin{pmatrix} -aq - q^{-1} & 0 \\ 0 & -a - 1 \end{pmatrix}$$

is Hermitian. Direct computation shows that  $V$  satisfies

$$\text{ad}_M V = MV - VM = \begin{pmatrix} -aq - q^{-1} & 0 \\ 0 & -a - 1 \end{pmatrix} = V$$

i.e.,  $V$  is a fixed point of  $\text{ad}_M$ .

On the other hand since  $A = e^{-\log(q)M}$ , we have  $\text{ad}_M^k K = \text{ad}_M^k (K - aA - A) = \text{ad}_M^k V$ , so we get that  $T(q^x)^{-1}F_0(q^x)T(q^x) = Vq^{-x}$ , which is a diagonal real matrix, hence Hermitian.

*Step 3. Boundary conditions.*

Since  $F_{-1}(z) = (z^{-1} - 1)A^{-1}$ , the first boundary condition  $F_{-1}(1)W(1) = 0$  holds.

To check the last boundary condition  $q^x(F_1(q^x)W(q^x) - W(q^x)F_1(q^x)^*) \rightarrow 0$  as  $x \rightarrow \infty$ , we calculate

$$A^x = \begin{pmatrix} q^x & -v(1 - q^x) \\ 0 & 1 \end{pmatrix}, \quad x \in \mathbb{Z}. \quad (5.7)$$

Then we have

$$\begin{aligned} q^x(F_1(q^x)W(q^x) - W(q^x)F_1(q^x)^*) &= q^x a^x \frac{(bq; q)_x}{(q; q)_x} (aq^{-x} - abq) (A^{x+1}(A^*)^x - A^x(A^*)^{x+1}) \\ &= q^{2x-1} a^x \frac{(bq; q)_x}{(q; q)_x} (aq^{-x} - abq) A \begin{pmatrix} 0 & -v(1 - q) \\ \bar{v}(1 - q) & 0 \end{pmatrix} A^*. \end{aligned} \quad (5.8)$$

Because  $a < q^{-1}$ , (5.8) tends to 0 if  $x \rightarrow \infty$ . It is easy to see that the second boundary condition  $q^{2x}F_1(q^x)W(q^x) \rightarrow 0$  as  $x \rightarrow \infty$  also holds.

We have proved that  $D: \mathbb{P}[z] \rightarrow \mathbb{P}[z]$  is symmetric with respect to  $W$  and that  $D$  is an operator that preserves polynomials and does not raise the degree. We are under the hypothesis of Theorem 3.4, and we can conclude that the orthogonal polynomials with respect to  $W$  are common eigenfunctions of  $D$ .

Finally by equating the coefficients in the equation  $DP_n = \Lambda_n P_n$ , for the monic sequence of orthogonal polynomials with respect to  $W$ , we obtain the expression (5.5) for the eigenvalues  $\Lambda_n = -q^{-n}A^{-1} + K - abq^{n+1}A$ .  $\square$

**Proposition 5.4.** The moments associated to (2.9) with  $W$  as in (5.4) are given by

$$M_n = \langle z^n I, I \rangle_W = \begin{pmatrix} m_n(aq^2, b) & -v(m_n(a, b) - m_n(aq, b)) \\ -\bar{v}(m_n(a, b) - m_n(aq, b)) & m_n(a, b) \end{pmatrix},$$

where  $m_n(a, b)$  are given in (2.6).

*Proof.* Using (5.7) we can write

$$A^x(A^*)^x = \begin{pmatrix} q^{2x} & -v(1 - q^x) \\ -\bar{v}(1 - q^x) & 1 \end{pmatrix}. \quad (5.9)$$

Substituting (5.9) in  $\langle x^n I, I \rangle_W = \sum_{x=0}^{\infty} q^x a^x \frac{(bq; q)_x}{(q; q)_x} q^{nx} A^x(A^*)^x$ , we get the result using (2.6).  $\square$

## 5.2 Explicit expression of $P_n$

In this section we give an explicit expression of  $P_n$  in terms of scalar little  $q$ -Jacobi polynomials by decoupling the matrix-valued  $q$ -difference operator (5.3).

