Certified Connection Tableaux Proofs
for HOL Light and TPTP

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Abstract
In the recent years, the Metis prover based on ordered paramodulation and model elimination has replaced the earlier built-in methods for general-purpose proof automation in HOL 4 and Isabelle/HOL. In the annual CASC competition, the leanCoP system based on connection tableaux has however performed better than Metis. In this paper we show how the leanCoP’s core algorithm can be implemented inside HOL Light. leanCoP’s flagship feature, namely its minimalistic core, results in a very simple proof system. This plays a crucial role in extending the MESON proof reconstruction mechanism to connection tableaux proofs, providing an implementation of leanCoP that certifies its proofs. We discuss the differences between our direct implementation using an explicit Prolog stack, to the continuation passing implementation of MESON present in HOL Light and compare their performance on all core HOL Light goals. The resulting prover can be also used as a general purpose TPTP prover. We compare its performance against the resolution based Metis on TPTP and other interesting datasets.

Categories and Subject Descriptors CR-number [subcategory]: third-level

Keywords keyword1, keyword2

1. Introduction and Related Work
The leanCoP[19] automated theorem prover (ATP) has an unusually good ratio of performance to its implementation size. While its core algorithm fits on some twenty lines of Prolog, starting with CASC-21[27] it has regularly beaten Otter[17] and Metis[1] in the FOF division of the CASC ATP competitions. In 2014, leanCoP solved 158 FOF problems in CASC-J7[8] while Prover9 solved 95 problems. On the large-theory (chainy) division of the MPTP Challenge benchmark[1], leanCoP’s goal-directed calculus beats also SPASS 2.2[31], and its further AI-style strengthening by integrating into leanCoP’s simple core learning-based guidance trained on such larger ITP corpora is an interesting possibility[34].

Compact ATP calculi such as leanTAP[2], and MESON[16] have been used for some time in Isabelle[20,21] and HOLs[4] as general first-order automation tactics for discharging goals that are already simple enough. With the arrival of large-theory “hammer” linkups[2,10,13,22] between ITPs, state-of-the-art ATPs such as Vampire[12] and E[24], and premise selection methods[14], such tactics also became used as a relatively cheap method for reconstructing the (minimized) proofs found by the stronger ATPs. In particular, Hurd’s Metis has been adopted as the main proof reconstruction tool used by Isabelle’s Sledgehammer link up[6], while Harrison’s version of MESON could reconstruct in 1 second about 80% of the minimized proofs found by E in the first experiments with the HOL’Hammer link up[4].

Since HOL Light already contains a lot of the necessary infrastructure for Prolog-style proof search and its reconstruction, integrating leanCoP into HOL Light in a similar way as MESON should not be too costly, while it could lead to interesting strengthening of HOL Light’s first-order automation and proof reconstruction methods. In this paper we describe how this was done, resulting in an Ocaml implementation of leanCoP and a general leanCoP first-order tactic in HOL Light. We compare their performance with MESON, Metis and the Prolog version of leanCoP in several scenarios, showing quite significant improvements over MESON and Metis.

2. leanCoP and Its Calculus

leanCoP is an automated theorem prover for classical first-order logic based on a compact Prolog implementation of the clausal connection (tableaux) calculus[15,19] with several simple strategies that significantly reduce the search space on many problems. In contrast to saturation-based calculi used in most of the state-of-the-art ATPs (E, Vampire, etc.), connection calculi implement goal-oriented proof search. Their main inference step connects a literal on the current path to a new literal with the same predicate symbol but different polarity. The formal definition (derived from Otten[18]) of the particular connection calculus relevant in leanCoP is as follows:

\[\exists x \in C, \exists \sigma \in L, \neg \sigma(x) \wedge C \wedge L \Rightarrow \sigma(x) \wedge L\]

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\( \sigma(L_1) = \sigma(L_2) \). The rules of the calculus are applied in an analytic (i.e., bottom-up) way. The term substitution \( \sigma \) is applied to the whole derivation.