**Theorem 5.5.** The monic matrix-valued orthogonal polynomials with respect to the matrix-valued inner product (2.9) and weight matrix (5.4) are of the form

$$P_n(z) = N_n^{-1} \begin{pmatrix} \kappa_{11}^n p_n(z; aq^2, b; q) & \kappa_{12}^n p_{n+1}(z; a, b; q) + \kappa_{11}^n (1 - z)vp_n(z; aq^2, b; q) \\ \kappa_{12}^n p_{n-1}(z; aq^2, b; q) & \kappa_{22}^n p_n(z; a, b; q) + \kappa_{21}^n (1 - z)vp_{n-1}(z; aq^2, b; q) \end{pmatrix},$$

where

$$N_n = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \quad \alpha = \frac{1 - q^n + aq^{n+1} - abq^{2n+2}}{1 - abq^{2n+2}}v = v \left( 1 + q^n \frac{aq - 1}{1 - abq^{2n+2}} \right),$$

$p_n(z; a, b; q)$  are the little  $q$ -Jacobi polynomials (2.4) and with coefficients

$$\kappa_{11}^n = (-1)^n q^{\binom{n}{2}} \frac{(aq^3; q)_n}{(abq^{n+3}; q)_n}, \quad \kappa_{12}^n = (-1)^{n+1} v q^{\binom{n+1}{2}} \frac{(aq; q)_{n+1}}{(abq^{n+2}; q)_{n+1}}, \quad (5.10)$$

$$\kappa_{21}^n = (-1)^n \xi_n a \bar{v} q^{\binom{n}{2} - n + 2} \frac{(1 - q^n)(1 - bq^n)}{(1 - aq)(1 - aq^2)} \frac{(aq; q)_n}{(abq^{n+1}; q)_n}, \quad \kappa_{22}^n = (-1)^n \xi_n q^{\binom{n}{2}} \frac{(aq; q)_n}{(abq^{n+1}; q)_n},$$

where

$$\xi_n = \left( 1 + aq|v|^2 \frac{(1-q^n)(1-bq^n)}{(1-abq^{n+1})(1-aq^{n+1})} \right)^{-1}.$$

*Proof.* Let us define  $\tilde{P}_n = N_n P_n$  and notice that  $(\tilde{\Lambda}_n)_{n \geq 0} = (N_n \Lambda_n N_n^{-1})_{n \geq 0}$  are diagonal. Then  $D\tilde{P}_n = \tilde{\Lambda}_n \tilde{P}_n = \text{diag}(-q^{-n-1} - abq^{n+2}, -q^{-n} - abq^{n+1}) \tilde{P}_n$ . Now write using (5.7)

$$\begin{aligned} \tilde{P}_n(q^x) &= N_n P_n(q^x) = \begin{pmatrix} \tilde{p}_{11}^n(q^x) & \tilde{p}_{12}^n(q^x) \\ \tilde{p}_{21}^n(q^x) & \tilde{p}_{22}^n(q^x) \end{pmatrix}, \\ Q_n(q^x) &= \tilde{P}_n(q^x) A^x = \begin{pmatrix} q^x \tilde{p}_{11}^n(q^x) & \tilde{p}_{12}^n - (1-q^x)v\tilde{p}_{11}^n(q^x) \\ q^x \tilde{p}_{21}^n(q^x) & \tilde{p}_{22}^n - (1-q^x)v\tilde{p}_{21}^n(q^x) \end{pmatrix} = \begin{pmatrix} r_{11}^n(q^x) & r_{12}^n(q^x) \\ r_{21}^n(q^x) & r_{22}^n(q^x) \end{pmatrix}, \\ \implies & \begin{pmatrix} \tilde{p}_{11}^n(q^x) & \tilde{p}_{12}^n(q^x) \\ \tilde{p}_{21}^n(q^x) & \tilde{p}_{22}^n(q^x) \end{pmatrix} = \begin{pmatrix} q^{-x} r_{11}^n(q^x) & r_{12}^n(q^x) + v(q^{-x} - 1)r_{11}^n(q^x) \\ q^{-x} r_{21}^n(q^x) & r_{22}^n(q^x) + v(q^{-x} - 1)r_{21}^n(q^x) \end{pmatrix}, \end{aligned} \quad (5.11)$$