The connection calculus is correct and complete \([13]\) in the following sense: A first-order formula \( M \) in clausal form is valid iff there is a connection proof for "\(-: M, \{\}\)", i.e., a derivation for "\(-: M, \{\} \)" in the connection calculus so that all leaves are axioms. The Prolog predicate `prove/5` implements the axiom, the reduction and the extension rule of the basic connection calculus in leanCoP:

```prolog
\[
prove([Lit|Cla], Path, PathLim, Lem, Set) :-
\]
\[
\}
\]
\[
member(NegL, Path),
\]
\[
\}
prove(Cla, Path, PathLim, Lem, Set).
\]
```

The tuple "\(C, M, Path\)" in connection calculus is here represented as follows:

- \(C\) representing the open subgoal is a Prolog list `Cla`;
- the active path \(P\) is a Prolog list `Path`;
- \(M\) is written into Prolog’s database before the actual proof search starts in a such way that for every clause \(C' \in M\) and for every literal \(\neg Lit\) \(\in\) \(\text{Indexing}_L\) \(\cup\) \(\text{Grnd}\), \(\text{Indexing}_L\) same as \(\text{L modulo all its variables which are fresh (there is no twice or more occurrences in } \text{Indexing}_L\text{ everywhere in } \text{Indexing}_L\text{ and it is used for fast finding the right fact in database without affecting the logically correct } \text{L by standard Prolog unification without occurs check.}\)
- \(\text{Atoms are represented by Prolog atoms, negation by "-".}\)
- \(\text{The substitution } \sigma \text{ is stored implicitly by Prolog.}\)

\(\text{PathLim}\) is the current limit used for iterative deepening, \(\text{Lem}\) is the list of usable (previously derived) lemmas, \(\text{Set}\) is a list of options, and \text{Proof} is the resulting proof. This predicate succeeds (using iterative deepening) iff there is a connection proof for the tuple represented by `Cla`, the DNF representation of the problem stored in Prolog’s database using the `lit` predicate, and a `Path` with \(\text{[Path} \leftarrow \text{Pathlim} \mid \text{where Pathlim is the maximum size of the active Path. The predicate works as follows:} \ldots\)

Line 18 implements the axiom, line 4 calculates the complement of the first literal `Lit` in `Cla`, which is used as the principal literal for the next reduction or extension step. The reduction rule is implemented in lines 7, 8 and 17. At line 7 and 8 it is checked whether the active path `Path` contains a literal `NegLit` that unifies with the complement `NegLit` of the principal literal `Lit`. In this case the alternative lines after the semicolon are skipped and the proof search for the premise of the reduction rule is invoked in line 17. The extension rule is implemented in lines 10, 11, 14 and 17. Lines 10 and 11 are used to find a clause that contains the complement `NegLit` of the principal literal `Lit`. `Cla1` is the remaining set of literals of the selected clause and the new open subgoal of the left premise. The proof search for the left premise of the extension rule, in which the active path `Path` is extended by the principal literal `Lit`, is invoked in line 14, and if successful, we again continue on line 17.

Compared to standard tableaux or sequent calculi, connection calculi are not confluent. To achieve completeness, an extensive use of backtracking is required. leanCoP uses two simple incomplete strategies (namely options `scut` and `cut`) for restricting backtracking that significantly reduces the search space \([18]\) without affecting the ability to find proofs in most tested cases (see Section 5).

Another major problem in connection calculi is the integration of equality. The paramodulation method that is widely used in saturation-based ATPs is not complete for goal-oriented approach of connection calculi. Therefore equality in leanCoP and similar ATPs is usually managed by adding the axioms of equality (reflexivity, symmetry, transitivity and substitutivity). To obtain the clausal form, leanCoP uses its own implementation of classifier introducing definitions (the `def` option), which seems to perform better with the leanCoP’s core prover than other standard classifiers (TPTP2X using the option `tclausify: tptp, FLOTTER and E`) or direct transformation into clausal form (`nodef` option in leanCoP) \([18]\).

In the following subsections, we summarize several further methods used by leanCoP that improve its performance.

### 2.1 Iterative deepening

Prolog uses a simple incomplete depth-first search strategy to explore the search space. This kind of incompleteness would result in a calculus that hardly proves any formula. In order to obtain a complete proof search in the connection calculus, iterative deepening on the proof depth, i.e., the size of the active path, is performed. It is achieved by inserting the following lines into the code:

\[
(12) \text{Grnd}= \text{true ; length(Path, M), }
\]
\[
(13) \text{\leftarrow pathlim } \leftarrow \text{true ; assert(pathlim, fail),}
\]
and the whole prover runs in the following iterative sense starting from `PathLimit`= 1:

\[
\text{prove(PathLimit, Set) :-}
\]
\[
\text{retract(pathlim) } \leftarrow \text{PathLimit } \leftarrow \text{PathLimit+1,}
\]
\[
\text{prove(PathLimit+1, Set).}
\]

When the extension rule is applied and the new clause is not ground, i.e., it does not contain any variable, it is checked whether the size \(k\) of the active path exceeds the current path limit `PathLimit`.

\(^5\) Bad choice of connection might end up in dead end
(line 12). In this case the dynamic predicate pathlim/0 is written into the Prolog’s database (line 13) indicating the need to increase the path limit if the proof search with the current path limit fails. If the proof search fails and the predicate pathlim can be found in the database, then PathLim is increased by one and the proof search starts again.

2.2 Regularity Condition Optimization

Definition 2. A connection proof is regular iff no literal occurs more than once in the active path.

Since the active path corresponds to the set of literals in a branch in the connection tableau representation, a connection tableau proof is regular if in the current branch no literal occurs more than once. The regularity condition is integrated into the connection calculus in Figure 1 by imposing the following restriction on the reduction and extension rule: ∀L′ ∈ C ∪ {L1} : σ(L′) ∉ σ(Path)

Lemma 2.1. A formula M in clausal form described above is valid iff there is a regular connection proof for ~♯, M, {♯}

Regularity is correct, since it only imposes a restriction on the applicability of the reduction and extension rules. The completeness proof can be found in [18, 19].

The Prolog predicate \+ Goal succeeds only if Goal cannot be proven. In line 3 the corresponding Goal succeeds if the open subgoal \[Lit[/Cla\] contains a literal LitC that is syntactically identical (built-in predicate ==/2 in Prolog) with a literal LitP in the active path Path. The built-in predicate member/2 is used to enumerate all elements of a list.

2.3 Lemmata optimization

The set of lemmata is represented by the list Lem. The lemma rule is implemented by inserting the following lines:

\begin{align*}
(3) & \text{\textbf{\textbackslash +}} \text{ member}(\text{LitC}, [\text{Lit}|\text{Cla}\]), \\
& \text{member}(\text{LitP}, \text{Path}), \\
& \text{LitC}=\text{LitP},
\end{align*}

The Prolog predicate \textbf{\textbackslash +} Goal succeeds only if Goal cannot be proven. In line 3 the corresponding Goal succeeds if the open subgoal \[Lit[/Cla\] contains a literal LitC that is syntactically identical (built-in predicate ==/2 in Prolog) with a literal LitP in the active path Path. The built-in predicate member/2 is used to enumerate all elements of a list.

The regularity condition must be checked whenever the reduction, extension or lemma rule is applied. The substitution σ is not modified, i.e. the regularity condition is satisfied if the open subgoal does not contain a literal that is syntactically identical with a literal in the active path. This is implemented by inserting the following line into the code:

\begin{align*}
(3) & \text{\textbf{\textbackslash +}} \text{ member}(\text{LitC}, [\text{Lit}|\text{Cla}\]), \\
& \text{member}(\text{LitP}, \text{Path}), \\
& \text{LitC}=\text{LitP},
\end{align*}

2.4 Restricted backtracking

In Prolog the cut (!) is used to cut off alternative solutions when Prolog tries to prove a goal. The Prolog cut is a built-in predicate, which succeeds immediately when first encountered as a goal. Any attempt to re-satisfy the cut fails for the parent goal, i.e. other alternative choices are discarded that have been made from the point when the parent goal was invoked. Consequently, restricted backtracking is achieved by inserting a Prolog cut after the lemma, reduction, or extension rule is applied. It is implemented by inserting the following line into the code:

\begin{align*}
(16) & \text{member(cut,Set) } \rightarrow \text{ ! } \text{true },
\end{align*}

Restricted backtracking is switched on if the list Set contains the option cut. The restricted start step is used if the list Set includes the option scut. In this case only the first matching clause to starting ¬♯ literal is used. Restricted backtracking and restricted start step lead to an incomplete proof search. In order to regain completeness, these strategies can be switched off when the search reaches a certain path limit. If the list Set contains the option comp (Limit), where Limit is a natural number, the proof search is stopped and started again without using incomplete search strategies.

3. OCaml Implementation

In this section, we first discuss our implementation of leanCoP in OCaml and its integration in HOL Light: the transformation of the higher-order goal to first order and the proof reconstruction. After that we compare our implementation to Harrison’s implementation of MESON.

3.1 leanCoP in OCaml

Otten’s implementation of leanCoP uses the Prolog search, backtracking, and indexing mechanisms to implement the connection tableaux proof search. This is a variation of the general idea of using the “Prolog technology theorem prover” (PTTP) proposed by Stickel [25], in which connection tableaux takes a number of advantages from its similarity to Prolog.