Taking into account (5.6) and the proof of step 2 in the proof of Theorem 5.2 we obtain

$$\begin{aligned} (D\tilde{P}_n)(q^x) A^x &= \tilde{P}_n(q^{x-1}) A^x (q^{-x} - 1) A^{-1} + \tilde{P}_n(q^x) A^x (A^{-x} K A^x - q^{-x} (A^{-1} + aA)) \\ &\quad + \tilde{P}_n(q^{x+1}) A^x (aq^{-x} - abq) A \\ &= (q^{-x} - 1) Q_n(q^{x-1}) + Q_n(q^x) q^{-x} \begin{pmatrix} -(q^{-1} + aq) & 0 \\ 0 & -(1+a) \end{pmatrix} + (aq^{-x} - abq) Q_n(q^{x+1}) \\ &= \text{diag}(-q^{-n-1} - abq^{n+2}, -q^{-n} - abq^{n+1}) Q_n(q^x). \end{aligned} \quad (5.12)$$

Since the eigenvalues as well as all the matrix coefficients involved are diagonal, (5.12) gives four uncoupled scalar-valued  $q$ -difference equations

$$\begin{aligned} r_{11}^n(q^{x-1})(q^{-x} - 1) - r_{11}^n(q^x) q^{-x} (q^{-1} + aq) + r_{11}^n(q^{x+1})(aq^{-x} - abq) &= -(q^{-n-1} + abq^{n+2}) r_{11}^n(q^x), \\ r_{21}^n(q^{x-1})(q^{-x} - 1) - r_{21}^n(q^x) q^{-x} (q^{-1} + aq) + r_{21}^n(q^{x+1})(aq^{-x} - abq) &= -(q^{-n} + abq^{n+1}) r_{21}^n(q^x), \\ r_{12}^n(q^{x-1})(q^{-x} - 1) - r_{12}^n(q^x) q^{-x} (1+a) + r_{12}^n(q^{x+1})(aq^{-x} - abq) &= -(q^{-n-1} + abq^{n+2}) r_{12}^n(q^x), \\ r_{22}^n(q^{x-1})(q^{-x} - 1) - r_{22}^n(q^x) q^{-x} (1+a) + r_{22}^n(q^{x+1})(aq^{-x} - abq) &= -(q^{-n} + abq^{n+1}) r_{22}^n(q^x), \end{aligned}$$

that can be solved using (2.7). Using the first column of the last equation of (5.11) we obtain recurrences for the polynomials  $\tilde{p}_{11}^n$  of degree  $n$  and  $\tilde{p}_{21}^n$  of degree  $n-1$ , which gives the first column in

$$\tilde{P}_n(z) = \begin{pmatrix} \kappa_{11}^n p_n(z; aq^2, b; q) & \kappa_{12}^n p_{n+1}(z; a, b; q) + \kappa_{11}^n (1-z) v p_n(z; aq^2, b; q) \\ \kappa_{21}^n p_{n-1}(z; aq^2, b; q) & \kappa_{22}^n p_n(z; a, b; q) + \kappa_{21}^n (1-z) v p_{n-1}(z; aq^2, b; q) \end{pmatrix}.$$

Since  $r_{12}^n$ , respectively  $r_{22}^n$ , are polynomial of degree  $n+1$ , respectively  $n$ , we find the explicit expression for  $r_{12}^n$  and  $r_{22}^n$  in terms of little  $q$ -Jacobi polynomials from (2.7), so that (5.11) gives the result.