In order to implement an equivalent program in a functional programming language, one needs to use either an explicit stack for keeping track of the current proof state (including the trail of variable bindings), or the continuation passing style. We choose to do the former, namely we add an explicit todo stack, subst (trail) and off (offset in the trail) arguments to the main prove function. The stack keeps a list of tuples that are given as arguments to the recursive invocations of prove, whose full OCaml declaration (takings the open subgoal as its last argument) then looks as follows:

\begin{align*}
\text{let rec prove off subst path limit lemmas todo = function} & \text{ begin... end } \\
& | () \rightarrow \text{... end} \\
& | ((\text{lit1 :: rest_clauses}) as \text{clause}) \rightarrow \text{... } ;
\end{align*}

The function performs the proof search to the given depth, and if a proof has not been found, it returns the unit. It takes special attention to traverse the tree in the same order as the Prolog version. In particular, when the global option "cut" (restricting backtracking) is off, it performs all the backtrackings explicitly, while if "cut" is on, the parts of backtracking avoided in Prolog are also omitted. When a proof is found, the exception 'Solved' is raised: no further search is performed and the function exits with this exception.

The OCaml version and the Prolog version (simplified and with symbols renamed for clarity of comparison) are displayed together in Fig 2. The algorithm proceeds as follows:

1. If nonempty, decompose the current open subgoal into the first literal lit and the rest cla.
2. Check for intersection between the current open subgoal and path.
3. Compute the negation of lit.

\footnote{Available online at\url{http://cl-informatik.uibk.ac.at/users/cek/cpp15}}
4. Check if lit is among the lemmas lem, if so try to prove cla.
   If cut is set, no other options are tried.
5. For each literal on the path, if neglit unifies with it, try to
   prove cla. If the unification succeeded and cut is set, no other
   options are tried.
6. For each clause in the matrix, try to find a literal that unify
   with neglit, and then try to prove the rest of the newly
   created subgoal and the rest of the current open subgoal. If
   the unification and the first proof succeeded and cut is set,
   no other options are tried.
7. When the current open subgoal is empty, the subproof is fin-
   ished (the axiom rule).

In Otten’s implementation, the behaviour of the program with
 cut set is enabled by the use of the Prolog cut (!). Implementing
 it in OCaml amounts to a different mechanism in each of the three
 cases. In point 4 in the enumeration above, given that a single
 lemma has been found, there is no need to check for other lemmas.
 Therefore a simple List.exists call is sufficient to emulate
 this behaviour in OCaml. No backtracking over other possible
 occurrences of the lemma is needed here, and it is not necessary
 to add in this case the literal again into the list of lemmas as is
done in the Prolog code (last-but-one line).

In point 5, multiple literals on the path may unify with differ-
ent substitutions. We therefore use a list fold mechanism which
changes the value whenever the unification is successful and cut
is set. In point 6, we need to change our behaviour in between two
 successive calls to prove. As the first call takes the arguments
to the second call on the stack, we additionally add a cut marker
on the stack and handle an exception that can be raised by the call
on the stack.

Whenever the clause becomes empty, all the tuples in the stack
list need to be processed. For each tuple, the first component is
the cut marker: if it is set, the Cut exception is raised with a
depth level. The exception is handled only at the appropriate level.

This directly corresponds to the semantics of the cut operator in
Prolog [26].

What remains to be implemented is efficient equality checking
and unification. Since we want to integrate our mechanism in HOL
Light, we reuse the first order logic representation used in the
implementation of HOL Light’s MESON procedure: the substitutions
are implemented as association lists, and applications of substitu-
tions are delayed until an actual equality check or a unification step.

3.2 leanCoP in HOL Light

In order to enable the OCaml version of leanCoP as a proof tactic
and procedure in HOL Light, we first need to transform a HOL
goal to a leanCoP problem and when a proof has been found
we replay the proof in higher-order logic. In order to transform a
problem in higher-order logic to first-order logic without equality,
we mostly reuse the steps of the transformation already used by
MESON, additionally ensuring that the conjecture is separated from
the axioms to preserve leanCoP’s goal-directed approach. The
transformation starts by assuming the duplicated copies of
polyomorphic theorems to match the existing assumptions. Next the
goal axioms → conjecture is transformed to (axioms ∧ δ) →
(conjecture ∧ δ) with the help of a special symbol, which we
define in HOL as: δ = T. Since the conjecture is refuted and the
problem is converted to CNF, the only positive occurrence of δ is
present in a singleton clause, and the negative occurrences of δ are
present in every clause originating from the conjecture. The CNF
clauses directly correspond to the DNF used by the Prolog version
of leanCoP. Since no steps distinguish between the positivity of
literals, the two can be used interchangeably in the proof procedure.
We start the FOL algorithm by trying to prove ¬δ.