From the expression of the leading coefficient of  $\tilde{P}_n$ ,  $N_n$ , the coefficients  $\kappa_{11}^n$  and  $\kappa_{12}^n$  are determined and we obtain (5.10). The expression of  $(N_n)_{22}$  gives the relation

$$\kappa_{22}^n = (-1)^n q^{\binom{n}{2}} \frac{(aq; q)_n}{(abq^{n+1}; q)_n} - \kappa_{21}^n v q^{n-1} \frac{(1-aq)(1-aq^2)}{(1-abq^{n+1})(1-aq^{n+1})}. \quad (5.13)$$

Now we use orthogonality to determine completely  $\kappa_{21}^n$  and  $\kappa_{22}^n$ ,

$$\begin{aligned} \langle \tilde{P}_m, \tilde{P}_n \rangle_W &= \sum_{x=0}^{\infty} (aq)^x \frac{(bq; q)_x}{(q; q)_x} \left( \tilde{P}_m(q^x) A^x \right) \left( \tilde{P}_n(q^x) A^x \right)^* \\ &= \sum_{x=0}^{\infty} (aq)^x \frac{(bq; q)_x}{(q; q)_x} Q_m(q^x) Q_n^*(q^x) = H_n \delta_{m,n}, \end{aligned} \quad (5.14)$$

where  $H_n$  is a strictly positive matrix and

$$Q_m(q^x) Q_n^*(q^x) = \begin{pmatrix} r_{11}^m(q^x) \overline{r_{11}^n(q^x)} + r_{12}^m(q^x) \overline{r_{12}^n(q^x)} & r_{11}^m(q^x) \overline{r_{21}^n(q^x)} + r_{12}^m(q^x) \overline{r_{22}^n(q^x)} \\ r_{21}^m(q^x) \overline{r_{11}^n(q^x)} + r_{22}^m(q^x) \overline{r_{12}^n(q^x)} & r_{21}^m(q^x) \overline{r_{21}^n(q^x)} + r_{22}^m(q^x) \overline{r_{22}^n(q^x)} \end{pmatrix}. \quad (5.15)$$

Combining (5.14) with entry (2, 1) of (5.15) we have

$$\begin{aligned}
(H_n)_{21}\delta_{m,n} &= \sum_{x=0}^{\infty} (aq)^x \frac{(bq; q)_x}{(q; q)_x} (r_{21}^m r_{11}^n + r_{22}^m r_{12}^n) = \langle r_{21}^m, r_{11}^n \rangle_{(a,b)} + \langle r_{22}^m, r_{12}^n \rangle_{(a,b)} \\
&= \langle \kappa_{21}^m z p_{m-1}(z; aq^2, b; q), \kappa_{11}^n z p_n(z; aq^2, b; q) \rangle_{(a,b)} + \langle \kappa_{22}^m p_m(z; a, b; q), \kappa_{12}^n p_{n+1}(z; a, b; q) \rangle_{(a,b)} \\
&= \kappa_{21}^m \kappa_{11}^n \langle p_{m-1}(z; aq^2; b; q), p_n(z; aq^2; b; q) \rangle_{(aq^2, b)} + \kappa_{22}^m \kappa_{12}^n \langle p_m(z; a, b; q), p_{n+1}(z; a, b; q) \rangle_{(a,b)}.
\end{aligned} \tag{5.16}$$

Taking  $(m, n) \mapsto (n, n-1)$  and using the orthogonality relations (2.5) gives a linear relation, which together with (5.13) determine  $\kappa_{22}^n$  and  $\kappa_{21}^n$  as given by (5.10). This completes the proof of the theorem.  $\square$

**Corollary 5.6.** For the matrix-valued polynomials  $(\tilde{P}_n)_{n \geq 0}$  as in the proof of Theorem 5.5 with diagonal eigenvalues we have

$$\langle \tilde{P}_m, \tilde{P}_n \rangle_W = H_n \delta_{m,n},$$

where  $H_n$  is the diagonal matrix

$$H_n = \text{diag}(|\kappa_{11}^n|^2 h_n(aq^2, b; q) + |\kappa_{12}^n|^2 h_n(a, b; q), |\kappa_{21}^n|^2 h_n(aq^2, b; q) + |\kappa_{22}^n|^2 h_n(a, b; q)),$$

and  $h_n(a, b; q)$  is defined in (2.5).