Since the final leanCoP proof may include references to lemm-
as, the reconstruction cannot be performed the same way as it is
done in MESON. There, a tree structure is used for finished proofs.
Each subgoal either closes the branch (the literal is a negation of
a literal already present on the path) or is a branch extension with

---

Here is a simplified OCaml and Prolog code side by side. The
explicit tag argument and the computation of the resulting proof have
been omitted for clarity, and some symbols were renamed to better
correspond to each other. White-space and order of clauses has
been modified to exemplify corresponding parts of the two implementa-
tions. Function substeq checks equality under the current context of
variable bindings. Note that the last-but-one line in the Prolog code was
merged into each of the three cases in the OCaml code. See the
function lprove in file leanCoP.ml on our web page for the actual (non-simplified)
OCaml code.

```
4.
5.
6.
7. when the current open subgoal is empty, the subproof is finished (the axiom rule).

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done in MESON. There, a tree structure is used for finished proofs.
Each subgoal either closes the branch (the literal is a negation of
a literal already present on the path) or is a branch extension with
```
a (possibly empty) list of subgoals. In leanCoP, each subgoal can refer to previous subgoals, so the order of the subgoals becomes important. We therefore flatten the tree to a list, which needs to be traversed in a linear order to reconstruct the proof.

We define a type of proof steps, one for each proof step in the calculus. Each application of a lemma step or path step constructs a proof step with an appropriate first-order term. For an application of a tableaux extension step we use Harrison’s contrapositive mechanism: we store the reference to the actual transformed HOL theorem whose conclusion is a disjunction together with the number of the disjunct that got resolved.  

\[
\text{fun mexpand rules ancestors g cont (env,n,k)} =
\begin{cases}
\text{if } n < 0 \text{ then failwith } "\text{too deep}" \text{ else } & \\
\text{try tryfind } (\text{fun } a \to \text{cont (unify_literals env } (g,\text{negate } a),n,k)) & \\
\text{ancestors} & \\
\text{with Failure } \to \text{ tryfind } (\text{fun rule } \to ) & \\
\text{let } (\text{asm,c), k } = \text{renamerule } k \text{ rule in } & \\
\text{itlist } (\text{mexpand rules } (g:ancestors)) \text{ asm cont } & \\
\{ \text{unify_literals env } (g,c),n \text{ - length asm, k } \} & \\
\text{rules}; & \\
\text{let puremeson fm = } & \\
\text{let cls = simpnf(specialize(pnf fm )) in } & \\
\text{let rules = itlist } (\{0\} \times \text{contrapositives}) \text{ cls [] in } & \\
\text{deepen } (\text{fun } n \to \text{mexpand rules } []) \text{ False } (\text{fun } x \to x) & \\
\{ \text{undefined,n,0} \}; & \\
\text{n}\); & \\
\text{0}; & \\
\text{The toplevel puremeson function proceeds by turning the input formula into a clausal form, making contrapositives } (\text{rules}) & \\
\text{from the clauses, and then repeatedly calling the mexpand function with these rules using iterative deepening over the number of nodes permitted in the proof tree. } & \\
\text{The mexpand function takes as its arguments the rules, an } & \\
\text{(initially empty) list of goal ancestors, the goal } g \text{ to prove } & \\
\text{(initially False, which was also added to all-negative clauses when creating contrapositives), a continuation function cont for } & \\
\text{solving the rest of the subgoals (initially the identity), and a tuple consisting of the current trail } \text{env}, \text{the number } n \text{ of additional nodes in the proof tree permitted, and a counter } k \text{ for variable renaming. } & \\
\text{If the allowed node count is not negative, mexpand first tries to } & \\
\text{unify the current goal with a negated ancestor, followed by calling } & \\
\text{the current continuation (trying to solve the remaining goals) with the } & \\
\text{extended trail. If all such unification/continuation attempts fail } & \\
\text{(i.e., they throw Failure), an extension step is tried with all rules. } & \\
\text{This means that the head of a (renamed) rule is unified with the } & \\
\text{goal } g, \text{ the goal is appended to the ancestors and the } & \\
mexpand \text{ is called (again using list folding with the subsequently modified trail and continuation) for all the assumptions of the rule, decreasing the } & \\
\text{allowed node count for the recursive calls. } & \\
\text{The full HOL Light version of MESON additionally uses a smarter (divide-and-conquer) policy for the size limit, checks the } & \\
\text{goal for being already among the ancestors, caches continuations, } & \\
\text{and uses simple indexing. Below we enumerate some of the most } & \\
\text{important differences between the leanCoP algorithm and MESON } & \\
\text{and their implementations in HOL Light. Their practical effect is } & \\
\text{measured in Section } 3 & \\
\cdot \text{leanCoP computes and uses lemmas. The literals that correspond to closed branches are stored in a list. Each call to the } & \\
\text{main prove function additionally looks for the first literal in } & \\
\text{the list of lemmas. This can cost a linear number of equality checks if no parts of the proof are reused, but it saves computations } & \\
\text{if there are repetitions. } & \\
\cdot \text{Both algorithms use iterative deepening; however the depth and } & \\
\text{termination conditions are computed differently. } & \\
\cdot \text{MESON is implemented in the continuation-passing-style, so it } & \\
\text{can use as an additional optimization caching of the continuations. If any continuations are repeated (at the same depth level), the } & \\
\text{subproof is not retried. Otten’s leanCoP uses a direct Prolog implementation which cannot (without further tricks) do such repetition elimination. The implementation of leanCoP in OCaml behaves the same. } & \\
\cdot \text{leanCoP } 5 & \\
\text{may use the cut after the lemma step, path step or } & \\
\text{successful branch closing in the extension step. Implementing this } & \\
\text{behaviour in OCaml exactly requires multiple Cut exceptions } & \\
\text{– one for each depth of the proof. } & \\
\text{The checking for repetitions is done in a coarser way in } & \\
\text{leanCoP than in MESON, allowing leanCoP to skip some work } & \\
\text{done by MESON. } & \\
\text{The search is started differently in leanCoP and in MESON. } & \\
\text{leanCoP starts with a conjunct clause, which likely contributes } & \\
\text{to its relatively good performance on larger problems. } & \\
\end{cases}
\]
For HOL Light, we evaluate (with 5 second time limit) on two sets of problems. First, we look at 872 MESON-solved HOL Light goals that were made harder by removing splitting. In this scenario the tactic is applied to a subgoal of a proof, which is a bit similar to the Judgement-Day evaluation used for Isabelle/Sledgehammer, where the goals are however restricted to the solvable ones. Table 1 shows the results. Second, we evaluate on the top-level goals (with their dependencies already minimized) that have been solved with the HOL Light system [11], i.e., by using the strongest available ATPs. This set is important because tactics such as MESON, Metis and now also leanCoP can be tried as a first cheap method for reconstructing the proofs found by the stronger ATPs. The results are shown in Table 2. In both cases, the OCaml implementation of leanCoP performs best, improving the MESON’s performance in the first case by about 11%, and improving on Metis on the second set of problems by about 45%.