*Proof.* For  $m = n$  (5.16) shows  $(H_n)_{21} = 0$ . Similarly we compute  $(H_n)_{12} = 0$ . The entries (1, 1) and (2, 2) can be found by straightforward calculations similar to entries (1, 2) and (2, 1).  $\square$

### 5.3 The matrix-valued $q$ -hypergeometric equation

Write  $\tilde{P}_{i,n}$  for the  $i$ -th row of the matrix-valued polynomial  $\tilde{P}_n$ . The equation  $D\tilde{P}_n = \tilde{\Lambda}_n \tilde{P}_n$  can be written as two decoupled row equations

$$D\tilde{P}_{i,n}(z) = \tilde{P}_{i,n}(q^{-1}z)F_{-1}(z) + \tilde{P}_{i,n}(z)F_0(z) + \tilde{P}_{i,n}(qz)F_1(z) = \tilde{\lambda}_{i,n}\tilde{P}_{i,n}, \tag{5.17}$$

where  $i = 1, 2$ ,  $\tilde{\lambda}_{1,n} = -q^{-n-1} - abq^{n+2}$ ,  $\tilde{\lambda}_{2,n} = -q^{-n} - abq^{n+1}$  and  $\tilde{P}_{i,n}$  are the rows of the matrix polynomials  $\tilde{P}_n$ . We rewrite (5.17) by multiplying on the right by  $zA$

$$\tilde{P}_{i,n}(q^{-1}z)(1-z) + \tilde{P}_{i,n}(z)(z(K - \lambda_{i,n}I)A - (I + aA^2)) + \tilde{P}_{i,n}(qz)((a - abqz)A^2) = 0. \tag{5.18}$$

**Proposition 5.7.** The solution of (5.18) is

$$\tilde{P}_{i,n}(z) = \tilde{P}_{i,n}(0) {}_2\eta_1 \left[ \begin{matrix} KA - \tilde{\lambda}_{i,n}A, -abqA^2 \\ aA^2 \end{matrix}; q, qz \right]. \tag{5.19}$$

*Proof.* Since  $0 < a < q^{-1}$  we have  $\sigma(aA^2) \cap q^{-\mathbb{N} \setminus \{0\}} = \{a, aq^2\} \cap q^{-\mathbb{N} \setminus \{0\}} = \emptyset$ , so that we can apply Theorem 4.2 on (5.18) to get (5.19).  $\square$

Because  $\tilde{P}_{i,n}$  are not only analytic row-vector-valued, but actually polynomials, we find conditions on  $\tilde{P}_{i,n}(0)$  in order for the series (5.19) to terminate. Writing  $\tilde{P}_{i,n}(z) = \sum_{k=0}^{\infty} G_i^k z^k$  we have

$$G_i^k = \frac{q}{1 - q^k} G_i^{k-1} (I - q^{k-1}(K - \tilde{\lambda}_{i,n})A + abq^{2k-1}A^2)(I - aq^k A^2)^{-1}, \tag{5.20}$$

and we must have  $G_i^n \neq 0$  and  $G_i^{n+1} = 0$ . Therefore

$$(G_i^n)^t \in \ker((I - q^n(K - \tilde{\lambda}_{i,n})A + abq^{2n+1}A^2)^t). \tag{5.21}$$

The matrix is upper triangular and the (1, 1)-entry vanishes for  $\lambda_{1,n}$  and the (2, 2)-entry vanishes for  $\lambda_{2,n}$ . Using the definition of  $G_i^k$  we can determine  $G_2^0$  completely up to a scalar, since all the matrices in (5.20) are invertible for  $1 \leq k \leq n$ . Because  $\tilde{\lambda}_{1,n-1} = \tilde{\lambda}_{2,n}$  it is not possible to determine  $G_1^0$ , since the kernel in (5.21) is also non-trivial for  $n$  replaced by  $n-1$ . However adding the orthogonality relation (5.14), we can determine  $G_1^0$ . Therefore the coefficients of  $\tilde{P}_{i,n}(0)$  are completely determined up to a scalar by the fact that it is a orthogonal polynomial solution to (5.18).

Proposition 5.7 gives a way to write the orthogonal polynomials in a closed form. The matrix-valued basic hypergeometric series expression of the polynomials might be useful to generalize the polynomials to higher dimensions.

## 5.4 The three term recurrence relation and the Rodrigues formula

The first goal in this section is to find the three term recurrence relation for  $\tilde{P}_n$ ;

$$z\tilde{P}_n(z) = A_n\tilde{P}_{n+1}(z) + B_n\tilde{P}_n(z) + C_n\tilde{P}_{n-1}(z). \quad (5.22)$$

By comparing the leading coefficients of (5.22) we read off

$$A_n = N_n N_{n+1}^{-1} = \begin{pmatrix} 1 & -\frac{q^n(1-q)(1-aq)(1+abq^{2n+3})}{(abq^{2n+2}; q^2)_2} v \\ 0 & 1 \end{pmatrix}.$$

By a well-known argument

$$C_n = \langle \tilde{P}_n, \tilde{P}_n \rangle_W A_{n-1}^* \langle \tilde{P}_{n-1}, \tilde{P}_{n-1} \rangle_W^{-1}.$$

Therefore by Corollary 5.6 we can write  $C_n = H_n A_{n-1}^* H_{n-1}^{-1}$ . To find  $B_n$  we first remark that

$$\tilde{P}_n(0) = \begin{pmatrix} \kappa_{11}^n & \kappa_{12}^n + \kappa_{11}^n v \\ \kappa_{21}^n & \kappa_{22}^n + \kappa_{21}^n v \end{pmatrix}$$

and  $\det(\tilde{P}_n(0)) = \kappa_{11}^n \kappa_{22}^n - \kappa_{21}^n \kappa_{12}^n > 0$ , because both terms are positive by Theorem 5.5. If we plug in  $z = 0$  in (5.22) we find

$$B_n = -A_n \tilde{P}_{n+1}(0) (\tilde{P}_n(0))^{-1} - C_n \tilde{P}_{n-1}(0) (\tilde{P}_n(0))^{-1}.$$

Theorem 5.8 gives a Rodrigues formula for the matrix-valued little  $q$ -Jacobi polynomials.

**Theorem 5.8.** The expression

$$P_n(x) = q^{-x} D_q^n \left( \frac{a^x q^{(n+1)x} (bq; q)_x}{(q; q)_{x-n}} T(q^x) R(n) T(q^x)^* \right) W(q^x)^{-1}, \quad (5.23)$$

defines a sequence of matrix-valued orthogonal polynomials with respect to (2.9) with weight matrix (5.4), where

$$R(n) = \begin{pmatrix} \frac{(1-aq^{n+2})(1-abq^{n+3}) + av^2 q^2 (1-q^n)(1-bq^{n+1})}{1-abq^{2n+3}} & 0 \\ -\frac{(1-q^n)avq^2}{1-abq^{2n+3}} & 1-aq^{n+2} \end{pmatrix}.$$

*Proof.* Since the proof contains a couple of lengthy but direct calculations, we only give a sketch and leave the details to the reader.

To see that (5.23) defines a family of orthogonal polynomials two things need to be proved. (1) For all  $n \geq 0$ , (5.23) defines a matrix-valued polynomial of degree  $n$  with non-singular coefficients. (2) The polynomials defined by (5.23) are orthogonal with respect to the  $q$ -weight given by (5.4).