Table 1. Core HOL Light MESON calls without splitting (872 goals), 5sec per goal

<table>
<thead>
<tr>
<th>Prover</th>
<th>Theorem (%)</th>
<th>Unique</th>
</tr>
</thead>
<tbody>
<tr>
<td>mleancop-cut-comp</td>
<td>759 (87.04)</td>
<td>2</td>
</tr>
<tr>
<td>mleancop-nocut</td>
<td>759 (87.04)</td>
<td>2</td>
</tr>
<tr>
<td>mleancop-cut</td>
<td>752 (86.23)</td>
<td>0</td>
</tr>
<tr>
<td>mleancop-nc</td>
<td>751 (86.12)</td>
<td>0</td>
</tr>
<tr>
<td>meson-23</td>
<td>708 (81.19)</td>
<td>26</td>
</tr>
<tr>
<td>meson</td>
<td>683 (78.32)</td>
<td>4</td>
</tr>
<tr>
<td>any</td>
<td>832 (95.41)</td>
<td></td>
</tr>
</tbody>
</table>

Table 2. HOL Light dependencies (1556 goals, 5sec)

<table>
<thead>
<tr>
<th>Prover</th>
<th>Theorem (%)</th>
<th>Unique</th>
</tr>
</thead>
<tbody>
<tr>
<td>mleancop-cut-comp</td>
<td>1178 (75.70)</td>
<td>12</td>
</tr>
<tr>
<td>mleancop-nocut</td>
<td>1162 (74.67)</td>
<td>0</td>
</tr>
<tr>
<td>meson</td>
<td>1110 (71.33)</td>
<td>39</td>
</tr>
<tr>
<td>mleancop-nc</td>
<td>1085 (69.73)</td>
<td>0</td>
</tr>
<tr>
<td>mleancop-cut</td>
<td>1084 (69.66)</td>
<td>0</td>
</tr>
<tr>
<td>meson-23</td>
<td>814 (52.31)</td>
<td>16</td>
</tr>
<tr>
<td>any</td>
<td>1260 (80.97)</td>
<td></td>
</tr>
</tbody>
</table>

For the general set of problems from the TPTP library, and the large-theory problems extracted from Mizar [28], the first set of problems is important, however since these problems come from HOL Light itself, they are likely naturally biased towards MESON. The Mizar problems come from the two MPTP-based benchmarks: the MPTP Challenge and MPTP2078 [1]. These are large-theory problems coming from a different ITP, hence they do not introduce the implicit bias as the HOL Light problems, while coming from a more relevant application domain than the general TPTP problems.

Table 3 shows the results of the evaluation on all 7036 FOF problems, coming from TPTP 6.0.0, using 10 second time limit. Here the difference to Metis is not so significant, probably because Metis implements ordered paramodulation, which is useful for many TPTP problems containing equality. The improvement over MESON is about 17%. Table 4 and Table 5 show the results on the small (heuristically minimized) and large MPTP Challenge problems. The best version of the OCaml implementation of leanCoP improves by 54% on Metis and by 90% on MESON on the small problems, and by 88% on Metis and 100% on MESON on the large problems. Here the goal directness of leanCoP is probably the main factor.

Finally, to get a comparison also with the best ATPs on a larger ITP-oriented benchmark (using different hardware), we have done a 10s evaluation of several systems on the newer MPTP2078 benchmark (used in the 2012 CASC@Turing competition), see Table 6 and Table 7. The difference to Metis and MESON on the small problems is still quite significant (40% improvement over MESON), while on the large problems the goal-directedness again shows even more (about 90% improvement). While Vampire’s (version 2.6) SInE heuristic [6] helps a lot on the larger problems [29], the difference there between E (1.8) and our version of leanCoP is not so great as one could imagine given the several orders of magnitude difference in the size of their implementations.

Table 3. TPTP (7036 goals with at least one conjecture, 10sec)

<table>
<thead>
<tr>
<th>Prover</th>
<th>Theorem (%)</th>
<th>Unique</th>
</tr>
</thead>
<tbody>
<tr>
<td>mleancop-cut-conj</td>
<td>1669 (23.72)</td>
<td>73</td>
</tr>
<tr>
<td>plean-cut-conj</td>
<td>1648 (23.42)</td>
<td>21</td>
</tr>
<tr>
<td>mleancop-cut</td>
<td>1622 (23.05)</td>
<td>34</td>
</tr>
<tr>
<td>mleancop-nocut</td>
<td>1571 (22.32)</td>
<td>9</td>
</tr>
<tr>
<td>metis-23</td>
<td>1562 (22.20)</td>
<td>261</td>
</tr>
<tr>
<td>meson</td>
<td>1430 (20.32)</td>
<td>28</td>
</tr>
<tr>
<td>plean-nocut</td>
<td>1358 (19.30)</td>
<td>25</td>
</tr>
<tr>
<td>mleancop-nocut</td>
<td>1158 (16.45)</td>
<td>3</td>
</tr>
<tr>
<td>any</td>
<td>2433 (34.57)</td>
<td></td>
</tr>
</tbody>
</table>

Table 4. Bushy (small) MPTP Challenge problems (252 in total), 10sec)

<table>
<thead>
<tr>
<th>Prover</th>
<th>Theorem (%)</th>
<th>Unique</th>
</tr>
</thead>
<tbody>
<tr>
<td>plean-cut-conj</td>
<td>103 (40.8730)</td>
<td>2</td>
</tr>
<tr>
<td>plean-cut</td>
<td>99 (39.2857)</td>
<td>8</td>
</tr>
<tr>
<td>mleancop-cut-conj</td>
<td>91 (36.1111)</td>
<td>2</td>
</tr>
<tr>
<td>mleancop-cut</td>
<td>79 (31.3492)</td>
<td>0</td>
</tr>
<tr>
<td>mleancop-nocut</td>
<td>76 (30.1587)</td>
<td>0</td>
</tr>
<tr>
<td>plean-nocut</td>
<td>62 (24.6037)</td>
<td>1</td>
</tr>
<tr>
<td>metis-23</td>
<td>59 (23.4172)</td>
<td>3</td>
</tr>
<tr>
<td>meson-infer</td>
<td>48 (19.0476)</td>
<td>0</td>
</tr>
<tr>
<td>any</td>
<td>124 (49.2063)</td>
<td></td>
</tr>
</tbody>
</table>

Table 5. Chainy (large) MPTP Challenge problems (252 in total), 10sec)