*First step.*

Let us write  $q^x W(q^x) = \rho(q^x) T(q^x) T(q^x)^*$ , where  $\rho(q^x)$  is the weight associated to the scalar little  $q$ -Jacobi polynomials with parameters  $a$  and  $b$ . Using the  $q$ -Leibniz rule (2.1) and that  $T(q^x) R(n) T(q^x)^*$  is a matrix-valued polynomial of degree 2 in  $q^x$  we can write

$$\begin{aligned} & D_q^n \left( \frac{a^x q^{(n+1)x} (bq; q)_x}{(q; q)_{x-n}} T(q^x) R(n) T(q^x)^* \right) W(q^x)^{-1} \\ &= D_q^n \left( \frac{a^x q^{(n+1)x} (bq; q)_x}{(q; q)_{x-n}} \right) (\rho(x))^{-1} T(q^x) R(n) T(q^x)^* (T(q^x) T(q^x)^*)^{-1} \\ &+ \begin{bmatrix} n \\ 1 \end{bmatrix}_q D_q^{n-1} \left( \frac{a^{x+1} q^{(n+1)(x+1)} (bq; q)_{x+1}}{(q; q)_{x-n+1}} \right) (\rho(x))^{-1} D_q (T(q^x) R(n) T(q^x)^*) (T(q^x) T(q^x)^*)^{-1} \\ &+ \begin{bmatrix} n \\ 2 \end{bmatrix}_q D_q^{n-2} \left( \frac{a^{x+2} q^{(n+1)(x+2)} (bq; q)_{x+2}}{(q; q)_{x-n+2}} \right) (\rho(x))^{-1} D_q^2 (T(q^x) R(n) T(q^x)^*) (T(q^x) T(q^x)^*)^{-1}. \end{aligned} \quad (5.24)$$

With the use of (2.2) and some lengthy calculations we can find polynomials  $t_n, r_n$  and  $s_n$  of degree  $n$  in  $q^x$  such that

$$\begin{aligned} t_n(q^x) &= D_q^n \left( \frac{a^x q^{(n+1)x} (bq; q)_x}{(q; q)_{x-n}} \right) \rho(q^x)^{-1}, \\ q^x r_n(q^x) &= D_q^{n-1} \left( \frac{a^{x+1} q^{(n+1)(x+1)} (bq; q)_{x+1}}{(q; q)_{x+1-n}} \right) \rho(q^x)^{-1}, \\ q^{2x} s_n(q^x) &= D_q^{n-2} \left( \frac{a^{x+2} q^{(n+1)(x+2)} (bq; q)_{x+2}}{(q; q)_{x+2-n}} \right) \rho(q^x)^{-1}. \end{aligned}$$

Let us now focus on the matrix part of (5.24). By using the  $q$ -Leibniz rule, (2.2) and the fact that  $T(q^{x+1}) = T(q^x)A$  we can write

$$\begin{aligned} D_q(T(q^x)R(n)T(q^x)^*)(T(q^x)^*)^{-1}T(q^x)^{-1} &= \frac{1}{(1-q)q^x} T(q^x)R_1(n)T(q^x)^{-1} \\ D_q^2(T(q^x)R(n)T(q^x)^*)(T(q^x)^*)^{-1}T(q^x)^{-1} &= \frac{1}{(1-q)^2 q^{2x}} T(q^x)R_2(n)T(q^x)^{-1}, \end{aligned}$$

where

$$R_1(n) = R(n) - AR(n)A^*, \quad R_2(n) = R(n) - (1 + q^{-1})AR(n)A^* + q^{-1}A^2R(n)(A^*)^2.$$

In (5.24) we can now write

$$\begin{aligned} (T(q^x)R(n)T(q^x)^*)(T(q^x)T(q^x)^*)^{-1} &= q^{-x}A_0(n) + B_0(n) + q^x C_0(n) \\ D_q(T(q^x)R(n)T(q^x)^*)(T(q^x)T(q^x)^*)^{-1} &= q^{-2x}A_1(n) + q^{-x}B_1(n) + C_1(n), \\ D_q^2(T(q^x)R(n)T(q^x)^*)(T(q^x)T(q^x)^*)^{-1} &= q^{-3x}A_2(n) + q^{-2x}B_2(n) + q^{-x}C_2(n). \end{aligned}$$