<table>
<thead>
<tr>
<th>Prover</th>
<th>Theorem (%)</th>
<th>Unique</th>
</tr>
</thead>
<tbody>
<tr>
<td>plean-cut-conj</td>
<td>61 (24.2065)</td>
<td>5</td>
</tr>
<tr>
<td>mleancop-cut-conj</td>
<td>60 (23.8095)</td>
<td>9</td>
</tr>
<tr>
<td>plean-cut</td>
<td>57 (22.6105)</td>
<td>4</td>
</tr>
<tr>
<td>mleancop-nocut</td>
<td>47 (18.6507)</td>
<td>0</td>
</tr>
<tr>
<td>mleancop-cut</td>
<td>47 (18.6507)</td>
<td>0</td>
</tr>
<tr>
<td>metis-23</td>
<td>32 (12.6984)</td>
<td>3</td>
</tr>
<tr>
<td>meson-infer</td>
<td>30 (11.9047)</td>
<td>0</td>
</tr>
<tr>
<td>plean-nocut</td>
<td>26 (10.3174)</td>
<td>0</td>
</tr>
<tr>
<td>any</td>
<td>83 (32.9365)</td>
<td></td>
</tr>
</tbody>
</table>
Table 6. Bushy (small) MPTP2078 problems (2078 in total), 10sec

<table>
<thead>
<tr>
<th>Prover</th>
<th>Theorem (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vampire</td>
<td>1198 (57.65)</td>
</tr>
<tr>
<td>e18</td>
<td>1022 (49.18)</td>
</tr>
<tr>
<td>mleancop-cut-conj</td>
<td>613 (29.49)</td>
</tr>
<tr>
<td>pllean-cut-conj</td>
<td>597 (28.72)</td>
</tr>
<tr>
<td>metis-23</td>
<td>564 (27.14)</td>
</tr>
<tr>
<td>mleancop-cut</td>
<td>559 (26.90)</td>
</tr>
<tr>
<td>pllean-cut</td>
<td>544 (26.17)</td>
</tr>
<tr>
<td>pllean-comp7</td>
<td>539 (25.93)</td>
</tr>
<tr>
<td>mleancop-nocut</td>
<td>521 (25.07)</td>
</tr>
<tr>
<td>pllean-nc</td>
<td>454 (21.84)</td>
</tr>
<tr>
<td>meson-infer</td>
<td>438 (21.07)</td>
</tr>
<tr>
<td>any</td>
<td>1277 (61.45)</td>
</tr>
</tbody>
</table>

Table 7. Chainy (large) MPTP2078 problems (2078 in total), 10sec

<table>
<thead>
<tr>
<th>Prover</th>
<th>Theorem (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vampire</td>
<td>634 (30.51)</td>
</tr>
<tr>
<td>e18</td>
<td>317 (15.25)</td>
</tr>
<tr>
<td>mleancop-cut-conj</td>
<td>243 (11.69)</td>
</tr>
<tr>
<td>pllean-cut-conj</td>
<td>196 (9.43)</td>
</tr>
<tr>
<td>pllean-cut</td>
<td>170 (8.18)</td>
</tr>
<tr>
<td>pllean-comp7</td>
<td>159 (7.65)</td>
</tr>
<tr>
<td>mleancop-nocut</td>
<td>150 (7.21)</td>
</tr>
<tr>
<td>mleancop-cut</td>
<td>146 (7.02)</td>
</tr>
<tr>
<td>meson-infer</td>
<td>145 (6.97)</td>
</tr>
<tr>
<td>metis-23</td>
<td>138 (6.64)</td>
</tr>
<tr>
<td>pllean-nc</td>
<td>126 (6.06)</td>
</tr>
<tr>
<td>any</td>
<td>693 (33.34)</td>
</tr>
</tbody>
</table>

5. Conclusion

We have implemented an OCaml version of the leanCoP compact connection prover, and the reconstruction of its proofs inside HOL Light. This proof-reconstruction functionality can be also used to certify in HOL Light an arbitrary TPTP proof produced by leanCoP, thus turning leanCoP into one of the few ATPs whose proofs enjoy LCF-style verification in one of the safest LCF-based systems. The performance of the OCaml version on the benchmarks is comparable to the Prolog version, while it always outperforms Metis and MESON, sometimes very significantly on the relevant ITP-related benchmarks.

We provide a HOL Light interface that is identical to the one offered by MESON, namely we provide two tactics and a rule. LEANCOP_TAC and ASM_LEANCOP_TAC are given a list of helper theorems, and then try to solve the given goal together (or the given goal and assumptions, respectively). The LEANCOP rule, given a list of helper theorems acts as a conversion, i.e., given a term statement it tries to prove a theorem whose conclusion is identical to that of the term. The benchmarks show that these are likely the strongest single-step proof-reconstructing first-order tactics available today in any ITP system.

Acknowledgments

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References