Tedious, although straightforward calculations, show that  $t_n^0 A_0(n) + r_n^0 A_1(n) + s_n^0 A_2(n) = 0$ ,  $t_n^n C_0(n) + r_n^n C_1(n) + s_n^n C_2(n) = 0$  and  $t_n^n B_0(n) + r_n^n B_1(n) + s_n^n B_2(n) + r_n^{n-1} C_1(n) + s_n^{n-1} C_2(n)$  is non-singular. This shows that (5.23) is a matrix-valued polynomial of degree  $n$  with non-singular leading coefficient. *Second step.*

To prove that the sequence of polynomials given by (5.23) is orthogonal, we must prove that for  $n \geq 1$  and  $0 \leq m < n$ ,  $\langle P_n, x^m I \rangle_W = 0$  holds.

In order to prove this we use Lemma 5.9, which will be proved later.

**Lemma 5.9.** For  $1 < k < n$ ,

$$D_q^{n-k} \left( \frac{a^{x+k-1} q^{(n+1)(x+k-1)} (bq; q)_{x+k-1} T(q^{x+k-1})R(n)T(q^{x+k-1})^*}{(q; q)_{x+k-n-1}} \right) D_q^k(q^{xm})$$

is zero for  $x = 0$  and  $x \rightarrow \infty$ .

By using the  $q$ -Leibniz rule (2.1), the formal identity given in (2.3) and Lemma 5.9, we get

$$\begin{aligned} \langle P_n, z^m \rangle_W &= \sum_{x=0}^{\infty} D_q^n \left( \frac{a^x q^{(n+1)x} (bq; q)_x T(q^x)R(n)T(q^x)^*}{(q; q)_{x-n}} \right) q^{xm} \\ &= D_q^{n-1} \left( \frac{a^x q^{(n+1)x} (bq; q)_x T(q^x)R(n)T(q^x)^*}{(q; q)_{x-n}} \right) D_q(q^{xm})|_{x=0}^{\infty} \\ &\quad + \sum_{x=0}^{\infty} D_q^{n-1} \left( \frac{a^{x+1} q^{(n+1)(x+1)} (bq; q)_{x+1} T(q^x)R(n)T(q^x)^*}{(q; q)_{x+1-n}} \right) D_q(q^{xm}) \\ &= \sum_{x=0}^{\infty} D_q^{n-1} \left( \frac{a^{x+1} q^{(n+1)(x+1)} (bq; q)_{x+1} T(q^x)R(n)T(q^x)^*}{(q; q)_{x+1-n}} \right) D_q(q^{xm}), \end{aligned}$$

By repeating this process we obtain

$$\langle P_n, z^m \rangle = \sum_{x=0}^{\infty} D_q^{n-m-1} \left( \frac{a^{x+m+1} q^{n(x+m+2)} (bq; q)_{x+m+1} T(q^x) R(n) T(q^x)^*}{(q; q)_{x+m+1-n}} \right) D_q^{m+1} (q^{x^m}) q^x = 0$$

because  $D_q^{m+1}(q^{x^m}) = 0$ . This gives the desired result.  $\square$

*Proof of Lemma 5.9.* To see that the first boundary condition at  $x = 0$  holds, we use the expression

$$\frac{1}{(q; q)_{x-n}} = \frac{(q^{x-n+1}; q)_n}{(q; q)_x},$$

which vanishes at  $x = 0$ . Any other quantity involved is bounded in  $x = 0$ , hence Lemma 5.9 holds in this case.

For  $x \rightarrow \infty$ , use that  $a < q^{-1}$  and so  $a^{x+k-1} q^{(n+1)(x+k-1)}$  tends to 0 when  $x$  tends to  $\infty$ . Since all the other quantities remain bounded when  $x$  tends to  $\infty$ , we obtain the desired result.  $\square$

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